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Rough lacunary statistical convergence via ideals in $\mathcal{L}$-fuzzy normed spaces

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Abstract. The present article aims to explain and investigate the concept of rough lacunary statistical convergence of sequences through ideals within $\mathcal{L}$-fuzzy normed spaces. We introduce key concepts such as rough lacunary ideal statistical limit points and rough lacunary ideal cluster points in $\mathcal{L}$-fuzzy normed spaces. Additionally, we examine the theory of boundedness, convexity, and closedness related to rough lacunary ideal statistical limit points.

1. Introduction

The evolution of mathematical structures allow more flexibility and realism in modeling problems where uncertainty and imprecision are inherent. The study of $\mathcal{L}$-fuzzy normed spaces [10] brings novel aspects in the field of functional analysis, particularly in areas involving lattice-valued logic, fuzzy topology, and the general theory of fuzzy sets. The foundation of this theory builds upon Zadeh's [30] fuzzy sets introduced in 1965 which also led to subsequent developments in fuzzy analysis. While classical fuzzy sets assign membership degrees from the unit interval [0,1], many real-world situations demand more complex structures for expressing degrees of membership or uncertainty. The introduction of $\mathcal{L}$-fuzzy sets [13], where membership degrees are taken from a complete lattice $\mathcal{L}$, has provided this additional flexibility and generality. In $\mathcal{L}$-fuzzy normed spaces a complete lattice $\mathcal{L}$ is considered instead of the unit interval $[0, 1]$ which make it a profound generalization of fuzzy normed spaces. This generalisation offers a more flexible and comprehensive framework for examining problems where uncertainty and ordered structures co-exist. First, the lattice structure allows for more nuanced representations of uncertainty and imprecision. Second, the framework accommodates various special cases, including classical fuzzy normed spaces and intuitionistic fuzzy normed spaces, making it a unifying theory. Recent research has demonstrated the potential of $\mathcal{L}$-fuzzy normed spaces in various domains([15, 23, 28, 29]), particularly where classical fuzzy approaches prove insufficient. The lattice-valued approach provides additional degrees of freedom in

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modeling uncertainty, making it especially suitable for problems involving multiple criteria or hierarchical structures.

The convergence of sequences is an important part of real analysis and functional analysis. However, in many practical situations, which involves experimental data or approximate computations, sequences may not converge in the traditional sense due to irregular fluctuations or noise. To address such limitations, the concept of statistical convergence has been introduced as a generalized summability approach to usual convergence, which was established by Fast [11] in 1951. A sequence $y = \{y_i\}$ in $\mathbb{R}$ is statistical convergence to $y_0 \in \mathbb{R}$ if to each $\varepsilon > 0$, set $\{i \in \mathbb{N} : |y_i - y_0| \geq \varepsilon\}$ has natural density zero. The natural density of set K_n, characterized as $d(K_n) = \lim_{n\to\infty} \frac{|K_n|}{n}$, where $|K_n|$ indicates elements present in $K_n = \{m \in \mathbb{N} : m \leq n\}$. Statistical convergence's notion has been extended by many researchers for the development of new methodologies by considering various mathematical structures, like normed spaces, topological spaces, and in fuzzy and probabilistic settings.

Kostryko *et al*. [18] has proposed the idea of ideal convergence ($\mathcal{I}$-convergence) by generalizing the concept of statistical convergence in 2000. This convergence has inspired researchers to explore in this direction, including extending ideal convergence from sequences of real numbers to sequences of functions, broadening its scope for sequences, and various other related advancements [5, 6, 17, 19].

On the other hand, Phu [25] introduced the concept of rough convergence in 2001 for sequences for the finite-dimensional normed linear spaces, which was later extended to infinite-dimensional normed linear spaces [26]. In 2008, Aytar [4] further advanced the field by proposing rough statistical convergence as a generalized form of convergence. The concept of rough convergence has since inspired extensive research on various sequence spaces, including double sequences [20, 21], triple sequences [9], lacunary sequences [16], and ideals [22, 24], among others [1–3, 7, 8, 14].

The notion of statistical convergence in $\mathcal{L}$-fuzzy normed spaces was explored by Yapali [28]. Subsequently, Yapali *et al*. [29] expanded this concept by studying lacunary statistical convergence sequences, a refined version of statistical convergence, within the framework of $\mathcal{L}$-fuzzy normed spaces, which extend traditional fuzzy spaces. Their work introduced definitions of lacunary statistical Cauchy sequences, completeness in $\mathcal{L}$-fuzzy normed spaces, and relevant theorems. Aykut *et al*. [23] explored rough convergence and rough statistical convergence in $\mathcal{L}$-fuzzy normed spaces, introducing new concepts such as rough statistical limit points and rough statistical cluster points for sequences in these spaces. Additionally, Khan *et al*. [15] investigated ideal convergence within the context of $\mathcal{L}$-fuzzy normed spaces.

The goal of this article is to define and investigate the fundamental properties of rough ideal lacunary statistical convergence within the framework of $\mathcal{L}$-fuzzy normed spaces, offering a novel perspective for analyzing sequences in environments characterized by both imprecision and irregularity. Compared to previous studies, the results are presented in a more generalized form. The initial section will include basic definitions and concepts relevant to this study.

2. Preliminaries

The essential terminology and concept for ideals and $\mathcal{L}$-fuzzy normed spaces are reviewed in the present section.

Definition 2.1. *[18] Let $\Upsilon \neq \emptyset$. A family $\mathcal{I} \subset P(\Upsilon)$ of subsets from Υ is called an ideal in Υ provided, (i) $\emptyset \in \mathcal{I}$, (ii) if $A, B \in \mathcal{I}$ then $A \cup B \in \mathcal{I}$, and (iii) if $A \in \mathcal{I}, B \subset A$ then $B \in \mathcal{I}$.*
Here, $P(\Upsilon)$ represents the power set of Υ, which is the collection of all the subsets of Υ.

An admissible ideal in Υ is defined as non-trivial ideal ($\mathcal{I} \neq P(\Upsilon)$) that is a proper subset of $P(\Upsilon)$ and encompasses all the singleton sets. This ideal must be distinct from $P(\Upsilon)$ itself.
Consider $\mathcal{I}$ as non-trivial admissible ideal in the set of natural numbers throughout the article.

Definition 2.2. *[18] Let $\Upsilon \neq \emptyset$. A non-empty family $\mathcal{F} \subset P(\Upsilon)$ is said to be a filter in Υ provided, (i) $\emptyset \notin \mathcal{F}$, (ii) if $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$, and (iii) if $B \in \mathcal{F}, A \subset B$ then $B \in \mathcal{F}$.*

Every ideal $\mathcal{I}$ is connected with a filter $\mathcal{F}(\mathcal{I})$ defined by $\mathcal{F}(\mathcal{I}) = \{M \subseteq \Upsilon : M^c \in \mathcal{I}\}$ where $M^c = \Upsilon - M$.

Definition 2.3. *[18] A sequence $y = \{y_m\} \in \Upsilon$ is said to be ideal convergent ($\mathcal{I}$ -convergent) to ζ^* if for every $\varepsilon > 0$, we have $A(\varepsilon) = \{n \in \mathbb{N} : |y_m - \zeta^*| \geq \varepsilon\} \in \mathcal{I}$. Here, ζ^* is termed as $\mathcal{I}$-limit of the sequence $y = \{y_m\}$.*

Example 2.4. *The minimal ideal of $\mathbb{N}$ is $\mathcal{I}_0 = \{\emptyset\}$. A sequence $y = \{y_m\}$ is $\mathcal{I}_0$-convergent iff it is constant.*

Now we discuss the lacunary convergence which can detect the patterns in sequences that might be missed by other forms of convergence. It is particularly useful when dealing with the sequences that exhibit irregular behaviour or when studying a phenomena that occur over sparse sub-sequences. There are several types of lacunary convergence, each with its own characteristics and applications.

Definition 2.5. *[12] The sequence $\theta = \theta_u = \{i_u\}$ is said to be lacunary sequence if there is a sequence of integers satisfying $i_0 = 0$, $h_u = i_u - i_{u-1} \to \infty$ as $u \to \infty$. Let $z_u = i_u$ and θ_u is determined by $\varrho_u = \{i : i_{u-1} < i \leq i_u,\}$, $q_u = \frac{i_u}{i_u - 1}$.*

Definition 2.6. *[12] A sequence $y = \{y_m\}$ is said to be lacunary statistically convergent (St_θ-convergent) to a number ζ^* if for each $\varepsilon > 0$,*

$$\lim_{u\to\infty} \frac{1}{h_u} \left|\{m \in \varrho_u : |y_m - \zeta^*| \geq \varepsilon\}\right| = 0,$$

$$i.e \;\; d_\theta(\{m \in \mathbb{N} : |y_m - \zeta^*| \geq \varepsilon\}) = 0.$$

It is denoted by $y_m \xrightarrow{St_\theta} \zeta^*$ or $St_\theta - \lim_{m\to\infty} y_m = \zeta^*$.

Further, we would like to highlight the concept of rough convergence, which can be delineated as follows:

Definition 2.7. *[25]Let $(\Upsilon, \|.\|)$ be a normed linear space. A sequence $y = \{y_m\} \in \Upsilon$ is said to be rough convergent (r-convergent) to $\zeta^* \in \Upsilon$ for some non-negative real number r if there exists $m_0 \in \mathbb{N}$ for every $\varepsilon > 0$ such that $\|y_m - \xi\| < r + \varepsilon$ for all $m \geq m_0$.*
It is denoted by $y_m \xrightarrow{r} \zeta^*$ or $r - \lim_{m\to\infty} y_m = \zeta^*$, where r is called roughness degree of rough convergence of sequence $y = \{y_m\}$.

Definition 2.8. *[24] A sequence $y = \{y_m\}$ is said to be rough ideal convergent ($r - \mathcal{I}$-convergent) to ζ^*, where r is a non-negative real number, if for every $\varepsilon > 0$,*

$$\{m \in \mathbb{N} : |y_m - \zeta^*| \geq r + \varepsilon\} \in \mathcal{I}.$$

We will discuss $\mathcal{L}$-fuzzy normed spaces and convergence of sequences in these spaces through the various setups.

Definition 2.9. *[27] Consider a complete lattice $\mathcal{L} = (L, \preccurlyeq)$ and set A referred as universe. A L-fuzzy set on A is characterized by the mapping $X : A \to L$. The collection of all the L-sets for a given set A is represented by L^A.*
The intersection and union of two L-sets S_1 and S_2 on A can be depicted as follows,

$$(S_1 \cap S_2)(y) = S_1(y) \wedge S_2(y),$$

$$(S_1 \cup S_2)(y) = S_1(y) \vee S_2(y),$$

for each $y \in A$.
In a complete lattice $\mathcal{L}$, 0_L and 1_L represent the minimum and maximum elements, respectively. For the given lattice $(L, \preccurlyeq)$, we use symbols $\succeq$, $\succ$ and $\prec$ with their conventional meanings.

Definition 2.10. *[27] Let $\mathcal{L} = (L, \preccurlyeq)$ be a complete lattice. A function $\Omega : L \times L \to L$ is t-norm if satisfies the following for all $p, q, r, s \in L$:*

(i) $\Omega(p, q) = \Omega(q, p)$,

(ii) $\Omega(\Omega(p,q),r) = \Omega(p,\Omega(q,r))$,
(iii) $\Omega(p,1_L) = \Omega(1_L,p) = p$,
(iv) $p \preccurlyeq q$ *and* $r \preccurlyeq s$, *then* $\Omega(p,r) \preccurlyeq \Omega(q,s)$.

Definition 2.11. *[27] A function $\mathcal{N} : L \longrightarrow L$ is said to be a negator on $\mathcal{L} = (L, \preccurlyeq)$ if,*

(i) $\mathcal{N}(0_L) = 1_L$,
(ii) $\mathcal{N}(1_L) = 0_L$,
(iii) $p \preccurlyeq q$ *implies* $\mathcal{N}(q) \preccurlyeq \mathcal{N}(p)$ *for all* $p,q \in L$.
If in addition,
(iv) $\mathcal{N}(\mathcal{N}(p)) = p$ *for all* $p \in L$.
Then, $\mathcal{N}$ *is known as an involutive negator.*

Definition 2.12. *[27]Let $\mathcal{L} = (L, \preccurlyeq)$ be a complete lattice and Υ be a real vector space. Ω be a continuous t-norm on L and μ be a L-set on $\Upsilon \times (0,\infty)$ satisfying*

(i) $\mu(p,s) \succ 0_L$ *for all* $p \in \Upsilon$, $s > 0$,
(ii) $\mu(p,s) = 1_L$ *for all* $s > 0$ *if and only if* $p = \theta$,
(iii) $\mu(\alpha p,s) = \mu\left(p, \frac{s}{|\alpha|}\right)$ *for all* $p \in \Upsilon$, $s > 0$ *and* $\alpha \in R - \{\theta\}$,
(iv) $\Omega(\mu(p,s),\mu(q,t)) \preccurlyeq \mu(p+q,s+t)$, *for all* $p,q \in \Upsilon$ *and* $s,t > 0$,
(v) $\lim\limits_{s\to\infty} \mu(p,s) = 1_L$ *and* $\lim\limits_{s\to 0} \mu(p,s) = 0_L$ *for all* $p \in \Upsilon - \{\theta\}$,
(vi) *Functions* $f_p : (0,\infty) \to L$ *such that* $f_p(s) = \mu(p,s)$ *are continuous.*

Then, triplet (Υ, μ, Ω) is called a $\mathcal{L}$-fuzzy normed space or $\mathcal{L}$-normed space.

Example 2.13. *[23]Let $(\mathbb{R}, |\,.\,|)$ be a normed space, Ω be a continuous t-norm given by $\Omega(a_1,a_2) = a_1a_2$ for $a_1,a_2 \in L$, and μ be a L-fuzzy set on $\mathbb{R} \times (0,\infty)$ defined by $\mu(a,s) = \frac{s}{s+|a|}$ for all $s > 0$ and $a \in \mathbb{R}$. Then, triplet $(\mathbb{R},\mu,\Omega)$ is a $\mathcal{L}$-fuzzy normed space.*

Definition 2.14. *[27] Let (Υ, μ, Ω) be a $\mathcal{L}$-fuzzy normed space. A sequence $y = \{y_m\} \in \Upsilon$ is said to be convergent to $\zeta^* \in \Upsilon$ with respect to fuzzy norm μ if for every $\varepsilon \in L - \{0_L\}$ and $s \in R^+$, there exists m_0 such that $\mu(y_m - \zeta^*; s) \succ \mathcal{N}(\varepsilon)$ for all $m \geq m_0$.*

Definition 2.15. *[28] Let (Υ, μ, Ω) be a $\mathcal{L}$-fuzzy normed space. A sequence $y = \{y_m\} \in \Upsilon$ is said to be statistically convergent to $\zeta^* \in \Upsilon$ with respect to fuzzy norm μ if for every $\varepsilon \in L - \{0_L\}$ and $s \in R^+$, we have $d(\{m \in \mathbb{N} : \mu(y_m - \zeta^*; s) \not\succ \mathcal{N}(\varepsilon)\}) = 0$.*
It is denoted by $y_m \xrightarrow{St_{\mathcal{L}}} \zeta^*$ or $St_{\mathcal{L}} - \lim\limits_{m\to\infty} y_m = \zeta^*$.

Definition 2.16. *[29] Let (Υ, μ, Ω) be a $\mathcal{L}$-fuzzy normed space. A sequence $y = \{y_m\} \in \Upsilon$ is said to be lacunary statistically convergent to $\zeta^* \in \Upsilon$ with respect to fuzzy norm μ if for every $\varepsilon \in L - \{0_L\}$ and $s \in R^+$,*

$$\lim_{u\to\infty} \frac{1}{h_u} |\{m \in \varrho_u : \mu(y_m - \zeta^*; s) \not\succ \mathcal{N}(\varepsilon)\}| = 0,$$
$$i.e \;\; d_\theta(\{m \in \mathbb{N} : \mu(y_m - \zeta^*; s) \not\succ \mathcal{N}(\varepsilon)\}) = 0.$$

It is denoted by $y_m \xrightarrow{St_\theta^{\mathcal{L}}} \zeta^*$ or $St_\theta^{\mathcal{L}} - \lim\limits_{m\to\infty} y_m = \zeta^*$.

Example 2.17. *[29] Let $\Upsilon = \mathbb{R}$ and $\mathcal{L} = (P(\mathbb{R}^+), \subseteq)$, the lattice of all subsets of the set of non-negative real numbers. Ω be a continuous t-norm defined by $\Omega(a_1,a_2) = a_1a_2$ for $a_1,a_2 \in L$. Take a function $\mu : \mathbb{R} \times (0,\infty) \to P(\mathbb{R}^+)$ such that $\mu(a,s) = \{r \in \mathbb{R}^+ : |a| < \frac{s}{r}\}$. As fuzzy norm μ satisfies the above conditions, then triplet $(\mathbb{R},\mu,\Omega)$ becomes a $\mathcal{L}$-fuzzy normed space. Consider a sequence $y = \{y_m\}$ such that*

$$y_m = \begin{cases} 1 & m \in (i_u - \ln(h_u), i_u] \\ 0 & otherwise. \end{cases}, \; u \in \mathbb{N}$$

Then, $\lim\limits_{u\to\infty} \dfrac{\ln(h_u)}{h_u} = 0$.

Definition 2.18. *[23] Let (Υ, μ, Ω) be a $\mathcal{L}$-fuzzy normed space. A sequence $y = \{y_m\} \in \Upsilon$ is said to be rough convergent to $\zeta^* \in \Upsilon$ for some $r \geq 0$ with respect to fuzzy norm μ if for every $\varepsilon \in L - \{0_L\}$ and $s \in R^+$, there exists m_0 such that $\mu(y_m - \zeta^*; r + s) \succ \mathcal{N}(\varepsilon)$ for all $m \geq m_0$.*

Definition 2.19. *[23] Let (Υ, μ, Ω) be a $\mathcal{L}$-fuzzy normed space. A sequence $y = \{y_m\} \in \Upsilon$ is said to be rough statistically convergent to $\zeta^* \in \Upsilon$ for some $r \geq 0$ with respect to fuzzy norm μ if for every $\varepsilon \in L - \{0_L\}$ and $s \in R^+$, we have $d(\{m \in \mathbb{N} : \mu(y_m - \zeta^*; r + s) \nsucc \mathcal{N}(\varepsilon)\}) = 0$.*
It is denoted by $y_m \xrightarrow{r-St_{\mathcal{L}}} \zeta^*$ or $r - St_{\mathcal{L}} - \lim\limits_{m\to\infty} y_m = \zeta^*$.

Definition 2.20. *[15] Let (Υ, μ, Ω) be a $\mathcal{L}$-fuzzy normed space. A sequence $y = \{y_m\} \in \Upsilon$ is said to be ideal convergent to $\zeta^* \in \Upsilon$ with respect to fuzzy norm μ if for every $\varepsilon \in L - \{0_L\}$ and $s \in R^+$, we have $\{m \leq n : \mu(y_m - \zeta^*; s) \nsucc \mathcal{N}(\varepsilon)\} \in \mathcal{I}$.*
It is denoted by $y_m \xrightarrow{\mathcal{I}_{\mathcal{L}}} \zeta^*$ or $\mathcal{I}_{\mathcal{L}} - \lim\limits_{m\to\infty} y_m = \zeta^*$.

3. Main Results

In this section, we will discuss rough lacunary ideal statistical convergence in a $\mathcal{L}$-Fuzzy normed space and demonstrate some of its important properties.

Definition 3.1. *Let (Υ, μ, Ω) be a $\mathcal{L}$-fuzzy normed space. A sequence $y = \{y_m\} \in \Upsilon$ is said to be ideal statistically convergent to $\zeta^* \in \Upsilon$ with respect to fuzzy norm μ if for every $\varepsilon \in L - \{0_L\}$, $\delta > 0$ and $s \in R^+$,*

$$\left\{n \in \mathbb{N} : \frac{1}{n} \left|\{m \leq n : \mu(y_m - \zeta^*; s) \nsucc \mathcal{N}(\varepsilon)\}\right| > \delta\right\} \in \mathcal{I}.$$

It is denoted by $y_m \xrightarrow{\mathcal{I}-St_{\mathcal{L}}} \zeta^*$ or $\mathcal{I} - St_{\mathcal{L}} - \lim\limits_{m\to\infty} y_m = \zeta^*$.

Definition 3.2. *Let (Υ, μ, Ω) be a $\mathcal{L}$-fuzzy normed space. A sequence $y = \{y_m\} \in \Upsilon$ is said to be lacunary ideal statistically convergent to $\zeta^* \in \Upsilon$ with respect to fuzzy norm μ if for every $\varepsilon \in \mathcal{L} - \{0_{\mathcal{L}}\}$, $\delta > 0$ and $s \in R^+$,*

$$\left\{u \in \mathbb{N} : \frac{1}{h_u} \left|\{m \in \varrho_u : \mu(y_m - \zeta^*; s) \nsucc \mathcal{N}(\varepsilon)\}\right| \geq \delta\right\} \in \mathcal{I}.$$

It is denoted by $y_m \xrightarrow{\mathcal{I}-St^{\mathcal{L}}_{\theta}} \zeta^*$ or $\mathcal{I} - St^{\mathcal{L}}_{\theta} - \lim\limits_{m\to\infty} y_m = \zeta^*$.

Definition 3.3. *Let (Υ, μ, Ω) be a $\mathcal{L}$-fuzzy normed space and $r \geq 0$. A sequence $y = \{y_m\} \in \Upsilon$ is said to be rough ideal statistically convergent to $\zeta^* \in \Upsilon$ with respect to fuzzy norm μ if for every $\varepsilon \in \mathcal{L} - \{0_{\mathcal{L}}\}$, $\delta > 0$ and $s \in R^+$,*

$$\left\{n \in \mathbb{N} : \frac{1}{n} \left|\{m \leq n : \mu(y_m - \zeta^*; r + s) \nsucc \mathcal{N}(\varepsilon)\}\right| \geq \delta\right\} \in \mathcal{I}.$$

It is denoted by $y_m \xrightarrow{r-\mathcal{I}-St_{\mathcal{L}}} \zeta^*$ or $r - \mathcal{I} - St_{\mathcal{L}} - \lim\limits_{m\to\infty} y_m = \zeta^*$.

Remark 3.4. *For $r = 0$, notion of rough ideal statistical convergence agrees with ideal statistical convergence in a $\mathcal{L}$-fuzzy normed space.*

Definition 3.5. *Let (Υ, μ, Ω) be a $\mathcal{L}$-fuzzy normed space and $r \geq 0$. A sequence $y = \{y_m\} \in \Upsilon$ is said to be rough lacunary ideal statistical convergent to $\zeta^* \in \Upsilon$ with respect to fuzzy norm μ if for every $\varepsilon \in \mathcal{L} - \{0_{\mathcal{L}}\}$, $\delta > 0$ and $s \in R^+$,*

$$\left\{u \in \mathbb{N} : \frac{1}{h_u} \left|\{m \in \varrho_u : \mu(y_m - \zeta^*; r + s) \nsucc \mathcal{N}(\varepsilon)\}\right| \geq \delta\right\} \in \mathcal{I}.$$

It is denoted by $y_m \xrightarrow{r-\mathcal{I}-St^{\mathcal{L}}_{\theta}} \zeta^*$ or $r - \mathcal{I} - St^{\mathcal{L}}_{\theta} - \lim\limits_{m\to\infty} y_m = \zeta^*$.

In general, $r-\mathcal{I}-St_\theta^{\mathcal{L}}$ limit of a sequence may not be unique. Therefore, we consider the set $\mathcal{I}-St_\theta^{\mathcal{L}}-LIM_\mu^r(y)$ which represents the set of all rough lacunary ideal statistical limit points of a sequence $y=\{y_m\}$ with respect to fuzzy norm μ in $\mathcal{L}$-fuzzy normed space (Υ,μ,Ω) as

$$\mathcal{I}-St_\theta^{\mathcal{L}}-LIM_\mu^r(y)=\left\{\zeta^*: y_m \xrightarrow{r-\mathcal{I}-St_\theta^{\mathcal{L}}} \zeta^*\right\}.$$

Moreover, sequence $y=\{y_m\}$ is ζ^*-convergent, provided that $\mathcal{I}-St_\theta^{\mathcal{L}}-LIM_\mu^r(y)\neq\emptyset$. For a sequence $y=\{y_m\}$ of real numbers, the set of rough limit points is observed as

$$\mathcal{I}-St_\theta^{\mathcal{L}}-LIM_\mu^r(y)=\left\{\mathcal{I}-St_\theta^{\mathcal{L}}-\limsup_{m\to\infty} y_m - r,\ \mathcal{I}-St_\theta^{\mathcal{L}}-\liminf_{m\to\infty} y_m + r\right\}.$$

Example 3.6. *Let* $(\mathbb{R},|.|)$ *be a normed space,* Ω *be a continuous t-norm defined by* $\Omega(a_1,a_2)=a_1a_2$ *for* $a_1,a_2\in L$, *and* μ *be a L-fuzzy set on* $\mathbb{R}\times(0,\infty)$ *defined by* $\mu(a,s)=\frac{s}{s+|a|}$ *for all* $s>0$ *and* $a\in\mathbb{R}$. *As* μ *satisfies above conditions then triplet* $(\mathbb{R},\mu,\Omega)$ *is a* $\mathcal{L}$*-fuzzy normed space. Consider ideal in* $\mathbb{N}$ *which consists of sets whose natural density are zero. Take a sequence* $y=\{y_m\}$ *such that*

$$y_m=\begin{cases}(-1)^m & i_u-[\sqrt{h_u}]+1\le m<i_u\\ m & otherwise.\end{cases},\ u\in\mathbb{N}$$

Then,

$$\mathcal{I}-St_\theta^{\mathcal{L}}-LIM_\mu^r(y)=\begin{cases}\emptyset & r<1\\ [1-r,r-1] & otherwise.\end{cases}$$

Definition 3.7. *A sequence* $y=\{y_m\}$ *in a* $\mathcal{L}$*-fuzzy normed space* (Υ,μ,Ω) *is* $\mathcal{I}-St_\theta^{\mathcal{L}}$*-bounded if for every* $\varepsilon\in\mathcal{L}-\{0_{\mathcal{L}}\}$ *and* $\delta>0$, *there exists* $H>0$ *such that*

$$\left\{u\in\mathbb{N}:\frac{1}{h_u}\left|\{m\in\varrho_u:\mu(y_m;H)\not\succ\mathcal{N}(\varepsilon)\}\right|\ge\delta\right\}\in\mathcal{I}.$$

Next, we demonstrate some of the important properties of lacunary rough ideal statistical convergence in the $\mathcal{L}$-fuzzy normed spaces.

Theorem 3.8. *Let* (Υ,μ,Ω) *be* $\mathcal{L}$*-fuzzy normed space. A sequence* $y=\{y_m\}$ *in* Υ *is lacunary ideal statistically bounded or* $\mathcal{I}-St_\theta^{\mathcal{L}}$*-bounded if and only if* $\mathcal{I}-St_\theta^{\mathcal{L}}-LIM_\mu^r(y)\neq\emptyset$ *for some* $r>0$.

Proof. Necessary Part:
Consider a sequence $y=\{y_m\}$ which is lacunary ideal statistically bounded or $\mathcal{I}-St_\theta^{\mathcal{L}}$-bounded on $\mathcal{L}$-fuzzy normed space. Then, for every $\varepsilon\in\mathcal{L}-\{0_{\mathcal{L}}\}$, $\delta>0$ and $s\in R^+$

$$\left\{u\in\mathbb{N}:\frac{1}{h_u}\left|\{m\in\varrho_u:\mu(y_m;s)\not\succ\mathcal{N}(\varepsilon)\}\right|\ge\delta\right\}\in\mathcal{I}.$$

Since $\mathcal{I}$ is an admissible ideal, then $M=\mathbb{N}\setminus G\neq\emptyset$, where

$$G=\left\{u\in\mathbb{N}:\frac{1}{h_u}\left|\{m\in\varrho_u:\mu(y_m;s)\not\succ\mathcal{N}(\varepsilon)\}\right|\ge\delta\right\}.$$

Choose $m\in M$, then

$$\left\{u\in\mathbb{N}:\frac{1}{h_u}\left|\{m\in\varrho_u:\mu(y_m;s)\not\succ\mathcal{N}(\varepsilon)\}\right|\right\}<\delta$$

$$\Longrightarrow\ \left\{u\in\mathbb{N}:\frac{1}{h_u}\left|\{m\in\varrho_u:\mu(y_m;s)\succ\mathcal{N}(\varepsilon)\}\right|\right\}\ge1-\delta. \tag{1}$$

Let $\mathbb{K} = \{m \in \varrho_u : \mu(y_m; s) \succ \mathcal{N}(\varepsilon)\}$.
Also,

$$\begin{aligned}\mu(y_m; r+s) &\geq \Omega\left((\mu(0,r), \mu(y_m, s)\right)\\ &= \Omega\left(1_{\mathcal{L}}, \mu(y_m; s)\right)\\ &\succ \mathcal{N}(\varepsilon).\end{aligned}$$

Thus, $\mathbb{K} \subset \{m \in \varrho_u : \mu(y_m; r+s) \succ \mathcal{N}(\varepsilon)\}$.
Using (1), this implies that $1 - \delta \leq \frac{|\mathbb{K}|}{h_u} \leq \frac{1}{h_u}\left|\{m \in \varrho_u : \mu(y_m; r+s) \succ \mathcal{N}(\varepsilon)\}\right|$.
Therefore,

$$\frac{1}{h_u}\left|\{m \in \varrho_u : \mu(y_m; r+s) \nsucc \mathcal{N}(\varepsilon)\}\right| < 1 - (1-\delta) < \delta.$$

Which gives

$$\left\{u \in \mathbb{N} : \frac{1}{h_u}\left|\{m \in \varrho_u : \mu(y_m; r+s) \nsucc \mathcal{N}(\varepsilon)\}\right| \geq \delta\right\} \subset G \in \mathcal{I}.$$

Hence, $0 \in \mathcal{I} - St_\theta^{\mathcal{L}} - LIM_\mu^r(y)$. Therefore, $\mathcal{I} - St_\theta^{\mathcal{L}} - LIM_\mu^r(y) \neq \emptyset$.

Sufficient Part:
Let $\mathcal{I} - St_\theta^{\mathcal{L}} - LIM_\mu^r(y) \neq \emptyset$ for some $r > 0$. Then, there exists some $\xi \in \Upsilon$ such that $\xi \in \mathcal{I} - St_\theta^{\mathcal{L}} - LIM_\mu^r(y)$. Hence, for all $\varepsilon \in \mathcal{L} - \{0_{\mathcal{L}}\}$, $\delta > 0$ and $s \in R^+$, we have

$$\left\{u \in \mathbb{N} : \frac{1}{h_u}|\{m \in \varrho_u : \mu(y_m - \xi; s) \nsucc \mathcal{N}(\varepsilon)\}| \geq \delta\right\} \in \mathcal{I}.$$

Therefore, almost all y_m's are enclosed in some ball with a centre ξ in the $\mathcal{L}$-fuzzy norm space, which imply that $y = \{y_m\}$ is $\mathcal{I} - St_\theta^{\mathcal{L}}$-statistically bounded in a $\mathcal{L}$-fuzzy normed space. □

The algebraic characterization of $r - \mathcal{I} - St_\theta^{\mathcal{L}}$-convergence provides a useful tool for analyzing and understanding the behavior of sequences in $\mathcal{L}$-fuzzy normed space.
Next, we discuss a $r - \mathcal{I} - St_\theta^{\mathcal{L}}$-convergent sequence $y = \{y_m\} \subset \Upsilon$ can be characterized algebraically by the following results in $\mathcal{L}$-fuzzy normed spaces.

Theorem 3.9. *Let $\{y_m\}$ and $\{z_m\}$ be sequences in a $\mathcal{L}$-fuzzy normed space (Υ, μ, Ω) and $\mathcal{I}$ be an admissible ideal, then following results holds:*
(i) If $y_m \xrightarrow{r-\mathcal{I}-St_\theta^{\mathcal{L}}} y_0$ and $\alpha \in \mathbb{R}$ then $\alpha y_m \xrightarrow{r-\mathcal{I}-St_\theta^{\mathcal{L}}} \alpha y_0$.
(ii) If $y_m \xrightarrow{r-\mathcal{I}-St_\theta^{\mathcal{L}}} y_0$ and $z_m \xrightarrow{r-\mathcal{I}-St_\theta^{\mathcal{L}}} z_0$, then $(y_m + z_m) \xrightarrow{r-\mathcal{I}-St_\theta^{\mathcal{L}}} (y_0 + z_0)$.

Proof. (i) Assume $\alpha \neq 0$; otherwise, we would encounter with a zero sequence, then no proof left to establish. Since $y = \{y_m\}$ is a lacunary ideal statistically convergent to y_0 with respect to fuzzy norm μ *i.e* $y_m \xrightarrow{\mathcal{I}-St_\theta^{\mathcal{L}}} y_0$, then for $\varepsilon \in \mathcal{L} - \{0_{\mathcal{L}}\}$, $\delta > 0$ and $s \in R^+$,

$$B = \left\{u \in \mathbb{N} : \frac{1}{h_u}\left|\left\{m \in \varrho_u : \mu\left(y_m - y_0; \frac{r+s}{\alpha}\right) \nsucc \mathcal{N}(\varepsilon)\right\}\right| > \delta\right\} \in \mathcal{I}.$$

Since $\mathcal{I}$ is an admissible ideal so $G = \mathbb{N} \setminus B \neq \emptyset$, then for $m \in G^c$,

$$\frac{1}{h_u}\left|\left\{m \in \varrho_u : \mu\left(y_m - y_0; \frac{r+s}{\alpha}\right) \nsucc \mathcal{N}(\varepsilon)\right\}\right| < \delta.$$

$$\Rightarrow \frac{1}{h_u}\left|\left\{m \in \varrho_u : \mu\left(y_m - y_0; \frac{r+s}{\alpha}\right) \succ \mathcal{N}(\varepsilon)\right\}\right| \geq 1 - \delta.$$

Now,

$$\begin{aligned}\mu(\alpha y_m - \alpha y_0; r+s) &= \mu\left(y_m - y_0, \frac{r+s}{\alpha}\right)\\ &> \mathcal{N}(\varepsilon).\end{aligned}$$

This implies that

$$1-\delta \leq \frac{1}{h_u}\left|\{m \in \varrho_u : \mu\left(\alpha y_m - \alpha y_0; r+s\right) > \mathcal{N}(\varepsilon)\}\right|.$$

Therefore, $\frac{1}{h_u}\left|\{m \in \varrho_u : \mu\left(\alpha y_m - \alpha y_0; r+s\right) \not> \mathcal{N}(\varepsilon)\}\right| < 1-(1-\delta) = \delta$.
Then,

$$\left\{u \in \mathbb{N} : \frac{1}{h_u}\left|\{m \in \varrho_u : \mu\left(\alpha y_m - \alpha y_0; r+s\right) \not> \mathcal{N}(\varepsilon)\}\right| \geq \delta\right\} \subset B \in \mathcal{I},$$

which shows that $\alpha y_m \xrightarrow{r-\mathcal{I}-St_\theta^{\mathcal{L}}} \alpha y_0$.

(ii) If $y_m \xrightarrow{r-\mathcal{I}-St_\theta^{\mathcal{L}}} y_0$ and $z_m \xrightarrow{r-\mathcal{I}-St_\theta^{\mathcal{L}}} z_0$, then for $\varepsilon \in \mathcal{L} - \{0_{\mathcal{L}}\}$, $\delta > 0$ and $s \in R^+$, consider

$$A_0 = \left\{m \in \varrho_u : \mu\left(y_m - y_0; \frac{r+s}{2}\right) \not> \mathcal{N}(\varepsilon)\right\},$$

and

$$A_1 = \left\{m \in \varrho_u : \mu\left(z_m - z_0; \frac{r+s}{2}\right) \not> \mathcal{N}(\varepsilon)\right\}.$$

For $\delta > 0$, we have

$$\left\{m \in \varrho_u : \frac{1}{h_u}\left|\{m \in \varrho_u : m \in A_0 \cup A_1\}\right| \geq \delta\right\} \in \mathcal{I}.$$

Now, choose $0 < \delta_1 < 1$ such that $0 < 1-\delta_1 < \delta$. Then

$$A = \left\{u \in \mathbb{N} : \frac{1}{h_u}\left|\{m \in \varrho_u : m \in A_0 \cup A_1\}\right| \geq \delta_1\right\} \in \mathcal{I}.$$

Now, for $u \notin A$

$$\frac{1}{h_u}\left|\{m \in \varrho_u : m \in A_0 \cup A_1\}\right| < 1-\delta_1.$$

$$\frac{1}{h_u}\left|\{m \in \varrho_u : m \notin A_0 \cup A_1\}\right| \geq 1-(1-\delta_1) = \delta_1.$$

Thus, $\{m \in \varrho_u : m \notin A_0 \cup A_1\} \neq \emptyset$. Let $j \in (A_0 \cup A_1)^c = A_0^c \cap A_1^c$.
Then,

$$\begin{aligned}\mu\left(y_j + z_j - (y_0 + z_0); r+s\right) &\geq \Omega\left(\mu\left(y_{m_0} - y_0; \frac{r+s}{2}\right), \mu\left(z_{m_0} - z_0; \frac{r+s}{2}\right)\right)\\ &> \mathcal{N}(\varepsilon).\end{aligned}$$

This implies $A_0^c \cap A_1^c \subset {B_u}^c$ where $B_u = \{m \in \varrho_u : \mu\left((y_m + z_m) - (y_0 + z_0); r+s\right) \not> \mathcal{N}(\varepsilon)\}$.
For $u \notin A$,

$$\delta_1 \leq \frac{1}{h_u}\left|\{m \in \varrho_u : m \notin A_0 \cup A_1\}\right| \leq \frac{1}{h_u}\left|\{m \in \varrho_u : m \notin B_u\}\right|$$

or

$$\frac{1}{h_u}\left|\{m \in \varrho_u : m \in B_u\}\right| < 1-\delta_1 < \delta.$$

Hence, $A^c \subset \left\{u \in \mathbb{N} : \frac{1}{h_u}\left|\{m \in \varrho_u : m \in B_u\}\right| < \delta\right\}$.
Since $A^c \in \mathcal{F}(\mathcal{I})$, then $\left\{u \in \mathbb{N} : \frac{1}{h_u}\left|\{m \in \varrho_u : m \in B_u\}\right| < \delta\right\} \in \mathcal{F}(\mathcal{I})$, which implies that $\left\{u \in \mathbb{N} : \frac{1}{h_u}\left|\{m \in \varrho_u : m \in B_u\}\right| \geq \delta\right\} \in \mathcal{I}$. Therefore, $(y_m + z_m) \xrightarrow{r-\mathcal{I}-St_\theta^{\mathcal{L}}} (y_0 + z_0)$. $\square$

In the next theorem, we will show the set $\mathcal{I} - St_\theta^{\mathcal{L}} - LIM_\mu^r(y)$ is closed.

Theorem 3.10. *Let $y = \{y_m\}$ be a sequence and r is some non-negative real number. Then, the set of rough lacunary ideal limit points i.e $\mathcal{I} - St_\theta^{\mathcal{L}} - LIM_\mu^r(y)$ of a sequence $y = \{y_m\}$ in a $\mathcal{L}$-fuzzy normed space (Υ, μ, Ω) is a closed set.*

Proof. If $r = 0$ then the result is obvious as $\mathcal{I} - St_\theta^{\mathcal{L}} - LIM_\mu^r(y)$ is either empty set or singleton set.
Let $\mathcal{I} - St_\theta^{\mathcal{L}} - LIM_\mu^r(y) \neq \emptyset$ for some $r > 0$.
Let $x = \{x_m\}$ be a convergent sequence in (Υ, μ, Ω) with respect to fuzzy norm μ, which converges to $x_0 \in \Upsilon$.
For $\varepsilon \in \mathcal{L} - \{0_{\mathcal{L}}\}$ and $s \in R^+$ then there exists $m_0 \in \mathbb{N}$ such that

$$\mu\left(x_m - x_0; \frac{s}{2}\right) \succ \mathcal{N}(\varepsilon) \text{ for all } m \geq m_0.$$

Let us take $x_{m_1} \in \mathcal{I} - St_\theta^{\mathcal{L}} - LIM_\mu^r(y)$ and $\delta > 0$ then

$$A = \left\{u \in \mathbb{N} : \frac{1}{h_u}\left|\left\{m \in \varrho_u : \mu\left(y_m - x_{m_1}; r + \frac{s}{2}\right) \succ \mathcal{N}(\varepsilon)\right\}\right| \geq \delta\right\} \in \mathcal{I}.$$

Since $\mathcal{I}$ is an admissible ideal so $G = \mathbb{N} \setminus A \neq \emptyset$. Choose $m \in G$, then

$$\frac{1}{h_u}\left|\left\{m \in \varrho_u : \mu\left(y_m - x_{m_1}; r + \frac{s}{2}\right) \nsucc \mathcal{N}(\varepsilon)\right\}\right| < \delta$$

$$\Rightarrow \frac{1}{h_u}\left|\left\{m \in \varrho_u : \mu\left(y_m - x_{m_1}; r + \frac{s}{2}\right) \succ \mathcal{N}(\varepsilon)\right\}\right| \geq 1 - \delta.$$

Put $B_u = \left\{m \in \varrho_u : \mu\left(y_m - x_{m_1}; r + \frac{s}{2}\right) \succ \mathcal{N}(\varepsilon)\right\}$. Then, for $j \in B_u$, $j \geq m_0$, implies that

$$\begin{aligned}\mu(y_j - x_0; r + s) &\geq \Omega\left(\mu\left(y_j - x_{m_1}; r + \frac{s}{2}\right), \mu\left(x_{m_1} - x_0; \frac{s}{2}\right)\right)\\ &\succ \mathcal{N}(\varepsilon).\end{aligned}$$

Therefore,

$$j \in \{m \in \varrho_u : \mu(y_m - x_0; r + s) \succ \mathcal{N}(\varepsilon)\}.$$

Hence, $B_u \subset \{m \in \mathbb{N} : \mu(y_m - x_0; r + s) \succ \mathcal{N}(\varepsilon)\}$, which implies

$$1 - \delta \leq \frac{|B_u|}{h_u} \leq \frac{1}{h_u}\left|\{m \in \varrho_u : \mu(y_m - x_0; r + s) \succ \mathcal{N}(\varepsilon)\}\right|.$$

Therefore, $\frac{1}{h_u}\left|\{m \in \varrho_u : \mu(y_m - x_0; r + s) \nsucc \mathcal{N}(\varepsilon)\}\right| < \delta$.
Then,

$$\{u \in \mathbb{N} : \frac{1}{h_u}\left|\{m \in \varrho_u : \mu(y_m - x_0; r + s) \nsucc \mathcal{N}(\varepsilon)\}\right| \geq \delta\} \subset A \in \mathcal{I}.$$

Hence, $x_0 \in \mathcal{I} - st_{\mathcal{L}} - LIM_\mu^r(y)$ in (Υ, μ, Ω). $\square$

The convexity of the set $\mathcal{I} - st_{\mathcal{L}} - LIM_\mu^r(y)$ is explained in the next result.

Theorem 3.11. *Let $y = \{y_m\}$ be any sequence in a $\mathcal{L}$-fuzzy normed space (Υ, μ, Ω) then the set $\mathcal{I} - St_\theta^{\mathcal{L}} - LIM_\mu^r(y)$ is a convex set for some number $r > 0$.*

Proof. Let $\varphi_1, \varphi_2 \in \mathcal{I} - St_\theta^{\mathcal{L}} - LIM_\mu^r(y)$. For convexity we will show that $(1 - \omega)\varphi_1 + \omega\varphi_2 \in \mathcal{I} - St_\theta^{\mathcal{L}} - LIM_\mu^r(y)$ for any real number $\omega \in (0, 1)$.
As $\varphi_1, \varphi_2 \in \mathcal{I} - St_\theta^{\mathcal{L}} - LIM_\mu^r(y)$, then there exists $m \in \varrho_u$ for every $\varepsilon \in \mathcal{L} - \{0_{\mathcal{L}}\}$ and $s \in R^+$ such that

$$A_0 = \left\{m \in \varrho_u : \mu\left(y_m - \varphi_1; \frac{r + s}{2(1 - \omega)}\right) \nsucc \mathcal{N}(\varepsilon)\right\},$$

and

$$A_1 = \left\{ m \in \varrho_u : \mu\left(y_m - \varphi_2; \frac{r+s}{2\omega}\right) \nsucc \mathcal{N}(\varepsilon) \right\}.$$

For $\delta > 0$, we have

$$\left\{ m \in \varrho_u : \frac{1}{h_u} \left| \{ m \in \varrho_u : m \in A_0 \cup A_1 \} \right| \geq \delta \right\} \in \mathcal{I}.$$

Now, choose $0 < \delta_1 < 1$ such that $0 < 1 - \delta_1 < \delta$. Then

$$A = \left\{ u \in \mathbb{N} : \frac{1}{h_u} \left| \{ m \in \varrho_u : m \in A_0 \cup A_1 \} \right| \geq \delta_1 \right\} \in \mathcal{I}.$$

Now, for $u \notin A$

$$\frac{1}{h_u} \left| \{ m \in \varrho_u : m \in A_0 \cup A_1 \} \right| < 1 - \delta_1.$$

$$\frac{1}{h_u} \left| \{ m \in \varrho_u : m \notin A_0 \cup A_1 \} \right| \geq 1 - (1 - \delta_1) = \delta_1.$$

Thus, $\{ m \in \varrho_u : m \notin A_0 \cup A_1 \} \neq \emptyset$. Let $m_0 \in (A_0 \cup A_1)^c = A_0^c \cap A_1^c$.
Then,

$$\begin{aligned}
\mu\left(y_{m_0} - [(1-\omega)\varphi_1 + \omega\varphi_2]; r+s\right) &= \mu[(1-\omega)(y_{m_0} - \varphi_1) + \omega(y_{m_0} - \varphi_2); r+s] \\
&\geq \Omega\left(\mu\left((1-\omega)(y_{m_0} - \varphi_1); \frac{r+s}{2}\right), \mu\left(\omega(y_{m_0} - \varphi_2); \frac{r+s}{2}\right)\right) \\
&= \Omega\left(\mu\left(y_{m_0} - \varphi_1; \frac{r+\varepsilon}{2(1-\omega)}\right), \mu\left(y_{m_0} - \varphi_2; \frac{r+\varepsilon}{2\omega}\right)\right) \\
&\succ \mathcal{N}(\varepsilon).
\end{aligned}$$

This implies $A_0^c \cap A_1^c \subset {B_u}^c$ where $B_u = \{ m \in \varrho_u : \mu\left(y_{m_0} - [(1-\omega)\varphi_1 + \omega\varphi_2]; r+s\right) \nsucc \mathcal{N}(\varepsilon) \}$.
For $u \notin A$,

$$\delta_1 \leq \frac{1}{h_u} \left| \{ m \in \varrho_u : m \notin A_0 \cup A_1 \} \right| \leq \frac{1}{h_u} \left| \{ m \in \varrho_u : m \notin B_u \} \right|$$

or

$$\frac{1}{h_u} \left| \{ m \in \varrho_u : m \in B_u \} \right| < 1 - \delta_1 < \delta.$$

Hence, $A^c \subset \left\{ u \in \mathbb{N} : \frac{1}{h_u} \left| \{ m \in \varrho_u : m \in B_u \} \right| < \delta \right\}$.
Since $A^c \in \mathcal{F}(\mathcal{I})$, then $\left\{ u \in \mathbb{N} : \frac{1}{h_u} \left| \{ m \in \varrho_u : m \in B_u \} \right| < \delta \right\} \in \mathcal{F}(\mathcal{I})$, which implies that $\left\{ u \in \mathbb{N} : \frac{1}{h_u} \left| \{ m \in \varrho_u : m \in B_u \} \right| \geq \delta \right\} \in \mathcal{I}$. Therefore, $\mathcal{I} - St_\theta^{\mathcal{L}} - LIM_\mu^r(y)$ is a convex set. □

Theorem 3.12. *A sequence $y = \{y_m\}$ in a $\mathcal{L}$-fuzzy normed space (Υ, μ, Ω) is rough lacunary ideal statistically convergent to $\zeta^* \in \Upsilon$ with respect to the norm μ for some $r > 0$ if there exists a sequence $z = \{z_m\}$ in Υ such that $\mathcal{I} - St_\theta^{\mathcal{L}} - \lim_{m\to\infty} z_m = \zeta^*$ in Υ and for every $\varepsilon \in \mathcal{L} - \{0_\mathcal{L}\}$ and $s \in R^+$ have $\mu(y_m - z_m; r+s) \succ \mathcal{N}(\varepsilon)$ for all $m \in \mathbb{N}$.*

Proof. Consider $z = \{z_m\}$ be a sequence from Υ, which is a lacunary $\mathcal{I}$-statistically convergent to ζ^* and $\mu(y_m - z_m; r+s) \succ \mathcal{N}(\varepsilon)$.
For $\varepsilon \in \mathcal{L} - \{0_\mathcal{L}\}$, $\delta > 0$ and $s \in R^+$ the set

$$M = \left\{ u \in \mathbb{N} : \frac{1}{h_u} \left| \{ m \in \varrho_u : \mu(z_m - \zeta^*; s) \nsucc \mathcal{N}(\varepsilon) \} \right| \geq \delta \right\} \in \mathcal{I}.$$

Define

$$A_1 = \{ m \in \varrho_u : \mu(z_m - \zeta^*; s) \nsucc \mathcal{N}(\varepsilon) \}$$

$$A_2 = \{m \in \varrho_u : \mu(y_m - z_m; r) \not\succ \mathcal{N}(\varepsilon)\}.$$

For $\delta > 0$, we have

$$\left\{u \in \mathbb{N} : \frac{1}{h_u} \left|\{m \in \varrho_u : m \in A_1 \cup A_2\}\right| \geq \delta\right\} \in \mathcal{I}.$$

Now, choose $0 < \delta_1 < 1$ such that $0 < 1 - \delta_1 < \delta$. Then

$$A = \left\{u \in \mathbb{N} : \frac{1}{n} \left|\{m \in \varrho_u : m \in A_1 \cup A_2\}\right| \geq \delta_1\right\} \in \mathcal{I}.$$

Now, for $u \notin A$

$$\frac{1}{h_u} \left|\{m \in \varrho_u : m \in A_1 \cup A_2\}\right| < 1 - \delta_1$$

$$\frac{1}{h_u} \left|\{m \in \varrho_u : m \notin A_1 \cup A_2\}\right| \geq \delta_1.$$

Thus, $\{m \in \varrho_u : m \notin A_1 \cup A_2\} \neq \emptyset$.
Let $m \in (A_1 \cup A_2)^c = A_1^c \cap A_2^c$.
Then,

$$\begin{aligned} \mu\left(y_m - \zeta^*; r + s\right) &\geq \Omega\left(\mu(y_m - z_m; r), \mu(z_m - \zeta^*; s)\right) \\ &\succ \mathcal{N}(\varepsilon). \end{aligned}$$

Which gives $A_1^c \cap A_2^c \subset B_u^c$, where

$$B_u = \{m \in \varrho_u : \mu\left(y_m - \zeta^*; r + s\right) \not\succ \mathcal{N}(\varepsilon)\}.$$

So, for $u \notin A$,

$$\delta_1 \leq \frac{1}{h_u} \left|\{m \in \varrho_u : m \notin A_1 \cup A_2\}\right| \leq \frac{1}{h_u} \left|\{m \in \varrho_u : m \notin B_u\}\right|$$

or

$$\frac{1}{h_u} \left|\{m \in \varrho_u : m \in B_u\}\right| < 1 - \delta_1 < \delta.$$

Therefore, $A^c \subset \left\{u \in \mathbb{N} : \frac{1}{h_u} \left|m \in \varrho_u : m \in B_u\right| < \delta\right\}$. Since $A^c \in \mathcal{F}(\mathcal{I})$ then $\left\{u \in \mathbb{N} : \frac{1}{h_u} \left|m \in \varrho_u : m \in B_u\right| < \delta\right\} \in \mathcal{F}(\mathcal{I})$, i.e $\left\{u \in \mathbb{N} : \frac{1}{h_u} \left|m \in \varrho_u : m \in B_u\right| \geq \delta\right\} \in \mathcal{I}$. Hence, $y_m \xrightarrow{r-\mathcal{I}-St_\theta^{\mathcal{L}}} \zeta^*$. □

Theorem 3.13. *Let $y = \{y_m\}$ be a sequence in a $\mathcal{L}$-fuzzy normed space (Υ, μ, η) then there does not exist two elements $\beta_1, \beta_2 \in \mathcal{I} - St_\theta^{\mathcal{L}} - LIM_\mu^r(y)$ for $r > 0$ and $\varepsilon \in \mathcal{L} - \{0_\mathcal{L}\}$ such that $\mu(\beta_1 - \beta_2; cr) \succ \mathcal{N}(\varepsilon)$ for $c > 2$.*

Proof. We shall use contradiction to support our conclusion. Assume there exist two elements $\beta_1, \beta_2 \in \mathcal{I} - St_\theta^{\mathcal{L}} - LIM_\mu^r(y)$ such that

$$\mu(\beta_1 - \beta_2; cr) \not\succ \mathcal{N}(\varepsilon) \text{ for } c > 2. \tag{2}$$

As $\beta_1, \beta_2 \in \mathcal{I} - St_\theta^{\mathcal{L}} - LIM_\mu^r(y)$ then for every $\varepsilon \in \mathcal{L} - \{0_\mathcal{L}\}$ and $s \in R^+$. Define,

$$A_1 = \left\{m \in \varrho_u : \mu\left(y_m - \beta_1; r + \frac{s}{2}\right) \not\succ \mathcal{N}(\varepsilon)\right\}$$

$$A_2 = \left\{m \in \varrho_u : \mu\left(y_m - \beta_2; r + \frac{s}{2}\right) \not\succ \mathcal{N}(\varepsilon)\right\}.$$

Then,

$$\frac{1}{h_u} \left|\{m \in \varrho_u : m \in A_1 \cup A_2\}\right| \leq \frac{1}{h_u} \left|\{m \in \varrho_u : m \in A_1\}\right| + \frac{1}{h_u} \left|\{m \in \varrho_u : m \in A_2\}\right|.$$

By the property of $\mathcal{I}$-convergence,

$$\mathcal{I}-\lim_{u\to\infty}\frac{1}{h_u}\left|\{m\in\varrho_u : m\in A_1\cup A_2\}\right| \leq \mathcal{I}-\lim_{u\to\infty}\frac{1}{h_u}\left|\{m\in\varrho_u : m\in A_1\}\right| + \mathcal{I}-\lim_{u\to\infty}\frac{1}{h_u}\left|\{m\in\varrho_u : m\in A_2\}\right| = 0.$$

Thus,

$$\left\{u\in\mathbb{N} : \frac{1}{h_u}\left|\{m\in\varrho_u : m\in A_1\cup A_2|\right\} \geq \delta\right\} \in \mathcal{I}, \text{ for all } \delta > 0.$$

Now, choose $0<\delta_1<1$ such that $0<1-\delta_1<\delta$.
Let

$$K=\left\{u\in\mathbb{N} : \frac{1}{h_u}\left|\{m\in\varrho_u : m\in A_1\cup A_2\}\right| \geq \delta_1\right\} \in \mathcal{I}.$$

Now, for $u\notin K$

$$\frac{1}{h_u}\left|\{m\in\varrho_u : m\in A_1\cup A_2\}\right| < 1-\delta_1,$$

$$\frac{1}{h_u}\left|\{m\in\varrho_u : m\notin A_1\cup A_2\}\right| \geq 1-(1-\delta_1)=\delta_1.$$

This implies $\{m\in\varrho_u : m\notin A_1\cup A_2\}\neq\emptyset$. Then, for $m\in A_1^c\cap A_2^c$ the following holds

$$\mu\left(\beta_1-\beta_2; 2r+s\right) \geq \Omega\left(\mu\left(y_m-\beta_2 : r+\frac{s}{2}\right), \mu\left(y_m-\beta_1; r+\frac{s}{2}\right)\right) \succ \mathcal{N}(\varepsilon).$$

Hence,

$$\mu\left(\beta_1-\beta_2; 2r+s\right) \succ \mathcal{N}(\varepsilon). \tag{3}$$

Then, from (3) implies that $\mu\left(\beta_1-\beta_2; cr\right) \succ \mathcal{N}(\varepsilon)$ for $c>2$. which is contradiction to (2). Therefore, there does not exist two elements such that $\mu(\beta_1-\beta_2; cr) \nsucc \mathcal{N}(\varepsilon)$ for $c>2$. □

Definition 3.14. *Let (Υ,μ,Ω) be a $\mathcal{L}$-fuzzy normed space. Then $\gamma\in\Upsilon$ is said to be lacunary rough $\mathcal{I}$-statistical cluster point of the sequence $y=\{y_m\}\in\Upsilon$ with respect to fuzzy norm μ for some $r>0$ if for every $\varepsilon\in\mathcal{L}-\{0_{\mathcal{L}}\}$, $\delta>0$ and $s\in R^+$*

$$\left\{u\in\mathbb{N} : \frac{1}{h_u}\left|\{m\in\varrho_u : \mu(y_m-\gamma; r+s)\nsucc\mathcal{N}(\varepsilon)\}\right| < \delta\right\} \notin \mathcal{I}.$$

In this case, γ is known as lacunary rough ideal statistically cluster point of a sequence $y=\{y_m\}$.

Let $\Gamma^{r-\mathcal{I}}_{St^{\mathcal{L}}_{\theta}(\mu)}(y)$ denotes the set of all lacunary rough ideal statistically cluster points of a sequence $y=\{y_m\}$ with respect to fuzzy norm μ of a sequence $y=\{y_m\}$ in a $\mathcal{L}$-fuzzy normed space (Υ,μ,Ω). If $r=0$, then we get lacunary ideal statistically cluster point with respect to fuzzy norm μ in a $\mathcal{L}$-fuzzy normed space (Υ,μ,Ω) *i.e* $\Gamma^{r-\mathcal{I}}_{St^{\mathcal{L}}_{\theta}(\mu)}(y)=\Gamma^{\mathcal{I}}_{St^{\mathcal{L}}_{\theta}(\mu)}(y)$.

Theorem 3.15. *Let (Υ,μ,Ω) be a $\mathcal{L}$-fuzzy normed space. Then $\Gamma^{r-\mathcal{I}}_{St^{\mathcal{L}}_{\theta}(\mu)}(x)$, the set of all r-$\mathcal{I}$-statistical-cluster points with respect to fuzzy norm μ of a sequence $x=\{x_m\}$ is closed for some $r>0$.*

Proof. If $\Gamma^{r-\mathcal{I}}_{St^{\mathcal{L}}_{\theta}(\mu)}(x)=\emptyset$, then we have nothing to prove.

Assume, $\Gamma^{r-\mathcal{I}}_{St^{\mathcal{L}}_{\theta}(\mu)}(x)\neq\emptyset$. Take a sequence $y=\{y_m\}\subseteq\Gamma^{r-\mathcal{I}}_{St^{\mathcal{L}}_{\theta}(\mu)}(x)$ such that $y_m\xrightarrow{\mu}y_0$. It is sufficient to show that $y_0\in\Gamma^{r-\mathcal{I}}_{st(\mu)}(x)$.

As $y_m \xrightarrow{\mu} y_0$, then for every $\varepsilon \in \mathcal{L} - \{0_\mathcal{L}\}$ and $s \in R^+$, there exists $m_\varepsilon \in \mathbb{N}$ such that $\mu\left(y_m - y_0; \frac{s}{2}\right) \succ \mathcal{N}(\varepsilon)$ for $m \geq m_\varepsilon$.

Let us choose some $m_0 \in \mathbb{N}$ such that $m_0 \geq m_\varepsilon$. Then, we have $\mu\left(y_{m_0} - y_0; \frac{s}{2}\right) \succ \mathcal{N}(\varepsilon)$.

Again as $y = \{y_m\} \subseteq \Gamma^{r-\mathcal{I}}_{st(\mu)}(x)$ which implies that $y_{m_0} \in \Gamma^{r-\mathcal{I}}_{st(\mu)}(x)$.

$$\implies \left\{u \in \mathbb{N} : \frac{1}{h_u}\left|\left\{m \in \varrho_u : \mu(x_m - y_{m_0}; r + \frac{s}{2}) \nsucc \mathcal{N}(\varepsilon)\right\}\right| < \delta\right\} \notin \mathcal{I}. \tag{4}$$

Consider $G = \left\{m \in \varrho_u : \mu(x_m - y_{m_0}; r + \frac{s}{2}) \nsucc \mathcal{N}(\varepsilon)\right\}$. Choose $j \in G^c$ such that $\mu(x_j - y_{m_0}; r + \frac{s}{2}) \succ \mathcal{N}(\varepsilon)$.
Now,

$$\begin{aligned}\mu\left(x_j - y_0; r + s\right) &\geq \Omega\left(\mu\left(x_j - y_{m_0}; r + \frac{s}{2}\right), \mu\left(y_{m_0} - y_0; r + \frac{s}{2}\right)\right)\\ &\succ \mathcal{N}(\varepsilon).\end{aligned}$$

Hence,
$\left\{m \in \varrho_u : \mu(x_m - y_{m_0}; r + \frac{s}{2}) \succ \mathcal{N}(\varepsilon)\right\} \subseteq \{m \in \varrho_u : \mu(x_m - y_0; r + s) \succ \mathcal{N}(\varepsilon)\}$.
$\implies \{m \in \varrho_u : \mu(x_m - y_0; r + s) \nsucc \mathcal{N}(\varepsilon)\} \subseteq \left\{m \leq n : \mu(x_m - y_{m_0}; r + \frac{s}{2}) \nsucc \mathcal{N}(\varepsilon)\right\}$.
Thus,

$$\begin{aligned}&\{u \in \mathbb{N} : \frac{1}{h_u}\left|\{m \in \varrho_u : \mu(x_m - y_0; r + s) \nsucc \mathcal{N}(\varepsilon)\}\right| < \delta\}\\ &\qquad \subseteq \left\{u \in \mathbb{N} : \frac{1}{h_u}\left|\left\{m \in \varrho_u : \mu(x_m - y_{m_0}; r + \frac{s}{2}) \nsucc \mathcal{N}(\varepsilon)\right\}\right| < \delta\right\}.\end{aligned}$$

From (4), we have

$$\left\{u \in \mathbb{N} : \frac{1}{h_u}\left|\{m \in \varrho_u : \mu(x_m - y_0; r + \varepsilon) \nsucc \mathcal{N}(\varepsilon)\}\right| < \delta\right\} \notin \mathcal{I}.$$

Therefore, $y_0 \in \Gamma^{r(\mathcal{I})}_{st(\mu)}(x)$. Hence, the result proved. □

Theorem 3.16. *Let $y = \{y_m\}$ be a sequence in a $\mathcal{L}$-fuzzy normed space (Υ, μ, Ω), which is lacunary ideal statistically convergent to ζ^* and $\overline{B(\zeta^*, \varepsilon, r)} = \{y \in \Upsilon : \mu(y - \zeta^*; r) \nsucc \mathcal{N}(\varepsilon)\}$ is a closed ball for some $r > 0$ and $\varepsilon \in \mathcal{L} - \{0_\mathcal{L}\}$ then $\overline{B(\zeta^*, \varepsilon, r)} \subset \mathcal{I} - St^{\mathcal{L}}_\theta - LIM^r_\mu(y)$.*

Proof. Since $y = \{y_m\}$ is a lacunary ideal statistically convergent to ζ^* with respect to fuzzy norm μ *i.e* $y_m \xrightarrow{\mathcal{I}-St^{\mathcal{L}}_\theta} \zeta^*$, then for $\varepsilon \in \mathcal{L} - \{0_\mathcal{L}\}$, $\delta > 0$ and $s \in R^+$

$$A = \left\{u \in \mathbb{N} : \frac{1}{h_u}\left|\{m \in \varrho_u : \mu(y_m - \zeta^*; s) \nsucc \mathcal{N}(\varepsilon)\}\right| > \delta\right\} \in \mathcal{I}.$$

Since $\mathcal{I}$ is an admissible ideal then $G = \mathbb{N} \setminus A \neq \emptyset$, then for $m \in G^c$,

$$\frac{1}{h_u}\left|\{m \in \varrho_u : \mu(y_m - \zeta^*; s) \nsucc \mathcal{N}(\varepsilon)\}\right| < \delta.$$

$$\Rightarrow \frac{1}{h_u}\left|\{m \in \varrho_u : \mu(y_m - \zeta^*; s) \succ \mathcal{N}(\varepsilon)\}\right| \geq 1 - \delta.$$

Put $B_u = \{m \in \varrho_u : \mu(y_m - \zeta^*; s) \succ \mathcal{N}(\varepsilon)\}$ for $j \geq m$. For $j \in B_u$, $\mu\left(y_j - \zeta^*; s\right) \succ \mathcal{N}(\varepsilon)$.
Let $y \in \overline{B_u(\zeta^*, \lambda, r)}$. We will prove $y \in \mathcal{I} - St^{\mathcal{L}}_\theta - LIM^r_y$. As

$$\begin{aligned}\mu(y_j - y; r + s) &\geq \Omega\left(\mu(y_j - \zeta^*, s), \mu(y - \zeta^*, r)\right)\\ &\succ \mathcal{N}(\varepsilon).\end{aligned}$$

Hence, $B_u \subset \{m \in \varrho_u : \mu(y_m - y; r+s) \succ \mathcal{N}(\varepsilon)\}$, which implies that

$$1 - \delta \leq \frac{|B_u|}{h_u} \leq \frac{1}{h_u}\left|\{m \in \varrho_u : \mu(y_m - y; r+s) \succ \mathcal{N}(\varepsilon)\}\right|.$$

Therefore, $\frac{1}{h_u}\left|\{m \in \varrho_u : \mu(y_m - y; r+s) \nsucc \mathcal{N}(\varepsilon)\}\right| < 1 - (1-\delta) = \delta$.
Then,

$$\left\{u \in \mathbb{N} : \frac{1}{h_u}\left|\{m \in \varrho_u : \mu(y_m - y; r+s) \nsucc \mathcal{N}(\varepsilon)\}\right| \geq \delta\right\} \subset A \in \mathcal{I}.$$

which shows that $y \in \mathcal{I} - St_\theta^{\mathcal{L}} - LIM_\mu^r(y)$ in (Υ, μ, Ω). $\square$

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