



On the $g\pi$ -Hirano invertibility in Banach algebras

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Abstract. In a Banach algebra, we introduce a new type of generalized inverse called $g\pi$ -Hirano inverse. Firstly, several existence criteria and the equivalent definition of this inverse are investigated. Then, we discuss the relationship between the $g\pi$ -Hirano invertibility of a , b and that of the sum $a + b$ under some weaker conditions. Finally, as applications to the previous additive results, some equivalent characterizations for the $g\pi$ -Hirano invertibility of the anti-triangular matrix over Banach algebras are obtained.

1. Introduction

Let $\mathcal{A}$ be a complex Banach algebra with unit 1. For $a \in \mathcal{A}$, denote the spectrum and the spectral radius of a by $\sigma(a)$ and $r(a)$, respectively. $\mathcal{A}^{qnil}$ and $\mathcal{A}^{nil}$ stand for the sets of all quasinilpotent (i.e. for all element $a \in \mathcal{A}$ such that $\sigma(a) = \{0\}$) and nilpotent elements in $\mathcal{A}$, respectively. It is well known that $a \in \mathcal{A}^{qnil}$ if and only if $r(a) = 0$. The double commutant of an element $a \in \mathcal{A}$ is defined by $\text{comm}^2(a) = \{b \in \mathcal{A} : bc = cb, \text{ for any } c \in \mathcal{A} \text{ satisfying } ca = ac\}$.

As is known to all, Drazin inverse [5] is a kind of classic generalized inverse and has many applications. Until now, there have been many types of generalized inverses related to the Drazin inverse. Here we list some of them as follows.

The generalized Drazin inverse (or g -Drazin inverse) of $a \in \mathcal{A}$ [7] is the element $x \in \mathcal{A}$ which satisfies

$$xax = x, \quad ax = xa \quad \text{and} \quad a - a^2x \in \mathcal{A}^{qnil}.$$

Such x , if it exists, is unique and will be denoted by a^d .

An element $x \in \mathcal{A}$ is called the generalized strong Drazin inverse (or gs -Drazin inverse) of $a \in \mathcal{A}$ [10] if it satisfies

$$xax = x, \quad ax = xa \quad \text{and} \quad a - ax \in \mathcal{A}^{qnil}.$$

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Recently, the notion of π -Hirano inverse [6] was introduced in Banach algebras. Namely, the π -Hirano inverse of $a \in \mathcal{A}$ is the unique element x satisfying

$$xax = x, \quad ax = xa \quad \text{and} \quad a - a^{n+2}x \in \mathcal{A}^{nil},$$

for some $n \in \mathbb{N}$. Motivated by this notion, we give the definition of the generalized π -Hirano inverse as follows.

An element $x \in \mathcal{A}$ is called the generalized π -Hirano inverse (or $g\pi$ -Hirano inverse) of $a \in \mathcal{A}$ if it satisfies

$$xax = x, \quad ax = xa \quad \text{and} \quad a - a^{n+2}x \in \mathcal{A}^{qnil},$$

for some $n \in \mathbb{N}$.

All the time different types of generalized inverses were investigated in several directions (existences, sums, block matrices, reverse order laws, applications etc.) and in different settings (operator algebras, C^* -algebras, Banach algebras, rings etc.). For example, Drazin [5] proved that $a \in R$ is Drazin invertible if and only if it is strongly π -regular (i.e. $a^m \in a^{m+1}R \cap Ra^{m+1}$, for some $m \in \mathbb{N}$) in a ring R . Meanwhile, the Drazin invertibility of the sum $a + b$ was studied under the condition $ab = ba = 0$. Later, in a Banach algebra Koliha [7] claimed that $a \in \mathcal{A}$ has the generalized Drazin inverse if and only if 0 is not an accumulation point of $\sigma(a)$. For the ring case, Koliha and Patrício [8] showed that $a \in R$ is generalized Drazin invertible if and only if a is quasipolar. In [6], the authors considered the π -Hirano invertibility of a 2×2 operator matrix. More results on the generalized inverses related to this paper can be found in [12, 13, 16, 17].

All the results mentioned above served as motivation for further consideration of the $g\pi$ -Hirano inverse in Banach algebras. This paper is composed of four sections. In Section 2, we characterize the $g\pi$ -Hirano inverse by means of the quasinilpotent elements. Then, the equivalent definition of this inverse is given. In Section 3, sufficient and necessary conditions for the $g\pi$ -Hirano invertibility of the sum $a + b$ are obtained under some weaker conditions. In Section 4, we investigated the $g\pi$ -Hirano invertibility of several kinds of anti-triangular matrices over Banach algebras.

Next, we introduce some well-known lemmas, which are related to the quasinilpotency in a Banach algebra.

Lemma 1.1. [3, Lemma 2.1] *Let $a, b \in \mathcal{A}$ be such that $ab = ba$. The following hold:*

- (1) *If $a \in \mathcal{A}^{qnil}$ (or $b \in \mathcal{A}^{qnil}$), then $ab \in \mathcal{A}^{qnil}$;*
- (2) *If $a, b \in \mathcal{A}^{qnil}$, then $a + b \in \mathcal{A}^{qnil}$.*

Lemma 1.2. [1, Lemma 2.4] *Let $a, b \in \mathcal{A}^{qnil}$. If $ab = 0$, then $a + b \in \mathcal{A}^{qnil}$.*

Lemma 1.3. [11, Lemma 1.1] *Let $n \in \mathbb{N}$. Then, $a \in \mathcal{A}^{qnil}$ if and only if $a^n \in \mathcal{A}^{qnil}$.*

Lemma 1.4. [2, Lemma 2.2] *Let $a \in \mathcal{A}$. Then, a is gs -Drazin invertible if and only if $a - a^2 \in \mathcal{A}^{qnil}$.*

2. Characterizations for the $g\pi$ -Hirano invertibility

In this section, we investigate the existence criterion for the $g\pi$ -Hirano inverse in terms of quasinilpotent elements in Banach algebras. Then, using this characterization, we obtain the equivalent definition for the $g\pi$ -Hirano inverse.

Firstly, we give the relationship between the $g\pi$ -Hirano inverse and the g -Drazin inverse. Let $\mathcal{A}^d$ and $\mathcal{A}^{g\pi H}$ denote the sets of all g -Drazin and $g\pi$ -Hirano invertible elements in $\mathcal{A}$, respectively.

Proposition 2.1. *Let $a \in \mathcal{A}$. If x is the $g\pi$ -Hirano inverse of a , then $a \in \mathcal{A}^d$ and $a^d = x$.*

Proof. Suppose that x is the $g\pi$ -Hirano inverse of $a \in \mathcal{A}$, i.e.

$$xax = x, ax = xa \text{ and } a - a^{n+2}x \in \mathcal{A}^{qnil},$$

for some $n \in \mathbb{N}$. Hence, by Lemma 1.1(1), we get

$$a - a^2x = (a - a^{n+2}x)(1 - ax) \in \mathcal{A}^{qnil}.$$

So, $a \in \mathcal{A}^d$ and $a^d = x$. $\square$

From Proposition 2.1, we see that the $g\pi$ -Hirano inverse is a subclass of the g -Drazin inverse. According to the uniqueness of the g -Drazin inverse, we obtain that the $g\pi$ -Hirano inverse is unique if it exists. So, we use $a^{g\pi H}$ to denote the $g\pi$ -Hirano inverse of a in a Banach algebra.

In [6, Theorem 2.1], the authors investigated the existence of π -Hirano inverse by means of the nilpotent element. Inspired by this theorem, we obtain the corresponding result for the $g\pi$ -Hirano inverse with the help of Lemma 1.4.

Theorem 2.2. *Let $a \in \mathcal{A}$. Then the following are equivalent:*

- (1) $a \in \mathcal{A}^{g\pi H}$;
- (2) $a - a^{n+1} \in \mathcal{A}^{qnil}$, for some $n \in \mathbb{N}$;
- (3) $a^m - a^n \in \mathcal{A}^{qnil}$, for some $m, n \in \mathbb{N}$ such that $m \neq n$.

Proof. (1) $\Rightarrow$ (2). Suppose that $a \in \mathcal{A}^{g\pi H}$. Then, there exists $x \in \mathcal{A}$ such that

$$xax = x, ax = xa \text{ and } a - a^{n+2}x \in \mathcal{A}^{qnil},$$

for some $n \in \mathbb{N}$. Therefore, we have

$$a - a^{n+1} = (a - a^{n+2}x)(1 + a^{n+1}x - a^n) \in \mathcal{A}^{qnil}.$$

(2) $\Rightarrow$ (3). It is obvious.

(3) $\Rightarrow$ (1). Suppose that $n > m$. Note that

$$\begin{aligned} (a - a^{n-m+1})^m &= (a(1 - a^{n-m}))^m = a^m(1 - a^{n-m})(1 - a^{n-m})^{m-1} \\ &= (a^m - a^n)(1 - a^{n-m})^{m-1} \end{aligned}$$

and $a^m - a^n \in \mathcal{A}^{qnil}$. So, by Lemma 1.1(1) and Lemma 1.3, we get $a - a^{n-m+1} \in \mathcal{A}^{qnil}$, which gives $a^{n-m} - (a^{n-m})^2 = a^{n-m-1}(a - a^{n-m+1}) \in \mathcal{A}^{qnil}$. Thus, in view of Lemma 1.4 we obtain a^{n-m} is gs -Drazin invertible. Let x be the gs -Drazin inverse of a^{n-m} , i.e.

$$xa^{n-m}x = x, xa^{n-m} = a^{n-m}x \text{ and } a^{n-m} - a^{n-m}x \in \mathcal{A}^{qnil}.$$

Define $y = a^{n-m-1}x$. Next, we prove that $a \in \mathcal{A}^{g\pi H}$ and $a^{g\pi H} = y$. Observe the fact that $x \in \text{comm}^2(a^{n-m})$. So, $xa = ax$. Then, we have $ay = ya$ and $yay = y$. It is clear that

$$a - a^{(n-m)+2}y = (a - a^{2n-2m+1}) + (a^{2n-2m+1} - a^{2n-2m+1}x).$$

Since

$$a - a^{2n-2m+1} = (a - a^{n-m+1})(1 + a^{n-m}) \in \mathcal{A}^{qnil}$$

and

$$a^{2n-2m+1} - a^{2n-2m+1}x = a^{n-m+1}(a^{n-m} - a^{n-m}x) \in \mathcal{A}^{qnil},$$

from Lemma 1.1(2) it follows that $a - a^{(n-m)+2}y \in \mathcal{A}^{qnil}$. This completes the proof. $\square$

Applying Theorem 2.2, we get the following results.

Corollary 2.3. *Let $a \in \mathcal{A}$ and $k \in \mathbb{N}$. Then, $a \in \mathcal{A}^{g\pi H}$ if and only if $a^k \in \mathcal{A}^{g\pi H}$.*

Proof. Suppose that $a \in \mathcal{A}^{g\pi H}$. By Theorem 2.2(1)(2), we get $a - a^{n+1} \in \mathcal{A}^{qnil}$, for some $n \in \mathbb{N}$. Then, we deduce that

$$a^k - (a^k)^{n+1} = a^k - (a^{n+1})^k = \left(a - a^{n+1}\right) \sum_{i=0}^{k-1} a^{ni+k-1} \in \mathcal{A}^{qnil},$$

which implies $a^k \in \mathcal{A}^{g\pi H}$.

For the converse, we have $a^k - (a^k)^{m+1} \in \mathcal{A}^{qnil}$ for some $m \in \mathbb{N}$. So, $a^k - a^{km+k} \in \mathcal{A}^{qnil}$. Evidently, $k \neq km+k$. According to Theorem 2.2(1)(3), it follows that $a \in \mathcal{A}^{g\pi H}$. $\square$

Corollary 2.4. Let $\lambda \in \mathbb{C}$ be the root of unity. Then, $a \in \mathcal{A}^{g\pi H}$ if and only if $\lambda a \in \mathcal{A}^{g\pi H}$.

Proof. Let us suppose that $a \in \mathcal{A}^{g\pi H}$ and λ is the root of unity. Then, $a - a^{n+1} \in \mathcal{A}^{qnil}$ and $\lambda^k = 1$, for some $n, k \in \mathbb{N}$. Thus, we obtain

$$\lambda a - (\lambda a)^{kn+1} = \lambda \left(a - a^{n+1}\right) \left(a^{n(k-1)} + a^{n(k-2)} + \dots + 1\right) \in \mathcal{A}^{qnil}.$$

Thus, $\lambda a \in \mathcal{A}^{g\pi H}$. $\square$

Remark 2.5. In general, the condition $a \in \mathcal{A}^{g\pi H}$ and $\lambda \in \mathbb{C}$ do not imply $\lambda a \in \mathcal{A}^{g\pi H}$. For example, let $\mathcal{A} = \mathbb{C}$, $a = 1$ and $\lambda = 2$. Obviously, $a \in \mathcal{A}^{g\pi H}$. But, $2a = 2 \notin \mathcal{A}^{g\pi H}$.

Now, we are in the position to give the equivalent definition for the $g\pi$ -Hirano inverse in a Banach algebra.

Theorem 2.6. Let $a, x \in \mathcal{A}$. Then the following are equivalent:

- (1) $a \in \mathcal{A}^{g\pi H}$ and $a^{g\pi H} = x$;
- (2) $xax = x$, $xa = ax$ and $a^n - ax \in \mathcal{A}^{qnil}$, for some $n \in \mathbb{N}$;
- (3) $xax = x$, $xa = ax$ and $a^n - a^m x \in \mathcal{A}^{qnil}$, for some $m, n \in \mathbb{N}$ such that $m - n \neq 1$.

Proof. (1) $\Rightarrow$ (2). It is clear that

$$xax = x, xa = ax \text{ and } a - a^{n+2}x \in \mathcal{A}^{qnil},$$

for some $n \in \mathbb{N}$. According to the proof of the implication (1) $\Rightarrow$ (2) of Theorem 2.2 and Proposition 2.1, we see that $a - a^{n+1} \in \mathcal{A}^{qnil}$ and $a - a^2x \in \mathcal{A}^{qnil}$. Thus, by Lemma 1.1 we obtain

$$a^n - ax = a^{n-1}(a - a^2x) - (a - a^{n+1})x \in \mathcal{A}^{qnil}.$$

(2) $\Rightarrow$ (3). It is trivial.

(3) $\Rightarrow$ (1). Using item (3), we get

$$(a - a^2x)^n = a^n(1 - ax) = (a^n - a^m x)(1 - ax) \in \mathcal{A}^{qnil},$$

i.e. $a - a^2x \in \mathcal{A}^{qnil}$. Since $m - n \neq 1$, then we can consider the following two cases.

Case 1: Assume that $m - n \geq 2$. Then,

$$a^n - a^{m-1} = (a^n - a^m x) - a^{m-2}(a - a^2x) \in \mathcal{A}^{qnil}.$$

Thus, we deduce that

$$(a - a^{m-n})^n = (a^n - a^{m-1})(1 - a^{m-n-1})^{n-1} \in \mathcal{A}^{qnil},$$

i.e. $a - a^{m-n} \in \mathcal{A}^{qnil}$. So,

$$a - a^{(m-n-1)+2}x = (a - a^2x) + (a - a^{m-n})ax \in \mathcal{A}^{qnil}.$$

Case 2: Assume that $n - m \geq 0$. By the hypotheses $a^n - a^m x \in \mathcal{A}^{qnil}$ and $ax = xa$, we conclude that

$$(a^{n-m+1} - ax)^m = (a^n - a^m x)(a^{n-m} - x)^{m-1} \in \mathcal{A}^{qnil}.$$

Hence, we get $a^{n-m+1} - ax \in \mathcal{A}^{qnil}$, which yields

$$x - a^{n-m+1}x = -x(a^{n-m+1} - ax) \in \mathcal{A}^{qnil}.$$

Then,

$$a - a^{(n-m+1)+2}x = (a - a^2x) + (x - a^{n-m+1}x)a^2 \in \mathcal{A}^{qnil}.$$

Therefore, by these two cases we obtain $a \in \mathcal{A}^{g\pi H}$ and $a^{g\pi H} = x$. $\square$

Remark 2.7. (i) In [11], Mosić introduced the definition of the generalized n -strong Drazin inverse (or gns-Drazin inverse), where n is a fixed positive integer. By Theorem 2.6(1)(2), we can see that if $a \in \mathcal{A}$ is gns-Drazin invertible then a is $g\pi$ -Hirano invertible. Conversely, the $g\pi$ -Hirano inverse is a kind of the gns-Drazin inverse.

(ii) Analogously to item (3) of Theorem 2.6, for the case $m - n = 1$, we have $a \in \mathcal{A}^d$ with $a^d = x$ if and only if $xax = x$, $xa = ax$ and $a^n - a^{n+1}x \in \mathcal{A}^{qnil}$, for some $n \in \mathbb{N}$.

By Proposition 2.1 and Theorem 2.6, we immediately obtain the equivalent definition of the g -Drazin inverse in Banach algebras as follows.

Theorem 2.8. Let $a, x \in \mathcal{A}$. Then the following are equivalent:

- (1) $a \in \mathcal{A}^d$ and $a^d = x$;
- (2) $xax = x$, $ax = xa$ and $a - a^n x \in \mathcal{A}^{qnil}$, for some $n \in \mathbb{N}$;
- (3) $xax = x$, $ax = xa$ and $a^m - a^n x \in \mathcal{A}^{qnil}$, for some $m, n \in \mathbb{N}$.

Note that Theorem 2.8 is not a characterization for the $g\pi$ -Hirano invertible elements, which can be seen from the next example. Take $7 \in \mathcal{A}$, clearly, 7 is not $g\pi$ -Hirano invertible by Theorem 2.2. But, 7 is indeed invertible and hence it is g -Drazin invertible.

3. Additive results on the $g\pi$ -Hirano invertibility

Let $a, b \in \mathcal{A}$ and $k \in \mathbb{N}$. Then, the elements a, b are said to satisfy the “ $k\star$ ” condition if

$$ab \prod_{l=1}^k \alpha_l = 0, \text{ for any } \alpha_1, \alpha_2, \dots, \alpha_k \in \{a, b\}.$$

Obviously, if a, b satisfy the “ $k\star$ ” condition, then a, b satisfy the “ $(k+1)\star$ ” condition, but b, a do not satisfy the “ $k\star$ ” condition in general. Note that if $ab = 0$ then a, b satisfy the “ $k\star$ ” condition, for any $k \in \mathbb{N}$. Also, for $k = 1, 2, 3$, the “ $k\star$ ” condition becomes the following special cases, respectively.

- (1) $aba = ab^2 = 0$;
- (2) $abab = aba^2 = ab^2a = ab^3 = 0$;
- (3) $ababa = abab^2 = aba^3 = aba^2b = ab^2a^2 = ab^2ab = ab^3a = ab^4 = 0$.

The “ $k\star$ ” condition was introduced by Cvetković-Ilić [4]. For two Drazin invertible elements a, b in a ring, the author [4] studied the sufficient condition for the Drazin invertibility of the sum $a + b$ under the “ $k\star$ ” condition. Motivated by this, in this section we will consider the equivalence of the $g\pi$ -Hirano invertibility between the elements a, b and the sum $a + b$ under the “ $k\star$ ” condition in a Banach algebra.

We begin with the following crucial lemma.

Lemma 3.1. Let $k, i, j \in \mathbb{N}$. If $a, b \in \mathcal{A}$ satisfy the “ $k\star$ ” condition, then

$$(1) a, b \in \mathcal{A}^{qnil} \iff (2) a + b \in \mathcal{A}^{qnil} \iff (3) a^i + b^j \in \mathcal{A}^{qnil}.$$

Proof. (1) $\Rightarrow$ (2). Suppose that $a, b \in \mathcal{A}^{qnil}$. Note that

$$\begin{aligned}(a+b)^{k+2} &= a^2(a+b)^k + (ba(a+b)^k + b^2(a+b)^k) \\ &:= x_1 + (x_2 + x_3).\end{aligned}$$

Since $a, b \in \mathcal{A}$ satisfy the “ $k\star$ ” condition, we have $x_1(x_2 + x_3) = 0$, $x_2x_3 = 0$ and $x_2^2 = 0$. Obviously, $x_1 = a^{k+2} + \sum_{r=0}^{k-1} a^{k-r}(ab)(a+b)^r$. Observe that $\left(\sum_{r=0}^{k-1} a^{k-r}(ab)(a+b)^r\right)^2 = 0$, $a^{k+2} \in \mathcal{A}^{qnil}$ and $\left(\sum_{r=0}^{k-1} a^{k-r}(ab)(a+b)^r\right)a^{k+2} = 0$. Thus, from Lemma 1.2 we obtain $x_1 \in \mathcal{A}^{qnil}$. Similarly, we can get $x_3 \in \mathcal{A}^{qnil}$. So, we have $(a+b)^{k+2} \in \mathcal{A}^{qnil}$, i.e. $a+b \in \mathcal{A}^{qnil}$.

(2) $\Rightarrow$ (1). Let us suppose that $a+b \in \mathcal{A}^{qnil}$. Applying the “ $k\star$ ” condition, we have the following equations:

$$a(a+b)^m a^k = a^{m+k+1} \text{ and } b(a+b)^m b^{k+1} = b^{m+k+2}, \text{ for any } m \in \mathbb{N}.$$

Therefore, we get

$$\|a^{m+k+1}\| = \|a(a+b)^m a^k\| \leq \|a\|^{k+1} \|(a+b)^m\|,$$

which together with $a+b \in \mathcal{A}^{qnil}$ imply that

$$r(a) = \lim_{m \rightarrow \infty} \left(\|a^{m+k+1}\|^{1/m} \right)^{\frac{m}{m+k+1}} = \lim_{m \rightarrow \infty} \|a^{m+k+1}\|^{1/m} \leq \lim_{m \rightarrow \infty} \|a\|^{\frac{k+1}{m}} \|(a+b)^m\|^{1/m} = 0.$$

Therefore, $a \in \mathcal{A}^{qnil}$. Similarly, we can verify $b \in \mathcal{A}^{qnil}$.

(1) $\Leftrightarrow$ (3). Observe that a^i, b^j satisfy the “ $k\star$ ” condition. Then, according to the equivalence of item (1) and item (2), we have $a^i + b^j \in \mathcal{A}^{qnil}$ if and only if $a^i, b^j \in \mathcal{A}^{qnil}$. In view of Lemma 1.3, we get that $a^i + b^j \in \mathcal{A}^{qnil}$ if and only if $a, b \in \mathcal{A}^{qnil}$. $\square$

Now, we give the relationship between the $g\pi$ -Hirano invertibility of a, b and that of the sum $a+b$ under the “ $k\star$ ” condition in a Banach algebra as follows.

Theorem 3.2. Let $k, i, j \in \mathbb{N}$. If $a, b \in \mathcal{A}$ satisfy the “ $k\star$ ” condition, then

$$(1) a, b \in \mathcal{A}^{g\pi H} \iff (2) a+b \in \mathcal{A}^{g\pi H} \iff (3) a^i + b^j \in \mathcal{A}^{g\pi H}.$$

Proof. (1) $\Rightarrow$ (2). Suppose that $a, b \in \mathcal{A}^{g\pi H}$. So, there exist $m_1, m_2 \in \mathbb{N}$ such that $a - a^{m_1+1} \in \mathcal{A}^{qnil}$ and $b - b^{m_2+1} \in \mathcal{A}^{qnil}$. Take $m = km_1m_2$. Then, we get $m \geq k$ and

$$a - a^{m+1} = (a - a^{m_1+1}) \left(1 + a^{m_1} + a^{2m_1} + \dots + a^{(km_2-1)m_1} \right) \in \mathcal{A}^{qnil}.$$

Similarly, $b - b^{m+1} \in \mathcal{A}^{qnil}$. Let $\Sigma_1 = \sum_{r=0}^{m-1} (-a^{m-1-r})ab(a+b)^r$ and $\Sigma_2 = \sum_{r=0}^{m-1} (-b^{m-1-r})ba(a+b)^r$. Note that

$$\begin{aligned}x &:= (a+b) - (a+b)^{m+1} \\ &= (a - a^{m+1}) + (b - b^{m+1}) + (\Sigma_1 + \Sigma_2) \\ &:= x_1 + x_2 + x_3.\end{aligned}$$

Since a, b satisfy the “ $k\star$ ” condition, by computation we conclude that

$$(\Sigma_1)^2 = 0, (\Sigma_2)^3 = 0 \text{ and } \Sigma_1 \Sigma_2 = 0.$$

Hence, $x_3 \in \mathcal{A}^{qnil}$. Clearly, x_3 and x_2 satisfy the “ $k\star$ ” condition. Then, by Lemma 3.1 it follows that $x_2 + x_3 \in \mathcal{A}^{qnil}$. In addition, note that x_1 and $x_2 + x_3$ also satisfy the “ $k\star$ ” condition. Applying Lemma 3.1 again, we derive $x \in \mathcal{A}^{qnil}$, which yields $a + b \in \mathcal{A}^{g\pi H}$.

(2) $\Rightarrow$ (1). Suppose that $a + b \in \mathcal{A}^{g\pi H}$. Then, applying the above similar strategy, we can prove $a, b \in \mathcal{A}^{g\pi H}$ in terms of Lemma 3.1.

(1) $\Leftrightarrow$ (3). By Corollary 2.3 and (1) $\Leftrightarrow$ (2), we conclude that $a, b \in \mathcal{A}^{g\pi H}$ if and only if $a^i + b^j \in \mathcal{A}^{g\pi H}$. $\square$

The following corollary can be directly obtained from Theorem 3.2.

Corollary 3.3. Let $i, j \in \mathbb{N}$ and $a, b \in \mathcal{A}$. If $ab = 0$, then

$$(1) a, b \in \mathcal{A}^{g\pi H} \Leftrightarrow (2) a + b \in \mathcal{A}^{g\pi H} \Leftrightarrow (3) a^i + b^j \in \mathcal{A}^{g\pi H}.$$

Following the same way as in the proof of Theorem 3.2, we have

Theorem 3.4. Let $k, i, j \in \mathbb{N}$. If $a, b \in \mathcal{A}$ satisfy $\left(\prod_{l=1}^k \alpha_l\right)ab = 0$ for any $\alpha_1, \alpha_2, \dots, \alpha_k \in \{a, b\}$, then

$$(1) a, b \in \mathcal{A}^{g\pi H} \Leftrightarrow (2) a + b \in \mathcal{A}^{g\pi H} \Leftrightarrow (3) a^i + b^j \in \mathcal{A}^{g\pi H}.$$

Remark 3.5. Let us compare the g -Drazin invertibility with $g\pi$ -Hirano invertibility for the sum $a + b$ under the condition $ab = 0$. It is well known that the following holds: for $a, b \in \mathcal{A}$ satisfying $ab = 0$, then

$$a, b \in \mathcal{A}^d \Rightarrow a + b \in \mathcal{A}^d.$$

Until now, we do not know whether the converse of the above implication holds or not. However, the author [14] recently proved that if $ab = ba = 0$, then

$$a + b \in \mathcal{A}^d \Rightarrow a, b \in \mathcal{A}^d.$$

Next, we continue to consider this problem about the g -Drazin invertibility.

Theorem 3.6. Let $m, n, i, j \in \mathbb{N}$ and $a, b \in \mathcal{A}$.

(i) If $a^m b = ba^n = 0$, then the following hold:

$$a + b \in \mathcal{A}^d \Rightarrow a \in \mathcal{A}^d.$$

Furthermore, $a^d = a^k \left((a + b)^d \right)^{k+1}$, where $k = \max\{m, n\}$.

(ii) If $a^m b = 0$, then the following hold:

$$a + b \in \mathcal{A}^d \text{ and } a(a + b)^d = (a + b)^d a \Rightarrow a \in \mathcal{A}^d.$$

Furthermore, $a^d = a^m \left((a + b)^d \right)^{m+1}$.

(iii) If $ab = 0$, then the following hold:

$$a + b \in \mathcal{A}^d \text{ and } a(a + b)^d = (a + b)^d a \Rightarrow a, b \in \mathcal{A}^d.$$

Furthermore, $a^d = a \left((a + b)^d \right)^2$ and $b^d = b \left((a + b)^d \right)^2$.

Proof. (i). Assume that $\max\{m, n\} = k \geq 2$ and let $u = (a + b)^d$. Then, setting $x_1 = a^k u^{k-1}$ and $x_2 = a - a^k u^{k-1}$, so we have $a = x_1 + x_2$. Next, we will show that $x_1 \in \mathcal{A}^d$ and $x_2 \in \mathcal{A}^{qnil}$. Let $y = a^k u^{k+1}$. By the hypothesis $a^m b = b a^n = 0$, we obtain $a^k(a + b) = (a + b)a^k$, which yields $a^k u = u a^k$. Thus, $x_1 y = y x_1$. Also, we have

$$\begin{aligned} y x_1 y &= a^k u^{k+1} a^k u^{k-1} a^k u^{k+1} \\ &= a^{3k} u^{3k+1} \\ &= a^k (a + b)^{2k} u^{3k+1} \\ &= a^k u^{k+1} \\ &= y \end{aligned}$$

and

$$\begin{aligned} x_1 - x_1^2 y &= a^k u^{k-1} - a^{3k} u^{3k-1} \\ &= a^k u^{k-1} - a^k (a + b)^{2k} u^{3k-1} \\ &= a^k u^{k-1} - a^k u^{k-1} \\ &= 0. \end{aligned}$$

Thus, $x_1 \in \mathcal{A}^d$ and $x_1^d = y$. Note the following equation:

$$\begin{aligned} x_2^k \left((a + b) - (a + b)^2 u \right) &= \left(a - a^k u^{k-1} \right)^k \left((a + b) - (a + b)^2 u \right) \\ &= \left(\prod_{l=1}^k \left(1 - a^{l-1} u^{k-1} a^{k-l} \right) \right) a^k \left((a + b) - (a + b)^2 u \right) \\ &= \left(\prod_{l=1}^k \left(1 - a^{l-1} u^{k-1} a^{k-l} \right) \right) \left(a^{k+1} - a^{k+2} u \right) \\ &= \left(\prod_{l=1}^k \left(1 - a^{l-1} u^{k-1} a^{k-l} \right) \right) \left(a^{k+1} - a^{k+2} (a + b)^{k-2} u^{k-1} \right) \\ &= \left(\prod_{l=1}^k \left(1 - a^{l-1} u^{k-1} a^{k-l} \right) \right) \left(a^{k+1} - a^{2k} u^{k-1} \right) \\ &= \left(\prod_{l=1}^k \left(1 - a^{l-1} u^{k-1} a^{k-l} \right) \right) a^k \left(a - a^k u^{k-1} \right) \\ &= x_2^{k+1}. \end{aligned}$$

Similarly, $\left((a + b) - (a + b)^2 u \right) x_2^k = x_2^{k+1}$. Since $(a + b) - (a + b)^2 u \in \mathcal{A}^{qnil}$, we get $x_2 \in \mathcal{A}^{qnil}$ by Lemma 1.1(1) and Lemma 1.3. In addition, we obtain

$$\begin{aligned} x_1 x_2 &= u^{k-1} a^{k+1} - u^{2k-2} a^{2k} \\ &= u^{k-1} a^{k+1} - u^{2k-2} (a + b)^{k-1} a^{k+1} \\ &= u^{k-1} a^{k+1} - u^{k-1} a^{k+1} \\ &= 0. \end{aligned}$$

Similarly, $x_2 x_1 = 0$. Thus, in view of [7, Theorem 5.7] we deduce that $a \in \mathcal{A}^d$ and $a^d = x_1^d + x_2^d = y$.

For the case $\max\{m, n\} = 1$, we have $a^2 b = b a^2 = 0$. Thus, this case turns into the previous case.

(ii). Let $u = (a + b)^d$ and $z = a^m u^{m+1}$. Next, using the definition of the g-Drazin inverse we prove that $a \in \mathcal{A}^d$ and $a^d = z$. Since $au = ua$, we get $az = za$ and

$$z a z = a^{2m+1} u^{2m+2} = a^m (a + b)^{m+1} u^{2m+2} = a^m u^{m+1} = z.$$

In addition, for any $l \in \mathbb{N}$, we have

$$\begin{aligned} a^m \left((a+b) - (a+b)^2 u \right)^l &= a^m (a+b)^l (1 - (a+b)u) \\ &= a^{m+l} - a^{m+l+1}u \\ &= a^{m+l} - a^{m+l+1}(a+b)u^{m+1} \\ &= a^{m+l} - a^{m+l+1}a^m u^{m+1} \\ &= a^{m+l} - a^{m+l+1}z \\ &= (a - a^2z)^{m+l}. \end{aligned}$$

So, we get

$$\begin{aligned} r(a - a^2z) &= \lim_{l \rightarrow \infty} \|(a - a^2z)^{m+l}\|^{\frac{1}{m+l}} \\ &= \lim_{l \rightarrow \infty} \|(a - a^2z)^{m+l}\|^{\frac{1}{l}} \\ &\leq \lim_{l \rightarrow \infty} \|a^m\|^{\frac{1}{l}} \lim_{l \rightarrow \infty} \|(a+b) - (a+b)^2u\|^{\frac{1}{l}} \\ &= 0, \end{aligned}$$

which means $a - a^2z \in \mathcal{A}^{qnil}$, as required.

(iii). Let $m = 1$ in item (ii), we obtain $a \in \mathcal{A}^d$ and $a^d = a((a+b)^d)^2$. Note that $b(a+b)^d = (a+b)^d b$. Similarly, by the definition of the g-Drazin inverse we can deduce that $b \in \mathcal{A}^d$ and $b^d = b((a+b)^d)^2$. $\square$

By any item of Theorem 3.6 and [7, Theorem 5.7], we immediately infer

Corollary 3.7. [14, Corollary 3.7] Let $i, j \in \mathbb{N}$ and $a, b \in \mathcal{A}$. If $ab = ba = 0$, then

$$(i) \ a, b \in \mathcal{A}^d \iff (ii) \ a + b \in \mathcal{A}^d \iff (iii) \ a^i + b^j \in \mathcal{A}^d.$$

In order to continue considering the topic on the $g\pi$ -Hirano invertibility of the sum $a + b$, we need to prepare the following.

At the first we present a new condition. Namely, for $a, b \in \mathcal{A}$ and $k \in \mathbb{N}$, we say that a, b satisfy the “ $k*$ ” condition, i.e.

$$\left(\prod_{l=1}^k \alpha_l \right) ab = \left(\prod_{l=1}^k \alpha_l \right) ba, \text{ for any } \alpha_1, \alpha_2, \dots, \alpha_k \in \{a, b\}.$$

We can see that if a, b satisfy the “ $k*$ ” condition then a, b satisfy the “ $(k+1)*$ ” condition. In addition, the “ $k*$ ” condition contains the following specializations:

- (1) $ab = ba$;
- (2) $a^2b = aba$ and $b^2a = bab$; ($k = 1$)
- (3) $a^3b = a^2ba$, $ba^2b = (ba)^2$, $ab^2a = (ab)^2$ and $b^2ab = b^3a$. ($k = 2$)

Let $p \in \mathcal{A}$ be an idempotent ($p^2 = p$). Then we can represent element $a \in \mathcal{A}$ as

$$a = \begin{pmatrix} a_1 & a_3 \\ a_4 & a_2 \end{pmatrix}_p,$$

where $a_1 = pap$, $a_2 = (1-p)a(1-p)$, $a_3 = pa(1-p)$ and $a_4 = (1-p)ap$.

In what follows, by $\mathcal{A}_1, \mathcal{A}_2$ we denote the algebra $p\mathcal{A}p, (1-p)\mathcal{A}(1-p)$, where $p^2 = p \in \mathcal{A}$, respectively. The following lemmas play an important role in the sequel.

Lemma 3.8. Let $x, y \in \mathcal{A}$ and $p^2 = p \in \mathcal{A}$. If x and y have the representations

$$x = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}_p \text{ and } y = \begin{pmatrix} b & 0 \\ c & a \end{pmatrix}_{1-p},$$

then the following hold:

- (1) $a \in \mathcal{A}_1^{qnil}$ and $b \in \mathcal{A}_2^{qnil} \iff x \in \mathcal{A}^{qnil}$ (resp. $y \in \mathcal{A}^{qnil}$);
- (2) $a \in \mathcal{A}_1^{g\pi H}$ and $b \in \mathcal{A}_2^{g\pi H} \iff x \in \mathcal{A}^{g\pi H}$ (resp. $y \in \mathcal{A}^{g\pi H}$).

Proof. (1). Assume that $a \in \mathcal{A}_1^{qnil}$ and $b \in \mathcal{A}_2^{qnil}$. Since $\sigma_{\mathcal{A}}(x) \subseteq \sigma_{\mathcal{A}_1}(a) \cup \sigma_{\mathcal{A}_2}(b)$, then we get $\sigma_{\mathcal{A}}(x) = \{0\}$, i.e. $x \in \mathcal{A}^{qnil}$.

On the converse, note that $(1-p)xp = 0$, i.e. $pxp = xp$. Then, by induction we obtain that $a^m = (pxp)^m = px^mp$ for any $m \in \mathbb{N}$. Using the condition $x \in \mathcal{A}^{qnil}$, we conclude

$$r(a) = \lim_{m \rightarrow \infty} \|a^m\|^{\frac{1}{m}} = \lim_{m \rightarrow \infty} \|px^mp\|^{\frac{1}{m}} \leq \lim_{m \rightarrow \infty} \|p\|^{\frac{1}{m}} \|x^m\|^{\frac{1}{m}} \|p\|^{\frac{1}{m}} = 0.$$

So, $a \in \mathcal{A}_1^{qnil}$. Also, by $\sigma_{\mathcal{A}_2}(b) \subseteq \sigma_{\mathcal{A}_1}(a) \cup \sigma_{\mathcal{A}}(x)$ it follows that $\sigma_{\mathcal{A}_2}(b) = \{0\}$, i.e. $b \in \mathcal{A}_2^{qnil}$.

(2). For any $n \in \mathbb{N}$, we have

$$x - x^{n+1} = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}_p - \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}_p^{n+1} = \begin{pmatrix} a - a^{n+1} & \Delta \\ 0 & b - b^{n+1} \end{pmatrix}_p.$$

Suppose that $a \in \mathcal{A}_1^{g\pi H}$ and $b \in \mathcal{A}_2^{g\pi H}$. Then, there exists $m \in \mathbb{N}$ such that $a - a^{m+1} \in \mathcal{A}_1^{qnil}$ and $b - b^{m+1} \in \mathcal{A}_2^{qnil}$. So, $x - x^{m+1} \in \mathcal{A}^{qnil}$ by item (1). Therefore, we get $x \in \mathcal{A}^{g\pi H}$. The sufficiency can be proved similarly. $\square$

Remark 3.9. Item (2) of Lemma 3.8 is somewhat different from the g -Drazin inverse case, namely, if $a \in \mathcal{A}_1^d$, then $b \in \mathcal{A}_2^d$ if and only if $x \in \mathcal{A}^d$ ([1, Theorem 2.3]).

Lemma 3.10. Let $k \in \mathbb{N}$. If $a, b \in \mathcal{A}$ satisfy the “ $k*$ ” condition, then the following hold:

- (1) If $a \in \mathcal{A}^{qnil}$ (or $b \in \mathcal{A}^{qnil}$), then $ab \in \mathcal{A}^{qnil}$;
- (2) If $a \in \mathcal{A}^{qnil}$, then $b \in \mathcal{A}^{qnil}$ if and only if $a + b \in \mathcal{A}^{qnil}$;
- (3) If $a, b \in \mathcal{A}^{g\pi H}$, then $ab \in \mathcal{A}^{g\pi H}$;
- (4) If $a \in \mathcal{A}^{qnil}$, then $b \in \mathcal{A}^{g\pi H}$ if and only if $a + b \in \mathcal{A}^{g\pi H}$.

Proof. (1). Since a, b satisfy the “ $k*$ ” condition, then we conclude that $(ab)^{n+k} = (ab)^k a^n b^n$ for any $n \in \mathbb{N}$. Applying the hypothesis $a \in \mathcal{A}^{qnil}$ or $b \in \mathcal{A}^{qnil}$, we obtain

$$r(ab) = \lim_{n \rightarrow \infty} \|(ab)^{n+k}\|^{\frac{1}{n}} \leq \lim_{n \rightarrow \infty} \|(ab)^k\|^{\frac{1}{n}} \lim_{n \rightarrow \infty} \|a^n\|^{\frac{1}{n}} \lim_{n \rightarrow \infty} \|b^n\|^{\frac{1}{n}} = 0,$$

which implies $ab \in \mathcal{A}^{qnil}$.

(2). Suppose that $a, b \in \mathcal{A}^{qnil}$. Note that $(a+b)^{k+1} = (a+b)^k a + (a+b)^k b$. Let $c = (a+b)^k a$ and $d = (a+b)^k b$. From the “ $k*$ ” condition, we get $cd = dc$ and $c^n = (a+b)^{kn} a^n$ for any $n \in \mathbb{N}$. So,

$$r(c) = \lim_{n \rightarrow \infty} \|c^n\|^{\frac{1}{n}} \leq \|a+b\|^k \lim_{n \rightarrow \infty} \|a^n\|^{\frac{1}{n}} = 0,$$

which means $c \in \mathcal{A}^{qnil}$. Similarly, we have $d \in \mathcal{A}^{qnil}$. Applying Lemma 1.1(2), it follows that $(a+b)^{k+1} \in \mathcal{A}^{qnil}$, i.e. $a+b \in \mathcal{A}^{qnil}$.

To prove the converse, let us suppose that $a, a+b \in \mathcal{A}^{qnil}$. Obviously, $-a, a+b$ satisfy the “ $k*$ ” condition. Then, we deduce that $b = -a + (a+b) \in \mathcal{A}^{qnil}$ by the proof of the necessity of item (2).

(3). In view of the condition $a, b \in \mathcal{A}^{g\pi H}$, we obtain $a - a^{m+1} \in \mathcal{A}^{qnil}$ and $b - b^{m+1} \in \mathcal{A}^{qnil}$ for some $m \in \mathbb{N}$. By the “ $k*$ ” condition, we get

$$(ab)^{k+1} - (ab)^{m+k+1} = (ab)^k(a - a^{m+1})b + (ab)^k a^{m+1}(b - b^{m+1}).$$

Setting $s = (ab)^k(a - a^{m+1})b$ and $t = (ab)^k a^{m+1}(b - b^{m+1})$. Applying the “ $k*$ ” condition again, we obtain $s^n = (ab)^{kn}(a - a^{m+1})^n b^n$ for any $n \in \mathbb{N}$, which implies

$$r(s) = \lim_{n \rightarrow \infty} \|s^n\|^{\frac{1}{n}} \leq \|ab\|^k \|b\| \lim_{n \rightarrow \infty} \|(a - a^{m+1})^n\|^{\frac{1}{n}} = 0.$$

So, $s \in \mathcal{A}^{qnil}$. Similarly, $t \in \mathcal{A}^{qnil}$. Note that $st = ts$. Therefore, $s + t \in \mathcal{A}^{qnil}$, which gives $ab \in \mathcal{A}^{g\pi H}$ by Theorem 2.2(1)(3).

(4). Suppose that $b \in \mathcal{A}^{g\pi H}$ and let $p = bb^{g\pi H}$. Then, we consider the matrix representations of a and b relative to the idempotent p :

$$a = \begin{pmatrix} a_1 & a_3 \\ a_4 & a_2 \end{pmatrix}_p \text{ and } b = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}_p.$$

Obviously, $b_1 \in \mathcal{A}_1^{-1}$ with $(b_1)_{\mathcal{A}_1}^{-1} = b^{g\pi H}$. Also, by item (3) we get $b_1 = b(bb^{g\pi H}) \in \mathcal{A}_1^{g\pi H}$. From Proposition 2.1, it follows that $b_2 = b - b^2 b^{g\pi H} = b - b^2 b^d \in \mathcal{A}_2^{qnil}$.

Note that

$$\begin{pmatrix} b_1^k a_1 b_1 & b_1^k a_3 b_2 \\ b_2^k a_4 b_1 & b_2^k a_2 b_2 \end{pmatrix}_p = b^k ab = b^{k+1} a = \begin{pmatrix} b_1^{k+1} a_1 & b_1^{k+1} a_3 \\ b_2^{k+1} a_4 & b_2^{k+1} a_2 \end{pmatrix}_p.$$

Thus, we have $b_1^k a_3 b_2 = b_1^{k+1} a_3$, so, $a_3 = b_1^{-1} a_3 b_2$, which implies $a_3 = b_1^{-n} a_3 b_2^n$ for any $n \in \mathbb{N}$. Since $b_2 \in \mathcal{A}_2^{qnil}$, we have

$$\lim_{n \rightarrow \infty} \|a_3\|^{\frac{1}{n}} \leq \|b_1^{-1}\| \lim_{n \rightarrow \infty} \|a_3\|^{\frac{1}{n}} \lim_{n \rightarrow \infty} \|b_2^n\|^{\frac{1}{n}} = 0.$$

Hence, $a_3 = 0$. In addition, it is easy to see that $a_1 b_1 = b_1 a_1$ and a_2, b_2 satisfy the “ $k*$ ” condition.

Now, we have

$$a = \begin{pmatrix} a_1 & 0 \\ a_4 & a_2 \end{pmatrix}_p \text{ and } a + b = \begin{pmatrix} a_1 + b_1 & 0 \\ a_4 & a_2 + b_2 \end{pmatrix}_p.$$

From the condition $a \in \mathcal{A}^{qnil}$, it follows that $a_1 \in \mathcal{A}_1^{qnil}$ and $a_2 \in \mathcal{A}_2^{qnil}$ by Lemma 3.8(1). Applying item (2), we can obtain $a_2 + b_2 \in \mathcal{A}_2^{qnil}$, which implies $a_2 + b_2 \in \mathcal{A}_2^{g\pi H}$. Note that $(p + b_1^{-1} a_1) - (p + b_1^{-1} a_1)^2 = -a_1(b_1^{-1} + a_1 b_1^{-2}) \in \mathcal{A}_1^{qnil}$ by Lemma 1.1(1). Hence, we conclude $p + b_1^{-1} a_1 \in \mathcal{A}_1^{g\pi H}$. Then in view of item (3), we obtain $a_1 + b_1 = b_1(p + b_1^{-1} a_1) \in \mathcal{A}_1^{g\pi H}$. Finally, by Lemma 3.8(2) we deduce $a + b \in \mathcal{A}^{g\pi H}$.

The sufficiency of item (4) can be proved by the equality $b = -a + (a + b)$ and the necessity of item (4). $\square$

Now, we present the equivalent characterization for the $g\pi$ -Hirano invertibility of the sum $\lambda a + \mu b$ under the “ $k*$ ” condition.

Theorem 3.11. Let $k, i, j \in \mathbb{N}$, $\mu \in \mathbb{C}$ be the root of unity and $\lambda \in \mathbb{C}$. If $a, b \in \mathcal{A}^{g\pi H}$ satisfy the “ $k*$ ” condition, then

$$(1) \lambda a + \mu b \in \mathcal{A}^{g\pi H} \iff (2) a^i(\lambda a + \mu b)a^j \in \mathcal{A}^{g\pi H} \iff (3) a^i(\lambda 1 + \mu a^{g\pi H} b)a^j \in \mathcal{A}^{g\pi H}.$$

Proof. (1) $\Rightarrow$ (2). Evidently, $a^i, \lambda a + \mu b$ satisfy the “ $k*$ ” condition and $a^i \in \mathcal{A}^{g\pi H}$. Hence, by Lemma 3.10(3) we get $a^i(\lambda a + \mu b) \in \mathcal{A}^{g\pi H}$. Since $a^i(\lambda a + \mu b), a^j$ also satisfy the “ $k*$ ” condition, then $a^i(\lambda a + \mu b)a^j \in \mathcal{A}^{g\pi H}$.

(2) $\Rightarrow$ (1). Let $p = aa^{g\pi H}$. Then, as in the proof of Lemma 3.10(4) we have

$$a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}_p, \quad b = \begin{pmatrix} b_1 & 0 \\ b_4 & b_2 \end{pmatrix}_p \text{ and } \lambda a + \mu b = \begin{pmatrix} \lambda a_1 + \mu b_1 & 0 \\ \mu b_4 & \lambda a_2 + \mu b_2 \end{pmatrix}_p,$$

where $a_1 \in \mathcal{A}_1^{-1} \cap \mathcal{A}_1^{g\pi H}$, $(a_1)^{-1}_{\mathcal{A}_1} = a^{g\pi H}$ and $a_2 \in \mathcal{A}_2^{qnil}$. In addition, we have $a_1 b_1 = b_1 a_1$, also, a_2, b_2 satisfy the “ $k*$ ” condition. Using $b \in \mathcal{A}^{g\pi H}$, by Lemma 3.8(2) we get $b_1 \in \mathcal{A}_1^{g\pi H}$ and $b_2 \in \mathcal{A}_2^{g\pi H}$. So, from Corollary 2.4 it follows that $\mu b_2 \in \mathcal{A}_2^{g\pi H}$. Note that $\lambda a_2 \in \mathcal{A}_2^{qnil}$ and $\lambda a_2, \mu b_2$ satisfy the “ $k*$ ” condition. In view of Lemma 3.10(4), we get $\lambda a_2 + \mu b_2 \in \mathcal{A}_2^{g\pi H}$. From Lemma 3.8(2), we claim that $\lambda a + \mu b \in \mathcal{A}^{g\pi H}$ if and only if $\lambda a_1 + \mu b_1 \in \mathcal{A}_1^{g\pi H}$. So, we only need to prove $\lambda a_1 + \mu b_1 \in \mathcal{A}_1^{g\pi H}$.

Note that

$$a^i(\lambda a + \mu b)a^j = \begin{pmatrix} a_1^{i+j}(\lambda a_1 + \mu b_1) & 0 \\ \mu a_2^i b_2 a_1^j & a_2^i(\lambda a_2 + \mu b_2)a_2^j \end{pmatrix}_p.$$

Since $a^i(\lambda a + \mu b)a^j \in \mathcal{A}^{g\pi H}$, then $a_1^{i+j}(\lambda a_1 + \mu b_1) \in \mathcal{A}_1^{g\pi H}$. In addition, according to the hypothesis $a \in \mathcal{A}^{g\pi H}$, we have $a - a^{m+1} \in \mathcal{A}^{qnil}$ for some $m \in \mathbb{N}$. Then we get

$$a_1^{-1} - a_1^{-m-1} = a^{g\pi H} - (a^{g\pi H})^{m+1} = -(a^{g\pi H})^{m+2}(a - a^{m+1}) \in \mathcal{A}^{qnil},$$

which implies $a_1^{-1} \in \mathcal{A}_1^{g\pi H}$. So, we get that $\lambda a_1 + \mu b_1 = (a_1^{-1})^{i+j}(a_1^{i+j}(\lambda a_1 + \mu b_1)) \in \mathcal{A}_1^{g\pi H}$ by Lemma 3.10(3), which is what we need.

(2) $\Leftrightarrow$ (3). It is clear that

$$a^i(\lambda 1 + \mu a^{g\pi H} b)a^j = \begin{pmatrix} a_1^{i+j}(\lambda p + \mu a_1^{-1} b_1) & 0 \\ 0 & \lambda a_2^{i+j} \end{pmatrix}_p.$$

Note that $\lambda a_2^{i+j} \in \mathcal{A}_2^{qnil}$ and by Lemma 3.10(1) we have $a_2^i(\lambda a_2 + \mu b_2)a_2^j \in \mathcal{A}_2^{qnil}$. On the other hand, it is clear to see $a_1^{i+j}(\lambda a_1 + \mu b_1) \in \mathcal{A}_1^{g\pi H}$ if and only if $a_1^{i+j}(\lambda p + \mu a_1^{-1} b_1) \in \mathcal{A}_1^{g\pi H}$. Thus, we conclude that $a^i(\lambda a + \mu b)a^j \in \mathcal{A}^{g\pi H}$ is equivalent to $a^i(\lambda 1 + \mu a^{g\pi H} b)a^j \in \mathcal{A}^{g\pi H}$.

Remark 3.12. (i) One can see that (1) $\Rightarrow$ (2) and (2) $\Leftrightarrow$ (3) in Theorem 3.11 are valid even if b is not $g\pi$ -Hirano invertible. However, the following example shows that (2) $\Rightarrow$ (1) does not hold in general if b is not $g\pi$ -Hirano invertible. Let $\mathcal{A} = M_2(\mathbb{C})$, $i = j = 2 \in \mathbb{N}$, $\lambda = \mu = 1 \in \mathbb{C}$, $a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. Clearly,

$a^i(\lambda a + \mu b)a^j = 0 \in \mathcal{A}^{g\pi H}$. But, $\lambda a + \mu b = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \notin \mathcal{A}^{g\pi H}$.

(ii) By the proof of Theorem 3.11, it is clear that the condition $i, j \in \mathbb{N}$ can be replaced by $i, j \in \mathbb{N} \cup \{0\}$ and $i + j \neq 0$. For the case $i = j = 0$, the implication (3) $\Rightarrow$ (2) also holds, that is to say, if $\mu \in \mathbb{C}$ is the root of unity, $\lambda \in \mathbb{C}$, and $a, b \in \mathcal{A}^{g\pi H}$ satisfy the “ $k*$ ” condition, then we have

$$\lambda 1 + \mu a^{g\pi H} b \in \mathcal{A}^{g\pi H} \implies \lambda a + \mu b \in \mathcal{A}^{g\pi H}.$$

But, in general the converse of the above implication is not true. For example, take $\mathcal{A} = \mathbb{C}$, $a = 0$, $b = 1$, $\lambda = 2$ and $\mu = 1$. Then, $\lambda a + \mu b = 1 \in \mathcal{A}^{g\pi H}$. But, $\lambda 1 + \mu a^{g\pi H} b = 2 \notin \mathcal{A}^{g\pi H}$. If we assume that both λ and μ are the roots of unity, then we have the following result: let $\lambda, \mu \in \mathbb{C}$ be the roots of unity and $a, b \in \mathcal{A}^{g\pi H}$ satisfy the “ $k*$ ” condition, then

$$\lambda 1 + \mu a^{g\pi H} b \in \mathcal{A}^{g\pi H} \iff \lambda a + \mu b \in \mathcal{A}^{g\pi H}.$$

Dual to Theorem 3.11, we have the following result.

Theorem 3.13. Let $k, i, j \in \mathbb{N}$, $\mu \in \mathbb{C}$ be the root of unity and $\lambda \in \mathbb{C}$. If $a, b \in \mathcal{A}^{g\pi H}$ satisfy $ab \prod_{l=1}^k \alpha_l = ba \prod_{l=1}^k \alpha_l$ for any $\alpha_1, \alpha_2, \dots, \alpha_k \in \{a, b\}$, then

$$(1) \lambda a + \mu b \in \mathcal{A}^{g\pi H} \iff (2) a^i(\lambda a + \mu b)a^j \in \mathcal{A}^{g\pi H} \iff (3) a^i(\lambda 1 + \mu a^{g\pi H} b)a^j \in \mathcal{A}^{g\pi H}.$$

4. Anti-triangular matrices involving the $g\pi$ -Hirano inverse

In this section, we mainly consider some sufficient and necessary conditions for anti-triangular matrices $\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$ over Banach algebras to be $g\pi$ -Hirano invertible.

The authors [15] found the anti-triangular matrix $\begin{pmatrix} 1 & 1 \\ c & 0 \end{pmatrix} \in M_2(\mathcal{A})$ is Drazin invertible if and only if $c \in \mathcal{A}$ is Drazin invertible. But, for the $g\pi$ -Hirano case, we do not have the corresponding result, which can be seen from the following example. Let $\mathcal{A} = \mathbb{C}$ and $c = 1$. Obviously, $1 \in \mathcal{A}^{g\pi H}$. But, $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \notin M_2(\mathcal{A})^{g\pi H}$, since $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n+1} \notin M_2(\mathcal{A})^{qnil}$, for any $n \in \mathbb{N}$. So, what is the equivalent conditions for $\begin{pmatrix} 1 & 1 \\ c & 0 \end{pmatrix} \in M_2(\mathcal{A})$ to be $g\pi$ -Hirano invertible?

At the beginning, we consider the equivalent conditions for the matrix $\begin{pmatrix} 1 & 1 \\ c & 0 \end{pmatrix} \in M_2(\mathcal{A})$ to be g -Hirano invertible. The definition of the g -Hirano inverse was introduced by Chen and Sheibani [2], namely an element $a \in \mathcal{A}$ has g -Hirano inverse if there exists $x \in \mathcal{A}$ such that

$$xax = x, \quad ax = xa \quad \text{and} \quad a^2 - ax \in \mathcal{A}^{qnil}.$$

Clearly, by Theorem 2.6 we see that the g -Hirano inverse is a subclass of the $g\pi$ -Hirano inverse. Denote by $\mathcal{A}^{gH}$ the set of all g -Hirano invertible elements in $\mathcal{A}$.

Theorem 4.1. Let $M = \begin{pmatrix} 1 & 1 \\ c & 0 \end{pmatrix} \in M_2(\mathcal{A})$. Then,

$$c \in \mathcal{A}^{qnil} \iff M \in M_2(\mathcal{A})^{gH}.$$

Proof. From [2, Theorem 2.4] it follows that $M \in M_2(\mathcal{A})^{gH}$ if and only if $N := M - M^3 = -\begin{pmatrix} 2c & c \\ c^2 & c \end{pmatrix} \in M_2(\mathcal{A})^{qnil}$.

Therefore, we only need to prove that $c \in \mathcal{A}^{qnil}$ is equivalent to $N \in M_2(\mathcal{A})^{qnil}$.

Suppose that $c \in \mathcal{A}^{qnil}$. Then, $N = -\begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} 2 & 1 \\ c & 1 \end{pmatrix} \in M_2(\mathcal{A})^{qnil}$ by Lemma 1.1(1).

On the contrary, by $N \in M_2(\mathcal{A})^{qnil}$ we get that $\begin{pmatrix} 2c - \lambda & c \\ c^2 & c - \lambda \end{pmatrix}$ is invertible, for any $\lambda \in \mathbb{C} \setminus \{0\}$. Since

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2c - \lambda & c \\ c^2 & c - \lambda \end{pmatrix} = \begin{pmatrix} -c^2 + 2c - \lambda & \lambda \\ c^2 & c - \lambda \end{pmatrix},$$

we deduce that $\begin{pmatrix} -c^2 + 2c - \lambda & \lambda \\ c^2 & c - \lambda \end{pmatrix}$ is invertible for any $\lambda \in \mathbb{C} \setminus \{0\}$. Hence, there exists $\begin{pmatrix} x & y \\ z & w \end{pmatrix} \in M_2(\mathcal{A})$ such that

$$\begin{pmatrix} -c^2 + 2c - \lambda & \lambda \\ c^2 & c - \lambda \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} -c^2 + 2c - \lambda & \lambda \\ c^2 & c - \lambda \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

So, we can obtain the following equations

$$(-c^2 + 2c - \lambda)y + \lambda w = 0, \quad (1)$$

$$c^2y + (c - \lambda)w = 1, \quad (2)$$

$$x(-c^2 + 2c - \lambda) + yc^2 = 1, \quad (3)$$

$$\lambda x + y(a - \lambda) = 0. \quad (4)$$

By the equation (1), we have $w = \frac{1}{\lambda}(c^2 - 2c + \lambda)y$, which together with (2) imply

$$1 = c^2y + \frac{1}{\lambda}(c - \lambda)(c^2 - 2c + \lambda)y = \frac{1}{\lambda}(c^3 - 2c^2 + 3\lambda c - \lambda^2)y.$$

So, y is left invertible. Similarly, using (3) and (4) we conclude that y is right invertible. Therefore, y is invertible and $y^{-1} = \frac{1}{\lambda}(c^3 - 2c^2 + 3\lambda c - \lambda^2)$. So, $c^3 - 2c^2 + 3\lambda c - \lambda^2$ is invertible. Hence, $0 \notin \sigma(c^3 - 2c^2 + 3\lambda c - \lambda^2)$ for any $\lambda \in \mathbb{C} \setminus \{0\}$.

Now, assume that there exists $t \in \mathbb{C} \setminus \{0\}$ such that $t \in \sigma(c)$. Then, we can find $\lambda_0 \in \mathbb{C} \setminus \{0\}$ satisfying $t^3 - 2t^2 + 3\lambda_0 t - \lambda_0^2 = 0$. So, $0 \in \sigma(c^3 - 2c^2 + 3\lambda_0 c - \lambda_0^2)$, which contradicts with $0 \notin \sigma(c^3 - 2c^2 + 3\lambda c - \lambda^2)$ for any $\lambda \in \mathbb{C} \setminus \{0\}$. Hence, $\sigma(c) = \{0\}$, i.e. $c \in \mathcal{A}^{qnil}$. $\square$

Remark 4.2. By Theorem 4.1, we get

$$c \in \mathcal{A}^{qnil} \implies M = \begin{pmatrix} 1 & 1 \\ c & 0 \end{pmatrix} \in M_2(\mathcal{A})^{g\pi H}.$$

However, in general the converse of the above implication does not hold, which can be seen from the following example:

Example 4.3. Let $\mathcal{A} = \mathbb{C}$ and $c = -1$. Observe that $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^6 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Therefore, we get that $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \in M_2(\mathcal{A})^{g\pi H}$ and $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{g\pi H} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$. However, $-1 \notin \mathcal{A}^{qnil}$.

Next, we will consider the $g\pi$ -Hirano invertibility for the anti-triangular matrix $\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$ over Banach algebras. For future reference we state two lemmas as follows.

Lemma 4.4. Let $a, b \in \mathcal{A}$. Then, $ab \in \mathcal{A}^{g\pi H}$ if and only if $ba \in \mathcal{A}^{g\pi H}$.

Proof. If $ab \in \mathcal{A}^{g\pi H}$, then $ab - (ab)^{m+1} \in \mathcal{A}^{qnil}$, for some $m \in \mathbb{N}$. By induction, we have $((ba)^2 - (ba)^{m+2})^n = b(ab - (ab)^{m+1})^n(ab)^{n-1}a$, for any $n \in \mathbb{N}$. Thus,

$$\lim_{n \rightarrow \infty} \|((ba)^2 - (ba)^{m+2})^n\|^{\frac{1}{n}} \leq \|a\| \|b\| \lim_{n \rightarrow \infty} \|(ab - (ab)^{m+1})^n\|^{\frac{1}{n}} = 0.$$

So, $(ba)^2 - (ba)^{m+2} \in \mathcal{A}^{qnil}$, which means $ba \in \mathcal{A}^{g\pi H}$. $\square$

Lemma 4.5. Let $M = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}$ (or $\begin{pmatrix} a & 0 \\ d & b \end{pmatrix}$) $\in M_2(\mathcal{A})$. Then,

$$(1) \ a \in \mathcal{A}^{qnil} \text{ and } b \in \mathcal{A}^{qnil} \iff M \in M_2(\mathcal{A})^{qnil};$$

$$(2) \ a \in \mathcal{A}^{g\pi H} \text{ and } b \in \mathcal{A}^{g\pi H} \iff M \in M_2(\mathcal{A})^{g\pi H}.$$

Proof. (1). Suppose that $M = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \in M_2(\mathcal{A})^{qnil}$. Let $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(\mathcal{A})$. Then we have the following matrix representation of M relative to the idempotent P :

$$M = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}_P, \text{ where } A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \text{ and } C = \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}.$$

By Lemma 3.8(1), we obtain $A \in (PM_2(\mathcal{A})P)^{qnil}$ and $B \in ((I - P)M_2(\mathcal{A})(I - P))^{qnil}$, i.e. $\sigma_{PM_2(\mathcal{A})P}(A) = \{0\}$ and $\sigma_{(I-P)M_2(\mathcal{A})(I-P)}(B) = \{0\}$. Note that $\sigma_{PM_2(\mathcal{A})P}(A) \cup \{0\} = \sigma_{M_2(\mathcal{A})}(A)$. So, $\sigma_{M_2(\mathcal{A})}(A) = \{0\}$, which implies that $\lambda I - A$ is invertible, for any $\lambda \neq 0$. Hence, $\lambda I - a$ is invertible, so $\sigma_{\mathcal{A}}(a) = \{0\}$. Therefore, $a \in \mathcal{A}^{qnil}$. Similarly, we can get $b \in \mathcal{A}^{qnil}$. On the contrary, applying $\sigma_{M_2(\mathcal{A})}(M) \subseteq \sigma_{\mathcal{A}}(a) \cup \sigma_{\mathcal{A}}(b)$ we deduce $M \in M_2(\mathcal{A})^{qnil}$.

(2). By item (1) and Theorem 2.2, item (2) holds directly. $\square$

Now, we are ready to present an existence criterion for the $g\pi$ -Hirano inverse of the anti-triangular matrix under the “ $k\star$ ” condition as follows.

Theorem 4.6. Let $M = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \in M_2(\mathcal{A})$ and $k \in \mathbb{N}$. If a, bc satisfy the “ $k\star$ ” condition, then

$$a, bc \in \mathcal{A}^{g\pi H} \iff M \in M_2(\mathcal{A})^{g\pi H}.$$

Proof. Note that

$$M = \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}.$$

By Lemma 4.4, we have

$$M \in M_2(\mathcal{A})^{g\pi H} \iff N := \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & bc \\ 1 & 0 \end{pmatrix} \in M_2(\mathcal{A})^{g\pi H}.$$

Consider the following decomposition:

$$N^2 = \begin{pmatrix} a^2 & 0 \\ a & 0 \end{pmatrix} + \begin{pmatrix} bc & abc \\ 0 & bc \end{pmatrix} := N_1 + N_2.$$

Since a, bc satisfy the “ $k\star$ ” condition, so do N_1 and N_2 . Therefore, by using Corollary 2.3, Theorem 3.2 and Lemma 4.5(2) we deduce that

$$N \in M_2(\mathcal{A})^{g\pi H} \iff N^2 \in M_2(\mathcal{A})^{g\pi H} \iff N_1, N_2 \in M_2(\mathcal{A})^{g\pi H} \iff a, bc \in \mathcal{A}^{g\pi H},$$

as required. $\square$

It is easy to see that if the hypothesis a, bc satisfy the “ $k\star$ ” condition in Theorem 4.6 is replaced by the hypothesis bc, a satisfy the “ $k\star$ ” condition then this theorem still holds. So, we immediately obtain the following corollary.

Corollary 4.7. Let $M = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \in M_2(\mathcal{A})$. If $abc = 0$ (or $bca = 0$), then

$$a, bc \in \mathcal{A}^{g\pi H} \iff M \in M_2(\mathcal{A})^{g\pi H}.$$

Let us remark that in Corollary 4.7 the condition $abc = 0$ or $bca = 0$ in general can not be substituted by $acb = 0$ or $cab = 0$, which can be seen from the following examples.

Example 4.8. Let $\mathcal{H}$ be an infinite-dimensional Hilbert space and $\mathcal{A} = \mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators. Take $M = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \in M_2(\mathcal{A})$, where A, B and C are defined as follows. Let $A \in \mathcal{B}(\mathcal{H}) \setminus \{0\}$ be a quasinilpotent operator. Since $A \neq 0$, then $A - 3I \neq -3I$. So, there exists $x \neq 0$ such that $(A - 3I)x \neq -3x$. Let $t = (A - 3I)x$. Then, $t \neq -3x$. Let us prove that $t \notin \text{Span}\{x\}$. If we suppose that $t = \lambda x$, then $(A - 3I)x = \lambda x$, i.e. $(A - (3 + \lambda)I)x = 0$, which is possible only when $\lambda = -3$, since A is quasinilpotent. Hence, $t = -3x$, which is a contradiction. Now, take $y \neq 0$ and define $B \in \mathcal{B}(\mathcal{H})$ satisfying $By = -t$ and $B \equiv 0$ on $\text{Span}\{y\}^\perp$. Next, let $C \in \mathcal{B}(\mathcal{H})$ be such that $Cx = 3y$,

$Ct = 0$ and $C \equiv 0$ on $\text{Span}\{x, t\}^\perp$. Now, since $R(B) = \text{Span}\{t\}$, we get $CB = 0$. So, $ACB = 0$ and using the fact that $\sigma(BC) \cup \{0\} = \sigma(CB) \cup \{0\}$, we get $\sigma(BC) = \{0\}$, i.e. BC is quasinilpotent. Clearly, we have $A, BC \in \mathcal{A}^{g\pi H}$. Now, by $\begin{pmatrix} A-3I & B \\ C & -3I \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (A-3I)x + By \\ Cx - 3y \end{pmatrix} = \begin{pmatrix} t - t \\ 3y - 3y \end{pmatrix} = 0$, we get that $0 \neq \begin{pmatrix} x \\ y \end{pmatrix} \in N \begin{pmatrix} A-3I & B \\ C & -3I \end{pmatrix}$. Then, $3 \in \sigma(M)$. Thus, $3 - 3^{n+1} \in \sigma(M - M^{n+1})$, for any $n \in \mathbb{N}$. Note that $3 - 3^{n+1} \neq 0$, for any $n \in \mathbb{N}$. Thus, $M - M^{n+1} \notin \mathcal{A}^{qnil}$ for any $n \in \mathbb{N}$, so, $M \notin M_2(\mathcal{A})^{g\pi H}$.

Example 4.9. Let $\mathcal{A} = M_2(\mathbb{C})$ and $M = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \in M_2(\mathcal{A})$. If we take $a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $b = c = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, then $a, bc \in \mathcal{A}^{g\pi H}$ and $cab = 0$. But, $M \notin M_2(\mathcal{A})^{g\pi H}$.

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