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Bounds for the Berezin number inequalities of 2×2 operator matrices and applications

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Abstract. In this paper, we give several upper bounds for the Berezin number for 2×2 off-diagonal operator matrices. These upper bounds refine and generalize some earlier existing ones. As an application of one of our results, we give new upper bounds for the sum of Berezin numbers of two bounded operators.

1. Introduction

Let $\mathcal{B}(\mathcal{H})$ denote the C^*- algebra of all bounded linear operators on a non trivial complex Hilbert space $\mathcal{H}$ with inner product $\langle .,. \rangle$ and associated norm $\|.\|$. Let Ω be a nonempty set. A functional Hilbert space $\mathcal{H} = \mathcal{H}(\Omega)$ is a Hilbert space of complex valued functions, which has the property that point evaluations are continuous, i.e., for each $\lambda \in \Omega$ the map $f \longmapsto f(\lambda)$ is a continuous linear functional on $\mathcal{H}$. The Riesz representation theorem ensues that for each $\lambda \in \Omega$ there exists a unique element $k_\lambda \in \mathcal{H}$ such that $f(\lambda) = \langle f, k_\lambda \rangle$ for all $f \in \mathcal{H}$. The set $\{k_\lambda : \lambda \in \Omega\}$ is called the reproducing kernel of the space $\mathcal{H}$. If $\{e_n\}_{n \geq 0}$ is an orthonormal basis for a functional Hilbert space $\mathcal{H}$, then the reproducing kernel of $\mathcal{H}$ is given by $k_\lambda(z) = \sum_{n=0}^{+\infty} \overline{e_n(\lambda)} e_n(z)$ (see [25, 28]). For $\lambda \in \Omega$, let $\hat{k}_\lambda = \frac{k_\lambda}{\|k_\lambda\|}$ be the normalized reproducing kernel of $\mathcal{H}$. Let A a bounded linear operator on $\mathcal{H}$, the Berezin symbol of A, which firstly have been introduced by Berezin [10, 11] is the function $\tilde{A}$ on Ω defined by

$$\tilde{A}(\lambda) := \left\langle A\hat{k}_\lambda, \hat{k}_\lambda \right\rangle.$$

The Berezin set and the Berezin number of the operator A are defined respectively by:

$$\mathbf{Ber}(A) := \left\{ \left\langle A\hat{k}_\lambda, \hat{k}_\lambda \right\rangle : \lambda \in \Omega \right\}$$

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and

$$\mathbf{ber}(A) := \sup\left\{\left|\left\langle A\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right| : \lambda \in \Omega\right\}.$$

It is clear that the Berezin symbol $\tilde{A}$ is the bounded function on Ω whose value lies in the numerical range of the operator A and hence for any $A \in \mathcal{B}(\mathcal{H})$,

$$\mathbf{Ber}(A) \subset W(A) \text{ and } \mathbf{ber}(A) \leq \omega(A),$$

where

$$W(A) = \{\langle Ax, x\rangle : x \in \mathcal{H},\ \|x\| = 1\}$$

is the numerical range of the operator A and

$$\omega(A) = \sup\{|\langle Ax, x\rangle| : x \in \mathcal{H},\ \|x\| = 1\}$$

is the numerical radius of A.

Moreover, the Berezin number of an operator A satisfies the following properties:
(i) $\mathbf{ber}(A) = \mathbf{ber}(A^*)$.
(ii) $\mathbf{ber}(\mu A) = |\mu|\,\mathbf{ber}(A)$ for all $\mu \in \mathbb{C}$.
(iii) $\mathbf{ber}(A + B) \leq \mathbf{ber}(A) + \mathbf{ber}(B)$ for all $A, B \in \mathcal{B}(\mathcal{H})$.
Ceralry,

$$\mathbf{ber}(A) \leq \|A\|. \tag{1}$$

Notice that, in general, the Berezin number does not define a norm. However, if $\mathcal{H}$ is a reproducing kernel Hilbert space of analytic functions, (for instance on the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$), then $\mathbf{ber}(.)$ defines a norm on $\mathcal{B}(\mathcal{H}(D))$ (see [26, 27, 30, 31, 34]).

The Berezin symbol has been studied in detail for Toeplitz and Hankel operators on Hardy and Bergman spaces (see [33]). A nice property of the Berezin symbol is mentioned next. If $\tilde{A}(\lambda) = \tilde{B}(\lambda)$ for all $\lambda \in \Omega$, then $A = B$. Therefore, the Berezin symbol uniquely determines the operator. The Berezin symbol and Berezin number have been studied by many authors over the years, a few of them are [1, 4, 7, 14, 16–18, 20, 24, 35, 36].

The direct sum of two copies of $\mathcal{H}$ is denoted by $\mathcal{H}^2 = \mathcal{H} \oplus \mathcal{H}$. If $A, B, C, D \in \mathcal{B}(\mathcal{H})$, then the operator matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ can be considered as an operator in $\mathcal{B}(\mathcal{H} \oplus \mathcal{H})$, which is defined by $Tx = \begin{pmatrix} Ax_1 + Bx_2 \\ Cx_1 + Dx_2 \end{pmatrix}$ for every vector $x = (x_1, x_2) \in \mathcal{H} \oplus \mathcal{H}$. Recently, inspired by the numerical radius inequalities for operator matrices in [8, 9, 12, 29]. Very recently, some inequalities for the Berezin number of operator matrices have been presented in [5, 7, 13, 19, 23].

Our goal in this paper is to introduce several new upper bounds for the Berezin number of 2×2 block matrices. More precisely, we prove some upper bounds for the the Berezin number of the off-diagonal operator matrix $\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$ defined on $\mathcal{H} \oplus \mathcal{H}$.

2. Main results

In order to prove our results, we need the following lemmas.

Lemma 2.1. *[3] Let $A, B \in \mathcal{B}(\mathcal{H})$. Then*
(i) $\mathbf{ber}\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) \leq max\{\mathbf{ber}(A), \mathbf{ber}(B)\}$.
(ii) $\mathbf{ber}\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \leq \frac{\|A\|+\|B\|}{2}$.

Lemma 2.2. *[32] Let $A \in \mathcal{B}(\mathcal{H})$ be a positive operator and let $x \in \mathcal{H}$ with $\|x\| = 1$. Then*

$$\langle Ax, x\rangle^r \leq \langle A^r x, x\rangle \text{ for all } r \geq 1.$$

Lemma 2.3. *[29] Let $A \in \mathcal{B}(\mathcal{H})$, and let f and g be non-negative functions on $[0, \infty)$ which are continuous and that satisfy the relation $f(t)\, g(t) = t$ for all $t \in [0, \infty)$. Then*

$$\left|Ax, y\right| \leq \left\|f(|A|)\, x\right\| \left\|g(|A^*|)\, y\right\| \text{ for all } x, y \in \mathcal{H},$$

where $|A| = (A^*A)^{\frac{1}{2}}$ is the absolute value of A.

In particular, if $f(t) = g(t) = \sqrt{t}$, then

$$\left|\langle Ax, y\rangle\right|^2 \leq \langle |A|\, x, x\rangle \langle |A^*|\, y, y\rangle.$$

Lemma 2.4. *[15] Let $x, y, z \in \mathcal{H}$ with $\|z\| = 1$. Then*

$$\left|\langle x, z\rangle \langle z, y\rangle\right| \leq \frac{1}{2}\left(\|x\| \left\|y\right\| + \left|\langle x, y\rangle\right|\right).$$

Lemma 2.5. *[2] Let $x, y \in \mathcal{H}$. Then for any $\alpha \geq 0$,*

$$\left|\langle x, y\rangle\right|^2 \leq \frac{1}{1+\alpha} \|x\| \left\|y\right\| \left|\langle x, y\rangle\right| + \frac{\alpha}{1+\alpha} \|x\|^2 \left\|y\right\|^2 \leq \|x\|^2 \|x\|^2.$$

Lemma 2.6. *[2] Let $x, y, z \in \mathcal{H}$ with $\|z\| = 1$ and let $0 \leq \alpha \leq 1$. Then*

$$\left|\langle x, z\rangle \langle z, y\rangle\right|^2 \leq \frac{2\alpha+1}{2\alpha+2} \|x\|^2 \left\|y\right\|^2 + \frac{1}{2\alpha+2} \|x\| \left\|y\right\| \left|\langle x, y\rangle\right|.$$

Lemma 2.7. *[6] Let $A, B \in \mathcal{B}(\mathcal{H})$. Then*

$$\frac{1}{2}\mathbf{ber}(A+B) \leq \mathbf{ber}\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right).$$

Our first main result in this section reads as follows.

Theorem 2.8. *Let $A, B \in \mathcal{B}(\mathcal{H})$ and let f and g be non-negative functions on $[0, \infty)$ which are continuous and that satisfy the relation $f(t)\, g(t) = t$ for all $t \in [0, \infty)$. Then*

$$\begin{aligned} \mathbf{ber}^2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \quad \leq \quad & \frac{1}{4} \max\left\{\mathbf{ber}\left(f^4(|B|) + f^4(|A^*|)\right), \mathbf{ber}\left(f^4(|A|) + f^4(|B^*|)\right)\right\} \\ & + \frac{1}{2} \max\left\{\mathbf{ber}\left(|A|^2\right), \mathbf{ber}\left(|B|^2\right)\right\}. \end{aligned}$$

Proof. Let $X = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$. For any $(\lambda_1, \lambda_2) \in \Omega \times \Omega$, let $\hat{k}_{(\lambda_1,\lambda_2)} = (k_{\lambda_1}, k_{\lambda_2})$ be the normalized reproducing kernel in $\mathcal{H} \oplus \mathcal{H}$. Then, we have

$$\begin{aligned} & \left|\left\langle X\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|^2 \\ = \quad & \left|\left\langle X\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right| \left|\left\langle X\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right| \\ \leq \quad & \left|\left\langle X\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right| \left\|X\hat{k}_{(\lambda_1,\lambda_2)}\right\| \\ \leq \quad & \frac{1}{2}\left(\left|\left\langle X\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|^2 + \left\|X\hat{k}_{(\lambda_1,\lambda_2)}\right\|^2\right) \end{aligned}$$

$$
\begin{aligned}
& \text{(by the arithmetic-geometric mean inequality)} \\
\leq\ & \frac{1}{2}\left(\left\|f\left(|X|\right)\hat{k}_{(\lambda_1,\lambda_2)}\right\|^2\left\|g\left(|X^*|\right)\hat{k}_{(\lambda_1,\lambda_2)}\right\|^2+\left|\left\langle |X|^2\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|\right) \\
& \text{(by Lemma 2.3)} \\
=\ & \frac{1}{2}\left|\left\langle f^2\left(|X|\right)\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|\left|\left\langle g^2\left(|X^*|\right)\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right| \\
& +\frac{1}{2}\left|\left\langle |X|^2\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right| \\
\leq\ & \frac{1}{4}\left(\left\langle f^2\left(|X|\right)\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle^2+\left\langle g^2\left(|X^*|\right)\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle^2\right) \\
& +\frac{1}{2}\left|\left\langle |X|^2\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right| \\
& \text{(by the arithmetic-geometric mean inequality)} \\
\leq\ & \frac{1}{4}\left(\left\langle f^4\left(|X|\right)\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle+\left\langle g^4\left(|X^*|\right)\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right) \\
& +\frac{1}{2}\left|\left\langle |X|^2\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right| \\
& \text{(by Lemma 2.2)} \\
=\ & \frac{1}{4}\left\langle\left(f^4\left(|X|\right)+g^4\left(|X^*|\right)\right)\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle+\frac{1}{2}\left|\left\langle |X|^2\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right| \\
=\ & \frac{1}{4}\left\langle\left(\begin{bmatrix} f^4\left(|B|\right)+g^4\left(|A^*|\right) & 0 \\ 0 & f^4\left(|A|\right)+g^4\left(|B^*|\right)\end{bmatrix}\right)\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle \\
& +\frac{1}{2}\left|\left\langle\left(\begin{bmatrix} |B|^2 & 0 \\ 0 & |A|^2\end{bmatrix}\right)\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right| \\
\leq\ & \frac{1}{4}\mathbf{ber}\left(\begin{bmatrix} f^4\left(|B|\right)+g^4\left(|A^*|\right) & 0 \\ 0 & f^4\left(|A|\right)+g^4\left(|B^*|\right)\end{bmatrix}\right) \\
& +\frac{1}{2}\mathbf{ber}\left(\begin{bmatrix} |B|^2 & 0 \\ 0 & |A|^2\end{bmatrix}\right) \\
\leq\ & \frac{1}{4}\max\left\{\mathbf{ber}\left(f^4\left(|B|\right)+g^4\left(|A^*|\right)\right),\mathbf{ber}\left(f^4\left(|A|\right)+g^4\left(|B^*|\right)\right)\right\} \\
& +\frac{1}{2}\max\left\{\mathbf{ber}\left(|A|^2\right),\mathbf{ber}\left(|B|^2\right)\right\} \\
& \text{(by Lemma 2.1(i)).}
\end{aligned}
$$

Consequently, we obtain

$$
\begin{aligned}
\left|\left\langle X\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|^2 \leq\ & \frac{1}{4}\max\left\{\mathbf{ber}\left(f^4\left(|B|\right)+g^4\left(|A^*|\right)\right),\mathbf{ber}\left(f^4\left(|A|\right)+g^4\left(|B^*|\right)\right)\right\} \\
& +\frac{1}{2}\max\left\{\mathbf{ber}\left(|A|^2\right),\mathbf{ber}\left(|B|^2\right)\right\}.
\end{aligned}
$$

Taking the supremum over all $(\lambda_1, \lambda_2) \in \Omega \times \Omega$, we get

$$
\begin{aligned}
\mathbf{ber}^2\left(\begin{bmatrix} 0 & A \\ B & 0\end{bmatrix}\right) \leq\ & \frac{1}{4}\max\left\{\mathbf{ber}\left(f^4\left(|B|\right)+g^4\left(|A^*|\right)\right),\mathbf{ber}\left(f^4\left(|A|\right)+g^4\left(|B^*|\right)\right)\right\} \\
& +\frac{1}{2}\max\left\{\mathbf{ber}\left(|A|^2\right),\mathbf{ber}\left(|B|^2\right)\right\}.
\end{aligned}
$$

This completes the proof. □

The following corollaries follow from Theorem 2.8.

Corollary 2.9. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then*

$$\begin{aligned} \mathbf{ber}^2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) &\leq \frac{1}{4}\max\left\{\mathbf{ber}\left(|B|^2+|A^*|^2\right), \mathbf{ber}\left(|A|^2+|B^*|^2\right)\right\} \\ &+\frac{1}{2}\max\left\{\mathbf{ber}\left(|A|^2\right), \mathbf{ber}\left(|B|^2\right)\right\}. \end{aligned}$$

Proof. The result follows from Theorem 2.8 for the case $f(t) = g(t) = t^{\frac{1}{2}}$. □

Corollary 2.10. *Let $A, B \in \mathcal{B}(\mathcal{H})$ and let f and g as in Theorem 2.8. Then*

$$\begin{aligned} \mathbf{ber}^2(A+B) &\leq \max\left\{\mathbf{ber}\left(f^4(|B|)+g^4(|A^*|)\right), \mathbf{ber}\left(f^4(|A|)+g^4(|B^*|)\right)\right\} \\ &+2\max\left\{\mathbf{ber}\left(|A|^2\right), \mathbf{ber}\left(|B|^2\right)\right\}. \end{aligned}$$

Proof. Using Lemma 2.7 the result follows immediately. □

If we take $A = B$ in Corollary 2.10, then we get the following corollary.

Corollary 2.11. *Let $A \in \mathcal{B}(\mathcal{H})$ and let f and g as in Theorem 2.8. Then*

$$\mathbf{ber}^2(A) \leq \frac{1}{4}\mathbf{ber}\left(f^4(|A|)+g^4(|A^*|)\right)+\frac{1}{2}\mathbf{ber}\left(|A|^2\right).$$

Corollary 2.12. *Let $A \in \mathcal{B}(\mathcal{H})$. Then*

$$\mathbf{ber}^2(A) \leq \frac{1}{4}\mathbf{ber}\left(|A|^2+|A^*|^2\right)+\frac{1}{2}\mathbf{ber}\left(|A|^2\right).$$

Proof. The result follows from Corollary 2.11 for the case $f(t) = g(t) = t^{\frac{1}{2}}$. □

Remark 2.13. *We note that the inequality obtained in Corollary 2.12 refines the inequality (1). Indeed, by using the fact $\mathbf{ber}(T) \leq \|T\|$ for every $T \in \mathcal{B}(\mathcal{H})$, it follows that*

$$\begin{aligned} \mathbf{ber}^2(A) &\leq \frac{1}{4}\mathbf{ber}\left(|A|^2+|A^*|^2\right)+\frac{1}{2}\mathbf{ber}\left(|A|^2\right) \\ &\leq \frac{1}{4}\left\||A|^2+|A^*|^2\right\|+\frac{1}{2}\left\||A|^2\right\| \\ &\leq \frac{1}{4}\left(\left\||A|^2\right\|+\left\||A^*|^2\right\|\right)+\frac{1}{2}\left\||A|^2\right\| \\ &= \frac{1}{4}\left(\|A\|^2+\|A\|^2\right)+\frac{1}{2}\|A\|^2 \\ &= \|A\|^2. \end{aligned}$$

Therefore, $\mathbf{ber}^2(A) \leq \|A\|^2$, this implies that $\mathbf{ber}(A) \leq \|A\|$.

In the next theorem, we obtain another upper bound for Berezin number of $\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$.

Theorem 2.14. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then*

$$\begin{aligned} \mathbf{ber}^{2r}\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) &\leq \frac{1}{2}\max\left\{\sqrt{\mathbf{ber}\left(|B|^{2r}\right)}, \sqrt{\mathbf{ber}\left(|A|^{2r}\right)}\right\} \\ &\times\max\left\{\sqrt{\mathbf{ber}\left(|A^*|^{2r}\right)}, \sqrt{\mathbf{ber}\left(|B^*|^{2r}\right)}\right\} \\ &+\frac{1}{2}\max\left\{\mathbf{ber}^r(AB), \mathbf{ber}^r(BA)\right\} \end{aligned}$$

for any $r \geq 1$.

Proof. Let $X=\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$. For any $(\lambda_1,\lambda_2)\in\Omega\times\Omega$, let $\hat{k}_{(\lambda_1,\lambda_2)}=(k_{\lambda_1},k_{\lambda_2})$ be the normalized reproducing kernel in $\mathcal{H}\oplus\mathcal{H}$. Then, we have

$$\begin{aligned}
&\left|\left\langle X\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|^{2r}\\
&= \left(\left|\left\langle X\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|\left|\left\langle \hat{k}_{(\lambda_1,\lambda_2)},X^*\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|\right)^r\\
&\le \frac{1}{2}\left(\left\|X\hat{k}_{(\lambda_1,\lambda_2)}\right\|\left\|X^*\hat{k}_{(\lambda_1,\lambda_2)}\right\|+\left|\left\langle X\hat{k}_{(\lambda_1,\lambda_2)},X^*\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|\right)^r\\
&\quad \text{(by Lemma 2.4)}\\
&\le \frac{1}{2}\left(\left\|X\hat{k}_{(\lambda_1,\lambda_2)}\right\|^r\left\|X^*\hat{k}_{(\lambda_1,\lambda_2)}\right\|^r+\left|\left\langle X\hat{k}_{(\lambda_1,\lambda_2)},X^*\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|^r\right)\\
\text{(by convexity of } f(t) &= t^r \text{ for } r\geq 1)\\
&= \frac{1}{2}\left\langle |X|^2\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle^{\frac{r}{2}}\left\langle |X^*|^2\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle^{\frac{r}{2}}\\
&\quad +\frac{1}{2}\left|\left\langle X^2\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|^r\\
&\le \frac{1}{2}\left\langle |X|^{2r}\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle^{\frac{1}{2}}\left\langle |X^*|^{2r}\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle^{\frac{1}{2}}\\
&\quad +\left|\left\langle X^2\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|^r\\
&= \frac{1}{2}\left\langle\left(\begin{bmatrix} |B|^{2r} & 0 \\ 0 & |A|^{2r}\end{bmatrix}\right)\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle^{\frac{1}{2}}\\
&\quad \times\left\langle\left(\begin{bmatrix} |A^*|^{2r} & 0 \\ 0 & |B^*|^{2r}\end{bmatrix}\right)\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle^{\frac{1}{2}}\\
&\quad +\frac{1}{2}\left|\left\langle\left(\begin{bmatrix} AB & 0 \\ 0 & BA\end{bmatrix}\right)\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|^r\\
&\le \frac{1}{2}\mathbf{ber}^{\frac{1}{2}}\left(\begin{bmatrix} |B|^{2r} & 0 \\ 0 & |A|^{2r}\end{bmatrix}\right)\mathbf{ber}^{\frac{1}{2}}\left(\begin{bmatrix} |A^*|^{2r} & 0 \\ 0 & |B^*|^{2r}\end{bmatrix}\right)\\
&\quad +\frac{1}{2}\mathbf{ber}^r\left(\begin{bmatrix} AB & 0 \\ 0 & BA\end{bmatrix}\right)
\end{aligned}$$

$$\begin{aligned}
&\le \frac{1}{2}\max\left\{\sqrt{\mathbf{ber}\left(|B|^{2r}\right)},\sqrt{\mathbf{ber}\left(|A|^{2r}\right)}\right\}\\
&\quad \times\max\left\{\sqrt{\mathbf{ber}\left(|A^*|^{2r}\right)},\sqrt{\mathbf{ber}\left(|B^*|^{2r}\right)}\right\}\\
&\quad +\frac{1}{2}\max\left\{\mathbf{ber}^r(AB),\mathbf{ber}^r(BA)\right\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\left|\left\langle X\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|^{2r} &\le \frac{1}{2}\max\left\{\sqrt{\mathbf{ber}\left(|B|^{2r}\right)},\sqrt{\mathbf{ber}\left(|A|^{2r}\right)}\right\}\\
&\quad \times\max\left\{\sqrt{\mathbf{ber}\left(|A^*|^{2r}\right)},\sqrt{\mathbf{ber}\left(|B^*|^{2r}\right)}\right\}\\
&\quad +\frac{1}{2}\max\left\{\mathbf{ber}^r(AB),\mathbf{ber}^r(BA)\right\}.
\end{aligned}$$

Taking the supremum over all $(\lambda_1, \lambda_2) \in \Omega \times \Omega$, we get

$$\begin{aligned} \mathbf{ber}^{2r}\left(\left[\begin{array}{cc} 0 & A \\ B & 0 \end{array}\right]\right) &\leq \frac{1}{2}\max\left\{\sqrt{\mathbf{ber}\left(|B|^{2r}\right)}, \sqrt{\mathbf{ber}\left(|A|^{2r}\right)}\right\} \\ &\quad\times\max\left\{\sqrt{\mathbf{ber}\left(|A^*|^{2r}\right)}, \sqrt{\mathbf{ber}\left(|B^*|^{2r}\right)}\right\} \\ &\quad+\frac{1}{2}\max\left\{\mathbf{ber}^r(AB), \mathbf{ber}^r(BA)\right\}. \end{aligned}$$

This completes the proof. □

From Theorem 2.14, we get the following estimate for Berezin number of the sum of two bounded operators.

Corollary 2.15. *If $A, B \in \mathcal{B}(\mathcal{H})$, then*

$$\begin{aligned} \mathbf{ber}^{2r}(A+B) &\leq 2^{2r-1}\max\left\{\sqrt{\mathbf{ber}\left(|B|^{2r}\right)}, \sqrt{\mathbf{ber}\left(|A|^{2r}\right)}\right\}\max\left\{\sqrt{\mathbf{ber}\left(|A^*|^{2r}\right)}, \sqrt{\mathbf{ber}\left(|B^*|^{2r}\right)}\right\} \\ &\quad+2^{2r-1}\max\left\{\mathbf{ber}^r(AB), \mathbf{ber}^r(BA)\right\} \end{aligned}$$

for any $r \geq 1$.

Proof. The result follows immediately from Lemma 2.7. □

In particular, considering $A = B$ in Corollary 2.15, we get the following corollary.

Corollary 2.16. *Let $A \in \mathcal{B}(\mathcal{H})$. Then*

$$\mathbf{ber}^{2r}(A) \leq \frac{1}{2}\sqrt{\mathbf{ber}\left(|A|^{2r}\right)\mathbf{ber}\left(|A^*|^{2r}\right)} + \frac{1}{2}\mathbf{ber}^r\left(A^2\right)$$

for any $r \geq 1$.

Remark 2.17. *It is not difficult to check that the inequality in Corollary 2.16 is an improvement of the inequality (1).*

Next, we state now another upper bound for Berezin number of $\left[\begin{array}{cc} 0 & A \\ B & 0 \end{array}\right]$.

Theorem 2.18. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then for any $r \geq 1$,*

$$\begin{aligned} &\mathbf{ber}^{2r}\left(\left[\begin{array}{cc} 0 & A \\ B & 0 \end{array}\right]\right) \\ \leq\ &\frac{1}{4}\max\left\{\mathbf{ber}\left(|B|^{2r}+|A^*|^{2r}\right), \mathbf{ber}\left(|A|^{2r}+|B^*|^{2r}\right)\right\} \\ &+\frac{1}{2}\left\{\mathbf{ber}\left(|A^*|^r|B|^r\right), \mathbf{ber}\left(|B^*|^r|A|^r\right)\right\}. \end{aligned}$$

Proof. Let $X = \left[\begin{array}{cc} 0 & A \\ B & 0 \end{array}\right]$. For any $(\lambda_1, \lambda_2) \in \Omega \times \Omega$, let $\hat{k}_{(\lambda_1,\lambda_2)} = (k_{\lambda_1}, k_{\lambda_2})$ be the normalized reproducing kernel in $\mathcal{H} \oplus \mathcal{H}$. Then, we have

$$\begin{aligned} &\left|\left\langle X\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|^{2r} \\ \leq\ &\left(\left|\left\langle |X|\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|\left|\left\langle |X^*|\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|\right)^r \end{aligned}$$

$$
\begin{aligned}
& \quad \text{(by Lemma 2.3)} \\
\leq & \left|\left\langle |X|^{r}\,\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|\left|\left\langle |X^{*}|^{r}\,\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right| \\
& \quad \text{(by Lemma 2.2)} \\
= & \left|\left\langle |X|^{r}\,\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|\left|\left\langle \hat{k}_{(\lambda_1,\lambda_2)},|X^{*}|^{r}\,\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right| \\
\leq & \frac{1}{2}\left(\left\||X|^{r}\,\hat{k}_{(\lambda_1,\lambda_2)}\right\|\left\||X^{*}|^{r}\,\hat{k}_{(\lambda_1,\lambda_2)}\right\|+\left|\left\langle |X|^{r}\,\hat{k}_{(\lambda_1,\lambda_2)},|X^{*}|^{r}\,\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|\right) \\
& \quad \text{(by Lemma 2.4)} \\
\leq & \frac{1}{4}\left(\left\||X|^{r}\,\hat{k}_{(\lambda_1,\lambda_2)}\right\|^{2}+\left\||X^{*}|^{r}\,\hat{k}_{(\lambda_1,\lambda_2)}\right\|^{2}\right)+\frac{1}{2}\left|\left\langle |X^{*}|^{r}\,|X|^{r}\,\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right| \\
& \quad \text{(by the arithmetic-geometric mean inequality)} \\
= & \frac{1}{4}\left\langle\left(|X|^{2r}+|X^{*}|^{2r}\right)\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle+\frac{1}{2}\left|\left\langle |X^{*}|^{r}\,|X|^{r}\,\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right| \\
= & \frac{1}{4}\left\langle\left(\begin{bmatrix} |B|^{2r}+|A^{*}|^{2r} & 0 \\ 0 & |A|^{2r}+|B^{*}|^{2r}\end{bmatrix}\right)\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle \\
& +\frac{1}{2}\left|\left\langle\left(\begin{bmatrix} |A^{*}|^{r}\,|B|^{r} & 0 \\ 0 & |B^{*}|^{r}\,|A|^{r}\end{bmatrix}\right)\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right| \\
\leq & \frac{1}{4}\mathbf{ber}\left(\begin{bmatrix} |B|^{2r}+|A^{*}|^{2r} & 0 \\ 0 & |A|^{2r}+|B^{*}|^{2r}\end{bmatrix}\right) \\
& +\frac{1}{2}\mathbf{ber}\left(\begin{bmatrix} |A^{*}|^{r}\,|B|^{r} & 0 \\ 0 & |B^{*}|^{r}\,|A|^{r}\end{bmatrix}\right) \\
\leq & \frac{1}{4}\max\left\{\mathbf{ber}\left(|B|^{2r}+|A^{*}|^{2r}\right),\mathbf{ber}\left(|A|^{2r}+|B^{*}|^{2r}\right)\right\} \\
& +\frac{1}{2}\left\{\mathbf{ber}\left(|A^{*}|^{r}\,|B|^{r}\right),\mathbf{ber}\left(|B^{*}|^{r}\,|A|^{r}\right)\right\}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\left|\left\langle X\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|^{2r} \leq\ & \frac{1}{4}\max\left\{\mathbf{ber}\left(|B|^{2r}+|A^{*}|^{2r}\right),\mathbf{ber}\left(|A|^{2r}+|B^{*}|^{2r}\right)\right\} \\
& +\frac{1}{2}\left\{\mathbf{ber}\left(|A^{*}|^{r}\,|B|^{r}\right),\mathbf{ber}\left(|B^{*}|^{r}\,|A|^{r}\right)\right\}.
\end{aligned}
$$

Taking the supremum over all $(\lambda_1,\lambda_2)\in\Omega\times\Omega$, we get

$$
\begin{aligned}
\mathbf{ber}^{2r}\left(\begin{bmatrix} 0 & A \\ B & 0\end{bmatrix}\right) \leq\ & \frac{1}{4}\max\left\{\mathbf{ber}\left(|B|^{2r}+|A^{*}|^{2r}\right),\mathbf{ber}\left(|A|^{2r}+|B^{*}|^{2r}\right)\right\} \\
& +\frac{1}{2}\mathbf{ber}\left\{\left(|A^{*}|^{r}\,|B|^{r}\right),\mathbf{ber}\left(|B^{*}|^{r}\,|A|^{r}\right)\right\}.
\end{aligned}
$$

This completes the proof. □

By considering $r=1$ in Theorem 2.18, we get the following corollary.

Corollary 2.19. *Let $A,B\in\mathcal{B}(\mathcal{H})$. Then*

$$
\begin{aligned}
\mathbf{ber}^{2}\left(\begin{bmatrix} 0 & A \\ B & 0\end{bmatrix}\right) \leq\ & \frac{1}{4}\max\left\{\mathbf{ber}\left(|B|^{2}+|A^{*}|^{2}\right),\mathbf{ber}\left(|A|^{2}+|B^{*}|^{2}\right)\right\} \\
& +\frac{1}{2}\max\left\{\mathbf{ber}\left(|A^{*}|\,|B|\right),\mathbf{ber}\left(|B^{*}|\,|A|\right)\right\}.
\end{aligned}
$$

Remark 2.20. *In [13] the authors showed the following inequality:*

$$\mathbf{ber}^2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \leq \frac{1}{2}\max\left\{\mathbf{ber}\left(|B|^2+|A^*|^2\right), \mathbf{ber}\left(|A|^2+|B^*|^2\right)\right\}. \tag{2}$$

If we take, $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, then from Corollary 2.19, we get $\mathbf{ber}\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \leq 0.5$, whereas the inequality (2) gives $\mathbf{ber}\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \leq 0.7$. Hence the bound obtained in Corollary 2.19 is better than that given in (2).

Remark 2.21. *We observe that the inequality obtained in Corollary 2.19 refines the inequality Lemma 2.1 (ii). Indeed, we have*

$$\begin{aligned}
&\mathbf{ber}^2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \\
\leq\ & \frac{1}{4}\max\left\{\mathbf{ber}\left(|B|^2+|A^*|^2\right), \mathbf{ber}\left(|A|^2+|B^*|^2\right)\right\} \\
&+\frac{1}{2}\left\{\mathbf{ber}\left(|A^*|\,|B|\right), \mathbf{ber}\left(|B^*|\,|A|\right)\right\} \\
\leq\ & \frac{1}{4}\max\left\{\left\||B|^2+|A^*|^2\right\|, \left\||A|^2+|B^*|^2\right\|\right\} \\
&+\frac{1}{2}\max\left\{\||A^*|\,|B|\|, \||B^*|\,|A|\|\right\} \\
\leq\ & \frac{1}{4}\max\left\{\left\||B|^2\right\|+\left\||A^*|^2\right\|, \left\||A|^2\right\|+\left\||B^*|^2\right\|\right\} \\
&+\frac{1}{2}\max\left\{\||A^*|\|\,\||B|\|, \||B^*|\|\,\||A|\|\right\} \\
=\ & \frac{1}{4}\left(\|A\|^2+\|B\|^2\right)+\frac{1}{2}\|A\|\,\|B\| \\
=\ & \frac{1}{4}\left(\|A\|^2+\|B\|^2+2\|A\|\,\|B\|\right) \\
=\ & \frac{1}{4}\left(\|A\|+\|B\|\right)^2.
\end{aligned}$$

Therefore,

$$\mathbf{ber}\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \leq \frac{\|A\|+\|B\|}{2}.$$

Corollary 2.22. *Let $A, B \in \mathcal{B}(\mathcal{H})$ and let $r \geq 1$. Then*

$$\begin{aligned}
\mathbf{ber}^{2r}(A+B) \leq\ & 2^{2r-2}\max\left\{\mathbf{ber}\left(|B|^{2r}+|A^*|^{2r}\right), \mathbf{ber}\left(|A|^{2r}+|B^*|^{2r}\right)\right\} \\
&+2^{2r-1}\max\left\{\mathbf{ber}\left(|A^*|^r\,|B|^r\right), \mathbf{ber}\left(|B^*|^r\,|A|^r\right)\right\}.
\end{aligned}$$

Proof. By using Lemma 2.7, the result follows directly. □

Remark 2.23. *If we take $A = B$ in Corollary 2.22, then we get the following inequality*

$$\mathbf{ber}^{2r}(A) \leq \frac{1}{4}\mathbf{ber}\left(|A|^{2r}+|A^*|^2\right)+\frac{1}{2}\mathbf{ber}\left(|A^{*r}|\,|A|^r\right),$$

which is obtined in [14].

In the next theorem, we establish an upper bound of $\mathbf{ber}^4\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right)$.

Theorem 2.24. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then*

$$\begin{aligned}
\mathbf{ber}^4\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \leq\ & \frac{1}{16}\max\left\{\mathbf{ber}\left(|B|^4+|A^*|^4\right), \mathbf{ber}\left(|A|^4+|B^*|^4\right)\right\} \\
&+\frac{1}{8}\max\left\{\mathbf{ber}\left(|A^*|^2|B|^2\right), \mathbf{ber}\left(|B^*|^2|A|^2\right)\right\} \\
&+\frac{1}{4}\max\left\{\mathbf{ber}\left(|B|^2+|A^*|^2\right), \mathbf{ber}\left(|A|^2+|B^*|^2\right)\right\} \\
&\times\max\left\{\mathbf{ber}(AB), \mathbf{ber}(BA)\right\}+\frac{1}{4}\max\left\{\mathbf{ber}^2(AB), \mathbf{ber}^2(BA)\right\}.
\end{aligned}$$

Proof. Let $X=\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$. For any $(\lambda_1,\lambda_2)\in\Omega\times\Omega$, let $\hat{k}_{(\lambda_1,\lambda_2)}=(k_{\lambda_1},k_{\lambda_2})$ be the normalized reproducing kernel in $\mathcal{H}\oplus\mathcal{H}$. Then, we have

$$\begin{aligned}
& \left\|X\hat{k}_{(\lambda_1,\lambda_2)}\right\|^2\left\|X^*\hat{k}_{(\lambda_1,\lambda_2)}\right\|^2 \\
= & \left|\left\langle |X|^2\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|\left|\left\langle |X^*|^2\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right| \\
= & \left|\left\langle |X|^2\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|\left|\left\langle \hat{k}_{(\lambda_1,\lambda_2)}, |X^*|^2\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right| \\
\leq & \frac{1}{2}\left(\left\||X|^2\hat{k}_{(\lambda_1,\lambda_2)}\right\|\left\||X^*|^2\hat{k}_{(\lambda_1,\lambda_2)}\right\|+\left\langle |X|^2\hat{k}_{(\lambda_1,\lambda_2)}, |X^*|^2\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right) \\
& \text{(by Lemma 2.4)} \\
\leq & \frac{1}{4}\left(\left\||X|^2\hat{k}_{(\lambda_1,\lambda_2)}\right\|^2+\left\||X^*|^2\hat{k}_{(\lambda_1,\lambda_2)}\right\|^2\right)+\frac{1}{2}\left\langle |X^*|^2|X|^2\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle \\
& \text{(by the arithmetic-geometric mean inequality)} \\
\leq & \frac{1}{4}\left\langle \left(|X|^4+|X^*|^4\right)\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle+\frac{1}{2}\left\langle |X^*|^2|X|^2\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle.
\end{aligned}$$

Therefore, we infer that

$$\left\|X\hat{k}_{(\lambda_1,\lambda_2)}\right\|^2\left\|X^*\hat{k}_{(\lambda_1,\lambda_2)}\right\|^2 \leq \frac{1}{4}\left\langle \left(|X|^4+|X^*|^4\right)\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle+\frac{1}{2}\left\langle |X^*|^2|X|^2\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle. \tag{3}$$

On the other hand, we have

$$\begin{aligned}
& \left|\left\langle X\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|^4 \\
= & \left(\left|\left\langle X\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|\left|\left\langle \hat{k}_{(\lambda_1,\lambda_2)}, X^*\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|\right)^2 \\
\leq & \frac{1}{4}\left(\left\|X\hat{k}_{(\lambda_1,\lambda_2)}\right\|\left\|X^*\hat{k}_{(\lambda_1,\lambda_2)}\right\|+\left\langle X\hat{k}_{(\lambda_1,\lambda_2)}, X^*\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right)^2 \\
& \text{(by Lemma 2.4)} \\
= & \frac{1}{4}\left\|X\hat{k}_{(\lambda_1,\lambda_2)}\right\|^2\left\|X^*\hat{k}_{(\lambda_1,\lambda_2)}\right\|^2 \\
& +\frac{1}{2}\left\|X\hat{k}_{(\lambda_1,\lambda_2)}\right\|\left\|X^*\hat{k}_{(\lambda_1,\lambda_2)}\right\|\left\langle X^2\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle+\frac{1}{4}\left|\left\langle X^2\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|^2 \\
\leq & \frac{1}{4}\left\|X\hat{k}_{(\lambda_1,\lambda_2)}\right\|^2\left\|X^*\hat{k}_{(\lambda_1,\lambda_2)}\right\|^2
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{4}\left(\left\|X\hat{k}_{(\lambda_1,\lambda_2)}\right\|^2+\left\|X^*\hat{k}_{(\lambda_1,\lambda_2)}\right\|^2\right)\left\langle X^2\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle+\frac{1}{4}\left|\left\langle X^2\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|^2\\
&\text{(by the arithmetic-geometric mean inequality)}\\
= &\frac{1}{4}\left\|X\hat{k}_{(\lambda_1,\lambda_2)}\right\|^2\left\|X^*\hat{k}_{(\lambda_1,\lambda_2)}\right\|^2\\
&+\frac{1}{4}\left\langle\left(|X|^2+|X^*|^2\right)\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\left\langle X^2\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle+\frac{1}{4}\left|\left\langle X^2\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|^2\\
\leq &\frac{1}{16}\left\langle\left(|X|^4+|X^*|^4\right)\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle+\frac{1}{8}\left\langle|X^*|^2|X|^2\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\\
&+\frac{1}{4}\left\langle\left(|X|^2+|X^*|^2\right)\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\left\langle X^2\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle+\frac{1}{4}\left|\left\langle X^2\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|^2\\
&\text{(by the inequality (3))}\\
= &\frac{1}{16}\left\langle\left(\begin{bmatrix}|B|^4+|A^*|^4 & 0\\ 0 & |A|^4+|B^*|^4\end{bmatrix}\right)\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\\
&+\frac{1}{8}\left\langle\left(\begin{bmatrix}|A^*|^2|B|^2 & 0\\ 0 & |B^*|^2|A|^2\end{bmatrix}\right)\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\\
&+\frac{1}{4}\left\langle\left(\begin{bmatrix}|B|^2+|A^*|^2 & 0\\ 0 & |A|^2+|B^*|^2\end{bmatrix}\right)\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\\
&\times\left|\left\langle\left(\begin{bmatrix}AB & 0\\ 0 & BA\end{bmatrix}\right)\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|+\frac{1}{4}\left|\left\langle\left(\begin{bmatrix}AB & 0\\ 0 & BA\end{bmatrix}\right)\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|^2\\
\leq &\frac{1}{16}\mathbf{ber}\left(\begin{bmatrix}|B|^4+|A^*|^4 & 0\\ 0 & |A|^4+|B^*|^4\end{bmatrix}\right)+\frac{1}{8}\mathbf{ber}\left(\begin{bmatrix}|A^*|^2|B|^2 & 0\\ 0 & |B^*|^2|A|^2\end{bmatrix}\right)\\
&+\frac{1}{4}\mathbf{ber}\left(\begin{bmatrix}|B|^2+|A^*|^2 & 0\\ 0 & |A|^2+|B^*|^2\end{bmatrix}\right)\times\mathbf{ber}\left(\begin{bmatrix}AB & 0\\ 0 & BA\end{bmatrix}\right)+\frac{1}{4}\mathbf{ber}^2\left(\begin{bmatrix}AB & 0\\ 0 & BA\end{bmatrix}\right)\\
\leq &\frac{1}{16}\max\left\{\mathbf{ber}\left(|B|^4+|A^*|^4\right),\mathbf{ber}\left(|A|^4+|B^*|^4\right)\right\}+\frac{1}{8}\max\left\{\mathbf{ber}\left(|A^*|^2|B|^2\right),\mathbf{ber}\left(|B^*|^2|A|^2\right)\right\}\\
&+\frac{1}{4}\max\left\{\mathbf{ber}\left(|B|^2+|A^*|^2\right),\mathbf{ber}\left(|A|^2+|B^*|^2\right)\right\}\times\max\left\{\mathbf{ber}(AB),\mathbf{ber}(BA)\right\}\\
&+\frac{1}{4}\max\left\{\mathbf{ber}^2(AB),\mathbf{ber}^2(BA)\right\}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\left|\left\langle X\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|^4 \leq\ &\frac{1}{16}\max\left\{\mathbf{ber}\left(|B|^4+|A^*|^4\right),\mathbf{ber}\left(|A|^4+|B^*|^4\right)\right\}\\
&+\frac{1}{8}\max\left\{\mathbf{ber}\left(|A^*|^2|B|^2\right),\mathbf{ber}\left(|B^*|^2|A|^2\right)\right\}\\
&+\frac{1}{4}\max\left\{\mathbf{ber}\left(|B|^2+|A^*|^2\right),\mathbf{ber}\left(|A|^2+|B^*|^2\right)\right\}\\
&\times\max\left\{\mathbf{ber}(AB),\mathbf{ber}(BA)\right\}+\frac{1}{4}\max\left\{\mathbf{ber}^2(AB),\mathbf{ber}^2(BA)\right\}.
\end{aligned}
$$

Taking the supremum over all $(\lambda_1,\lambda_2)\in\Omega\times\Omega$, we get the desired result. □

Remark 2.25. *Very recently in [21, Theorem 4] the authors showed the following inequality:*

$$
\begin{aligned}
\mathbf{ber}^4\left(\begin{bmatrix}0 & A\\ B & 0\end{bmatrix}\right) \leq\ &\frac{1}{4}\max\left\{\mathbf{ber}\left(|B|^4+|A^*|^4\right),\mathbf{ber}\left(|A|^4+|B^*|^4\right)\right\}\\
&+\frac{1}{2}\max\left\{\mathbf{ber}\left(|A^*|^2|B|^2\right),\mathbf{ber}\left(|B^*|^2|A|^2\right)\right\}.
\end{aligned}\tag{4}
$$

If we take, $A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$, then from Theorem 2.24, we get $\mathbf{ber}\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \le 1.5$, whereas the inequality (4) gives $\mathbf{ber}\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \le 2.12$. Hence the bound obtained in Theorem 2.24 is better than that given in (4).

The following corollaries are immediate consequences of Theorem 2.24.

Corollary 2.26. *If $A, B \in \mathcal{B}(\mathcal{H})$ are normal operators, then*

$$\begin{aligned} \mathbf{ber}^4\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \le\ & \frac{1}{16}\mathbf{ber}\left(|A|^4 + |B|^4\right) + \frac{1}{8}\max\left\{\mathbf{ber}\left(|A|^2|B|^2\right), \mathbf{ber}\left(|B|^2|A|^2\right)\right\} \\ & + \frac{1}{4}\mathbf{ber}\left(|A|^2 + |B|^2\right)\max\{\mathbf{ber}(AB), \mathbf{ber}(BA)\} \\ & + \frac{1}{4}\max\left\{\mathbf{ber}^2(AB), \mathbf{ber}^2(BA)\right\}. \end{aligned}$$

Corollary 2.27. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then*

$$\begin{aligned} \mathbf{ber}^4(A+B) \le\ & \max\left\{\mathbf{ber}\left(|B|^4 + |A^*|^4\right), \mathbf{ber}\left(|A|^4 + |B^*|^4\right)\right\} \\ & + 2\max\left\{\mathbf{ber}\left(|A^*|^2|B|^2\right), \mathbf{ber}\left(|B^*|^2|A|^2\right)\right\} \\ & + 4\max\left\{\mathbf{ber}\left(|B|^2 + |A^*|^2\right), \mathbf{ber}\left(|A|^2 + |B^*|^2\right)\right\}\max\{\mathbf{ber}(AB), \mathbf{ber}(BA)\} \\ & + 4\max\left\{\mathbf{ber}^2(AB), \mathbf{ber}^2(BA)\right\}. \end{aligned}$$

Proof. By using Lemma 2.7, the result follows directly. □

If we take $A = B$ in Corollary 2.27, then we get the following corollary.

Corollary 2.28. *Let $A \in \mathcal{B}(\mathcal{H})$. Then*

$$\begin{aligned} \mathbf{ber}^4(A) \le\ & \frac{1}{16}\mathbf{ber}\left(|A|^4 + |A^*|^4\right) + \frac{1}{8}\mathbf{ber}\left(|A^*|^2|A|^2\right) \\ & + \frac{1}{4}\mathbf{ber}\left(|A|^2 + |A^*|^2\right)\mathbf{ber}\left(A^2\right) + \frac{1}{4}\mathbf{ber}^2\left(A^2\right). \end{aligned}$$

Remark 2.29. *It is not difficult to check that the inequality in Corollary 2.28 is an improvement of the inequality (1).*

In the next theorem we obtain another upper bound of $\mathbf{ber}^4\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right)$.

Theorem 2.30. *Let $A, B \in \mathcal{B}(\mathcal{H})$ and let $0 \le \alpha \le 1$. Then*

$$\begin{aligned} \mathbf{ber}^4\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \le\ & \frac{2\alpha+1}{4\alpha+4}\max\left\{\mathbf{ber}\left(|B|^4 + |A^*|^4\right), \mathbf{ber}\left(|A|^4 + |B^*|^4\right)\right\} \\ & + \frac{1}{4\alpha+4}\max\left\{\mathbf{ber}\left(|B|^2 + |A^*|^2\right), \mathbf{ber}\left(|A|^2 + |B^*|^2\right)\right\} \\ & \times\max\{\mathbf{ber}(AB), \mathbf{ber}(BA)\}. \end{aligned}$$

Proof. Let $X = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$. For any $(\lambda_1, \lambda_2) \in \Omega \times \Omega$, let $\hat{k}_{(\lambda_1,\lambda_2)} = (k_{\lambda_1}, k_{\lambda_2})$ be the normalized reproducing kernel in $\mathcal{H} \oplus \mathcal{H}$. Then, we have

$$\left|\left\langle X\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|^4$$

$$
\begin{aligned}
&= \left|\left\langle X\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|^2 \left|\left\langle X\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|^2 \\
&= \left|\left\langle X\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|^2 \left|\left\langle \hat{k}_{(\lambda_1,\lambda_2)}, X^*\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|^2 \\
&\leq \frac{2\alpha+1}{2\alpha+2}\left\|X\hat{k}_{(\lambda_1,\lambda_2)}\right\|^2\left\|X^*\hat{k}_{(\lambda_1,\lambda_2)}\right\|^2 + \frac{1}{2\alpha+2}\left\|X\hat{k}_{(\lambda_1,\lambda_2)}\right\|\left\|X^*\hat{k}_{(\lambda_1,\lambda_2)}\right\|\left|\left\langle X\hat{k}_{(\lambda_1,\lambda_2)}, X^*\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right| \\
&\quad \text{(by Lemma 2.6)} \\
&= \frac{2\alpha+1}{2\alpha+2}\left\langle |X|^2\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\left\langle |X^*|^2\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle \\
&\quad + \frac{1}{2\alpha+2}\left\langle |X|^2\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle^{\frac{1}{2}}\left\langle |X^*|^2\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle^{\frac{1}{2}}\left|\left\langle X^2\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right| \\
&\leq \frac{2\alpha+1}{4\alpha+4}\left(\left\langle |X|^2\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle^2 + \left\langle |X^*|^2\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle^2\right) \\
&\quad + \frac{1}{4\alpha+4}\left(\left\langle |X|^2\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle + \left\langle |X^*|^2\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right)\left|\left\langle X^2\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right| \\
&\quad \text{(by the arithmetic-geometric mean inequality)} \\
&\leq \frac{2\alpha+1}{4\alpha+4}\left\langle\left(|X|^4 + |X^*|^4\right)\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle \\
&\quad + \frac{1}{4\alpha+4}\left\langle\left(|X|^2 + |X^*|^2\right)\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\left|\left\langle X^2\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right| \\
&\quad \text{(by Lemma 2.2)} \\
&= \frac{2\alpha+1}{4\alpha+4}\left\langle\left(\begin{bmatrix} |B|^4 + |A^*|^4 & 0 \\ 0 & |A|^4 + |B^*|^4 \end{bmatrix}\right)\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle \\
&\quad + \frac{1}{4\alpha+4}\left\langle\left(\begin{bmatrix} |B|^2 + |A^*|^2 & 0 \\ 0 & |A|^2 + |B^*|^2 \end{bmatrix}\right)\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle \\
&\quad \times\left|\left\langle\left(\begin{bmatrix} AB & 0 \\ 0 & BA \end{bmatrix}\right)\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right| \\
&\leq \frac{2\alpha+1}{4\alpha+4}\mathbf{ber}\left(\begin{bmatrix} |B|^4 + |A^*|^4 & 0 \\ 0 & |A|^4 + |B^*|^4 \end{bmatrix}\right) + \frac{1}{4\alpha+4}\mathbf{ber}\left(\begin{bmatrix} |B|^2 + |A^*|^2 & 0 \\ 0 & |A|^2 + |B^*|^2 \end{bmatrix}\right) \\
&\quad \times\mathbf{ber}\left(\begin{bmatrix} AB & 0 \\ 0 & BA \end{bmatrix}\right) \\
&\leq \frac{2\alpha+1}{4\alpha+4}\max\left\{\mathbf{ber}\left(|B|^4 + |A^*|^4\right), \mathbf{ber}\left(|A|^4 + |B^*|^4\right)\right\} \\
&\quad + \frac{1}{4\alpha+4}\max\left\{\mathbf{ber}\left(|B|^2 + |A^*|^2\right), \mathbf{ber}\left(|A|^2 + |B^*|^2\right)\right\}\max\left\{\mathbf{ber}(AB), \mathbf{ber}(BA)\right\}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\left|\left\langle X\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|^4 \leq\ & \frac{2\alpha+1}{4\alpha+4}\max\left\{\mathbf{ber}\left(|B|^4 + |A^*|^4\right), \mathbf{ber}\left(|A|^4 + |B^*|^4\right)\right\} \\
& + \frac{1}{4\alpha+4}\max\left\{\mathbf{ber}\left(|B|^2 + |A^*|^2\right), \mathbf{ber}\left(|A|^2 + |B^*|^2\right)\right\} \\
& \times\max\left\{\mathbf{ber}(AB), \mathbf{ber}(BA)\right\}.
\end{aligned}
$$

Taking the supremum over all $(\lambda_1, \lambda_2) \in \Omega \times \Omega$, we get the desired inequality. □

As applications of Theorem 2.30, we state the following two corollaries.

Corollary 2.31. *If $A, B \in \mathcal{B}(\mathcal{H})$ are normal operators such that $AB = BA$ and let $0 \leq \alpha \leq 1$, then*

$$\mathbf{ber}^4\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \leq \frac{2\alpha+1}{4\alpha+4}\mathbf{ber}\left(|A|^4 + |B|^4\right) + \frac{1}{4\alpha+4}\mathbf{ber}\left(|A|^2 + |B|^2\right)\mathbf{ber}\left(AB\right).$$

Corollary 2.32. *Let $A, B \in \mathcal{B}(\mathcal{H})$ and let $0 \leq \alpha \leq 1$. Then*

$$\begin{aligned} \mathbf{ber}^4\left(A+B\right) \leq\ & \frac{8\alpha+4}{\alpha+1}\max\left\{\mathbf{ber}\left(|B|^4 + |A^*|^4\right), \mathbf{ber}\left(|A|^4 + |B^*|^4\right)\right\} \\ & + \frac{4}{\alpha+1}\max\left\{\mathbf{ber}\left(|B|^2 + |A^*|^2\right), \mathbf{ber}\left(|A|^2 + |B^*|^2\right)\right\} \\ & \times \max\left\{\mathbf{ber}\left(AB\right), \mathbf{ber}\left(BA\right)\right\}. \end{aligned}$$

Proof. The result follows directly by Lemma 2.7. □

Corollary 2.33. *Let $A \in \mathcal{B}(\mathcal{H})$ and let $0 \leq \alpha \leq 1$. Then*

$$\mathbf{ber}^4\left(A\right) \leq \frac{2\alpha+1}{4\alpha+4}\mathbf{ber}\left(|A|^4 + |A^*|^4\right) + \frac{1}{4\alpha+4}\mathbf{ber}\left(|A|^2 + |A^*|^2\right)\mathbf{ber}\left(A^2\right).$$

Remark 2.34. *If we take $\alpha = 1$ in Corollary 2.33, then we get the following inequality*

$$\mathbf{ber}^4\left(A\right) \leq \frac{3}{8}\mathbf{ber}\left(|A|^4 + |A^*|^4\right) + \frac{1}{8}\mathbf{ber}\left(|A|^2 + |A^*|^2\right)\mathbf{ber}\left(A^2\right),$$

which obtined recently in [22].

Remark 2.35. *We note that the inequality obtained in Corollary 2.33 refines the inequality (1). Indeed, we have*

$$\begin{aligned} \mathbf{ber}^4\left(A\right) &\leq \frac{2\alpha+1}{4\alpha+4}\mathbf{ber}\left(|A|^4 + |A^*|^4\right) + \frac{1}{4\alpha+4}\mathbf{ber}\left(|A|^2 + |A^*|^2\right)\mathbf{ber}\left(A^2\right) \\ &\leq \frac{2\alpha+1}{4\alpha+4}\left\||A|^4 + |A^*|^4\right\| + \frac{1}{4\alpha+4}\left\||A|^2 + |A^*|^2\right\|\left\|A^2\right\| \\ &\leq \frac{2\alpha+1}{4\alpha+4}\left(\left\||A|^4\right\| + \left\||A^*|^4\right\|\right) + \frac{1}{4\alpha+4}\left(\left\||A|^2\right\| + \left\||A^*|^2\right\|\right)\|A\|^2 \\ &= \frac{2\alpha+1}{4\alpha+4}\left(2\|A\|^4\right) + \frac{1}{4\alpha+4}\left(2\|A\|^2\right)\|A\|^2 \\ &= \frac{4\alpha+2}{4\alpha+4}\|A\|^4 + \frac{2}{4\alpha+4}\|A\|^4 \\ &= \|A\|^4. \end{aligned}$$

Therefore,

$$\mathbf{ber}^4\left(A\right) \leq \|A\|^4.$$

This implies that

$$\mathbf{ber}\left(A\right) \leq \|A\|.$$

Declarations

Conflict of interest The authors declare that there is no conflict of interest.

Data availability No data was used for the research described in the article.

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