



Modified Rota-Baxter Lie triple systems with higher derivations

Yuanyuan Ke^{a,*}, Dingguo Wang^{b,c}

^a*School of Artificial Intelligence, Jiangnan University, Wuhan 430056, P. R. China*

^b*Department of General Education, Shandong Xiehe University, Jinan 250109, P. R. China*

^c*School of Mathematical Sciences, Qufu Normal University, Qufu 273165, P. R. China*

Abstract. In this paper, we consider modified Rota-Baxter Lie triple systems with higher derivations. A pair consisting of a modified Rota-Baxter Lie triple system and a higher derivation is called a modified Rota-Baxter LieTHDer pair. We investigate cohomologies of modified Rota-Baxter LieTHDer pairs with coefficients in a suitable representation. As applications, we study abelian extensions and formal one-parameter deformations of modified Rota-Baxter LieTHDer pairs.

1. Introduction

Lie triple systems first appeared in Cartan's research on Riemannian geometry and were developed for symmetric spaces and related spaces. Building on Jordan theory and quantum mechanics, Jacobson [11] introduced the concepts of Lie triple systems and Jordan triple systems, which can be regarded as subspaces of an enclosed ternary Lie algebra. Lister [13] gave a complete structure theory for Lie triple systems. Simpler axioms were given by Yamaguti [21]. More results on Lie triple systems are referred to [1, 9, 10, 17, 18, 20, 23].

Deformation serves as a method for analyzing mathematical objects by transforming them into parameterized families of similar structures. Algebraic deformation theory was introduced by Gerstenhaber for rings and algebras [5]. Kubo and Taniguchi in [12] introduced deformation theory for Lie triple systems. Various linear operators like relative Rota-Baxter operator (also called $\mathcal{O}$ -operator) [3, 14], Rota-Baxter operator [2], generalized Reynolds operator [16], modified Rota-Baxter operator [6], crossed homomorphism [8], derivation [7, 15, 19], higher derivation [4] and their induced cohomology, extensions and deformations are studied.

Recently, the authors have established the cohomology of modified Rota-Baxter Lie triple systems and analyzed the relationship among modified Rota-Baxter Lie triple systems, Rota-Baxter Lie triple systems,

2020 *Mathematics Subject Classification.* Primary 17A40; Secondary 17A36, 17B10, 17B56.

Keywords. Modified Rota-Baxter Lie triple system; higher derivation; representation; cohomology; abelian extension; deformation.

Received: 18 March 2025; Revised: 22 July 2025; Accepted: 27 July 2025

Communicated by Dijana Mosić

Research supported by the National Natural Science Foundation of China (Grant No. 12426643, 12426644, 12471036, 12271292), Scientific Research Project of Department of Education of Hubei Province (No. Q20234405) and the Research Fund of Jiangnan University (No. 2023JCYJ10).

* Corresponding author: Yuanyuan Ke

Email addresses: keyy086@126.com (Yuanyuan Ke), dingguo95@126.com (Dingguo Wang)

ORCID iDs: <https://orcid.org/0000-0003-1123-5289> (Yuanyuan Ke), <https://orcid.org/0000-0001-7788-7613> (Dingguo Wang)

and Nijenhuis Lie triple systems in [6]. It would be particularly meaningful to investigate modified Rota-Baxter Lie triple systems endowed with higher derivations (termed modified Rota-Baxter LieTHDer pairs), as higher derivations play a crucial role in the theory of automorphisms of complete local rings. Drawing on the prior work, we can proceed with this study, and the subsequent sections of the paper are organized as follows. In Section 2, we recall some basic definitions about modified Rota-Baxter Lie triple systems and their cohomology. Section 3 explores the cohomologies of modified Rota-Baxter LieTHDer pairs, with a focus on representations endowed with suitable coefficients. Section 4 then investigates abelian extensions of these pairs, analyzing their structural implications within the algebraic framework. Finally, Section 5 delves into the formal one-parameter deformations of modified Rota-Baxter LieTHDer pairs, examining their foundational properties in detail.

All vector spaces and linear maps in this paper are over a field $\mathbb{F}$.

2. Modified Rota-Baxter Lie triple systems

We commence by recalling the fundamental concepts of modified Rota-Baxter Lie triple systems essential for our study, with primary reference to the foundational work in [6, 12, 22].

Definition 2.1. A Lie triple system is a vector space T equipped with a trilinear map $[\cdot, \cdot, \cdot] : T \otimes T \otimes T \rightarrow T$ satisfying

$$\begin{aligned} (T1) \quad & [x, x, y] = 0, \\ (T2) \quad & [x, y, z] + [y, z, x] + [z, x, y] = 0, \\ (T3) \quad & [u, v, [x, y, z]] = [[u, v, x], y, z] + [x, [u, v, y], z] + [x, y, [u, v, z]], \end{aligned}$$

for all $x, y, z, u, v \in T$.

Example 2.2. Let $(T, [\cdot, \cdot])$ be a Lie algebra. We define $[\cdot, \cdot, \cdot] : T \otimes T \otimes T \rightarrow T$ by

$$[x, y, z] = [[x, y], z], \quad \forall x, y, z \in T.$$

Then $(T, [\cdot, \cdot, \cdot])$ becomes a Lie triple system.

Definition 2.3. A representation of a Lie triple system T is a pair (M, θ) consisting of a vector space M and a bilinear map $\theta : T \otimes T \rightarrow \text{End}(M)$ satisfying the following conditions:

$$\begin{aligned} \theta(z, u)\theta(x, y) - \theta(y, u)\theta(x, z) - \theta(x, [y, z, u]) + D(y, z)\theta(x, u) &= 0, \\ \theta(z, u)D(x, y) - D(x, y)\theta(z, u) + \theta([x, y, z], u) + \theta(z, [x, y, u]) &= 0, \end{aligned}$$

where $D(x, y) = \theta(y, x) - \theta(x, y)$, for all $x, y, z, u \in T$.

By direct computation, we get

$$[D(x, y), D(z, u)] = D([x, y, z], u) + D(z, [x, y, u]).$$

Definition 2.4. Let T be a Lie triple system. A modified Rota-Baxter operator on T is a linear map $R : T \rightarrow T$ satisfying

$$\begin{aligned} [Rx, Ry, Rz] \\ = R([Rx, Ry, z] + [x, Ry, Rz] + [Rx, y, Rz] + [x, y, z]) - [Rx, y, z] - [x, Ry, z] - [x, y, Rz], \end{aligned}$$

for any $x, y, z \in T$.

A modified Rota-Baxter Lie triple system is a Lie triple system T equipped with a modified Rota-Baxter operator $R : T \rightarrow T$. For simplicity, a modified Rota-Baxter Lie triple system is often abbreviated as (T, R) .

Definition 2.5. Let (T, R) and (T', R') be two modified Rota-Baxter Lie triple systems. A morphism of modified Rota-Baxter Lie triple systems is given by a Lie triple system homomorphism $\phi : T \rightarrow T'$ satisfying $R' \circ \phi = \phi \circ R$.

Definition 2.6. Let (T, R) be a modified Rota-Baxter Lie triple system and M be a vector space. We define a representation of modified Rota-Baxter Lie triple system as a triple (M, θ, S) , where (M, θ) is a representation of the Lie triple system T and S is a linear map on M satisfies the following condition

$$\begin{aligned} \theta(Rx, Ry)S(m) &= S(\theta(Rx, Ry)m + \theta(Rx, y)S(m) + \theta(x, Ry)S(m) + \theta(x, y)m) \\ &\quad - \theta(Rx, y)m - \theta(x, y)S(m) - \theta(x, Ry)m, \end{aligned}$$

for any $x, y \in T$ and $m \in M$.

When the action map θ is evident from the context, the representation described above can be succinctly denoted by (M, S) .

Let (M, θ) be a representation of a Lie triple system T . Denote the $(2n + 1)$ -cochains group of T with coefficients in M by $C^{2n+1}(T, M)$, where $f \in \text{Hom}(T^{2n+1}, M)$ satisfies

$$\begin{aligned} f(x_1, \dots, x_{n-2}, x, x, y) &= 0, \\ f(x_1, \dots, x_{2n-2}, x, y, z) + f(x_1, \dots, x_{2n-2}, y, z, x) + f(x_1, \dots, x_{2n-2}, z, x, y) &= 0, \end{aligned}$$

for all $x_i, x, y, z \in T$, and $C^1(T, M) = \text{Hom}(T, M)$.

The differential operator $d_{\text{Lie}T}^{2n-1} : C^{2n-1}(T, M) \rightarrow C^{2n+1}(T, M)$, $n \geq 1$ is given by:

$$\begin{aligned} d_{\text{Lie}T}^{2n-1} f(x_1, \dots, x_{2n+1}) &= \theta(x_{2n}, x_{2n+1})f(x_1, \dots, x_{2n-1}) - \theta(x_{2n-1}, x_{2n+1})f(x_1, \dots, x_{2n-2}, x_{2n}) \\ &\quad + \sum_{k=1}^n (-1)^{n+k} D(x_{2k-1}, x_{2k})f(x_1, \dots, \widehat{x_{2k-1}}, \widehat{x_{2k}}, \dots, x_{2n+1}) \\ &\quad + \sum_{k=1}^n \sum_{j=2k+1}^{2n+1} (-1)^{n+k+1} f(x_1, \dots, \widehat{x_{2k-1}}, \widehat{x_{2k}}, \dots, [x_{2k-1}, x_{2k}, x_j], \dots, x_{2n+1}), \end{aligned}$$

for any $f \in C^{2n-1}(T, M)$, $n \geq 1$, where the sign $\widehat{}$ indicates that the element below must be omitted.

Next, let (T, R) be a modified Rota-Baxter Lie triple system and (M, S) be a representation of it. Then there is another cochain complex $\{C^*(T, M), \partial^*\}$ on the same cochain groups with differential

$$\begin{aligned} \partial^{2n-1} f(x_1, \dots, x_{2n+1}) &= \theta(R(x_{2n}), R(x_{2n+1}))f(x_1, x_2, \dots, x_{2n-1}) - S(\theta(R(x_{2n}), x_{2n+1})f(x_1, x_2, \dots, x_{2n-1}) \\ &\quad + \theta(x_{2n}, R(x_{2n+1}))f(x_1, x_2, \dots, x_{2n-1})) + \theta(x_{2n}, x_{2n+1})f(x_1, x_2, \dots, x_{2n-1}) \\ &\quad - \theta(R(x_{2n-1}), R(x_{2n+1}))f(x_1, x_2, \dots, x_{2n-2}, x_{2n}) + S(\theta(R(x_{2n-1}), x_{2n+1})f(x_1, x_2, \dots, x_{2n-2}, x_{2n}) \\ &\quad + \theta(x_{2n-1}, R(x_{2n+1}))f(x_1, x_2, \dots, x_{2n-2}, x_{2n})) - \theta(x_{2n-1}, x_{2n+1})f(x_1, x_2, \dots, x_{2n-2}, x_{2n}) \\ &\quad + \sum_{i=1}^n (-1)^{i+n} \left(D(R(x_{2i-1}), R(x_{2i}))f(x_1, \dots, \widehat{x_{2i-1}}, \widehat{x_{2i}}, \dots, x_{2n+1}) - S(D(R(x_{2i-1}), x_{2i}) \right. \\ &\quad \left. f(x_1, \dots, \widehat{x_{2i-1}}, \widehat{x_{2i}}, \dots, x_{2n+1}) + D(x_{2i-1}, R(x_{2i}))f(x_1, \dots, \widehat{x_{2i-1}}, \widehat{x_{2i}}, \dots, x_{2n+1})) \right. \\ &\quad \left. + D(x_{2i-1}, x_{2i})f(x_1, \dots, \widehat{x_{2i-1}}, \widehat{x_{2i}}, \dots, x_{2n+1}) \right) \\ &\quad + \sum_{i=1}^n \sum_{j=2i+1}^{2n+1} (-1)^{n+i+1} f(x_1, \dots, \widehat{x_{2i-1}}, \widehat{x_{2i}}, \dots, [R(x_{2i-1}), R(x_{2i}), x_j] + [x_{2i-1}, R(x_{2i}), R(x_j)] \\ &\quad + [R(x_{2i-1}), x_{2i}, R(x_j)] + [x_{2i-1}, x_{2i}, x_j], \dots, x_{2n+1}), \end{aligned}$$

for any $x_1, x_2, \dots, x_{2n+1} \in T$. Combining the above two complexes, the authors in [6] introduced a new cochain complex $\{C_{\text{MLieT}}^*(T, M), \delta_{\text{MLieT}}\}$, where

$$C_{\text{MLieT}}^1(T, M) = C^1(T, M), \quad C_{\text{MLieT}}^{2n+1}(T, M) = C^{2n+1}(T, M) \oplus C^{2n-1}(T, M), \quad \forall n \geq 1.$$

For $n = 1$, $\delta_{\text{MLieT}}^1 : C_{\text{MLieT}}^1(T, M) \rightarrow C_{\text{MLieT}}^3(T, M)$ is given by

$$\delta_{\text{MLieT}}^1(f) = (\partial^1(f), -\Phi^1(f)),$$

for any $f \in C^1(T, M)$.

For $n \geq 2$, $\delta_{\text{MLieT}}^{2n-1} : C_{\text{MLieT}}^{2n-1}(T, M) \rightarrow C_{\text{MLieT}}^{2n+1}(T, M)$ is given by

$$\delta_{\text{MLieT}}^{2n-1}(f, g) = (d_{\text{LieT}}^{2n-1}(f), -\partial^{2n-3}(g) - \Phi^{2n-1}(f)),$$

for any $f \in C^{2n+1}(T, M)$ and $g \in C^{2n-1}(T, M)$, where $\Phi^{2n-1} : C^{2n-1}(T, M) \rightarrow C^{2n-1}(T, M)$ is given by

$$\begin{aligned} (\Phi^{2n-1}f)(x_1, \dots, x_{2n-1}) = & \sum_{i=1}^n \left(\sum_{1 \leq j_1 < \dots < j_{2i-1} \leq 2n-1} f(x_1, \dots, Rx_{j_1}, \dots, Rx_{j_{2i-1}}, \dots, x_{2n-1}) \right. \\ & \left. - \sum_{1 \leq j_1 < \dots < j_{2i-2} \leq 2n-1} Sf(x_1, \dots, Rx_{j_1}, \dots, Rx_{j_{2i-2}}, \dots, x_{2n-1}) \right). \end{aligned}$$

The cohomology of the cochain complex $\{C_{\text{MLieT}}^*(T, M), \delta_{\text{MLieT}}\}$ is called the cohomology of the modified Rota-Baxter Lie triple system (T, R) with coefficients in (M, S) . Denote by $H_{\text{MLieT}}^*(T, M)$.

3. Cohomology of modified Rota-Baxter LieTHDer pairs

In this section, our focus is on defining representations of modified Rota-Baxter LieTHDer pairs and exploring their cohomological structures.

Definition 3.1. Let T be a Lie triple system. A higher derivation (of rank N) on T is a tuple $\mathbf{d} = (d_1, \dots, d_N)$ of linear maps on T satisfying the following identities

$$d_l([x, y, z]) = \sum_{i+j+k=l} [d_i(x), d_j(y), d_k(z)], \quad (3.1)$$

for $l = 1, \dots, N$ and $x, y, z \in T$, with the convention that $d_0 = \text{id}_T$.

It follows from (3.1) that d_1 is a derivation on T .

Example 3.2. If d is a derivation on T , then $\mathbf{d} = (d_1 = 0, \dots, d_i = d, \dots, d_N = 0)$ is a higher derivation of rank N .

Example 3.3. Let d be a derivation on T . Then $\mathbf{d} = (d, \frac{d^2}{2!}, \dots, \frac{d^N}{N!})$ is a higher derivation of rank N .

Example 3.4. Let $\mathbf{d} = (d_1, \dots, d_N)$ be a higher derivation on T . For any $1 \leq q \leq N$, we define a new tuple $\mathbf{d}' = (d'_1, \dots, d'_N)$ of linear maps by $d'_l = \begin{cases} 0 & \text{if } q \nmid l, \\ d_s, & \text{if } l = sq. \end{cases}$ Then, $\mathbf{d}' = (d'_1, \dots, d'_N)$ is a higher derivation of rank N .

Within the context of this paper, the term ‘higher derivation’ is uniformly defined as a higher derivation of fixed rank N .

Definition 3.5. A modified Rota-Baxter LieTHDer pair consists of a modified Rota-Baxter Lie triple system (T, R) equipped with a higher derivation $d_l : T \rightarrow T$ such that

$$R \circ d_l = d_l \circ R, \quad l = 1, \dots, N.$$

We shall denote it by $(T, R, \mathbf{d})$.

Example 3.6. Let (T, R) be a modified Rota-Baxter Lie algebra with a higher derivation $\mathbf{d} = (d_1, \dots, d_N)$. By Example 3.2, $(T, R, \mathbf{d})$ is a modified Rota-Baxter LieTHDer pair.

Example 3.7. Consider the 2-dimensional Lie triple system T given with respect to a basis $\{e_1, e_2\}$ whose non-zero products are given as follows:

$$[e_1, e_2, e_2] = e_1.$$

By direct calculation, $R = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ is a modified Rota-Baxter operator, and $\mathbf{d} = (d_1, d_2)$ is a higher derivation, where

$$d_1(e_1) = e_1, \quad d_1(e_2) = -e_1, \quad d_2(e_1) = e_1, \quad d_2(e_2) = -e_1$$

Obviously,

$$R \circ d_1 = d_1 \circ R, \quad R \circ d_2 = d_2 \circ R.$$

Thus, $(T, R, \mathbf{d})$ is a modified Rota-Baxter LieTHDer pair.

Example 3.8. Consider the 4-dimensional Lie triple system T given with respect to a basis $\{e_1, e_2, e_3, e_4\}$, whose non-zero products are given as follows:

$$[e_1, e_2, e_1] = e_4.$$

Through straightforward computation, $R = \begin{pmatrix} a & 1 & 1 & 0 \\ b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c \end{pmatrix}$ is a modified Rota-Baxter operator, and $\mathbf{d} = (d_1, d_2, d_3, d_4)$

is a higher derivation, where

$$\begin{aligned} d_1(e_1) &= e_1, & d_1(e_2) &= e_2, & d_1(e_3) &= e_3, & d_1(e_4) &= 3e_4, \\ d_2(e_1) &= e_1, & d_2(e_2) &= e_2, & d_2(e_3) &= e_3, & d_2(e_4) &= 6e_4, \\ d_3(e_1) &= e_1, & d_3(e_2) &= e_2, & d_3(e_3) &= e_3, & d_3(e_4) &= 10e_4, \\ d_4(e_1) &= e_1, & d_4(e_2) &= e_2, & d_4(e_3) &= e_3, & d_4(e_4) &= 15e_4. \end{aligned}$$

Obviously,

$$R \circ d_i = d_i \circ R, \quad i = 1, 2, 3, 4.$$

Thus, $(T, R, \mathbf{d})$ is a modified Rota-Baxter LieTHDer pair.

Definition 3.9. Given modified Rota-Baxter LieTHDer pairs $(T, R, \mathbf{d})$ and $(T', R', \mathbf{d}')$, a homomorphism of modified Rota-Baxter LieTHDer pairs from $(T, R, \mathbf{d})$ to $(T', R', \mathbf{d}')$ is a modified Rota-Baxter Lie triple system homomorphism $\varphi : T \rightarrow T'$ such that $\varphi \circ d_l = d'_l \circ \varphi$, $l = 1, \dots, N$.

Definition 3.10. (i) Let $(T, R, \mathbf{d})$ be a modified Rota-Baxter LieTHDer pair, and let M be a vector space equipped with linear maps $\mathbf{d}^M = (d_1^M, \dots, d_N^M) : M \rightarrow M$. The triple $(M, \theta, \mathbf{d}^M)$ is referred to as a representation of $(T, R, \mathbf{d})$ if two conditions are satisfied: first, (M, θ) serves as a representation of T ; second, for each $l = 1, \dots, N$, the following equation holds:

$$d_l^M(\theta(x, y)m) = \sum_{i+j+k=l} \theta(d_i(x), d_j(y))d_k^M(m),$$

Hence, we adopt the convention that $d_0 = \text{id}_T$ and $d_0^M = \text{id}_M$.

(ii) Let $(U, \theta_M, \mathbf{d}^M)$ and $(N, \theta_N, \mathbf{d}^N)$ be two representations of a modified Rota-Baxter LieTHDer pair $(T, R, \mathbf{d})$, a homomorphism from $(U, \theta_M, \mathbf{d}^M)$ to $(N, \theta_N, \mathbf{d}^N)$ is a homomorphism of representations over modified Rota-Baxter Lie triple system $f : U \rightarrow V$ such that

$$f \circ d_l^M = d_l^N \circ f, \quad l = 1, \dots, N.$$

In the next, we introduce a cohomology for a modified Rota-Baxter LieTHDer pair with coefficients in a representation.

Let $(T, R, \mathbf{d})$ be a modified Rota-Baxter LieTHDer pair and $(M, S, \mathbf{d}^M)$ be a representation of it. For any $n \geq 1$, we define a new map $\Delta_l^{2n+1} : C_{\text{MLieT}}^{2n+1}(T, M) \rightarrow C_{\text{MLieT}}^{2n+1}(T, M)$, for $l = 1, \dots, N$, by

$$\Delta_l^{2n+1}(f_n, \bar{f}_n) = (\Delta_l^{2n+1}(f_n), \Delta_l^{2n+1}(\bar{f}_n)), \quad (f_n, \bar{f}_n) \in C_{\text{MLieT}}^{2n+1}(T, M),$$

where

$$\Delta_l^{2n+1} f_n := \sum_{i=1}^{2n+1} \sum_{i_1+\dots+i_{2n+1}=l} f_n \circ (d_{i_1} \otimes \dots \otimes d_{i_{2n+1}}) - d_l^M \circ f_n.$$

The subsequent lemmas serve as a foundation for constructing the coboundary operator in the cohomology of modified Rota-Baxter LieTHDer pairs.

Lemma 3.11. *The map Δ_l commutes with Φ , i.e., $\Phi \circ \Delta_l = \Delta_l \circ \Phi$, for $l = 1, \dots, N$.*

Proof. For $n \geq 1$, for any $f_n \in C_{\text{MLieT}}^{2n-1}(T, M)$ and $x_1, \dots, x_{2n-1} \in T$, $l = 1, \dots, N$, we have

$$\begin{aligned} & \Phi^{2n-1} \circ \Delta_l^{2n-1} f_n(x_1, \dots, x_{2n-1}) \\ &= \sum_{i=1}^n \left(\sum_{1 \leq j_1 < \dots < j_{2i-1} \leq 2n-1} \Delta_l^{2n-1}(f_n(x_1, \dots, Rx_{j_1}, \dots, Rx_{j_{2i-1}}, \dots, x_{2n-1})) \right. \\ & \quad \left. - \sum_{1 \leq j_1 < \dots < j_{2i-2} \leq 2n-1} S \Delta_l^{2n-1}(f_n(x_1, \dots, Rx_{j_1}, \dots, Rx_{j_{2i-2}}, \dots, x_{2n-1})) \right) \\ &= \sum_{i=1}^n \left(\sum_{1 \leq j_1 < \dots < j_{2i-1} \leq 2n-1} \left(\sum_{k=1}^{2n-1} \sum_{k_1+\dots+k_{2n-1}=l} f_n(d_{k_1}(x_1), \dots, d_{k_{j_1}}(Rx_{j_1}), \dots, d_{k_{j_{2i-1}}}(Rx_{j_{2i-1}}), \dots, d_{k_{2n-1}}(x_{2n-1})) \right. \right. \\ & \quad \left. - d_l^M \circ f_n(x_1, \dots, Rx_{j_1}, \dots, Rx_{j_{2i-1}}, \dots, x_{2n-1}) \right) \\ & \quad \left. - \sum_{1 \leq j_1 < \dots < j_{2i-2} \leq 2n-1} \left(\sum_{k=1}^{2n-1} \sum_{k_1+\dots+k_{2n-1}=l} S f_n(d_{k_1}(x_1), \dots, d_{k_{j_1}}(Rx_{j_1}), \dots, d_{k_{j_{2i-2}}}(Rx_{j_{2i-2}}), \dots, d_{k_{2n-1}}(x_{2n-1})) \right. \right. \\ & \quad \left. \left. - S d_l^M f_n(x_1, \dots, Rx_{j_1}, \dots, Rx_{j_{2i-2}}, \dots, x_{2n-1}) \right) \right) \\ &= \Delta_l^{2n-1} \left(\sum_{i=1}^n \left(\sum_{1 \leq j_1 < \dots < j_{2i-1} \leq 2n-1} f_n(x_1, \dots, Rx_{j_1}, \dots, Rx_{j_{2i-1}}, \dots, x_{2n-1}) \right. \right. \\ & \quad \left. \left. - \sum_{1 \leq j_1 < \dots < j_{2i-2} \leq 2n-1} S f_n(x_1, \dots, Rx_{j_1}, \dots, Rx_{j_{2i-2}}, \dots, x_{2n-1}) \right) \right) \\ &= \Delta_l^{2n-1} \circ \Phi^{2n-1} f_n(x_1, \dots, x_{2n-1}). \end{aligned}$$

This shows that $\Phi^{2n-1} \circ \Delta_l^{2n-1} = \Delta_l^{2n-1} \circ \Phi^{2n-1}$, for $l = 1, \dots, N$. And the proof is finished. $\square$

Recall that, from the cohomology of LieTHDer pair [7], for $l = 1, \dots, N$, we have

$$d_{\text{LieT}} \circ \Delta_l = \Delta_l \circ d_{\text{LieT}}.$$

Also since (T, R) is a Lie triple system and ∂ is its coboundary with respect to the new representation and $(T, R, \mathbf{d})$ is a modified Rota-Baxter LieTHDer pair, then we get

$$\partial \circ \Delta_l = \Delta_l \circ \partial.$$

Lemma 3.12. *The map Δ_l commutes with δ_{MLieT} , i.e., $\delta_{\text{MLieT}} \circ \Delta_l = \Delta_l \circ \delta_{\text{MLieT}}$, for $l = 1, \dots, N$.*

Proof. This is obviously true when $n = 1$. For $n \geq 2$, for any $(f_n, \bar{f}_n) \in C_{\text{MLieT}}^{2n-1}(T, M)$, we have

$$\begin{aligned} & \delta_{\text{MLieT}}^{2n-1} \circ \Delta_l^{2n-1}(f_n, \bar{f}_n) \\ &= \delta_{\text{MLieT}}^{2n-1}(\Delta_l^{2n-1}(f_n), \Delta_l^{2n-3}(\bar{f}_n)) \\ &= (\partial_l^{2n-1}(\Delta_l^{2n-1}(f_n)), -\partial_l^{2n-3}(\Delta_l^{2n-3}(\bar{f}_n)) - \Phi^{2n-1}(\Delta_l^{2n-1}(f_n))) \\ &= (\Delta_l^{2n+1}(\partial_l^{2n-1}(f_n)), -\Delta_l^{2n-1}(\partial_l^{2n-3}(\bar{f}_n)) - \Delta_l^{2n-1}(\Phi^{2n-1}(f_n))) \\ &= \Delta_l^{2n+1}(\partial_l^{2n-1}(f_n), -\partial_l^{2n-3}(\bar{f}_n) - \Phi^{2n-1}(f_n)) \\ &= \Delta_l^{2n+1} \circ \delta_{\text{MLieT}}^{2n-1}(f_n, \bar{f}_n). \end{aligned}$$

That is, $\delta_{\text{MLieT}}^{2n-1} \circ \Delta_l^{2n-1} = \Delta_l^{2n+1} \circ \delta_{\text{MLieT}}^{2n-1}$, for $l = 1, \dots, N$. And the proof is finished. $\square$

Equipped with the preceding results, we are ready to define the cohomology of the modified Rota-Baxter LieTHDer pair. Define the space

$$C_{\text{MLieTHDer}}^0(T, M) = 0, \quad C_{\text{MLieTHDer}}^1(T, M) = C_{\text{MLieT}}^1(T, M),$$

for $n \geq 1$,

$$C_{\text{MLieTHDer}}^{2n+1}(T, M) := C_{\text{MLieT}}^{2n+1}(T, M) \times C_{\text{MLieT}}^{2n-1}(T, M).$$

We define a map $\mathfrak{D} : C_{\text{MLieTHDer}}^{2n-1}(T, M) \rightarrow C_{\text{MLieTHDer}}^{2n+1}(T, M)$, for $l = 1, \dots, N$, by

$$\begin{aligned} \mathfrak{D}f &= (\delta_{\text{MLieT}}f, -\Delta_l f), \text{ for any } f \in C_{\text{MLieTHDer}}^1(T, M), \\ \mathfrak{D}((f_n, \bar{f}_n), (g_n, \bar{g}_n)) &= (\delta_{\text{MLieT}}(f_n, \bar{f}_n), \delta_{\text{MLieT}}(g_n, \bar{g}_n) + (-1)^n \Delta_l(f_n, \bar{f}_n)), \end{aligned}$$

for any $(f_n, \bar{f}_n) \in C_{\text{MLieT}}^{2n-1}(T, M)$ and $(g_n, \bar{g}_n) \in C_{\text{MLieT}}^{2n-3}(T, M)$.

Proposition 3.13. The map $\mathfrak{D}$ satisfies $\mathfrak{D} \circ \mathfrak{D} = 0$.

Proof. For any $f \in C_{\text{MLieTHDer}}^1(T, M)$, for $l = 1, \dots, N$, we have

$$(\mathfrak{D} \circ \mathfrak{D})f = \mathfrak{D}(\delta_{\text{MLieT}}(f), -\Delta_l(f)) = ((\delta_{\text{MLieT}} \circ \delta_{\text{MLieT}})f, -(\delta_{\text{MLieT}} \circ \Delta_l)f + (\Delta_l \circ \delta_{\text{MLieT}})f) = 0.$$

Similarly, for any $(f_n, \bar{f}_n) \in C_{\text{MLieT}}^{2n-1}(T, M)$ and $(g_n, \bar{g}_n) \in C_{\text{MLieT}}^{2n-3}(T, M)$, we get

$$\begin{aligned} & (\mathfrak{D} \circ \mathfrak{D})((f_n, \bar{f}_n), (g_n, \bar{g}_n)) \\ &= \mathfrak{D}(\delta_{\text{MLieT}}(f_n, \bar{f}_n), \delta_{\text{MLieT}}(g_n, \bar{g}_n) + (-1)^n \Delta_l(f_n, \bar{f}_n)) \\ &= (\delta_{\text{MLieT}}^2(f_n, \bar{f}_n), \delta_{\text{MLieT}}^2(g_n, \bar{g}_n) + (-1)^n \delta_{\text{MLieT}} \Delta_l f_n + (-1)^{n+1} \Delta_l \delta_{\text{MLieT}}(f_n, \bar{f}_n)) \\ &= 0. \end{aligned}$$

Therefore, this completes the proof. $\square$

Denote the set of all $(2n+1)$ -cocycles by $Z_{\text{MLieTHDer}}^{2n+1}(T, M)$ and all $(2n+1)$ -coboundaries by $B_{\text{MLieTHDer}}^{2n+1}(T, M)$. Define the corresponding $(2n+1)$ -cohomology group by

$$H_{\text{MLieTHDer}}^{2n+1}(T, M) = Z_{\text{MLieTHDer}}^{2n+1}(T, M) / B_{\text{MLieTHDer}}^{2n+1}(T, M),$$

which is called the $(2n+1)$ -cohomology group of the modified Rota-Baxter LieTHDer pair $(T, R, \mathbf{d})$ with coefficients in the representation $(M, \theta, \mathbf{d}^M)$.

4. Abelian extensions of modified Rota-Baxter LieTHDer pairs

Within this section, we investigate abelian extensions of modified Rota-Baxter LieTHDer pairs and demonstrate that their classification is given by the third cohomology, a result that aligns with the expectations of a well-behaved cohomology theory.

Let $(T, R, \mathbf{d})$ be a modified Rota-Baxter LieTHDer pair and $(M, S, \mathbf{d}^M)$ be an abelian modified Rota-Baxter LieTHDer pair, that is, the Lie triple system bracket of M is trivial.

Definition 4.1. An abelian extension $(T', R', \mathbf{d}')$ of $(T, R, \mathbf{d})$ by $(M, S, \mathbf{d}^M)$ is defined as follows: if there exists a short exact sequence of morphisms of modified Rota-Baxter LieTHDer pairs

$$\begin{array}{ccccccc} 0 & \longrightarrow & (M, \mathbf{d}^M) & \xrightarrow{i} & (T', \mathbf{d}') & \xrightarrow{p} & (T, \mathbf{d}) \longrightarrow 0 \\ & & \downarrow S & & \downarrow R' & & \downarrow R \\ 0 & \longrightarrow & (M, \mathbf{d}^M) & \xrightarrow{i'} & (T', \mathbf{d}') & \xrightarrow{q} & (T, \mathbf{d}) \longrightarrow 0. \end{array}$$

A section of an abelian extension $(T', R', \mathbf{d}')$ of the modified Rota-Baxter LieTHDer pair $(T, R, \mathbf{d})$ by $(M, S, \mathbf{d}^M)$ consists of a linear map $s : T \rightarrow T'$ such that $p \circ s = \text{id}_T$.

For any section s , we define a map $\theta : T \otimes T \rightarrow \text{End}(M)$ by

$$\theta(x, y)m := [m, s(x), s(y)]_{T'}, \quad \forall x, y \in T, m \in M.$$

Moreover,

$$D(x, y)m := \theta(y, x)m - \theta(x, y)m = [s(x), s(y), m]_{T'}.$$

Lemma 4.2. With the above notations, $(M, \theta, S, \mathbf{d}^M)$ is a representation of the modified Rota-Baxter LieTHDer pair $(T, R, \mathbf{d})$.

Proof. It has been observed in [7] that (M, θ) is a representation of Lie triple system T , for any $x, y \in T$ and $m \in M$, $l = 1, \dots, N$, we have

$$\begin{aligned} & \sum_{i+j+k=l} \theta(d_i(x), d_j(y))d_k^M(m) \\ &= \sum_{i+j+k=l} [d_k^M(m), s(d_i(x)), s(d_j(y))] \\ &= \sum_{i+j+k=l} [d_k^M(m), d'_i(s(x)), d'_j(s(y))] \\ &= d_l^M([m, s(x), s(y)]) \\ &= d_l^M(\theta(x, y)m), \end{aligned}$$

and

$$\begin{aligned} & \theta(Rx, Ry)S(m) \\ &= [S(m), s(R(x)), s(R(y))] \\ &= [S(m), R'(s(x)), R'(s(y))] \\ &= S([m, R'(s(x)), R'(s(y))] + [S(m), R'(s(x)), s(y)] + [S(m), s(x), R'(s(y))] + [m, s(x), s(y)]) \\ &\quad - [S(m), R'(s(x)), s(y)] - [S(m), s(x), s(y)] - [m, s(x), R'(s(y))]) \\ &= S([m, s(R(x)), s(R(y))] + [S(m), s(R(x)), s(y)] + [S(m), s(x), s(R(y))] + [m, s(x), s(y)]) \\ &\quad - [S(m), s(R(x)), s(y)] - [S(m), s(x), s(y)] - [m, s(x), s(R(y))]) \\ &= S(\theta(Rx, Ry)m + \theta(Rx, y)S(m) + \theta(x, Ry)S(m) + \theta(x, y)m) \\ &\quad - \theta(Rx, y)m - \theta(x, y)S(m) - \theta(x, Ry)m. \end{aligned}$$

Hence, the proof is thereby accomplished. $\square$

For any $x, y, z \in T$ and $m \in M$, for $l = 1, \dots, N$, define $\psi : T \otimes T \otimes T \rightarrow M$, $\chi : T \rightarrow M$ and $\xi : T \rightarrow M$ by

$$\begin{aligned}\psi(x, y, z) &:= [s(x), s(y), s(z)]_{T'} - s([x, y, z]_T), \\ \chi(x) &= d'_l(s(x)) - s(d_l(x)), \\ \xi(x) &= R'(s(x)) - s(R(x)).\end{aligned}$$

Further, for any $x \in T$ and $m \in M$, the linear maps on $T \oplus M$ are given by

$$\begin{aligned}R_\xi(x + m) &= R(x) + S(m) + \xi(x), \\ d_l^\chi(x + m) &= d_l(x) + d_l^M(m) + \chi(x).\end{aligned}$$

Proposition 4.3. *With the above notations, the triple $(T \oplus M, R_\Phi, \mathbf{d}^{T \oplus M})$, where*

$$[(x + m), (y + n), (z + p)]_\psi = [x, y, z]_T + \psi(x, y, z), \quad \forall x, y, z \in T,$$

is a modified Rota-Baxter LieTHDer pair if and only if (ψ, χ, ξ) is a 3-cocycle in the cohomology of the modified Rota-Baxter LieTHDer pair $(T, R, \mathbf{d})$ with coefficients in the trivial representation M .

Proof. If the pair $(T \oplus M, [\cdot, \cdot, \cdot]_\psi)$ is a Lie triple system, we have

$$\begin{aligned}[(x, m), (x, m), (y, n)]_\psi &= 0, \\ [(x, m), (y, n), (z, p)]_\psi + [(y, n), (z, p), (x, m)]_\psi + [(z, p), (x, m), (y, n)]_\psi &= 0, \\ [(x, m), (y, n), [(z, p), (v, k), (w, l)]]_\psi & \\ = [[(x, m), (y, n), (z, p)]_\psi, (v, k), (w, l)]_\psi + [(z, p), [(x, m), (y, n), (v, k)]]_\psi, (w, l)]_\psi + & \\ [(z, p), (v, k), [(x, m), (y, n), (w, l)]]_\psi &\end{aligned}$$

for all $x \oplus m, y \oplus n, z \oplus p, v \oplus k, w \oplus l \in T \oplus M$. Through direct computation, it follows that

$$\psi(x, x, y) = 0, \quad \psi(x, y, z) + \psi(y, z, x) + \psi(z, x, y) = 0, \quad (4.1)$$

$$\psi(x, y, [z, v, w]_T) = \psi([x, y, z]_T, v, w) + \psi(z, [x, y, v]_T, w) + \psi(z, v, [x, y, w]_T). \quad (4.2)$$

We establish the equivalence between (4.1) and (4.2) with respect to

$$d_{\text{LieT}}^3(\psi) = 0. \quad (4.3)$$

For R_ξ to be a modified Rota-Baxter operator on $(T \oplus M, [\cdot, \cdot, \cdot]_\psi)$ necessitates that

$$\begin{aligned}& [R_\xi(x + m), R_\xi(y + n), R_\xi(z + p)]_\psi \\ &= R_\xi([R_\xi(x + m), R_\xi(y + n), z + p]_\psi + [x + m, R_\xi(y + n), R_\xi(z + p)]_\psi \\ &\quad + [R_\xi(x + m), y + n, R_\xi(z + p)]_\psi + [x + m, y + n, z + p]_\psi) - [R_\xi(x + m), y + n, z + p]_\psi \\ &\quad - [x, R_\xi(y + n), z + p]_\psi - [x + m, y + n, R_\xi(z + p)]_\psi.\end{aligned}$$

By the direct calculation, R_ξ is a modified Rota-Baxter operator on $(T \oplus M, [\cdot, \cdot, \cdot]_\psi)$ if and only if

$$\begin{aligned}& \psi(R(x), R(y), R(z)) - S\psi(R(x), R(y), z) - S\psi(x, R(y), R(z)) - S\psi(R(x), y, R(z)) - S\psi(x, y, z) \\ &\quad + \psi(R(x), y, z) + \psi(x, R(y), z) + \psi(x, y, R(z)) - \xi([R(x), R(y), z] + [x, R(y), R(z)] \\ &\quad [R(x), y, R(z)] + [x, y, z]) = 0,\end{aligned}$$

which is exactly

$$\partial^1(\xi) + \Phi^3(\psi) = 0. \quad (4.4)$$

And $\mathbf{d}^X$ is a higher derivation on the Lie triple system $(T \oplus M, [\cdot, \cdot, \cdot]_\psi)$ if and only if

$$d_l^X[(x, m), (y, n), (z, p)]_\psi = \sum_{i+j+k=l} [d_i^X(x, m), d_j^X(y, n), d_k^X(z, p)]_\psi.$$

By means of direct calculation, we derive $\mathbf{d}^X$ is a higher derivation on the Lie triple system $(T \oplus M, [\cdot, \cdot, \cdot]_\psi)$ if and only if

$$d_l^M(\psi(x, y, z)) + \chi([x, y, z]_T) = \sum_{i+j+k=l} \psi(d_i(x), d_j(y), d_k(z)).$$

This is the same as

$$d_{\text{LeiT}}^1(\chi) + \Delta^3\psi = 0. \quad (4.5)$$

Finally, for $l = 1, \dots, N$, we get

$$\begin{aligned} & d_l^X \circ R_\xi(x + m) - R_\xi \circ d_l^X(x + m) \\ &= d_l^X(R(x) + S(m) + \xi(x)) - R_\xi(d_l(x) + d_l^M(m) + \chi(x)) \\ &= d_l(R(x)) + d_l^M(S(m)) + d_l^M(\xi(x)) + \chi(R(x)) - R(d_l(x)) - S(d_l^M(m)) - S(\chi(x)) - \xi(d_l(x)) \\ &= d_l^M(\xi(x)) - \xi(d_l(x)) + \chi(R(x)) - S(\chi(x)), \end{aligned}$$

then R_ξ and d_l^X commute if and only if

$$d_l^M(\xi(x)) - \xi(d_l(x)) + \chi(R(x)) - S(\chi(x)) = 0,$$

which is exactly

$$\Delta^1(\xi) - \phi^1(\chi) = 0. \quad (4.6)$$

Therefore, the triple $(T \oplus M, R_\Phi, \mathbf{d}^{T \oplus M})$ is a modified Rota-Baxter LieTHDer pair if and only if equations (4.3), (4.4), (4.5), (4.6) hold.

On the other hand, (ψ, χ, ξ) is a 3-cocycle in the cohomology of the modified Rota-Baxter LieTHDer pair $(T, R, \mathbf{d})$ with coefficients in the trivial representation M if and only if equations (4.3), (4.4), (4.5), (4.6) hold. The proof is thereby accomplished. $\square$

Definition 4.4. Let $(T_1, R_1, \mathbf{d}^1)$ and $(T_2, R_2, \mathbf{d}^2)$ be two abelian extensions of a modified Rota-Baxter LieTHDer pair $(T, R, \mathbf{d})$ by $(M, S, \mathbf{d}^M)$. They are said to be equivalent if there is a homomorphism of modified Rota-Baxter LieTHDer pairs

$$\tau : (T_1, R_1, \mathbf{d}^1) \rightarrow (T_2, R_2, \mathbf{d}^2)$$

such that the following diagram commutes:

$$\begin{array}{ccccccc} 0 & \longrightarrow & (M, S, \mathbf{d}^M) & \xrightarrow{i_1} & (T_1, R_1, \mathbf{d}^1) & \xrightarrow{p_1} & (T, R, \mathbf{d}) \longrightarrow 0 \\ & & \downarrow \text{id} & & \downarrow \tau & & \downarrow \text{id} \\ 0 & \longrightarrow & (M, S, \mathbf{d}^M) & \xrightarrow{i_2} & (T_1, R_2, \mathbf{d}^2) & \xrightarrow{p_2} & (T, R, \mathbf{d}) \longrightarrow 0. \end{array}$$

Theorem 4.5. Abelian extensions of a modified Rota-Baxter LieTHDer pair $(T, R, \mathbf{d})$ by $(M, S, \mathbf{d}^M)$ are classified by the third cohomology group $H_{\text{MLieTHDer}}^3(T, M)$.

Proof. Let $(T_1, R_1, \mathbf{d}^1)$ and $(T_2, R_2, \mathbf{d}^2)$ be two abelian extensions of a modified Rota-Baxter LieTHDer pair $(T, R, \mathbf{d})$ by $(M, S, \mathbf{d}^M)$. Let s be a section of it, where $s : T \rightarrow T'$, we obtain a 3-cocycle (ψ, ξ, χ) according to Proposition 4.3.

First, we prove that the cohomological class of (ψ, ξ, χ) does not depend on the choice of sections. Let s_1, s_2 be two sections of p . Define a map $u : T \rightarrow M$ by $u(x) := s_1(x) - s_2(x)$, for $l = 1, \dots, N$, we have

$$\begin{aligned}\psi_1(x, y, z) &= [s_1(x), s_1(y), s_1(z)]_{\hat{T}} - s_1([x, y, z]_T) \\ &= [s_2(x) + u(x), s_2(y) + u(y), s_2(z) + u(z)]_{\hat{T}} - s_2([x, y, z]_T) - u([x, y, z]_T) \\ &= \psi_2(x, y, z) + d_{\text{Lie}T}(u)(x, y, z),\end{aligned}$$

$$\begin{aligned}\xi_1(x) &= R'(s_1(x)) - s_1(R(x)) \\ &= R'(u(x) + s_2(x)) - u(R(x)) - s_2(R(x)) \\ &= \xi_2(x) + S(u(x)) - u(R(x)) \\ &= \xi_2(x) - \phi^1 u(x),\end{aligned}$$

and

$$\begin{aligned}\chi_1(x) &= d'_l(s_1(x)) - s_1(d_l(x)) \\ &= d'_l(s_2(x) + u(x)) - s_2(d_l(x)) - u(d_l(x)) \\ &= \chi_2(x) + d_l^M(u(x)) - u(d_l T(x)) \\ &= \chi_2(x) - \Delta^1 u(x).\end{aligned}$$

This shows that $(\psi_1, \xi_1, \chi_1) - (\psi_2, \xi_2, \chi_2) = \mathfrak{D}_{\text{MLieTHDer}} u$. Hence they correspond to the same cohomology class.

Next, we show that equivalent abelian extensions give rise to the same element in $H^3_{\text{MLieTHDer}}(T, M)$.

Let $(T_1, R_1, \mathbf{d}^1)$ and $(T_2, R_2, \mathbf{d}^2)$ be two equivalent abelian extensions and the isomorphism is given by $\tau : T_1 \rightarrow T_2$. Let $s : T \rightarrow T_1$ be a section of p . Then

$$p_2 \circ (\tau \circ s) = (p_2 \circ \tau) \circ s = p_1 \circ s = \text{id}_T.$$

This confirms that $s_2 := \tau \circ s_1$ is a section of p_2 . Since τ is a morphism of modified Rota-Baxter LieTHDer pairs, we have $\tau|_M = \text{id}_M$, for $l = 1, \dots, N$. Thus,

$$\begin{aligned}\psi_2(x, y, z) &= [s_2(x), s_2(y), s_2(z)]_{T_2} - s_2([x, y, z]_T) \\ &= \tau([s_1(x), s_1(y), s_1(z)]_{T_1} - [x, y, z]_T) = \psi_1(x, y, z),\end{aligned}$$

$$\begin{aligned}\xi_2(x) &= R_2 s_2(x) - s_2(R(x)) \\ &= R_2(\tau(s_1(x))) - \tau(s_1(R(x))) \\ &= \tau(R_1(s_1(x)) - s_1(R(x))) \\ &= \xi_1(x),\end{aligned}$$

and

$$\begin{aligned}\chi_2(x) &= d_l^2(s_2(x)) - s_2(d_l^1(x)) \\ &= d_l^2(\tau \circ s_1(x)) - \tau \circ s_1(d_l^1(x)) \\ &= \tau(d_l^1(s_1(x)) - s_1(d_l^1(x))) = \chi_1(x).\end{aligned}$$

Therefore, equivalent abelian extensions give rise to same 3-cocycle. Hence, correspond to same element in $H^3_{\text{MLieTHDer}}(T, M)$.

Conversely, let (ψ_1, ξ_1, χ_1) and (ψ_2, ξ_2, χ_2) be two cohomologous 3-cocycles. Then, there exists a map $\eta : T \rightarrow M$ such that

$$(\psi_1, \xi_1, \chi_1) - (\psi_2, \xi_2, \chi_2) = \mathfrak{D}_{\text{MLieTHDer}} \eta.$$

Define $\tau : T \oplus M \rightarrow T \oplus M$ by $\tau(x + m) = x + m + \eta(x)$. Thus τ is a homomorphism of these two abelian extensions. Hence the proof is completed. $\square$

5. Deformations of modified Rota-Baxter LieTHDer pairs

In this section, we study a one-parameter formal deformation of modified Rota-Baxter LieTHDer pair.

A formal one-parameter deformation of the modified Rota-Baxter LieTHDer pair $(T, R, \mathbf{d})$ is a triple of power series, for $l = 1, \dots, N$,

$$\mu_t = \sum_{i=0}^{\infty} t^i \mu_i \in \text{Hom}(T^{\otimes 3}, T)[[t]] \text{ with } \mu_0 = \mu,$$

$$R_t = \sum_{i=0}^{\infty} t^i R_i \in \text{Hom}(T, T)[[t]] \text{ with } R_0 = R,$$

$$d_{l,t} = \sum_{i=0}^{\infty} t^i d_{l,i} \in \text{Hom}(T, T)[[t]] \text{ with } d_{l,0} = d_l,$$

such that $(T[[t]], R_t, d_{l,t})$ is a modified Rota-Baxter LieTHDer pair.

Therefore, for $l = 1, \dots, N$, $(\mu_t, R_t, d_{l,t})$ will be a formal one-parameter deformation of a modified Rota-Baxter LieTHDer pair (T, μ, R, d_l) if and only if the following conditions are satisfied: for any $x, y, z \in T$

$$\mu_t(x, x, y) = 0, \quad (5.1)$$

$$\mu_t(x, y, z) + \mu_t(y, z, x) + \mu_t(z, x, y) = 0, \quad (5.2)$$

$$\mu_t(x, y, \mu_t(z, v, w)) = \mu_t(\mu_t(x, y, z), v, w) + \mu_t(z, \mu_t(x, y, v), w) + \mu_t(z, v, \mu_t(x, y, w)), \quad (5.3)$$

$$\begin{aligned} \mu_t(R_t(x), R_t(y), R_t(z)) &= R_t(\mu_t(R_t(x), R_t(y), z) + \mu_t(x, R_t(y), R_t(z)) + \mu_t(R_t(x), y, R_t(z)) \\ &\quad + \mu_t(x, y, z)) - \mu_t(R_t(x), y, z) - \mu_t(x, R_t(y), z) - \mu_t(x, y, R_t(z)), \end{aligned} \quad (5.4)$$

$$d_{l,t}(\mu_t(x, y, z)) = \sum_{i+j+k=l} \mu_t(d_{i,t}(x), d_{j,t}(y), d_{k,t}(z)). \quad (5.5)$$

Equalities (5.1) and (5.2) are equivalent to the following equations:

$$\mu_n(x, x, y) = 0,$$

$$\mu_n(x, y, z) + \mu_n(y, z, x) + \mu_n(z, x, y) = 0,$$

respectively, for $n = 0, 1, 2, \dots$. Conditions (5.3), (5.4) and (5.5) are equivalent to the following equations:

$$\begin{aligned} \sum_{i+j=n} \mu_i(x, y, \mu_j(z, v, w)) &= \sum_{i+j=n} \mu_i(\mu_j(x, y, z), v, w) + \mu_i(z, \mu_j(x, y, v), w) + \mu_i(z, v, \mu_j(x, y, w)), \\ \sum_{a+b+c+d=n} \mu_a(R_b(x), R_c(y), R_d(z)) &= \sum_{a+b+c+d=n} R_a(\mu_b(R_c(x), R_d(y), z) + \mu_b(x, R_c(y), R_d(z)) \\ &\quad + \mu_b(R_c(x), y, R_d(z))) + \sum_{a+b=n} R_a \mu_b(x, y, z) - \sum_{a+b=n} (\mu_a(R_b(x), y, z) + \mu_a(x, R_b(y), z) + \mu_a(x, y, R_b(z))), \\ \sum_{i+j=n} d_{l,i}(\mu_j(x, y, z)) &= \sum_{a+b+c+d=n} \sum_{i+j+k=l} \mu_a(d_{i,b}(x), d_{j,c}(y), d_{k,d}(z)), \\ \sum_{i+j=n} R_i \circ d_{l,j} &= \sum_{i+j=n} d_{l,i} \circ R_j. \end{aligned}$$

All the identities hold for $n = 0$ as $(T, R, \mathbf{d})$ is a modified Rota-Baxter LieTHDer pair. For $n = 1$, we have

$$\begin{aligned} & \mu_1(x, y, [z, v, w]) + [x, y, \mu_1(z, v, w)] \\ &= \mu_1([x, y, z], v, w) + [\mu_1(x, y, z), v, w] + [z, \mu_1(x, y, v), w] + \mu_1(z, [x, y, v], w) \\ & \quad + [z, v, \mu_1(x, y, w)] + \mu_1(z, v, [x, y, w]), \end{aligned} \quad (5.6)$$

$$\begin{aligned} & \mu_1(R(x), R(y), R(z)) + [R_1(x), R(y), R(z)] + [R(x), R_1(y), R(z)] + [R(x), R(y), R_1(z)] \\ &= R_1([R(x), R(y), z] + [x, R(y), R(z)] + [R(x), y, R(z)]) + R(\mu_1(R(x), R(y), z) + \mu_1(x, R(y), R(z)) \\ & \quad + \mu_1(R(x), y, R(z))) + R([R_1(x), R(y), z] + [x, R_1(y), R(z)] + [R_1(x), y, R(z)]) + R([R(x), R_1(y), z] \\ & \quad + [x, R(y), R_1(z)] + [R(x), y, R_1(z)]) + \mu_1(x, y, z) + R\mu_1(x, y, z) + R_1([x, y, z]) - \mu_1(R(x), y, z) \\ & \quad - \mu_1(x, R(y), z) - \mu_1(x, y, R(z)) - [R_1(x), y, z] - [x, R_1(y), z] - [x, y, R_1(z)], \end{aligned} \quad (5.7)$$

$$\begin{aligned} & d_{l,1}([x, y, z]) + d_l(\mu_1(x, y, z)) \\ &= \sum_{i+j+k=l} \left(\mu_1(d_i(x), d_j(y), d_k(z)) + [d_{i,1}(x), d_j(y), d_k(z)] + [d_i(x), d_{j,1}(y), d_k(z)] \right. \\ & \quad \left. + [d_i(x), d_j(y), d_{k,1}(z)] \right), \end{aligned} \quad (5.8)$$

$$R \circ d_{l,1} + R_1 \circ d_l = d_{l,1} \circ R + d_l \circ R_1. \quad (5.9)$$

The condition (5.6) is equivalent to

$$d_{\text{LieT}}^3(\mu_1) = 0. \quad (5.10)$$

Condition (5.7) can be rephrased as

$$\partial^1(R_1) + \Phi^3(\mu_1) = 0. \quad (5.11)$$

Condition (5.8) translates to the requirement that

$$d_{\text{LieT}}^1(d_{l,1}) + \Delta^3 \mu_1 = 0. \quad (5.12)$$

For condition (5.9), an equivalent formulation is

$$\Delta^1(R_1) - \phi^1(d_{l,1}) = 0, \quad l = 1, \dots, N. \quad (5.13)$$

Therefore, we have

$$\mathfrak{D}_{\text{MLieTHDer}}(\mu_1, R_1, d_{1,1}, \dots, d_{N,1}) = 0.$$

Hence, we have the following proposition.

Proposition 5.1. *Let $(\mu_t, R_t, d_{1,t}, \dots, d_{N,t})$ be a formal one-parameter deformation of a modified Rota-Baxter LieTHDer pair $(T, R, \mathbf{d})$. Then the linear term $(\mu_1, R_1, d_{1,1}, \dots, d_{N,1})$ is a 3-cocycle in the cohomology of the modified Rota-Baxter LieTHDer pair $(T, R, \mathbf{d})$ with coefficients in itself.*

Definition 5.2. *The 3-cocycle $(\mu_1, R_1, d_{1,1}, \dots, d_{N,1})$ is called the infinitesimal of the formal one-parameter deformation $(\mu_t, R_t, d_{1,t}, \dots, d_{N,t})$ of the modified Rota-Baxter LieTHDer pair $(T, \mu, R, \mathbf{d})$.*

Definition 5.3. *Let $(\mu_t, R_t, d_{1,t}, \dots, d_{N,t})$ and $(\mu'_t, R'_t, d'_{1,t}, \dots, d'_{N,t})$ be two formal one-parameter deformations of a modified Rota-Baxter LieTHDer pair $(T, R, \mathbf{d})$. A formal isomorphism between these two deformations is a power series $\Psi_t = \sum_{i=0}^{\infty} t^i \Psi_i : T[[t]] \rightarrow T[[t]]$ with $\Psi_0 = \text{id}_T$ such that the following conditions are satisfied:*

$$\Psi_t \circ \mu_t = \mu'_t \circ (\Psi_t \otimes \Psi_t \otimes \Psi_t), \quad \Psi_t \circ R_t = R'_t \circ \Psi_t, \quad \Psi_t \circ d_{l,t} = d'_{l,t} \circ \Psi_t, \quad \text{for } l = 1, \dots, N.$$

Now expanding previous three equations and equating the coefficients of t^n from both the sides we get

$$\begin{aligned}\sum_{i+j=n} \Psi_i \circ \mu_j &= \sum_{p+q+r+l=n} \mu'_p \circ (\Psi_q \otimes \Psi_r \otimes \Psi_l), \\ \sum_{i+j=n} \Psi_i \circ R_j &= \sum_{i+j=n} R'_i \circ \Psi_j, \\ \sum_{i+j=n} d'_{l,i} \circ \Psi_j &= \sum_{p+q=n} \Psi_p \circ d_{l,q}, \quad \text{for } l = 1, \dots, N.\end{aligned}$$

Now putting $n = 1$ in the above equations, we get

$$\begin{aligned}\mu_1 + \Psi_1 \circ \mu &= \mu'_1 + \mu \circ (\Psi_1 \otimes \text{id} \otimes \text{id}) + \mu \circ (\text{id} \otimes \text{id} \otimes \Psi_1) + \mu \circ (\text{id} \otimes \Psi_1 \otimes \text{id}), \\ R_1 + \Psi_1 \circ R &= R'_1 + R' \circ \Psi_1, \\ d'_l \circ \Psi_1 + d'_{l,1} &= d_{l,1} + \Psi_1 \circ d_l, \quad \text{for } l = 1, \dots, N.\end{aligned}$$

Therefore, we obtain

$$(\mu_1, R_1, d_{1,1}, \dots, d_{N,1}) - (\mu'_1, R'_1, d'_{1,1}, \dots, d'_{N,1}) = \mathfrak{D}_{\text{MLieTHDer}}(\Psi_1).$$

Thus, we have the following proposition.

Proposition 5.4. *The infinitesimals of two equivalent one-parameter formal deformations of a modified Rota-Baxter LieTHDer pair $(T, \mu, R, \mathbf{d})$ are in the same cohomology class.*

Definition 5.5. *A modified Rota-Baxter LieTHDer pair $(T, \mu, R, \mathbf{d})$ is called rigid, if every one-parameter deformation is trivial.*

Theorem 5.6. *Let $(T, \mu, R, \mathbf{d})$ be a modified Rota-Baxter LieTHDer pair. Then $(T, \mu, R, \mathbf{d})$ is rigid if and only if $H^3_{\text{MLieTHDer}}(T, T) = 0$.*

Proof. Let $(\mu_t, R_t, d_{1,t}, \dots, d_{N,t})$ be a deformation of the modified Rota-Baxter LieTHDer pair $(T, R, \mathbf{d})$. From Proposition 5.1, $(\mu_1, R_1, d_{1,1}, \dots, d_{N,1})$ is a 3-cocycle and $H^3_{\text{MLieTHDer}}(T, T) = 0$. Therefore, there exists a 1-cochain $\Psi_1 \in C^1_{\text{MLieTHDer}}(T, T)$ such that

$$(\mu_1, R_1, d_{1,1}, \dots, d_{N,1}) = \mathfrak{D}_{\text{MLieTHDer}}(\Psi_1).$$

Then setting $\Psi_t = \text{id}_T + t\Psi_1 : T[[t]] \rightarrow T[[t]]$, for $l = 1, \dots, N$, we have a deformation $(\mu'_t, R'_t, d'_{l,t})$, where

$$\mu'_t = \Psi_t^{-1} \circ \mu_t \circ (\Psi_t \otimes \Psi_t \otimes \Psi_t), \quad R'_t = \Psi_t^{-1} \circ R_t \circ \Psi_t, \quad d'_{l,t} = \Psi_t^{-1} \circ d_{l,t} \circ \Psi_t. \quad (5.14)$$

Hence, $(\mu'_t, R'_t, d'_{1,t}, \dots, d'_{N,t})$ is equivalent to $(\mu_t, R_t, d_{1,t}, \dots, d_{N,t})$. Moreover, by (5.14), we have

$$\begin{aligned}\mu'_t &= \mu + t^2 \mu'_2 + \dots, \\ R'_t &= R + t^2 R'_2 + \dots, \\ d'_{l,t} &= d_l + t^2 d'_{l,2} + \dots, \quad \text{for } l = 1, \dots, N.\end{aligned}$$

Finally, by repeating the arguments, we can show that $(\mu_t, R_t, d_{1,t}, \dots, d_{N,t})$ is equivalent to the trivial deformation. Hence, $(T, \mu, R, \mathbf{d})$ is rigid. $\square$

Acknowledgements

The first author is grateful to the financial support from the Mathematical Tianyuan Fund of the National Natural Science Foundation of China (NSFC). The authors are grateful to the referees for carefully reading the manuscript and for many valuable comments which largely improved the article.

References

- [1] Y. Cao, L.Y. Chen, *On the structure of split Leibniz triple systems*, Acta Math. Sin. **31**(10) (2008), 1629–1644.
- [2] S. Chen, Q. Lou, Q.X. Sun, *Cohomologies of Rota-Baxter Lie triple systems and applications*, Commun. Algebra **51** (10) (2023), 4299–4315.
- [3] T. Chtioui, A. Hajjaji, S. Mabrouka, A. Makhlouf, *Cohomologies and deformations of O -operators on Lie triple systems*, J. Math. Phys. **64** (2023), 081701.
- [4] A. Da, *Extensions and deformations of algebras with higher derivations*, Bull. Malays. Math. Sci. Soc. **45** (2022), 379–398.
- [5] M. Gerstenhaber, *On the deformation of rings and algebras*, Ann. Math. **79** (1964), 59–103.
- [6] S.J. Guo, W. Teng, *Cohomology and deformation of modified Rota-Baxter Lie triple systems*, In press. (2025).
- [7] S.J. Guo, *Central extensions and deformations of Lie triple systems with a derivation*, J. Math. Res. Appl. **42** (2022), 189–198.
- [8] A. Hajjaji, *Maurer-Cartan characterizations and cohomologies of crossed homomorphisms on Lie triple systems*, Commun. Algebra **52**(21) (2024), 825–844.
- [9] B. Harris, *Cohomology of Lie triple systems and Lie algebras with involution*, Trans. Amer. Math. Soc. **98**(1) (1961), 148–162.
- [10] T. Hodge, B. Parshall, *On the representation theory of Lie triple systems*, Trans. Amer. Math. Soc. **354**(11) (2002), 4359–4391.
- [11] N. Jacobson, *Lie and Jordan triple systems*, Amer. J. Math. Soc. **71**(1) (1949), 149–170.
- [12] F. Kubo, Y. Taniguchi, *A controlling cohomology of the deformation theory of Lie triple systems*, J. Algebra **278**(1) (2004), 242–250.
- [13] W.G. Lister, *A structure theory for Lie triple systems*, Trans. Amer. Math. Soc. **72** (1952), 217–242.
- [14] Y.Z. Li, D.G. Wang, *Relative Rota-Baxter operators on Hom-Lie triple systems*, Commun. Algebra **52**(3) (2024), 1163–1178.
- [15] Y.Z. Li, D.G. Wang, *Hom-Lie Algebras with derivations*, Front. Math. **19**(3) (2024), 535–550.
- [16] G. Rahma, S. Mabrouka, A. Makhlouf, *Maurer-Cartan type cohomology on generalized Reynolds operators and NS-structures on Lie triple systems*, In press, (2025).
- [17] E. Stitzinger, *On derivation algebras of Malcev algebras and Lie triple systems*, Proc. Amer. Math. Soc. **55**(1) (1976), 9–13.
- [18] X.R. Wu, Y. Ma, L.Y. Chen, *Derivation extensions on Leibniz triple systems*, Filomat **37**(23) (2023), 7905–7918.
- [19] X.R. Wu, Y. Ma, L.Y. Chen, *Abelian extensions of Lie triple systems with derivations*, Electron. Res. Arch. **30** (2022), 1087–1103.
- [20] R. Yadav, N. Behera, R. Bhutia, *Equivariant one-parameter deformations of Lie triple systems*, J. Algebra **568** (2021), 467–479.
- [21] K. Yamaguti, *On algebras of totally geodesic spaces (Lie triple systems)*, J. Sci. Hiroshima Univ. Ser. A **21** (1957-1958), 107–113.
- [22] K. Yamaguti, *On the cohomology space of Lie triple system*, Kumamoto J. Sci. Ser. A **5** (1960), 44–52.
- [23] T. Zhang, *Notes on cohomologies of Lie triple systems*, J. Lie Theory **24**(4) (2014), 909–929.