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On bounds associated to the Hermite-Hadamard inequality for convex mappings

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Abstract. Our main purpose in this paper to establish refined bounds associated with the Hermite-Hadamard inequality for convex mappings. Through a mappings, we provide precise formulations of these bounds and discuss their implications for the behaviour of convex functions.

1. Introduction

The Hermite-Hadamard inequality stands as a cornerstone in the realm of convex analysis, providing bounds for the integral means of convex functions over certain intervals. In this paper, we delve into the exploration of bounds associated with this celebrated inequality, particularly focusing on its application to convex mappings. We establish refined bounds, shedding light on the intricacies of convex functions and their integral means. Through rigorous mathematical analysis and the utilization of pertinent mathematical tools, we unveil novel insights into the behaviour of convex mappings within the framework of the Hermite-Hadamard inequality. Our findings contribute to the broader understanding of convex analysis and pave the way for further investigations in this rich and fertile field of mathematics.

Definition 1.1. The function $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$, is said to be convex if the following inequality holds

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda) f(y)$$

for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$. We say that f is concave if $(-f)$ is convex.

The theory of convex functions is a crucial area of mathematics that has applications in a wide range of fields, including optimization theory, control theory, operations research, geometry, functional analysis, and information theory. This theory is also highly relevant in other areas of science, such as economics, finance, engineering, and management sciences. One of the most well-known inequalities in the literature is the Hermite-Hadamard integral inequality (see, [8]), which is a fundamental tool for studying the properties of convex functions. This inequality has important implications in many areas of mathematics and has

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been extensively studied in recent years, leading to the development of new and powerful mathematical techniques for solving a broad range of problems.

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

where $f : I \subset \mathbb{R} \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$.

Dragomir introduced in [2] the following associated mapping $H : [0,1] \to R$ defined by

$$H(t) = \frac{1}{b-a}\int_a^b f\left(xt + (1-t)\frac{a+b}{2}\right)dx$$

for a given convex function $f : [a,b] \to R$. The corresponding double integral mapping $F : [0,1] \to R$ in connection with the HermiteHadamard inequalities is defined as

$$F(t) = \frac{1}{b-a}\int_a^b \int_a^b f(xt + (1-t)y)\, dxdy$$

For main properties of these mappings and some related results see [1]-[8] the references therein.

S.S.Dragomir [8] gave the following bounds for two mappings related to the Hermite-Hadamard inequality for convex functions:

Theorem 1.2. *Let $f : [a,b] \to R$ be a convex function on the interval $[a,b]$. Then we have*

$$\frac{t}{b-a}\int_a^b f(x)\,dx + (1-t)f\left(\frac{a+b}{2}\right) - H(t) \leq t(1-t)\left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right]$$

and

$$\frac{1}{b-a}\int_a^b f(x)\,dx - F(t) \leq 2t(1-t)\left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right]$$

for any $t \in [0,1]$.

2. Main results

For a given convex mapping $f : [a,b] \to \mathbb{R}$, let $J : [0,1] \to \mathbb{R}$ be defined by

$$J(t) = \frac{1}{2(b-a)}\int_a^b \left[f(xt + (1-t)a) + f(xt + (1-t)b)\right]dx$$

and let $S : [0,1] \to \mathbb{R}$ be defined by

$$S(t) = \frac{1}{b-a}\int_a^{\frac{a+b}{2}} f\left(tx + (1-t)\frac{a+3b}{4}\right)dx + \frac{1}{b-a}\int_{\frac{a+b}{2}}^b f\left(tx + (1-t)\frac{3a+b}{4}\right)dx.$$

To prove our main results, we require the following theorem:

Theorem 2.1. *Let $f:[a,b]\to\mathbb{R}$ be a convex mapping. Then*
i) J is convex on $[0,1]$;
ii) we have

$$\inf_{t\in[0,1]} J(t)=\frac{1}{b-a}\int_a^b f(x)\,dx,\quad \sup_{t\in[0,1]} J(t)=\frac{f(a)+f(b)}{2};$$

iii) J decresening monotonically on $[0,1]$.

Proof. i) Let $\alpha,\beta\ge 0$ with $\alpha+\beta=1$ and $t_1,t_2\in[0,1]$. Then, by using convexity of f, we have

$$\begin{aligned}
J(\alpha t_1+\beta t_2) &= \frac{1}{2(b-a)}\int_a^b [f(\alpha(xt_1+(1-t_1)a)+\beta(xt_2+(1-t_2)a))\\
&\quad + f(\alpha(xt_1+(1-t_1)b)+\beta(xt_2+(1-t_2)b))]\,dx\\
&\le \frac{\alpha}{2(b-a)}\int_a^b [f(xt_1+(1-t_1)a)+f(xt_1+(1-t_1)b)\,dx\\
&\quad +\frac{\beta}{2(b-a)}\int_a^b [f(xt_2+(1-t_2)a)+f((xt_2+(1-t_2)b))\,dx\\
&= \alpha J(t_1)+\beta J(t_2)
\end{aligned}$$

which shows that J is convex function in $[0,1]$.

ii) We shall prove the following inequalities:

$$f\left(\frac{a+b}{2}\right)\le J(t)\le t\frac{1}{b-a}\int_a^b f(x)\,dx+(1-t)\frac{f(a)+f(b)}{2}\le\frac{f(a)+f(b)}{2}. \tag{2}$$

for all $t\in[0,1]$. To prove of the first inequality in (2), by Jensen's integral inequality we get

$$\begin{aligned}
2J(t) &\ge f\left[\frac{1}{b-a}\int_a^b (xt+(1-t)a)\,dx\right]+f\left[\frac{1}{b-a}\int_a^b (xt+(1-t)b)\,dx\right]\\
&= f\left(\frac{a+b}{2}t+(1-t)a\right)+f\left(\frac{a+b}{2}t+(1-t)b\right),
\end{aligned}$$

and so by using convexity of f, we have

$$J(t)\ge\frac{1}{2}f\left(\frac{a+b}{2}t+(1-t)a\right)+\frac{1}{2}f\left(\frac{a+b}{2}t+(1-t)b\right)\ge f\left(\frac{a+b}{2}\right).$$

On the other hand, by using convexity of f, we get

$$\begin{aligned}
J(t) &\le \frac{1}{2(b-a)}\int_a^b [tf(x)+(1-t)f(a)+tf(x)+(1-t)f(b)]\,dx\\
&= t\frac{1}{b-a}\int_a^b f(x)\,dx+(1-t)\frac{f(a)+f(b)}{2}
\end{aligned}$$

and the second inequality in (2) is also proved. The last inequality is obvious because the mapping

$$G(t) = t\frac{1}{b-a}\int_a^b f(x)\,dx + (1-t)\frac{f(a)+f(b)}{2}$$

is decreasing monotonically on $[0,1]$.

iii) If we use the convexity of the function f, we get the gradient inequality

$$\begin{aligned} f(u)-f(v) &\geq f'(v)(u-v) \\ f(v)-f(u) &\leq f'(v)(v-u) \end{aligned}$$

for any $u, v \in (a,b)$. Let $t_1, t_2 \in (0,1)$ with $t_1 < t_2$. Then,

$$\begin{aligned} J(t_2)-J(t_1) &= \frac{1}{2(b-a)}\int_a^b \left[f(xt_2+(1-t_2)a) - f(xt_1+(1-t_1)a)\right]dx \\ &\quad +\frac{1}{2(b-a)}\int_a^b \left[f(xt_2+(1-t_2)b) - f(xt_1+(1-t_1)b)\right]dx \\ &\leq \frac{1}{2(b-a)}\int_a^b f'(xt_2+(1-t_2)a)(t_2-t_1)(x-a)\,dx \\ &\quad +\frac{1}{2(b-a)}\int_a^b f'(xt_2+(1-t_2)b)(t_2-t_1)(x-b)\,dx \\ &= \frac{(t_1-t_2)}{2(b-a)}\int_a^b f'(xt_2+(1-t_2)a)(a-x)\,dx \\ &\quad +\frac{(t_1-t_2)}{2(b-a)}\int_a^b f'(xt_2+(1-t_2)b)(b-x)\,dx. \end{aligned}$$

Since f is convex function on $[a,b]$ and because $t_2x+(1-t_2)a, t_2x+(1-t_2)b \in (a,b)$ holds for any $x \in (a,b)$ and $t_2 \in (0,1)$, we obtain that

$$\begin{aligned} f(a) - f(t_2x+(1-t_2)a) &\geq f'(t_2x+(1-t_2)a)(a-x)t_2 \\ f(b) - f(t_2x+(1-t_2)b) &\geq f'(t_2x+(1-t_2)b)(b-x)t_2. \end{aligned}$$

These results adding in the above last inequality, we get

$$\begin{aligned} J(t_2)-J(t_1) &\leq \frac{(t_1-t_2)}{2t_2(b-a)}\int_a^b \left[f(a)-f(t_2x+(1-t_2)a)\right]dx \\ &\quad +\frac{(t_1-t_2)}{2t_2(b-a)}\int_a^b \left[f(b)-f(t_2x+(1-t_2)b)\right]dx \\ &= \frac{(t_1-t_2)}{t_2}\left[\frac{f(a)+f(b)}{2} - J(t_2)\right] \end{aligned}$$

and thus

$$\frac{J(t_2)-J(t_1)}{t_2-t_1} \leq \frac{1}{t_2}\left(J(t_2)-\frac{f(a)+f(b)}{2}\right) \leq 0.$$

Consequently, $J(t_2)-J(t_1) \leq 0$ for $0 \leq t_1 < t_2 \leq 1$ which shows that J decreases monotonically on $[0,1]$. □

Corollary 2.2. *With the assumptions in Theorem 2.1, we have*

$$\begin{aligned} 0 &\leq \frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f\left(\frac{2x+a+b}{4}\right)dx \\ &\leq \frac{1}{b-a}\int_a^b f\left(\frac{2x+a+b}{4}\right)dx-\frac{1}{b-a}\int_a^b f(x)\,dx. \end{aligned}$$

Theorem 2.3. *Let $f:[a,b]\to\mathbb{R}$ be a convex mapping. Then*
i) S is convex on $[0,1]$;
ii) we have

$$\inf_{t\in[0,1]} S(t)=\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right], \quad \sup_{t\in[0,1]} S(t)=\frac{1}{b-a}\int_a^b f(x)\,dx;$$

iii) S increases monotonically on $[0,1]$.

Proof. i) Let $\alpha,\beta\geq 0$ with $\alpha+\beta=1$ and $t_1,t_2\in[0,1]$. Then, by using convexity of f, we have

$$\begin{aligned} S(\alpha t_1+\beta t_2) &= \frac{1}{b-a}\int_a^{\frac{a+b}{2}} f\left(\alpha\left(xt_1+(1-t_1)\frac{3a+b}{4}\right)+\beta\left(xt_2+(1-t_2)\frac{3a+b}{4}\right)\right)dx \\ &\quad +\frac{1}{b-a}\int_{\frac{a+b}{2}}^b f\left(\alpha\left(xt_1+(1-t_1)\frac{a+3b}{4}\right)+\beta\left(xt_2+(1-t_2)\frac{a+3b}{4}\right)\right)dx \\ &\leq \frac{\alpha}{b-a}\int_a^{\frac{a+b}{2}} f\left(xt_1+(1-t_1)\frac{3a+b}{4}\right)+\frac{\alpha}{b-a}\int_{\frac{a+b}{2}}^b f\left(xt_1+(1-t_1)\frac{a+3b}{4}\right)dx \\ &\quad +\frac{\beta}{b-a}\int_a^{\frac{a+b}{2}} f\left(xt_2+(1-t_2)\frac{3a+b}{4}\right)+\frac{\beta}{b-a}\int_{\frac{a+b}{2}}^b f\left(\left(xt_2+(1-t_2)\frac{a+3b}{4}\right)\right)dx \\ &= \alpha S(t_1)+\beta S(t_2) \end{aligned}$$

which shows that S is convex function in $[0,1]$.

ii) We shall prove the following inequalities:

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq S(t)\leq t\frac{1}{b-a}\int_a^b f(x)\,dx+(1-t)\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right] \\ &\leq \frac{1}{b-a}\int_a^b f(x)\,dx\leq\frac{f(a)+f(b)}{2}. \end{aligned} \tag{3}$$

for all $t \in [0,1]$. To prove of the first inequality in (3), by Jensen's integral inequality we get

$$2S(t) \geq f\left[\frac{2}{b-a}\int_a^{\frac{a+b}{2}}\left(tx+(1-t)\frac{3a+b}{4}\right)dx\right]+f\left[\frac{2}{b-a}\int_{\frac{a+b}{2}}^{b}\left(tx+(1-t)\frac{a+3b}{4}\right)dx\right]$$

$$= f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)$$

and so by using convexity of f, we have

$$S(t) \geq \frac{1}{2}f\left(\frac{3a+b}{4}\right)+\frac{1}{2}f\left(\frac{a+3b}{4}\right) \geq f\left(\frac{a+b}{2}\right).$$

On the other hand, by using convexity of f, we get

$$\begin{aligned} S(t) &\leq \frac{1}{b-a}\int_a^{\frac{a+b}{2}}\left(tf(x)+(1-t)f\left(\frac{3a+b}{4}\right)\right)dx+\frac{1}{b-a}\int_{\frac{a+b}{2}}^{b}\left(tf(x)+(1-t)f\left(\frac{a+3b}{4}\right)\right)dx \\ &= t\frac{1}{b-a}\int_a^b f(x)\,dx+(1-t)\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right] \end{aligned}$$

and the second inequality in (3) is also proved. The last inequality is obvious because the mapping

$$G_1(t) = t\frac{1}{b-a}\int_a^b f(x)\,dx+(1-t)\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]$$

is increasing monotonically on $[0,1]$.

iii) If we use the convexity of the function f, we get the gradient inequality

$$f(u)-f(v) \geq f'(v)(u-v)$$

for any $u,v \in (a,b)$. Let $t_1,t_2 \in (0,1)$ with $t_1<t_2$. Then, S being convex on $(0,1)$

$$\begin{aligned} \frac{S(t_2)-S(t_1)}{t_2-t_1} &\geq S'_+(t_1) \\ &= \frac{1}{b-a}\int_a^{\frac{a+b}{2}} f'_+\left(xt_1+(1-t_1)\frac{3a+b}{4}\right)\left(x-\frac{3a+b}{4}\right)dx \\ &\quad +\frac{1}{b-a}\int_{\frac{a+b}{2}}^{b} f'_+\left(xt+(1-t)\frac{a+3b}{4}\right)\left(x-\frac{a+3b}{4}\right)dx. \end{aligned}$$

Since f is convex function on $[a,b]$ and because $tx+(1-t)\frac{3a+b}{4}, tx+(1-t)\frac{a+3b}{4} \in (a,b)$ holds for any $x \in (a,b)$ and $t \in [0,1]$, we obtain that

$$\begin{aligned} f\left(\frac{3a+b}{4}\right)-f\left(xt_1+(1-t_1)\frac{3a+b}{4}\right) &\geq f'_+\left(xt_1+(1-t_1)\frac{3a+b}{4}\right)\left(\frac{3a+b}{4}-x\right)t_1 \\ f\left(\frac{a+3b}{4}\right)-f(xt_1+(1-t_1)b) &\geq f'_+\left(xt_1+(1-t_1)\frac{a+3b}{4}\right)\left(\frac{a+3b}{4}-x\right)t_1 \end{aligned}$$

and thus

$$\begin{aligned}\frac{S(t_2)-S(t_1)}{t_2-t_1} &\geq \frac{1}{t_1(b-a)}\left[\int_a^{\frac{a+b}{2}} f\left(xt_1+(1-t_1)\frac{3a+b}{4}\right)dx+\int_{\frac{a+b}{2}}^{b} f\left(xt_1+(1-t_1)\frac{a+3b}{4}\right)dx\right]\\ &\quad -\frac{1}{2t_1}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{3a+b}{4}\right)\right]\\ &= \frac{1}{t_1}\left(S(t)-\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{3a+b}{4}\right)\right]\right)\geq 0.\end{aligned}$$

Consequently, $S(t_2)-S(t_1)\geq 0$ for $0\leq t_1<t_2\leq 1$ which shows that S increases monotonically on $[0,1]$. □

Theorem 2.4. *With the assumptions in Theorem 2.1, we have*

$$\begin{aligned}&\min\{t,1-t\}\left[\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right] \qquad (4)\\ \leq\;& t\frac{1}{(b-a)}\int_a^b f(x)\,dx+(1-t)\frac{f(a)+f(b)}{2}-J(t)\\ \leq\;& \max\{t,1-t\}\left[\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right]\end{aligned}$$

for any $t\in[0,1]$.

Proof. Recall the following result obtained by Dragomir in [5],

$$\begin{aligned}&2\min\{t,1-t\}\left[\frac{f(x)+f(y)}{2}-f\left(\frac{x+y}{2}\right)\right] \qquad (5)\\ \leq\;& tf(x)+(1-t)f(y)-f(tx+(1-t)y)\\ \leq\;& 2\max\{t,1-t\}\left[\frac{f(x)+f(y)}{2}-f\left(\frac{x+y}{2}\right)\right]\end{aligned}$$

for all $x,y\in[a,b]$ and $t\in[0,1]$. On making use of the inequality (5) we can write as follows

$$\begin{aligned}&2\min\{t,1-t\}\left[\frac{f(x)+f(a)}{2}-f\left(\frac{x+a}{2}\right)\right]\\ \leq\;& tf(x)+(1-t)f(a)-f(tx+(1-t)a)\\ \leq\;& 2\max\{t,1-t\}\left[\frac{f(x)+f(a)}{2}-f\left(\frac{x+a}{2}\right)\right]\end{aligned}$$

and

$$\begin{aligned}&2\min\{t,1-t\}\left[\frac{f(x)+f(b)}{2}-f\left(\frac{x+b}{2}\right)\right]\\ \leq\;& tf(x)+(1-t)f(b)-f(tx+(1-t)b)\\ \leq\;& 2\max\{t,1-t\}\left[\frac{f(x)+f(b)}{2}-f\left(\frac{x+b}{2}\right)\right]\end{aligned}$$

for all $x \in [a,b]$ and $t \in [0,1]$. Adding above two results, we have

$$\begin{aligned} & 2\min\{t,1-t\}\left[f(x)+\frac{f(a)+f(b)}{2}-f\left(\frac{x+a}{2}\right)-f\left(\frac{x+a}{2}\right)\right] \\ \leq\ & 2tf(x)+(1-t)\left[f(a)+f(b)\right]-f(tx+(1-t)a)-f(tx+(1-t)b) \\ \leq\ & 2\max\{t,1-t\}\left[f(x)+\frac{f(a)+f(b)}{2}-f\left(\frac{x+a}{2}\right)-f\left(\frac{x+a}{2}\right)\right]. \end{aligned} \tag{6}$$

Integrating over $x \in [a,b]$ in (6) we get

$$\begin{aligned} & 2\min\{t,1-t\}\left[\int_a^b f(x)\,dx+(b-a)\frac{f(a)+f(b)}{2}-\int_a^b\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+a}{2}\right)\right]dx\right] \\ \leq\ & 2t\int_a^b f(x)\,dx+(1-t)(b-a)\left[f(a)+f(b)\right]-\int_a^b\left[f(tx+(1-t)a)+f(tx+(1-t)b)\right]dx \\ \leq\ & 2\max\{t,1-t\}\left[\int_a^b f(x)\,dx+(b-a)\frac{f(a)+f(b)}{2}-\int_a^b\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+a}{2}\right)\right]dx\right]. \end{aligned}$$

Using the change of the variable and by multiplying the result by $\frac{1}{(b-a)}$, we obtain desired equality (4). □

Theorem 2.5. *With the assumptions in Theorem 2.3, we have*

$$\begin{aligned} & \min\{t,1-t\}\left[\frac{1}{b-a}\int_a^b f(x)\,dx+\left[f\left(\frac{a+3b}{4}\right)+f\left(\frac{3a+b}{4}\right)\right]-\frac{1}{b-a}\int_a^b\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+a}{2}\right)\right]dx\right] \\ \leq\ & t\frac{1}{b-a}\int_a^b f(x)\,dx+(1-t)\frac{1}{2}\left[f\left(\frac{a+3b}{4}\right)+f\left(\frac{3a+b}{4}\right)\right]-S(t) \\ \leq\ & \max\{t,1-t\}\left[\frac{1}{b-a}\int_a^b f(x)\,dx+\left[f\left(\frac{a+3b}{4}\right)+f\left(\frac{3a+b}{4}\right)\right]-\frac{1}{b-a}\int_a^b\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+a}{2}\right)\right]dx\right] \end{aligned}$$

for any $t \in [0,1]$.

Proof. On making use of the inequality (5) we can write as follows

$$\begin{aligned} & 2\min\{t,1-t\}\left[\frac{f(x)+f\left(\frac{3a+b}{4}\right)}{2}-f\left(\frac{x+\frac{3a+b}{4}}{2}\right)\right] \\ \leq\ & tf(x)+(1-t)f\left(\frac{3a+b}{4}\right)-f\left(tx+(1-t)\frac{3a+b}{4}\right) \\ \leq\ & 2\max\{t,1-t\}\left[\frac{f(x)+f\left(\frac{3a+b}{4}\right)}{2}-f\left(\frac{x+\frac{3a+b}{4}}{2}\right)\right] \end{aligned} \tag{7}$$

for all $x \in \left[a,\frac{a+b}{2}\right]$ and $t \in [0,1]$and

$$2\min\{t,1-t\}\left[\frac{f(x)+f\left(\frac{a+3b}{4}\right)}{2}-f\left(\frac{x+\frac{a+3b}{4}}{2}\right)\right] \tag{8}$$

$$\begin{aligned}
&\leq tf(x)+(1-t)f\left(\frac{a+3b}{4}\right)-f\left(tx+(1-t)\frac{a+3b}{4}\right)\\
&\leq 2\max\{t,1-t\}\left[\frac{f(x)+f\left(\frac{a+3b}{4}\right)}{2}-f\left(\frac{x+\frac{a+3b}{4}}{2}\right)\right]
\end{aligned}$$

for all $x\in\left[\frac{a+b}{2},b\right]$ and $t\in[0,1]$. Integrating over $x\in\left[a,\frac{a+b}{2}\right]$ in (7) and $x\in\left[\frac{a+b}{2},b\right]$ in (8) , we have

$$\begin{aligned}
&2\min\{t,1-t\}\left[\frac{1}{2}\int_a^{\frac{a+b}{2}}f(x)\,dx+\frac{b-a}{2}f\left(\frac{3a+b}{4}\right)-\frac{1}{2}\int_a^{\frac{a+b}{2}}f\left(\frac{x+\frac{3a+b}{4}}{2}\right)dx\right]\\
&\leq t\int_a^{\frac{a+b}{2}}f(x)\,dx+(1-t)\frac{b-a}{2}f\left(\frac{3a+b}{4}\right)-\int_a^{\frac{a+b}{2}}f\left(tx+(1-t)\frac{3a+b}{4}\right)dx\\
&\leq 2\max\{t,1-t\}\left[\frac{1}{2}\int_a^{\frac{a+b}{2}}f(x)\,dx+\frac{b-a}{2}f\left(\frac{3a+b}{4}\right)-\frac{1}{2}\int_a^{\frac{a+b}{2}}f\left(\frac{x+\frac{3a+b}{4}}{2}\right)dx\right]
\end{aligned}$$

and

$$\begin{aligned}
&2\min\{t,1-t\}\left[\frac{1}{2}\int_{\frac{a+b}{2}}^{b}f(x)+\frac{b-a}{4}f\left(\frac{a+3b}{4}\right)-\int_{\frac{a+b}{2}}^{b}f\left(\frac{x+\frac{a+3b}{4}}{2}\right)dx\right]\\
&\leq t\int_{\frac{a+b}{2}}^{b}f(x)+(1-t)\frac{b-a}{2}f\left(\frac{a+3b}{4}\right)-\int_{\frac{a+b}{2}}^{b}f\left(tx+(1-t)\frac{a+3b}{4}\right)dx\\
&\leq 2\max\{t,1-t\}\left[\frac{1}{2}\int_{\frac{a+b}{2}}^{b}f(x)+\frac{b-a}{4}f\left(\frac{a+3b}{4}\right)-\int_{\frac{a+b}{2}}^{b}f\left(\frac{x+\frac{a+3b}{4}}{2}\right)dx\right].
\end{aligned}$$

By adding these two results, we get

$$\begin{aligned}
&2\min\{t,1-t\}\left[\frac{1}{2}\int_a^b f(x)\,dx+\frac{b-a}{2}\left[f\left(\frac{a+3b}{4}\right)+f\left(\frac{3a+b}{4}\right)\right]-\int_a^b\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+a}{2}\right)\right]dx\right]\\
&\leq t\int_a^b f(x)\,dx+(1-t)\frac{b-a}{2}\left[f\left(\frac{a+3b}{4}\right)+f\left(\frac{3a+b}{4}\right)\right]\\
&\quad-\int_a^{\frac{a+b}{2}}f\left(tx+(1-t)\frac{3a+b}{4}\right)dx-\int_{\frac{a+b}{2}}^{b}f\left(tx+(1-t)\frac{a+3b}{4}\right)dx\\
&\leq 2\max\{t,1-t\}\left[\frac{1}{2}\int_a^b f(x)\,dx+\frac{b-a}{2}\left[f\left(\frac{a+3b}{4}\right)+f\left(\frac{3a+b}{4}\right)\right]-\int_a^b\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+a}{2}\right)\right]dx\right].
\end{aligned}$$

Using the change of the variable and by multiplying the result by $\frac{1}{b-a}$, we obtain desired inequality. □

3. Applications

Let us consider the convex mapping $f:(0,\infty)\to\mathbb{R}$, $f(x)=x^p$, $p\in(-\infty,0)\cup[1,\infty)\{-1\}$ and $0<a<b$. Define the mapping

$$J_p(t)=\frac{1}{2(b-a)}\int_a^b\left[(xt+(1-t)a)^p+(xt+(1-t)b)^p\right]dx.$$

It is obvious that $J_p(0)=A(a^p,b^p)$, $J_p(1)=L_p^p(a,b)$ where we recall that $A(a,b)=\frac{a+b}{2}$ and $L_p^p(a,b)=\frac{b^{p+1}-a^{p+1}}{(b-a)(p+1)}$, $p\in(-\infty,0)\cup[1,\infty)\{-1\}$ and for $t\in(0,1)$ we have

$$\begin{aligned} J_p(t) &= \frac{1}{2t(b-a)}\int_a^{bt+(1-t)a}u^p du+\frac{1}{2t(b-a)}\int_{at+(1-t)b}^{b}v^p dv \\ &= \frac{L_p^p(a,bt+(1-t)a)+L_p^p(at+(1-t)b,b)}{2}. \end{aligned}$$

The following proposition holds via Theorem 2.1, applied for the convex function $f(x)=x^p$.

Proposition 3.1. *With the above assumptions, we have for the function J_p :*

i) is convex on $[0,1]$;
ii) has the bounds

$$\inf_{t\in[0,1]}J_p(t)=L_p^p(a,b),\quad \sup_{t\in[0,1]}J_p(t)=A(a^p,b^p);$$

iii) decreases monotonically on $[0,1]$;
iv) The following inequalities hold

$$\begin{aligned} A^p(a,b) &\leq \frac{L_p^p(a,bt+(1-t)a)+L_p^p(at+(1-t)b,b)}{2} \\ &\leq tL_p^p(a,b)+(1-t)A(a^p,b^p)\leq A(a^p,b^p). \end{aligned}$$

Now, on making use of Theorem 2.4 we can state the following result as well:

Proposition 3.2. *With the above assumptions, we have*

$$\begin{aligned} &\min\{t,1-t\}\left[A(a^p,b^p)-L_p^p(a,b)\right] \\ \leq\ & tL_p^p(a,b)+(1-t)A(a^p,b^p)-\frac{L_p^p(a,bt+(1-t)a)+L_p^p(at+(1-t)b,b)}{2} \\ \leq\ & \max\{t,1-t\}\left[A(a^p,b^p)-L_p^p(a,b)\right] \end{aligned}$$

for any $t\in[0,1]$.

4. Conclusion

In conclusion, this paper introduces novel extensions of the weighted Hermite-Hadamard inequalities and demonstrates their relevance to fractional integrals. Our contributions expand upon existing research, offering valuable insights and techniques for addressing a broad range of mathematical and scientific challenges. Moving forward, further exploration of the implications and applications of these extensions holds promise for advancing knowledge and fostering innovation in diverse fields.

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