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Non-null space curves related by a transformation of combescure

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Abstract. This study investigates non-null space curves in Minkowski 3-space that are related through the Combescure transformation, a geometric operation characterized by parallel tangent vectors at corresponding points. Using the Frenet apparatus, conditions for two non-null curves to exhibit this transformation are derived, with a focus on their tangent, principal normal, and binormal vectors. The relationships reveal significant properties of curve interactions, including parallelism of Frenet vectors and connections to Bertrand curve pairs. Additionally, the effects of the Combescure transformation on curvature and torsion are analyzed and illustrated through examples. Furthermore, as an application, the conditions under which a spatial curve associated with a non-null biharmonic curve via the Combescure transformation remains biharmonic are determined and supported by relevant examples.

1. Introduction

An important aspect of the geometry of a curve involves analyzing curve pairs through the relationships between their Frenet vectors, tangent, principal normal, and binormal. These vectors encapsulate critical geometric information, enabling the study of curve interactions. Special attention is given to cases where the Frenet vectors of one curve correspond to those of another, leading to six distinct geometric relationships [14]. Such alignments define significant curve classes, like Bertrand and Mannheim curve pairs, which exhibit unique geometric properties and dependencies [5].

For example, Bertrand curve pairs have parallel principal normals at corresponding points, implying a relationship between their osculating planes. Mannheim curve pairs, where the principal normal of one curve aligns with the binormal of another, highlight interactions between curvature and torsion. These classifications play a key role in differential geometry, with applications in robotics, computer-aided design, and structural analysis [1, 2, 10, 11, 18].

The Combescure transformation, a fundamental concept in differential geometry, relates two curves whose tangent vectors are parallel at corresponding points, ensuring the parallelism of their principal normal and binormal vectors. This transformation has been generalized to Riemannian manifolds, where it defines ”parallel tangent deformations” preserving normal vector alignments. Hayden demonstrated that flat spaces are the only Riemannian manifolds exhibiting the generalized Combescure (G.C.) property [7, 20].

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In recent studies, the Combescure transformation has been used to systematically generate special curves, such as Bertrand, Mannheim, and Salkowski curves, from a given curve. For instance, a Mannheim curve can be derived from an anti-Salkowski curve. These investigations provide a framework for identifying curve classes and exploring their geometric properties, enriching both theoretical and applied geometry [3, 13, 15, 16].

In this paper, the geometric properties of non-null space curve pairs in Minkowski 3-space, which possess parallel tangent vector fields or, equivalently, are associated through a Combescure transformation, are studied by considering the works mentioned above. First and foremost, these curve pairs emerge as having the same causal character and parallel Frenet vectors. It is also demonstrated that these curve pairs have equal curvature ratios. The study further investigates the properties of certain special curves, such as circular helices, general helices, and slant helices, which are connected through the Combescure transformation to their corresponding associated curves, as these curves hold significant importance in differential geometry. Additionally, the conditions for an associated curve connected via the Combescure transformation to be a biharmonic curve are presented under a separate section dedicated to biharmonic curves. All results are supported with illustrative examples.

2. Preliminaries

Minkowski space $\mathbb{E}_1^3$ is a three-dimensional affine space endowed with an indefinite flat metric $g(\cdot,\cdot)$ with signature $(-,+,+)$. This means that metric bilinear form can be written as

$$g(u,v) = -u_1v_1 + u_2v_2 + u_3v_3,$$

for and two vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ in $\mathbb{E}_1^3$. Recall that a vector $u \in \mathbb{E}_1^3\backslash\{0\}$ can be *spacelike* if $g(u,u) > 0$, *timelike* if $g(u,u) < 0$ and *null (lightlike)* if $g(u,u) = 0$ and $u \neq 0$. In particular, the vector $u = 0$ is spacelike. The norm of a vector u is given by $\|u\| = \sqrt{|g(u,u)|}$, and two vectors u and v are said to be orthogonal, if $g(u,v) = 0$. An arbitrary curve $\varphi(s)$ in $\mathbb{E}_1^3$, can locally be *spacelike*, *timelike*, or *null* (*lightlike*), if all its velocity vectors $\varphi'(s)$ are respectively spacelike, timelike, or null. A null curve φ is parameterized by pseudo-arc s if $g(\varphi''(s), \varphi''(s)) = 1$. A spacelike or a timelike curve $\varphi(s)$ has unit speed, if $g(\varphi'(s), \varphi'(s)) = \pm 1$ [12], [17].

Let $\{T, N, B\}$ denote the moving Frenet frame along a curve φ in $\mathbb{E}_1^3$, T, N and B represent the tangent, principal normal, and binormal vector fields, respectively. The form of the Frenet equations varies depending on the causal character of the curve φ.

If φ is a non-null curve, the Frenet equations are given by:

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \epsilon_2\kappa & 0 \\ -\epsilon_1\kappa & 0 & \epsilon_3\tau \\ 0 & -\epsilon_2\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{1}$$

where κ and τ are the first and the second curvature of the curve, respectively. Moreover, the following conditions hold:

$$\epsilon_1 = g(T,T) = \pm 1, \epsilon_2 = g(N,N) = \pm 1, \epsilon_3 = g(B,B) = \pm 1$$

$$g(T,N) = g(T,B) = g(N,B) = 0$$

where $\epsilon_1\epsilon_2\epsilon_3 = -1$ and $\exists i \in \{1,2,3\}, \epsilon_i = 1$.

3. Non-null Space Curves Related by a Transformation of Combescure

In this section, the properties of non-null space curves associated with Combescure transformation in Minkowski 3-space will be analysed. Firstly, let us give the Minkowski 3-space analogue of the definition given by Graustein [6] and [19] in Euclidean 3-space.

Definition 3.1. *Let $\varphi : I \subseteq \mathbb{R} \to \mathbb{E}_1^3$ and $\varphi^* : I^* \subseteq \mathbb{R} \to \mathbb{E}_1^3$ be unit speed non-null curves with the same causal characters in $\mathbb{E}_1^3$ with Frenet apparatus $\{T, N, B, \kappa, \tau\}$ and $\{T^*, N^*, B^*, \kappa^*, \tau^*\}$, respectively. If the tangent vectors at the corresponding points of φ and φ^* are equal, these curves are called curves related by a transformation of Combescure.*

Theorem 3.2. *Let $\varphi : I \subseteq \mathbb{R} \to \mathbb{E}_1^3$ and $\varphi^* : I^* \subseteq \mathbb{R} \to \mathbb{E}_1^3$ be unit speed non-null curves with the same causal characters in $\mathbb{E}_1^3$ with Frenet apparatus $\{T, N, B, \kappa, \tau\}$ and $\{T^*, N^*, B^*, \kappa^*, \tau^*\}$, respectively. Then the tangent vector T of φ is equal to the tangent vector T^* of φ^* at the corresponding points if and only if there exists $C : I \subseteq \mathbb{R} \to \mathbb{R}$ differentiable function such that*

$$\varphi^*(f(s)) = \varphi(s) + \frac{d}{dr}\left(C(s) - \varepsilon_1 \frac{d^2C(s)}{dt^2}\right)T - \varepsilon_3 \frac{d^2C(s)}{dt^2} N + \frac{dC(s)}{dt} B \tag{2}$$

where $t = \int \tau(s)\,ds$, $r = \int \kappa(s)\,ds$ and $f'(s) = ds^*/ds$ where s and s^* arclength parameters of φ and φ^*, respectively.

Proof. Let $\varphi : I \subseteq \mathbb{R} \to \mathbb{E}_1^3$ and $\varphi^* : I^* \subseteq \mathbb{R} \to \mathbb{E}_1^3$ be unit speed non-null curves in $\mathbb{E}_1^3$ with Frenet apparatus $\{T, N, B, \kappa, \tau\}$ and $\{T^*, N^*, B^*, \kappa^*, \tau^*\}$, respectively. Assume that $T = T^*$ and

$$\varphi^*(s^*) = \varphi^*(f(s)) = \varphi(s) + \mu_1(s)\,T(s) + \mu_2(s)\,N(s) + \mu_3(s)\,B(s) \tag{3}$$

where μ_1, μ_2 and μ_3 are differentiable functions on $I \subseteq \mathbb{R}$. Then differentiating (3) with respect to s and using (1), we obtain

$$\frac{d\varphi^*}{ds^*}\frac{ds^*}{ds} = \left(\mu_1' - \varepsilon_1\kappa\mu_2 + 1\right)T + \left(\mu_2' +_1 \varepsilon_2\kappa\mu_1 - \varepsilon_2\tau\mu_3\right)N + \left(\mu_3' + \varepsilon_3\tau\mu_2\right)B.$$

From $T = T^*$, we get

$$\left.\begin{array}{c} \mu_1' - \varepsilon_1\kappa\mu_2 + 1 = \frac{ds^*}{ds} \\ \mu_2' + \varepsilon_2\kappa\mu_1 - \varepsilon_2\tau\mu_3 = 0 \\ \mu_3' + \varepsilon_3\tau\mu_2 = 0 \end{array}\right\}. \tag{4}$$

In the solution of the above system, if we take

$$\mu_3 = \frac{dC(s)}{dt}, \tag{5}$$

where $t = \int \tau(s)\,ds$, which is the arclength element of the binormal indicatrix of φ, and using the third equation in (4), then we get

$$\begin{aligned} \mu_2 &= -\varepsilon_3 \frac{1}{\tau}\frac{d}{ds}\left(\frac{dC(s)}{dt}\right) \\ &= -\varepsilon_3 \frac{1}{\tau}\frac{d^2C(s)}{dt^2}\frac{dt}{ds} \\ &= -\varepsilon_3 \frac{d^2C(s)}{dt^2}. \end{aligned} \tag{6}$$

Substituting (5) and (6) in the second equation in (4), we obtain

$$\begin{aligned} \mu_1 &= \frac{1}{\kappa}\left(-\varepsilon_2\mu_2' + \tau\mu_3\right) \\ &= \frac{1}{\kappa}\left(\tau\frac{dC(s)}{dt} - \varepsilon_2\frac{d}{ds}\left(-\varepsilon_3\frac{d^2C(s)}{dt^2}\right)\right) \\ &= \frac{1}{\kappa}\left(\tau\frac{dC(s)}{dt} + \varepsilon_2\varepsilon_3\frac{d^3C(s)}{dt^3}\frac{dt}{ds}\right). \end{aligned}$$

If we use $r=\int\kappa(s)\,ds$, which is the arclength element of the tangent indicatrix of φ, and chain rule, we get

$$\begin{aligned}\mu_1 &= \frac{1}{\kappa}\left(\tau\frac{dC(s)}{dt}+\varepsilon_2\varepsilon_3\frac{d^3C(s)}{dt^3}\frac{dt}{ds}\right)\\ &= \frac{dC(s)}{dt}\frac{dt}{ds}\frac{ds}{dr}+\varepsilon_2\varepsilon_3\frac{d^3C(s)}{dt^3}\frac{dt}{ds}\frac{ds}{dr}\\ &= \frac{dC(s)}{dt}\frac{dt}{dr}+\varepsilon_2\varepsilon_3\frac{d^3C(s)}{dt^3}\frac{dt}{dr}\\ &= \frac{d}{dr}\left(C(s)+\varepsilon_2\varepsilon_3\frac{d^2C(s)}{dt^2}\right).\end{aligned}$$

Substituting μ_1,μ_2 and μ_3 in (3), we obtain

$$\varphi^*(f(s))=\varphi(s)+\frac{d}{dr}\left(C(s)+\varepsilon_2\varepsilon_3\frac{d^2C(s)}{dt^2}\right)T-\varepsilon_3\frac{d^2C(s)}{dt^2}N+\frac{dC(s)}{dt}B.$$

Conversely, assume that $\varphi: I\subseteq\mathbb{R}\to\mathbb{E}_1^3$ be a unit speed non-null curve in $\mathbb{E}_1^3$ with Frenet apparatus $\{T,N,B,\kappa,\tau\}$, $C: I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function and

$$\varphi^*(s^*)=\varphi^*(f(s))=\varphi(s)+\frac{d}{dr}\left(C(s)+\varepsilon_2\varepsilon_3\frac{d^2C(s)}{dt^2}\right)T-\varepsilon_3\frac{d^2C(s)}{dt^2}N+\frac{dC(s)}{dt}B. \tag{7}$$

Differentiating (7) with respect to s, we find

$$\begin{aligned}\frac{d\varphi^*}{ds^*}\frac{ds^*}{ds} &= \left\{1+\frac{d}{ds}\left(\frac{dC}{dr}-\varepsilon_1\frac{d^3C}{dt^3}\frac{dt}{dr}\right)-\varepsilon_2\frac{d^2C}{dt^2}\frac{dr}{ds}\right\}T\\ &\quad+\left\{\varepsilon_2\frac{dC}{dr}\frac{dr}{ds}+\varepsilon_3\frac{d^3C}{dt^3}\frac{dt}{dr}\frac{dr}{ds}-\varepsilon_3\frac{d^3C}{dt^3}\frac{dt}{ds}-\varepsilon_2\frac{dC}{dt}\frac{dt}{ds}\right\}N\\ &\quad+\left\{-\frac{d^2C}{dt^2}\frac{dt}{ds}+\frac{d^2C}{dt^2}\frac{dt}{ds}\right\}B.\end{aligned} \tag{8}$$

By using $t=\int\tau(s)\,ds$ and $r=\int\kappa(s)\,ds$ in (8), we get

$$T^*\frac{ds^*}{ds}=\left\{1+\frac{d}{ds}\left(\frac{dC}{dr}-\varepsilon_1\frac{d^3C}{dt^3}\frac{dt}{dr}\right)-\varepsilon_2\frac{d^2C}{dt^2}\frac{dr}{ds}\right\}T.$$

Since T^* and T are unit vectors from above equality, we get $T^*=T$. Thus φ^* and φ are non-null space curves are related by a transformation of Combescure. This completes the proof. □

Theorem 3.3. *Let $\varphi: I\subseteq\mathbb{R}\to\mathbb{E}_1^3$ and $\varphi^*: I^*\subseteq\mathbb{R}\to\mathbb{E}_1^3$ be unit speed non-null curves with the same causal characters in $\mathbb{E}_1^3$ with Frenet apparatus $\{T,N,B,\kappa,\tau\}$ and $\{T^*,N^*,B^*,\kappa^*,\tau^*\}$, respectively. The curves φ and φ^* are related by a transformation of Combescure if and only if $N=N^*$ and $B=B^*$ at the corresponding points of φ and φ^*.*

Proof. The proof is clear from the above proof of theorem. □

Corollary 3.4. *If two non-null space curves in $\mathbb{E}_1^3$ are connected related by a transformation of Combescure, then these curves have parallel Frenet vectors.*

Theorem 3.5. *Let $\varphi: I\subseteq\mathbb{R}\to\mathbb{E}_1^3$ and $\varphi^*: I^*\subseteq\mathbb{R}\to\mathbb{E}_1^3$ be unit speed non-null curves with the same causal characters in $\mathbb{E}_1^3$ with Frenet apparatus $\{T,N,B,\kappa,\tau\}$ and $\{T^*,N^*,B^*,\kappa^*,\tau^*\}$, respectively. Then the tangent vector T of φ is equal to the tangent vector T^* of φ^* if and only if*

$$\varphi^*(f(s))=f'(s)\,\varphi(s)-\int f''(s)\,\varphi(s)\,ds \tag{9}$$

for $f'(s)=ds^*/ds$ where s and s^* are arclength parameters of φ and φ^*, respectively.

Proof. Let $\varphi : I \subseteq \mathbb{R} \to \mathbb{E}_1^3$ and $\varphi^* : I^* \subseteq \mathbb{R} \to \mathbb{E}_1^3$ be unit speed non-null curves with the same causal characters in $\mathbb{E}_1^3$ with Frenet apparatus $\{T, N, B, \kappa, \tau\}$ and $\{T^*, N^*, B^*, \kappa^*, \tau^*\}$, respectively. Assume that the tangent vector T of φ is equal to the tangent vector T^* of φ^*. Then from Theorem 3.2, there exists $C : I \subseteq \mathbb{R} \to \mathbb{R}$ such that

$$\varphi^*(f(s)) = \varphi(s) + \frac{d}{dr}\left(C(s) - \varepsilon_1 \frac{d^2C(s)}{dt^2}\right)T - \varepsilon_3 \frac{d^2C(s)}{dt^2}N + \frac{dC(s)}{dt}B \tag{10}$$

where $t = \int \tau(s)\,ds$ and $r = \int \kappa(s)\,ds$. Differentiating (10) with respect to s and by using $T = T^*$, we get

$$\frac{ds^*}{ds} = f'(s) = 1 + \frac{d}{ds}\left(\frac{dC}{dr} - \varepsilon_1 \frac{d^3C}{dt^3}\frac{dt}{dr}\right) - \varepsilon_2 \frac{d^2C}{dt^2}\frac{dr}{ds}$$

and

$$T^* ds^* = f' T ds \tag{11}$$

By integrating the last equality, we have

$$\varphi^*(f(s)) = f'(s)\varphi(s) - \int f''(s)\varphi(s)\,ds. \tag{12}$$

Conversely we assume that equation (9) is satisfy. By differentiating (9) with respect to s, we get (11) .Which means that The tangent vectors of the curves φ and φ^* are equal. This completes the proof. □

Corollary 3.6. *Let $\varphi : I \subseteq \mathbb{R} \to \mathbb{E}_1^3$ and $\varphi^* : I^* \subseteq \mathbb{R} \to \mathbb{E}_1^3$ be unit speed non-null curves with the same causal characters in $\mathbb{E}_1^3$ with Frenet apparatus $\{T, N, B, \kappa, \tau\}$ and $\{T^*, N^*, B^*, \kappa^*, \tau^*\}$, respectively. If the tangent vector T of φ is equal to the tangent vector T^* of φ^* then we get*

$$f'(s) = \frac{ds^*}{ds} = \frac{\kappa}{\kappa^*} = \frac{\tau}{\tau^*}.$$

Proof. Let $\varphi : I \subseteq \mathbb{R} \to \mathbb{E}_1^3$ and $\varphi^* : I^* \subseteq \mathbb{R} \to \mathbb{E}_1^3$ be unit speed non-null curves with the same causal characters in $\mathbb{E}_1^3$ with Frenet apparatus $\{T, N, B, \kappa, \tau\}$ and $\{T^*, N^*, B^*, \kappa^*, \tau^*\}$, respectively. Assume that the tangent vector T of φ is equal to the tangent vector T^* of φ^*. Then differentiating $T = T^*$ with respect to s, we have

$$\frac{ds^*}{ds}\kappa^* N^* = \kappa N$$

which implies that

$$\frac{ds^*}{ds} = \frac{\kappa}{\kappa^*}.$$

Also differentiating $B = B^*$ with respect to s, we get

$$\frac{ds^*}{ds}\tau^* N^* = \tau N$$

which implies that

$$\frac{ds^*}{ds} = \frac{\tau}{\tau^*}.$$

□

As a consequence of the theorems and results stated and proven above, the relationships between certain special curves, which are extensively studied in differential geometry, and their corresponding conjugate curves under the Combescure transformation are presented below without proof.

Corollary 3.7. *Let $\varphi : I \subseteq \mathbb{R} \to \mathbb{E}_1^3$ and $\varphi^* : I^* \subseteq \mathbb{R} \to \mathbb{E}_1^3$ be unit speed non-null curves with the same causal characters in $\mathbb{E}_1^3$ with Frenet apparatus $\{T, N, B, \kappa, \tau\}$ and $\{T^*, N^*, B^*, \kappa^*, \tau^*\}$, respectively. If the tangent vectors at the corresponding points of φ and φ^* are equal, these curves are called curves related by a transformation of Combescure. Then we have the followings.*

(i) *The curvature ratios of the space curve pairs (φ, φ^*) are equal. In other words*

$$\frac{\kappa}{\tau} = \frac{\kappa^*}{\tau^*}$$

(ii) *The space curve pairs (φ, φ^*) is a Bertrand curve pairs.*

(iii) *If φ is a circular helix or general helix then φ^* is a circular helix or general helix.*

(iv) *If the curve φ is a non-null slant helix, then the necessary and sufficient condition for the curve φ^* to be a slant curve is that $f' = \dfrac{ds^*}{ds}$ is a nonzero constant.*

(v) *If φ is a non-null Salkowski curve, then φ^* cannot be a Salkowski curve. For the φ^* to be an anti-Salkowski curve, the torsion of the curve φ must be equal to f'. If φ is a non-null anti-Salkowski curve, then φ^* can be an anti-Salkowski curve if f' is a nonzero constant. For the φ^* to be a Salkowski curve, the curvature of the curve φ must be equal to f'.*

Example 3.8. *Let us consider the curve in $\mathbb{E}_1^3$ given by*

$$\varphi(s) = \left(\sinh s, \cosh s, \sqrt{2}s\right)$$

with the curvatures $\kappa = 1$ and $\tau = \sqrt{2}$ and the Frenet frame as

$$T(s) = \left(\cosh s, \sinh s, \sqrt{2}\right)$$

$$N(s) = (\sinh s, \cosh s, 0)$$

$$B(s) = \left(-\sqrt{2}\cosh s, -\sqrt{2}\sinh s, -1\right).$$

If we take $C(s) = \sqrt{2}s^2$ in Theorem 3.2, we obtain related by transformation of Combescure curve φ_1^* as follows

$$\varphi_1^*(f(s)) = \left(\left(1+\sqrt{2}\right)\sinh s, \left(1+\sqrt{2}\right)\cosh s, \sqrt{2}\left(1+\sqrt{2}\right)s\right)$$

with curvatures $\kappa_1^* = \frac{1}{1+\sqrt{2}}$ and $\tau_1^* = \frac{\sqrt{2}}{\sqrt{2}+1}$.
If we take $C(s) = \sqrt{2}s^3$ Theorem 3.2, we obtain related by transformation of Combescure curve φ_2^* as follows

$$\varphi_2^*(f(s)) = \left(-3\sqrt{2}\cosh s + \left(1+3\sqrt{2}s\right)\sinh s, -3\sqrt{2}\sinh s + \left(1+3\sqrt{2}s\right)\cosh s, 3s^2 + \sqrt{2}s - 6\right)$$

with curvatures $\kappa_2^* = \frac{1}{3\sqrt{2}s+1}$ and $\tau_2^* = \frac{\sqrt{2}}{3\sqrt{2}s+1}$.

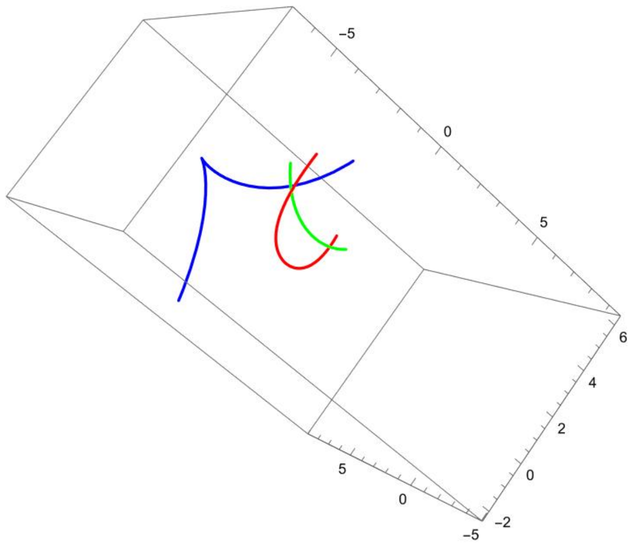

Figure 1: The curves φ (red), φ_1^* (green), and φ_2^* (blue)given in Example 1 are illustrated together.

Example 3.9. *Let us consider the curve in* $\mathbb{E}_1^3$ *given by*

$$\varphi(s) = \left(\sqrt{2}\sinh s, \sqrt{2}\cosh s, s\right)$$

withe the curvatures $\kappa = \sqrt{2}$ *and* $\tau = -1$ *and the Frenet frame as*

$$T(s) = \left(\sqrt{2}\cosh s, \sqrt{2}\sinh s, 1\right)$$

$$N(s) = (\sinh s, \cosh s, 0)$$

$$B(s) = \left(\cosh s, \sinh s, \sqrt{2}\right).$$

If we take $C(s) = \sinh s$ *Theorem 3.2, we obtain related by transformation of Combescure curve* φ_1^* *as follows*

$$\varphi_1^*(f(s)) = \left(\sqrt{2}\sinh s + 1, \sqrt{2}\cosh s, s\right)$$

with curvatures $\kappa_1^* = \frac{\sqrt{2}}{2\sqrt{2}\sinh s+1}$ *and* $\tau_1^* = \frac{-1}{2\sqrt{2}\sinh s+1}$.
If we take $C(s) = \sqrt{2}s^3$ *Theorem 3.2, we obtain related by transformation of Combescure curve* φ_2^* *as follows*

$$\varphi_2^*(f(s)) = \left(6\sqrt{2}\cosh s + \left(\sqrt{2} - 6\sqrt{2}s\right)\sinh s, 6\sqrt{2}\sinh s + \left(\sqrt{2} - 6\sqrt{2}s\right)\cosh s, -3s^2 + s + 6\right)$$

with curvatures $\kappa_2^* = \frac{\sqrt{2}}{12s+1}$ *and* $\tau_2^* = \frac{-1}{12s+1}$.

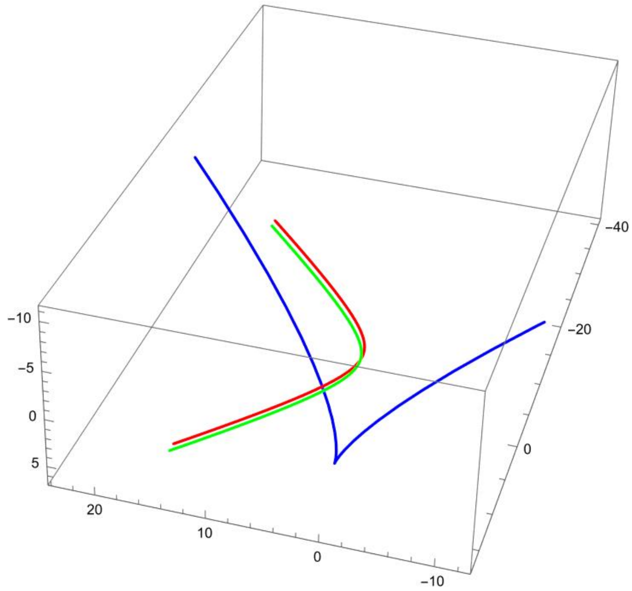

Figure 2: The curves φ (red), φ_1^* (green), and φ_2^* (blue) given in Example 3.9.

4. Biharmonic Curves Under Combescure Transformation

In this section, we investigate the conditions under which the counterpart of a non-null biharmonic curve, obtained via the Combescure transformation, remains a biharmonic curve. Before addressing this question, we briefly summarize biharmonic curves in Minkowski 3-space and the previously obtained results.

A unit-speed curve $\gamma : I \to M$ is biharmonic if the Laplacian of its mean curvature vector field satisfies $\Delta H = 0$. In semi-Euclidean 3-space, biharmonicity is equivalent to $\Delta\Delta\gamma = 0$. Chen and Ishikawa [4] demonstrated that all biharmonic curves in semi-Euclidean spaces $\mathbb{E}_v^n$ lie within three-dimensional totally geodesic subspaces. In Minkowski 3-space $\mathbb{E}_1^3$, Inoguchi classified all biharmonic curves as helices with curvature κ and torsion τ satisfying $\kappa^2 = \tau^2$ [8], [9]. We know that studying biharmonic curves helps in understanding the behavior of elastic curves in Lorentzian spaces, which differ significantly from Euclidean spaces due to the indefinite metric. The following result, provided by Inoguchi [8], expresses the necessary and sufficient conditions for a non-null curve in Minkowski 3-space to be a biharmonic curve.

Proposition 4.1. *Let γ be a unit speed curve in Minkowski 3-space $\mathbb{E}_1^3$. Then, γ is biharmonic if and only if γ is congruent to one of the following:*

1. *A spacelike or timelike line.*
2. *A spacelike curve such that $h(\gamma, \gamma) = 0$, given by*
$$\gamma(s) = (as^3 + bs^2, as^3 + bs^2, s),$$
where a and b are constants such that $a^2 + b^2 \neq 0$.
3. *A spacelike helix with a spacelike principal normal vector field satisfying $\kappa^2 = \tau^2 = a^2$, given by*
$$\gamma(s) = \left(\frac{a^2s^3}{6}, \frac{as^2}{2}, -\frac{a^2s^3}{6} + s\right).$$
4. *A timelike helix satisfying $\kappa^2 = \tau^2 = a^2$, given by*
$$\gamma(s) = \left(\frac{a^2s^3}{6} + s, \frac{as^2}{2}, \frac{a^2s^3}{6}\right).$$

The following theorem provides the conditions under which the associated curve of a non-null biharmonic curve in Minkowski 3-space, related through the comberscure transformation, is also a biharmonic curve.

Theorem 4.2. *Let $\varphi : I \subseteq \mathbb{R} \to \mathbb{E}_1^3$ and $\varphi^* : I^* \subseteq \mathbb{R} \to \mathbb{E}_1^3$ be unit speed non-null curves with the same causal characters in $\mathbb{E}_1^3$ with Frenet apparatus $\{T, N, B, \kappa, \tau\}$ and $\{T^*, N^*, B^*, \kappa^*, \tau^*\}$, respectively. When the curve $\varphi(s)$ is a biharmonic curve, a necessary and sufficient condition for the associated curve $\varphi^*(f(s))$, related to $\varphi(s)$ through the Combescure transformation, to also be a biharmonic curve is that one of the following conditions is satisfied*

(i) *$f'(s) = ds^*/ds$ is a constant different from 0 and 1.*

(ii) *The curve $\varphi^*(f(s))$ is given by $\varphi^*(f(s)) = f'(s)\varphi(s) + \overrightarrow{C}$,where $\overrightarrow{C} = (c_1, c_2, c_3) \in \mathbb{E}_1^3$ is a constant vector.*

(iii) *The differentiable function $C(s)$ in Theorem 3.2 given by $C(s) = \dfrac{(1-\lambda)\kappa\tau^2}{24}s^4 + \dfrac{c_1}{6}s^3 + \dfrac{c_2}{2}s^2 + c_3 s + c_4$. Where $f'(s) = \lambda = \frac{ds^*}{ds} \neq 1$ and non-zero constant, $c_i \in \mathbb{R}$ $(i = 1, 2, 3, 4)$.*

Proof. We assume that the curves $\varphi(s)$ and $\varphi^*(f(s))$ are related by a Combescure transformation. If the curve $\varphi(s)$ is a biharmonic curve, then according to the above proposition its curvature functions satisfy the equality $\kappa^2 = \tau^2 = a^2$ where $a \in \mathbb{R}_0$. On the other hand, if $\varphi^*(f(s))$ is a biharmonic curve, its curvature functions satisfy $(\kappa^*)^2 = (\tau^*)^2 = b^2$ where $b \in \mathbb{R}_0$. Since $\varphi(s)$ and $\varphi^*(f(s))$ are related by a Combescure transformation, Corollary 3.6 implies that

$$f'(s) = \frac{ds^*}{ds} = \frac{\kappa}{\kappa^*} = \frac{\tau}{\tau^*}.$$

Thus, $f'(s)$ is a non zero constant, satisfying condition (i).

Since $\varphi^*(f(s))$ is a biharmonic curve, the function $f'(s)$ is a non zero constant. From Theorem 3.5, we get

$$\varphi^*(f(s)) = f'(s)\varphi(s) + \overrightarrow{C},$$

where $\overrightarrow{C} = (c_1, c_2, c_3) \in \mathbb{E}_1^3$ is a constant vector. Therefore, condition (ii) is satisfied.

From Theorem 3.2, we obtain

$$f'(s) = 1 + \frac{d}{ds}\left(\frac{dC}{dr} - \varepsilon_1 \frac{d^3C}{dt^3}\frac{dt}{dr}\right) - \varepsilon_2 \frac{d^2C}{dt^2}\frac{dr}{ds}. \tag{13}$$

Since $\varphi^*(f(s))$ is a biharmonic curve, the function $f'(s) = \lambda$ is a non zero constant. By using $t = \int \tau(s)\,ds$, $r = \int \kappa(s)\,ds$, and $\kappa^2 = \tau^2 = a^2$ (where $a \in \mathbb{R}_0$) in equation (13), we get easily obtain

$$\frac{d^4C}{ds^4} = (1-\lambda)\kappa\tau^2$$

By integrating, we find

$$C(s) = \frac{(1-\lambda)\kappa\tau^2}{24}s^4 + \frac{c_1}{6}s^3 + \frac{c_2}{2}s^2 + c_3 s + c_4$$

where $f'(s) = \lambda = \frac{ds^*}{ds} \neq 1$ is a non-zero constant, and $c_i \in \mathbb{R}$ $(i = 1, 2, 3, 4)$. Thus, condition (iii) is satisfied.

Conversely, it is straightforward to prove that if the curve $\varphi(s)$ is a biharmonic curve and one of the conditions (i)-(iii) is satisfied, then the curve $\varphi^*(f(s))$ is also a biharmonic curve. $\square$

Corollary 4.3. *Using $f'(s)$ as a non-zero constant, the equation of the biharmonic curve $\varphi^*(f(s))$, which is Combescure related with the biharmonic curve $\varphi(s)$, is obtained as follows using Theorem 3.*

$$\varphi^*(f(s)) = f'(s)\,\varphi(s) + \overrightarrow{C} \tag{14}$$

where $\overrightarrow{C} = (c_1, c_2, c_3) \in \mathbb{E}_1^3$ is a constant vector. Since all curves in the family described by the above equality have the same curvatures, they are equivalent to the curve $\varphi^*(f(s)) = f'(s)\,\varphi(s)$ under the isometries of Minkowski 3- space.

Example 4.4. *Let us consider the spacelike biharmonic curve in $\mathbb{E}_1^3$ with a spacelike principal normal given by*

$$\varphi(s) = \left(-\frac{s^3}{6}, -\frac{s^3}{6} + s, \frac{s^2}{2}\right)$$

with the curvatures $\kappa = 1, \tau = -1$ and the Frenet frame as

$$T(s) = \left(-\frac{1}{2}s^2, 1 - \frac{1}{2}s^2, s\right)$$

$$N(s) = (-s, -s, 1)$$

$$B(s) = \left(-\frac{1}{2}s^2 - 1, -\frac{1}{2}s^2, s\right).$$

If we take $C(s) = -\frac{s^4}{12}$ in Theorem 3.2, we obtain related by transformation of Combescure curve φ^* as follows

$$\varphi^*(f(s)) = \left(-\frac{1}{2}s^3, -\frac{1}{2}s\left(s^2 - 6\right), \frac{3}{2}s^2\right)$$

with curvatures $\kappa^* = \frac{1}{3}$ and $\tau^* = -\frac{1}{3}$. Since $(\kappa^*)^2 = (\tau^*)^2 = \frac{1}{9}$, φ^* is a biharmonic curve.

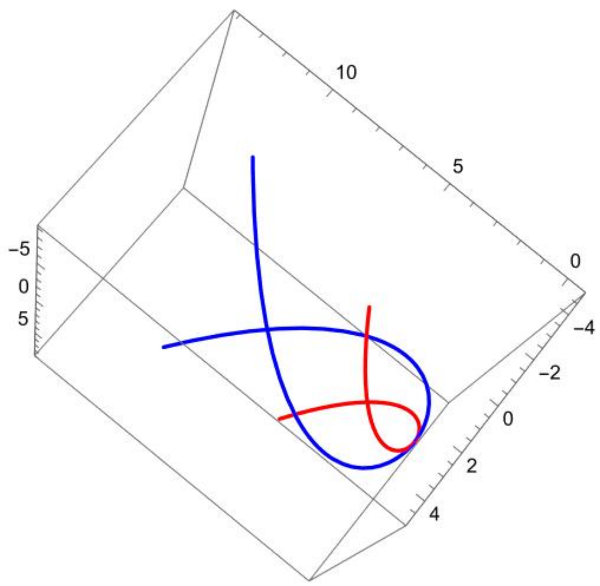

Figure 3: The curves φ (red), φ^* (blue) given in Example 4.4.

Example 4.5. *Let us consider the timelike biharmonic curve in $\mathbb{E}_1^3$ with a spacelike principal normal given by*

$$\varphi(s) = \left(s + \frac{2}{3}s^3, s^2, \frac{2}{3}s^3\right)$$

with the curvatures $\kappa = \tau = 2$ and the Frenet frame as

$$T(s) = \left(2s^2 + 1, 2s, 2s^2\right)$$

$$N(s) = (2s, 1, 2s)$$

$$B(s) = \left(-2s^2, -2s, 1 - 2s^2\right).$$

If we take $C(s) = s^4$ in Theorem 3.2, we obtain related by transformation of Combescure curve φ^* as follows

$$\varphi^*(f(s)) = \left(\frac{4}{3}s\left(2s^2 + 3\right), 4s^2, \frac{8}{3}s^3\right)$$

with curvatures $\kappa^* = \tau^* = \frac{1}{2}$. Since $(\kappa^*)^2 = (\tau^*)^2 = \frac{1}{4}$, φ^* is a biharmonic curve.

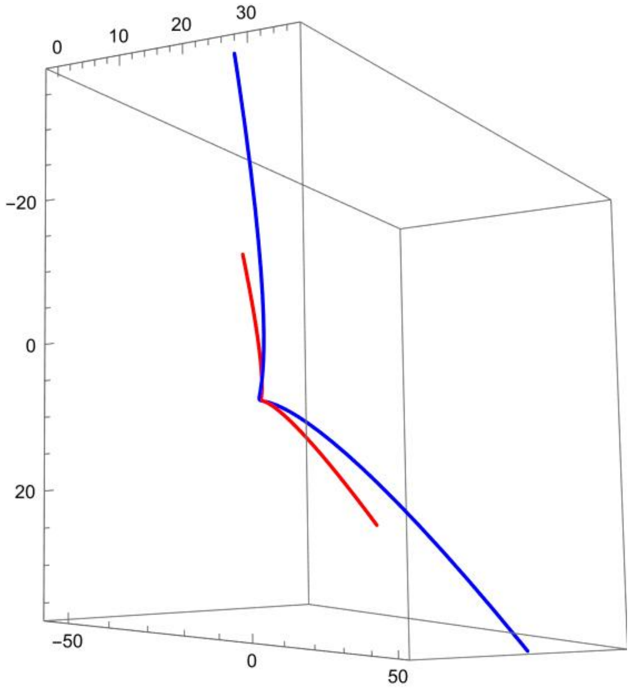

Figure 4: The curves φ (red), φ^* (blue) given in Example 4.5.

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