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The eigenvalues of some elliptic operators on cigar soliton

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Abstract. In the context of a bounded domain Ω within the framework of cigar solitons, we investigate the estimation of eigenvalues associated with certain elliptic operators. Our findings yield universal inequalities pertaining to these eigenvalues.

1. Introduction

Ricci solitons, which can be regarded as generalized Einstein metrics, represent self-similar solutions to the Ricci flow and were first introduced by Hamilton [8]. A Riemannian manifold denoted as (M^n, g) is classified as a Ricci soliton if there exists a vector field X and a constant ρ such that the equation $Ric + \frac{1}{2}L_X g = \rho g$ holds, where Ric refers to the Ricci tensor and L_X signifies the Lie derivative in the direction of X. In cases where X is expressed as $\nabla\psi$ for some smooth potential function ψ defined on M, the Ricci soliton simplifies to the form $Ric + Hess\psi = \rho g$, thereby being categorized as a gradient Ricci soliton. Furthermore, the classification of Ricci solitons into expanding, steady, and shrinking types is determined by the value of ρ, specifically when $\rho < 0$, $\rho = 0$, and $\rho > 0$, respectively. All compact steady Ricci solitons are Ricci flat. Hamilton [9] introduced a complete non-compact manifold which is a steady gradient soliton and this manifold is called a cigar soliton.

Let the local coordinates of $\mathbb{R}^2$ be denoted as $\{z^1, z^2\}$. The metric and potential function associated with a cigar soliton are expressed as $g = \frac{d(z^1)^2 + d(z^2)^2}{1+|\mathbf{z}|^2}$ and $\psi = -\log(1+|\mathbf{z}|^2)$, respectively, where $|\mathbf{z}|^2 = (z^1)^2 + (z^2)^2$. Furthermore, the cigar soliton is referred to as the Euclidean-Witten black hole in the context of the first-order Ricci flow of the world-sheet sigma model within closed string theory, specifically in a two-dimensional target space. It is important to note that the cigar soliton exhibits positive curvature and approaches a cylinder of finite circumference as it extends to infinity. Numerous researchers have explored the properties of the cigar soliton, as referenced in works such as [5, 7, 14, 16].

In the fields of mathematics and physics, acquiring accurate estimates for the eigenvalues of a geometric operator is of significant importance. This discussion will center on this particular issue.

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Payne, Polya, and Weinberger [13] derived the Dirichlet eigenvalues of the Laplacian on a domain $\Omega \subset \mathbb{R}^n$, which adhere to the following inequality:

$$\zeta_{k+1} - \zeta_k \leq \frac{4}{kn} \sum_{i=1}^{k} \zeta_i, \quad k = 1, 2, \cdots. \tag{1}$$

Numerous intriguing generalizations of the equation presented in (1) have been established over recent years. For example, Harrell et al. [11] expanded upon Yang's inequality concerning the Dirichlet eigenvalues of the Laplacian, applying it to bounded domains Ω within an n-dimensional complete Riemannian manifold M^n that is isometrically immersed in $\mathbb{R}^N$. Additionally, Wang and Xia, in their work [15], adapted Yang's methodology to derive universal inequalities for the eigenvalues associated with the clamped plate problem, which is defined by the following system of equations:

$$\begin{cases} \Delta^2 v = \zeta v & \mathit{in}\ \Omega, \\ v = \frac{\partial v}{\partial \nu} = 0 & \mathit{on}\ \partial\Omega, \end{cases} \tag{2}$$

which pertains to a bounded domain Ω featuring a smooth boundary within a complete Riemannian manifold M^n. Further references can be found in [2, 4, 8, 12, 13].

Let (M^n, g) represent a complete non-compact Riemannian manifold, and let Ω denote a bounded domain within M. Consider ζ_i as the i-th eigenvalue associated with the elliptic operator defined by the system of equations

$$\begin{cases} \mathcal{L} v = \zeta_i v, & \mathit{in}\ \Omega, \\ v = \frac{\partial v}{\partial \nu} = 0, & \mathit{on}\ \partial\Omega, \end{cases} \tag{3}$$

where $\mathcal{L} = a\Delta_\psi^2 - b\Delta_\psi$, with parameters $a, b \geq 0$ and $(a, b) \neq (0, 0)$. Here, ν represents the outward unit normal vector at the boundary $\partial\Omega$, while $\Delta_\psi = \Delta - \langle \nabla\psi, \nabla . \rangle$ denotes the bi-drifting Laplacian, with Δ being the standard Laplacian and ∇ the gradient operator corresponding to the metric g. In their work, Li et al. [6] derived two eigenvalue inequalities for the bi-drifting Laplacian expressed as

$$\begin{cases} \Delta_\psi^2 v = \zeta v, & \mathit{in}\ \Omega, \\ v = \frac{\partial v}{\partial \nu} = 0, & \mathit{on}\ \partial\Omega, \end{cases}$$

within the context of a bounded domain Ω situated in a cigar soliton $(\mathbb{R}^2, g, \psi)$. Inspired by the aforementioned studies and the findings presented in [1], this paper investigates the Dirichlet eigenvalues associated with the problem (3) on a cigar soliton. We establish universal inequalities pertaining to the eigenvalues of the problem (3) within the context of cigar solitons.

In this paper, we prove the validity of the subsequent theorems:

Theorem 1.1. *Let us consider that Ω represents a bounded domain within the context of the cigar soliton $(\mathbb{R}^2, g, \psi)$. Furthermore, let ζ_i, where $i = 1, 2, \ldots, k$, denote the i-th eigenvalue associated with the problem (3). then we get*

$$\begin{aligned} \sum_{i=1}^{k} (\zeta_{k+1} - \zeta_i)^2 \leq & \frac{1}{2} \left\{ \sum_{i=1}^{k} (\zeta_{k+1} - \zeta_i)^2 \left[\frac{16\sqrt{a}(1+\Gamma)}{1+\gamma} \zeta_i^{\frac{1}{2}} - \frac{24\gamma a + 16a}{1+\gamma} \right] \right\}^{\frac{1}{2}} \\ & \times \left\{ \sum_{i=1}^{k} \frac{(\zeta_{k+1} - \zeta_i)}{\delta} \left[\frac{\frac{2}{\sqrt{a}}(1+\Gamma)}{1+\gamma} \zeta_i^{\frac{1}{2}} - \frac{6\gamma + 4}{1+\gamma} \right] \right\}^{\frac{1}{2}} \end{aligned} \tag{4}$$

for $b = 0$ and

$$\sum_{i=1}^{k} (\zeta_{k+1} - \zeta_i)^2 \leq \frac{1}{2} \left\{ \sum_{i=1}^{k} (\zeta_{k+1} - \zeta_i)^2 \left[\frac{16\sqrt{a}(1+\Gamma)}{1+\gamma} \zeta_i^{\frac{1}{2}} - \frac{24\gamma a + 16a}{1+\gamma} \right.\right.$$

$$+\frac{8a(1+\Gamma)}{b(1+\gamma)}\zeta_i + \frac{4b(1+\Gamma)}{1+\gamma}\Bigg]\Bigg\}^{\frac{1}{2}}$$
$$\times\left\{\sum_{i=1}^{k}\frac{(\zeta_{k+1}-\zeta_i)}{\delta}\left[\frac{2}{b(1+\gamma)}(1+\Gamma)\zeta_i - \frac{6\gamma+4}{1+\gamma}\right]\right\}^{\frac{1}{2}} \tag{5}$$

for $b \neq 0$ and any $\delta > 0$ where $\gamma = \min_{z\in\Omega}|\mathbf{z}|^2$ and $\Gamma = \max_{z\in\Omega}|\mathbf{z}|^2$.

Theorem 1.2. *Let us consider that Ω represents a bounded domain within the context of the cigar soliton $(\mathbb{R}^2, g, \psi)$. Furthermore, let ζ_i, for $i = 1, 2, \ldots, k$, denote the i-th eigenvalue associated with the problem (3). We arrive at*

$$\sum_{\alpha=1}^{2}(\zeta_{\alpha+1}-\zeta_1)^{\frac{1}{2}} \le \frac{4}{1+\gamma}\left\{-3a\gamma - a + 2\sqrt{a}(1+\Gamma)\zeta_1^{\frac{1}{2}}\right\}^{\frac{1}{2}}\left\{-3\gamma - 2 + \frac{(1+\Gamma)}{\sqrt{a}}\zeta_1^{\frac{1}{2}}\right\}^{\frac{1}{2}} \tag{6}$$

for $b = 0$ and

$$\sum_{\alpha=1}^{2}(\zeta_{\alpha+1}-\zeta_1)^{\frac{1}{2}} \le \frac{1}{1+\gamma}\left\{-6a\gamma - 4a + \frac{2a(1+\Gamma)}{b}\zeta_1 + 2\sqrt{a}(1+\Gamma)\zeta_1^{\frac{1}{2}} + b(1+\Gamma)\right\}^{\frac{1}{2}}$$
$$\times\left\{-6\gamma - 4 + \frac{2(1+\Gamma)}{b}\zeta_1\right\}^{\frac{1}{2}}. \tag{7}$$

for $b \neq 0$ where $\gamma = \min_{z\in\Omega}|\mathbf{z}|^2$ and $\Gamma = \max_{z\in\Omega}|\mathbf{z}|^2$.

2. Proofs of results

In this section, we establish the validity of Theorems 1.1 and 1.2. To achieve this, we will initially present and demonstrate the following proposition.

Proposition 2.1. *Let $(M, \langle,\rangle)$ represent an n-dimensional non-compact complete Riemannian manifold, and let Ω be a bounded domain within M. We denote the outward unit normal vector on the boundary $\partial\Omega$ by ν. The eigenvalues of the associated problem are denoted as ζ_i for $i = 1, \ldots,$ with v_i representing the orthonormal eigenfunction corresponding to each eigenvalue ζ_i. This relationship is expressed through the following equations:*

$$\begin{cases}\mathcal{L}v_i = \zeta_i v_i & \text{in } \Omega\\ v_i = \frac{\partial v_i}{\partial \nu} = 0 & \text{on } \partial\Omega\\ \int_\Omega v_i v_j d\mu = \delta_{ij}, & \forall\, i, j = 1, \ldots, k.\end{cases} \tag{8}$$

For any smooth function $h : \Omega \to \mathbb{R}$, any positive integer k, and any $\delta > 0$, the following inequality holds:

$$\sum_{i=1}^{k}(\zeta_{k+1}-\zeta_i)^2 q_i \le \sum_{i=1}^{k}(\zeta_{k+1}-\zeta_i)\|p_i\|^2, \tag{9}$$

$$\sum_{i=1}^{k}(\zeta_{k+1}-\zeta_i)^2\int_\Omega v_i^2|\nabla h|^2 d\mu$$
$$\le \delta\sum_{i=1}^{k}(\zeta_{k+1}-\zeta_i)^2 q_i + \sum_{i=1}^{k}\frac{(\zeta_{k+1}-\zeta_i)}{\delta}\int_\Omega\left(\langle\nabla h, \nabla v_i\rangle + \frac{v_i\Delta_\psi h}{2}\right)^2 d\mu, \tag{10}$$

$$\sum_{i=1}^{k}(\zeta_{k+1}-\zeta_i)^2\int_\Omega v_i^2|\nabla h|^2 d\mu \tag{11}$$
$$\leq \delta\sum_{i=1}^{k}(\zeta_{k+1}-\zeta_i)\|p_i\|^2+\sum_{i=1}^{k}\frac{(\zeta_{k+1}-\zeta_i)}{\delta}\int_\Omega\left(\langle\nabla h,\nabla v_i\rangle+\frac{v_i\Delta_\psi h}{2}\right)^2 d\mu,$$

where

$$\begin{aligned} q_i &= \int_\Omega h v_i p_i d\mu,\quad s_{ij}=\int_\Omega v_j p_i d\mu,\\ p_i &= a\Big(\Delta_\psi(h)\Delta_\psi(v_i)+2\langle\nabla h,\nabla\Delta_\psi(v_i)\rangle+\Delta_\psi(v_i\Delta_\psi(h))+2\Delta_\psi(\langle\nabla v_i,\nabla h\rangle)\Big)-b\Big(v_i\Delta_\psi(h)+2\langle\nabla h,\nabla v_i\rangle\Big).\end{aligned}$$

Proof. For each integer i ranging from 1 to k, let us define the functions $\phi_i : M \to \mathbb{R}$ as $\phi_i = hv_i - \sum_{j=1}^{k} r_{ij}v_j$, where r_{ij} is expressed as $r_{ij} = \int_\Omega \rho h v_i v_j d\mu$. It is important to note that the boundary conditions are satisfied, specifically $\phi_i|_{\partial\Omega} = \frac{\partial\phi_i}{\partial\nu}|_{\partial\Omega} = 0$. Furthermore, the integral condition

$$\int_\Omega v_i\phi_j d\mu = 0,\quad \forall i, j = 1, ..., k,$$

leads us to conclude, based on the Rayleigh-Ritz inequality, that

$$\zeta_{k+1} \leq \frac{\int_\Omega \phi_i\mathcal{L}\phi_i d\mu}{\int_\Omega \rho\phi_i^2 d\mu}, \tag{12}$$

then

$$\zeta_{k+1}\int_\Omega \phi_i^2 d\mu \leq \int_\Omega \phi_i\mathcal{L}\phi_i d\mu. \tag{13}$$

Since

$$\Delta_\psi(\phi_i) = h\Delta_\psi(v_i)+v_i\Delta_\psi(h)+2\langle\nabla v_i,\nabla h\rangle-\sum_{j=1}^{k} r_{ij}\Delta_\psi(v_j),$$

$$\begin{aligned}\Delta_\psi^2(\phi_i) &= \Delta_\psi(h)\Delta_\psi(v_i)+h\Delta_\psi^2(v_i)+2\langle\nabla h,\nabla\Delta_\psi(v_i)\rangle+\Delta_\psi(v_i\Delta_\psi(h))\\ &\quad +2\Delta_\psi(\langle\nabla v_i,\nabla h\rangle)-\sum_{j=1}^{k} r_{ij}\Delta_\psi^2(v_j),\end{aligned}$$

and

$$V\phi_i = hVv_i-\sum_{j=1}^{k} r_{ij}Vv_j,$$

we have

$$\begin{aligned}\mathcal{L}\phi_i &= a\Big(\Delta_\psi(h)\Delta_\psi(v_i)+2\langle\nabla h,\nabla\Delta_\psi(v_i)\rangle+\Delta_\psi(v_i\Delta_\psi(h))+2\Delta_\psi(\langle\nabla v_i,\nabla h\rangle)\Big)\\ &\quad -b\Big(v_i\Delta_\psi(h)+2\langle\nabla h,\nabla v_i\rangle\Big)+h\lambda_i v_i-\sum_{j=1}^{k} r_{ij}\zeta_j v_j.\end{aligned}$$

Letting

$$p_i = a\Big(\Delta_\psi(h)\Delta_\psi(v_i) + 2\langle\nabla h, \nabla\Delta_\psi(v_i)\rangle + \Delta_\psi(v_i\Delta_\psi(h)) + 2\Delta_\psi(\langle\nabla v_i, \nabla h\rangle)\Big) - b\Big(v_i\Delta_\psi(h) + 2\langle\nabla h, \nabla v_i\rangle\Big),$$

we have

$$\begin{aligned}\int_\Omega \phi_i\mathcal{L}\phi_i d\mu &= \int_\Omega \phi_i p_i d\mu + \zeta_i\int_\Omega h\phi_i v_i d\mu - \sum_{j=1}^k r_{ij}\zeta_j\int_\Omega \phi_i v_j d\mu \\ &= \int_\Omega p_i[hv_i - \sum_{j=1}^k r_{ij}v_j]d\mu + \zeta_i\int_\Omega \rho\phi_i[\phi_i + \sum_{j=1}^k r_{ij}v_j]d\mu \\ &= \int_\Omega p_i h v_i d\mu - \sum_{j=1}^k r_{ij}\int_M p_i v_j d\mu + \zeta_i\int_\Omega \phi_i^2 d\mu.\end{aligned}$$

Therefore

$$\int_\Omega \phi_i\mathcal{L}\phi_i d\mu = \zeta_i\int_\Omega \phi_i^2 d\mu - \sum_{j=1}^k r_{ij}s_{ij} + q_i, \tag{14}$$

where

$$q_i = \int_\Omega hv_i p_i d\mu, \quad s_{ij} = \int_\Omega v_j p_i d\mu. \tag{15}$$

By applying the method of integration by parts, we can derive that

$$\begin{aligned}&a\int_\Omega v_j\Delta_\psi(\langle\nabla h, \nabla v_i\rangle)d\mu + a\int_\Omega v_j\langle\nabla h, \nabla\Delta_\psi(v_i)\rangle d\mu - b\int_\Omega v_j\langle\nabla h, \nabla v_i\rangle d\mu \\ &= a\int_\Omega \Delta_\psi(v_j)\langle\nabla h, \nabla v_i\rangle d\mu - a\int_\Omega \Delta_\psi(v_i)\langle\nabla h, \nabla v_j\rangle d\mu - a\int_\Omega v_j\Delta_\psi(h)\Delta_\psi(v_i)d\mu \\ &\quad + b\int_\Omega v_i\langle\nabla h, \nabla v_j\rangle d\mu + b\int_\Omega v_i v_j\Delta_\psi(h)d\mu.\end{aligned} \tag{16}$$

On the other hand

$$\begin{aligned}&a\int_\Omega \Delta_\psi(v_j)\langle\nabla h, \nabla v_i\rangle d\mu - a\int_\Omega \Delta_\psi(v_i)\langle\nabla h, \nabla v_j\rangle d\mu + b\int_\Omega v_i\langle\nabla h, \nabla v_j\rangle d\mu \\ &= -a\int_\Omega h\,\nabla(\Delta_\psi(v_j)\nabla v_i)d\mu + a\int_\Omega h\,\nabla(\Delta_\psi(v_i)\nabla v_j)d\mu - b\int_\Omega h\,\nabla(v_i\nabla v_j)d\mu \\ &= -a\int_\Omega \langle h\nabla\Delta_\psi(v_j), \nabla v_i\rangle d\mu + a\int_\Omega \langle h\nabla\Delta_\psi(v_i), \nabla v_j\rangle d\mu - b\int_\Omega h\,\langle\nabla v_i, \nabla v_j\rangle d\mu \\ &\quad - b\int_M hv_i\Delta_\psi(v_j)d\mu \\ &= a\int_\Omega v_i\nabla(h\nabla\Delta_\psi(v_j))d\mu - a\int_\Omega v_j\nabla(h\nabla\Delta_\psi(v_i))d\mu + b\int_\Omega v_j\,\langle\nabla h, \nabla v_i\rangle d\mu \\ &\quad + b\int_\Omega hv_j\Delta_\psi(v_i)d\mu - b\int_\Omega hv_i\Delta_\psi(v_j)d\mu \\ &= \int_\Omega \Big[v_i h\Big(a\Delta_\psi^2(v_j) - b\Delta_\psi(v_j)\Big) - v_j h\Big(a\Delta_\psi^2(v_i) - b\Delta_\psi(v_i)\Big)\Big]d\mu \\ &\quad + a\int_\Omega \Big[\langle v_i\nabla h, \nabla\Delta_\psi(v_j)\rangle - \langle v_j\nabla h, \nabla\Delta_\psi(v_i)\rangle\Big]d\mu + b\int_\Omega v_j\,\langle\nabla h, \nabla v_i\rangle d\mu\end{aligned}$$

$$
\begin{aligned}
&= (\zeta_j - \zeta_i)r_{ij} - a\int_M \Delta_\psi(v_j)\langle\nabla h, \nabla v_i)\rangle d\mu + a\int_\Omega \Delta_\psi(v_i)\langle\nabla h, \nabla v_j\rangle d\mu \\
&\quad - a\int_\Omega v_i\Delta_\psi(v_j)\Delta_\psi(h)d\mu + a\int_\Omega v_j\Delta_\psi(v_i)\Delta_\psi(h)d\mu - b\int_\Omega v_iv_j\Delta_\psi(h)d\mu \\
&\quad - b\int_\Omega v_i\langle\nabla h, \nabla v_j\rangle d\mu,
\end{aligned}
$$

therefore

$$
\begin{aligned}
&2a\int_\Omega \Delta_\psi(v_j)\langle\nabla h, \nabla v_i)\rangle d\mu - 2a\int_\Omega \Delta_\psi(v_i)\langle\nabla h, \nabla v_j\rangle d\mu + 2b\int_\Omega v_i\langle\nabla h, \nabla v_j\rangle d\mu \\
&= (\zeta_j - \zeta_i)r_{ij} - a\int_\Omega v_i\Delta_\psi(v_j)\Delta_\psi(h)d\mu + a\int_\Omega v_j\Delta_\psi(v_i)\Delta_\psi(h)d\mu - b\int_\Omega v_iv_j\Delta_\psi(h)d\mu.
\end{aligned} \tag{17}
$$

Inserting (17) into (16), it follows that

$$
\begin{aligned}
&2a\int_\Omega v_j\Delta_\psi(\langle\nabla h, \nabla v_i\rangle)d\mu + 2a\int_\Omega v_j\langle\nabla h, \nabla\Delta_\psi(v_i)\rangle d\mu - 2b\int_\Omega v_j\langle\nabla h, \nabla v_i\rangle d\mu \\
&\quad + a\int_\Omega v_j\Delta_\psi(h)\Delta_\psi(v_i)d\mu - b\int_\Omega v_iv_j\Delta_\psi(h)d\mu \\
&= (\zeta_j - \zeta_i)r_{ij} - a\int_\Omega v_i\Delta_\psi(v_j)\Delta_\psi(h)d\mu.
\end{aligned} \tag{18}
$$

Moreover

$$
\int_\Omega v_j\Delta_\psi(v_i\Delta_\psi(h))d\mu = \int_\Omega v_i\Delta_\psi(v_j)\Delta_\psi(h)d\mu, \tag{19}
$$

combining (18), (19) and $s_{ij} = \int_M p_iv_jd\mu$, we can write

$$
(\zeta_j - \zeta_i)r_{ij} = s_{ij}. \tag{20}
$$

It can be deduced from equations (13), (14), and (20) that the following inequality holds:

$$
(\zeta_{k+1} - \zeta_i)\int_\Omega \phi_i^2 d\mu \le q_i + \sum_{j=1}^{k}(\zeta_i - \zeta_j)r_{ij}^2. \tag{21}
$$

Additionally, we utilize the fact that $\int_\Omega \rho\phi_iv_jd\mu = 0$ to get

$$
\begin{aligned}
\left(\int_\Omega \phi_ip_id\mu\right)^2 &= \left(\int_\Omega \phi_i\left(p_i - \sum_{j=1}^{k}s_{ij}v_j\right)d\mu\right)^2 \\
&\le \|\phi_i\|^2\left(\|p_i\|^2 - \sum_{j=1}^{k}s_{ij}^2\right).
\end{aligned}
$$

Since

$$
(\zeta_{k+1} - \zeta_i)\int_\Omega \phi_i^2 d\mu \le \int_\Omega \phi_i\mathcal{L}\phi_id\mu - \zeta_i\int_\Omega \phi_i^2 d\mu \le \int_\Omega \phi_ip_id\mu,
$$

then

$$
(\zeta_{k+1} - \zeta_i)\left(\int_\Omega \phi_ip_id\mu\right) \le \|p_i\|^2 - \sum_{j=1}^{k}s_{ij}^2. \tag{22}
$$

By multiplying equation (22) by the term $(\zeta_{k+1}-\zeta_i)$ and subsequently summing over the index i from 1 to k, we derive the following expression:

$$\sum_{i=1}^{k}(\zeta_{k+1}-\zeta_i)^2\left(\int_\Omega \phi_i p_i d\mu\right) \leq \sum_{i=1}^{k}(\zeta_{k+1}-\zeta_i)\|p_i\|^2 - \sum_{i,j=1}^{k}(\zeta_{k+1}-\zeta_i)(\zeta_i-\zeta_j)^2 r_{ij}^2. \tag{23}$$

Furthermore, by multiplying the expression $\int_\Omega \phi_i p_i d\mu = q_i + \sum_{j=1}^{k}(\zeta_i-\zeta_j)r_{ij}^2$ by $(\zeta_{k+1}-\zeta_i)^2$ and summing over i from 1 to k, while noting that $r_{ij}=r_{ji}$, we infer

$$\begin{aligned}\sum_{i=1}^{k}(\zeta_{k+1}-\zeta_i)^2\left(\int_\Omega \phi_i p_i d\mu\right) &= \sum_{i=1}^{k}(\zeta_{k+1}-\zeta_i)^2 q_i + \sum_{i,j=1}^{k}(\zeta_{k+1}-\zeta_i)^2(\zeta_i-\zeta_j)r_{ij}^2 \\ &= \sum_{i=1}^{k}(\zeta_{k+1}-\zeta_i)^2 q_i - \sum_{i,j=1}^{k}(\zeta_{k+1}-\zeta_i)(\zeta_i-\zeta_j)^2 r_{ij}^2. \end{aligned}\tag{24}$$

Then (23) and (24) result that

$$\sum_{i=1}^{k}(\zeta_{k+1}-\zeta_i)^2 q_i \leq \sum_{i=1}^{k}(\zeta_{k+1}-\zeta_i)\|p_i\|^2. \tag{25}$$

The evidence presented indicates the validity of (9). To establish the truth of (10), we define t_{ij} as the integral $\int_\Omega v_j(\langle\nabla h,\nabla v_i\rangle + \frac{v_i\Delta_\psi h}{2})d\mu$. We have $t_{ij}=-t_{ji}$. Observe that

$$-2\int_\Omega h v_i\langle\nabla h,\nabla v_i\rangle d\mu = \int_\Omega v_i^2|\nabla h|^2 d\mu + \int_\Omega v_i^2 h\Delta_\psi h d\mu$$

and

$$\int_\Omega -2\phi_i(\langle\nabla h,\nabla v_i\rangle + \frac{v_i\Delta_\psi h}{2})d\mu = \int_\Omega v_i^2|\nabla h|^2 d\mu + 2\sum_{j=1}^{k} r_{ij}t_{ij}. \tag{26}$$

By multiplying equation (26) by $(\zeta_{k+1}-\zeta_i)^2$ and applying the Schwarz inequality for any $\delta>0$, we obtain

$$\begin{aligned}&(\zeta_{k+1}-\zeta_i)^2\left(\int_\Omega v_i^2|\nabla h|^2 d\mu + 2\sum_{j=1}^{k} r_{ij}t_{ij}\right)\\ &= (\zeta_{k+1}-\zeta_i)^2\int_\Omega(-2)\phi_i\left(\left(\langle\nabla h,\nabla v_i\rangle + \frac{v_i\Delta_\psi h}{2}\right) - \sum_{j=1}^{k} t_{ij}v_j\right)d\mu\\ &\leq \delta(\zeta_{k+1}-\zeta_i)^3\|\phi_i\|^2 + \frac{(\zeta_{k+1}-\zeta_i)}{\delta}\int_\Omega\left|\left(\langle\nabla h,\nabla v_i\rangle + \frac{v_i\Delta_\psi h}{2}\right) - \sum_{j=1}^{k} t_{ij}v_j\right|^2 d\mu\\ &= \delta(\zeta_{k+1}-\zeta_i)^3\|\phi_i\|^2 + \frac{(\zeta_{k+1}-\zeta_i)}{\delta}\left(\int_\Omega\left(\langle\nabla h,\nabla v_i\rangle + \frac{v_i\Delta_\psi h}{2}\right)^2 d\mu - \sum_{j=1}^{k} t_{ij}^2\right).\end{aligned}\tag{27}$$

Hence (21) implies that

$$(\zeta_{k+1}-\zeta_i)^2\left(\int_\Omega v_i^2|\nabla h|^2 d\mu + 2\sum_{j=1}^{k} r_{ij}t_{ij}\right)$$

$$\leq \delta(\zeta_{k+1}-\zeta_i)^2\left(q_i+\sum_{j=1}^{k}(\zeta_i-\zeta_j)r_{ij}^2\right)+\frac{(\zeta_{k+1}-\zeta_i)}{\delta}\left(\int_\Omega\left(\langle\nabla h,\nabla v_i\rangle+\frac{v_i\Delta_\psi h}{2}\right)^2 d\mu-\sum_{j=1}^{k}t_{ij}^2\right).$$

By summing over the index i from 1 to k and observing that $r_{ij}=r_{ji}$ and $t_{ij}=-t_{ji}$, we arrive at

$$\sum_{i=1}^{k}(\zeta_{k+1}-\zeta_i)^2\int_\Omega v_i^2|\nabla h|^2 d\mu-2\sum_{i,j=1}^{k}(\zeta_{k+1}-\zeta_i)(\zeta_i-\zeta_j)r_{ij}t_{ij} \tag{28}$$

$$\leq \delta\sum_{i=1}^{k}(\zeta_{k+1}-\zeta_i)^2 q_i+\delta\sum_{i,j=1}^{k}(\zeta_{k+1}-\zeta_i)^2(\zeta_i-\zeta_j)r_{ij}^2$$

$$+\sum_{i=1}^{k}\frac{(\zeta_{k+1}-\zeta_i)}{\delta}\int_\Omega\left(\langle\nabla h,\nabla v_i\rangle+\frac{v_i\Delta_\psi h}{2}\right)^2 d\mu-\sum_{i,j=1}^{k}\frac{(\zeta_{k+1}-\zeta_i)}{\delta}t_{ij}^2,$$

which leads to

$$\sum_{i=1}^{k}(\zeta_{k+1}-\zeta_i)^2\int_\Omega v_i^2|\nabla h|^2 d\mu$$

$$\leq \delta\sum_{i=1}^{k}(\zeta_{k+1}-\zeta_i)^2 q_i+\sum_{i=1}^{k}\frac{(\zeta_{k+1}-\zeta_i)}{\delta}\int_\Omega\left(\langle\nabla h,\nabla v_i\rangle+\frac{v_i\Delta_\psi h}{2}\right)^2 d\mu$$

$$-\sum_{i,j=1}^{k}(\zeta_{k+1}-\zeta_i)(\sqrt{\delta}(\zeta_i-\zeta_j)r_{ij}-\frac{1}{\sqrt{\delta}}t_{ij})^2 \tag{29}$$

$$\leq \delta\sum_{i=1}^{k}(\zeta_{k+1}-\zeta_i)^2 q_i+\sum_{i=1}^{k}\frac{(\zeta_{k+1}-\zeta_i)}{\delta}\int_\Omega\left(\langle\nabla h,\nabla v_i\rangle+\frac{v_i\Delta_\psi h}{2}\right)^2 d\mu$$

thus (10) is true. Substituting (25) into (29), we obtain

$$\sum_{i=1}^{k}(\zeta_{k+1}-\zeta_i)^2\int_\Omega v_i^2|\nabla h|^2 d\mu \tag{30}$$

$$\leq \delta\sum_{i=1}^{k}(\zeta_{k+1}-\zeta_i)\|p_i\|^2+\sum_{i=1}^{k}\frac{(\zeta_{k+1}-\zeta_i)}{\delta}\int_\Omega\left(\langle\nabla h,\nabla v_i\rangle+\frac{v_i\Delta_\psi h}{2}\right)^2 d\mu$$

which implies (11). □

Proof. [Proof of Theorem 1.1] Suppose that $h=z^\alpha$, where z^α is the i-th local coordinate of $p\in\Omega\subset\mathbb{R}^2$ and $\alpha=1,2$. We get

$$\sum_{i=1}^{k}(\zeta_{k+1}-\zeta_i)^2\int_\Omega v_i^2|\nabla z^\alpha|^2 d\mu$$

$$\leq \delta\sum_{i=1}^{k}(\zeta_{k+1}-\zeta_i)^2 q_i+\sum_{i=1}^{k}\frac{(\zeta_{k+1}-\zeta_i)}{\delta}\int_\Omega\left(\langle\nabla z^\alpha,\nabla v_i\rangle+\frac{v_i\Delta_\psi z^\alpha}{2}\right)^2 d\mu, \tag{31}$$

where

$$\begin{aligned} q_i &= \int_\Omega z^\alpha v_i p_i d\mu,\\ p_i &= a\Big(\Delta_\psi(z^\alpha)\Delta_\psi(v_i)+2\langle\nabla z^\alpha,\nabla\Delta_\psi(v_i)\rangle+\Delta_\psi(v_i\Delta_\psi(z^\alpha))+2\Delta_\psi(\langle\nabla v_i,\nabla z^\alpha\rangle)\Big)-b\Big(v_i\Delta_\psi(z^\alpha)+2\langle\nabla z^\alpha,\nabla v_i\rangle\Big). \end{aligned}$$

We have $\psi = -\log(1+|\mathbf{z}|^2)$ where $|\mathbf{z}|^2 = |z^1|^2+|z^2|^2$ and $g = \frac{d(z^1)^2+d(z^2)^2}{1+|\mathbf{z}|^2}$. Direct computation gives $|\nabla z^\alpha|^2 = 1+|\mathbf{z}|^2$, $\langle \nabla z^1, \nabla z^2\rangle = 0$ and $\Delta z^1 = \Delta z^2 = 0$. We also have

$$\Delta_\psi z^\alpha = 2z^\alpha,$$

and

$$\langle \nabla z^\alpha, \nabla v_i\rangle = (1+|\mathbf{z}|^2)\frac{\partial v_i}{\partial z^\alpha}.$$

Therefore

$$\sum_{\alpha=1}^{2}\langle \nabla z^\alpha, \nabla v_i\rangle^2 = (1+|\mathbf{z}|^2)|\nabla v_i|^2.$$

Taking $h = z^\alpha$ and the last equations in (10), we get

$$\begin{aligned}&\sum_{i=1}^{k}(\zeta_{k+1}-\zeta_i)^2\int_\Omega v_i^2|\nabla z^\alpha|^2 d\mu\\ &\leq \delta\sum_{i=1}^{k}(\zeta_{k+1}-\zeta_i)^2\int_\Omega z^\alpha v_i\bar{p}_i d\mu + \sum_{i=1}^{k}\frac{(\zeta_{k+1}-\zeta_i)}{\delta}\int_\Omega (\langle\nabla z^\alpha,\nabla v_i\rangle + v_i z^\alpha)^2\, d\mu,\end{aligned} \tag{32}$$

where

$$\bar{p}_i \;=\; a\Big(2x^\alpha\Delta_\psi(v_i) + 2\langle\nabla z^\alpha, \nabla\Delta_\psi(v_i)\rangle + 2\Delta_\psi(v_i z^\alpha) + 2\Delta_\psi(\langle\nabla v_i, \nabla z^\alpha\rangle)\Big) - b\Big(2u_i z^\alpha + 2\langle\nabla z^\alpha, \nabla v_i\rangle\Big).$$

Summing over α from 1 to 2, we infer

$$\begin{aligned}&2\sum_{i=1}^{k}(\zeta_{k+1}-\zeta_i)^2\int_\Omega v_i^2(1+|\mathbf{z}|^2)d\mu\\ &\leq \delta\sum_{i=1}^{k}(\zeta_{k+1}-\zeta_i)^2\int_\Omega\left\{a\left[-4(1+|\mathbf{z}|^2)v_i\Delta_\psi v_i + \sum_{\alpha=1}^{2}8u_i z^\alpha\langle\nabla z^\alpha,\nabla v_i\rangle\right.\right.\\ &\left.\left. +4(1+|\mathbf{z}|^2)|\nabla v_i|^2 + 4u_i^2|\mathbf{z}|^2\right] - b\left[2u_i^2|\mathbf{z}|^2 + \sum_{\alpha=1}^{2}2u_i z^\alpha\langle\nabla z^\alpha,\nabla v_i\rangle\right]\right\}d\mu\\ &+\sum_{i=1}^{k}\frac{(\zeta_{k+1}-\zeta_i)}{\delta}\int_\Omega\left[(1+|\mathbf{z}|^2)|\nabla v_i|^2 + \sum_{\alpha=1}^{2}2u_i z^\alpha\langle\nabla z^\alpha,\nabla v_i\rangle + v_i^2|\mathbf{z}|^2\right]d\mu.\end{aligned} \tag{33}$$

Since $\psi = -\log(1+|\mathbf{z}|^2)$ and $d\mu = e^{-\psi}dv = (1+|\mathbf{z}|^2)dv$, we have

$$\begin{aligned}&\sum_{\alpha=1}^{2}\int_\Omega v_i z^\alpha\langle\nabla z^\alpha,\nabla v_i\rangle d\mu\\ =\;& \sum_{\alpha=1}^{2}\int_\Omega v_i z^\alpha\langle\nabla z^\alpha,\nabla v_i\rangle(1+|\mathbf{z}|^2)dv\\ =\;& -\sum_{\alpha=1}^{2}\int_\Omega|\nabla z^\alpha|^2|v_i|^2 d\mu - \sum_{\alpha=1}^{2}\int_\Omega v_i z^\alpha\langle\nabla z^\alpha,\nabla v_i\rangle d\mu - \sum_{\alpha=1}^{2}\int_\Omega v_i^2 z^\alpha\langle\nabla z^\alpha,\nabla(1+|\mathbf{z}|^2)\rangle dv\\ =\;& -4\int_\Omega v_i^2|\mathbf{z}|^2 d\mu - \sum_{\alpha=1}^{2}\int_\Omega v_i z^\alpha\langle\nabla z^\alpha,\nabla v_i\rangle d\mu - 2,\end{aligned}$$

which yields

$$\sum_{\alpha=1}^{2}\int_{\Omega} v_i z^{\alpha}\langle\nabla z^{\alpha},\nabla v_i\rangle d\mu = -2\int_{\Omega} v_i^2|\mathbf{z}|^2 d\mu - 1. \tag{34}$$

Using Cauchy-Schwarz inequality, we deduce

$$\begin{aligned} -a\int_{\Omega}(1+|\mathbf{z}|^2)v_i\Delta_{\psi}v_i d\mu &\leq \left(\int_{\Omega} a(1+|\mathbf{z}|^2)^2 v_i^2 d\mu\right)^{\frac{1}{2}}\left(\int_{\Omega} a(\Delta_{\psi}v_i)^2 d\mu\right)^{\frac{1}{2}} \\ &\leq \left(\int_{\Omega} a(1+|\mathbf{z}|^2)^2 v_i^2 d\mu\right)^{\frac{1}{2}}\left(\int_{\Omega}\left(a(\Delta_{\psi}v_i)^2 + b|\nabla v_i|^2\right)d\mu\right)^{\frac{1}{2}} \\ &\leq \sqrt{a}(1+\Gamma)\zeta_i^{\frac{1}{2}}. \end{aligned} \tag{35}$$

Also, if $b = 0$ then

$$\begin{aligned} \int_{\Omega}|\nabla v_i|^2 d\mu &= -\int_{\Omega} v_i\Delta_{\psi}v_i d\mu \\ &\leq \left(\int_{\Omega} v_i^2 d\mu\right)^{\frac{1}{2}}\left(\int_{\Omega}(\Delta_{\psi}v_i)^2 d\mu\right)^{\frac{1}{2}} \\ &= \frac{1}{\sqrt{a}}\left(\int_{\Omega} a(\Delta_{\psi}v_i)^2 d\mu\right)^{\frac{1}{2}} \\ &\leq \frac{1}{\sqrt{a}}\left(\int_{\Omega}\left(a(\Delta_{\psi}v_i)^2 + b|\nabla v_i|^2\right)d\mu\right)^{\frac{1}{2}} \\ &= \frac{1}{\sqrt{a}}\zeta_i^{\frac{1}{2}}, \end{aligned} \tag{36}$$

and if $b \neq 0$ then

$$\begin{aligned} \int_{\Omega}|\nabla v_i|^2 d\mu &= \frac{1}{b}\int_{\Omega} b|\nabla v_i|^2 d\mu \\ &\leq \frac{1}{b}\int_{\Omega}\left(a(\Delta_{\psi}v_i)^2 + b|\nabla v_i|^2\right)d\mu \\ &= \frac{1}{b}\zeta_i. \end{aligned} \tag{37}$$

On the other hand, we have

$$2(1+\gamma)\sum_{i=1}^{k}(\zeta_{k+1}-\zeta_i)^2 \leq 2\sum_{i=1}^{k}(\zeta_{k+1}-\zeta_i)^2\int_{\Omega} v_i^2(1+|\mathbf{z}|^2)d\mu.$$

Hence, if $b = 0$ then

$$\begin{aligned} &2(1+\gamma)\sum_{i=1}^{k}(\zeta_{k+1}-\zeta_i)^2 \\ &\leq \delta\sum_{i=1}^{k}(\zeta_{k+1}-\zeta_i)^2\left\{8\sqrt{a}(1+\Gamma)\zeta_i^{\frac{1}{2}} - 12\gamma a - 8a\right\} + \sum_{i=1}^{k}\frac{(\zeta_{k+1}-\zeta_i)}{\delta}\left[\frac{1}{\sqrt{a}}(1+\Gamma)\zeta_i^{\frac{1}{2}} - 3\gamma - 2\right] \end{aligned}$$

and if $b \neq 0$ then

$$
\begin{aligned}
&2(1+\gamma)\sum_{i=1}^{k}(\zeta_{k+1}-\zeta_i)^2 \\
&\leq \delta\sum_{i=1}^{k}(\zeta_{k+1}-\zeta_i)^2\left\{8\sqrt{a}(1+\Gamma)\zeta_i^{\frac{1}{2}}-12\gamma a-8a+\frac{4a}{b}(1+\Gamma)\zeta_i+2b(1+\Gamma)\right\} \\
&+\sum_{i=1}^{k}\frac{(\zeta_{k+1}-\zeta_i)}{\delta}\left[\frac{1}{b}(1+\Gamma)\zeta_i-3\gamma-2\right].
\end{aligned}
$$

Therefore, for $b = 0$, by taking

$$
\delta=\frac{\left\{\sum_{i=1}^{k}(\zeta_{k+1}-\zeta_i)\left[\frac{1}{\sqrt{a}}(1+\Gamma)\zeta_i^{\frac{1}{2}}-3\gamma-2\right]\right\}^{\frac{1}{2}}}{\left\{\sum_{i=1}^{k}(\zeta_{k+1}-\zeta_i)^2\left[8\sqrt{a}(1+\Gamma)\zeta_i^{\frac{1}{2}}-12\gamma a-8a\right]\right\}^{\frac{1}{2}}},
$$

we obtain (4). For $b \neq 0$, by taking

$$
\delta=\frac{\left\{\sum_{i=1}^{k}(\zeta_{k+1}-\zeta_i)\left[\frac{1}{b}(1+\Gamma)\zeta_i-3\gamma-2\right]\right\}^{\frac{1}{2}}}{\left\{\sum_{i=1}^{k}(\zeta_{k+1}-\zeta_i)^2\left[8\sqrt{a}(1+\Gamma)\zeta_i^{\frac{1}{2}}-12\gamma a-8a+\frac{4a}{b}(1+\Gamma)\zeta_i+2b(1+\Gamma)\right]\right\}^{\frac{1}{2}}},
$$

we arrive at (5). □

Remark 2.2. *Notice that if $a = 1$ and $b = 0$ then Theorem 1.1 reduces to Theorem 1.1 of [6].*

Now using method of [3], we prove Theorem 1.2.

Proof. [Proof of Theorem 1.2] We consider a 2×2-matrix A as $A := (a_{\alpha\beta})$ where $a_{\alpha\beta} = \int_\Omega z^\alpha v_1 v_{\beta+1}d\mu$. By applying the orthogonalization of Gram and Schmidt, we can obtain an upper triangle matrix $B = (b_{\alpha\beta})$ and an orthogonal matrix $C = (c_{\alpha\beta})$ such that $B = CA$. Then $b_{\alpha\beta} = \int_\Omega \sum_{\gamma=1}^{2} c_{\alpha\gamma}z^\gamma v_1 v_{\beta+1}d\mu = 0$ for $1 \leq \beta < \alpha \leq 2$. Let $y^\alpha = \sum_{\gamma=1}^{2} c_{\alpha\gamma}z^\gamma$ and $\mathbf{y} = (y^1, y^2)$, then $|\mathbf{y}|^2 = |\mathbf{z}|^2$ and $\int_\Omega y^\alpha v_1 v_{\beta+1}d\mu = 0$. Also, we have $|\nabla y^\alpha|^2 = 1 + |\mathbf{z}|^2$, $\sum_{\alpha=1}^{2}\langle\nabla y^\alpha, \nabla v_1\rangle^2 = (1 + |\mathbf{z}|^2)|\nabla v_1|^2$, and $\Delta_\psi y^\alpha = 2y^\alpha$ for $\alpha = 1, 2$. Let

$$
\eta^\alpha := (y^\alpha - z^\alpha)v_1,\ \ z^\alpha := \int_\Omega y^\alpha v_1^2 d\mu,\ \ \alpha = 1, 2.
$$

Then $\int_\Omega \eta^\alpha v_{\beta+1}d\mu = 0$ for $0 \leq \beta < \alpha \leq 2$. Using the Rayleigh-Ritz inequality, we obtain

$$
\zeta_{\alpha+1} \leq \frac{\int_\Omega \eta^\alpha \mathcal{L}\eta^\alpha d\mu}{\int_\Omega (\eta^\alpha)^2 d\mu},\ \ \alpha = 1, 2. \tag{38}
$$

By a direct computation, we obtain

$$
\begin{aligned}
\int_\Omega \eta^\alpha\mathcal{L}\eta^\alpha d\mu &= \int_\Omega \eta^\alpha\mathcal{L}(y^\alpha v_1 - z^\alpha v_1)d\mu \\
&= a\int_\Omega \eta^\alpha\left[v_1\Delta_\psi^2 y^\alpha + 2\Delta_\psi y^\alpha\Delta_\psi v_1 + 2\langle\nabla v_1, \nabla\Delta_\psi y^\alpha\rangle + 2\langle\nabla y^\alpha, \nabla\Delta_\psi v_1\rangle\right. \\
&\quad \left.+2\Delta_\psi\langle\nabla v_1, \nabla y^\alpha\rangle\right]d\mu - b\int_\Omega \eta^\alpha\left[v_1\Delta_\psi y^\alpha + 2\langle\nabla v_1, \nabla y^\alpha\rangle\right]d\mu
\end{aligned}
$$

$$
\begin{aligned}
&+\zeta_1 \int_\Omega \eta^\alpha y^\alpha v_1 d\mu \\
= \quad & a \int_\Omega \eta^\alpha \left[4v_1 y^\alpha + 4y^\alpha \Delta_\psi v_1 + 4\langle \nabla v_1, \nabla y^\alpha \rangle + 2\langle \nabla y^\alpha, \nabla \Delta_\psi v_1 \rangle \right. \\
&\left. +2\Delta_\psi \langle \nabla v_1, \nabla y^\alpha \rangle \right] d\mu - b \int_\Omega \eta^\alpha \left[2v_1 y^\alpha + 2\langle \nabla v_1, \nabla y^\alpha \rangle\right] d\mu \\
&+\zeta_1 \int_\Omega (\eta^\alpha)^2 d\mu.
\end{aligned}
$$

By (38), we conclude

$$
(\zeta_{\alpha+1} - \zeta_1) \int_\Omega (\eta^\alpha)^2 d\mu \leq a \int_\Omega \eta^\alpha P d\mu - 2b \int_\Omega \eta^\alpha Q d\mu. \tag{39}
$$

where

$$
P = 4v_1 y^\alpha + 4y^\alpha \Delta_\psi v_1 + 4\langle \nabla v_1, \nabla y^\alpha \rangle + 2\langle \nabla y^\alpha, \nabla \Delta_\psi v_1 \rangle + 2\Delta_\psi \langle \nabla v_1, \nabla y^\alpha \rangle
$$

and

$$
Q = v_1 y^\alpha + \langle \nabla v_1, \nabla y^\alpha \rangle.
$$

Using the integration by parts, we obtain the following equations

$$
\begin{aligned}
&\int_\Omega v_1 P d\mu = \int_\Omega v_1 Q d\mu = 0, \\
&\int_\Omega y^\alpha v_1 \langle \nabla y^\alpha, \nabla \Delta_\psi v_1 \rangle d\mu = -\int_\Omega |\nabla y^\alpha|^2 v_1 \Delta_\psi v_1 d\mu - \int_\Omega \Delta_\psi v_1 \langle \nabla y^\alpha, \nabla v_1 \rangle d\mu \\
&-2 \int_\Omega (y^\alpha)^2 v_1 \Delta_\psi v_1 d\mu, \\
&\int_\Omega y^\alpha v_1 \Delta_\psi \langle \nabla y^\alpha, \nabla v_1 \rangle d\mu = 2 \int_\Omega y^\alpha v_1 \langle \nabla y^\alpha, \nabla v_1 \rangle d\mu + \int_\Omega y^\alpha \Delta_\psi v_1 \langle \nabla y^\alpha, \nabla v_1 \rangle d\mu \\
&+2 \int_\Omega \langle \nabla y^\alpha, \nabla v_1 \rangle^2 d\mu.
\end{aligned}
$$

Therefore, we have

$$
\int_\Omega y^\alpha v_1 P d\mu = 4 \int_\Omega Q^2 d\mu - 2 \int_\Omega |\nabla y^\alpha|^2 v_1 \Delta_\psi v_1 d\mu \tag{40}
$$

and

$$
(\zeta_{\alpha+1} - \zeta_1) \int_\Omega (\eta^\alpha)^2 d\mu \quad \leq \quad 4a \int_\Omega Q^2 d\mu - 2a \int_\Omega |\nabla y^\alpha|^2 v_1 \Delta_\psi v_1 d\mu - 2b \int_\Omega v_1 y^\alpha Q d\mu.
$$

On the other hand,

$$
\int_\Omega (y^\alpha)^2 v_1^2 d\mu = \frac{1}{2} \int_\Omega y^\alpha v_1^2 \Delta_\psi y^\alpha d\mu = -\frac{1}{2} \int_\Omega |v_1 \nabla y^\alpha|^2 d\mu - \int_\Omega y^\alpha v_1 \langle \nabla y^\alpha, \nabla v_1 \rangle d\mu
$$

and

$$
\int_\Omega \eta^\alpha Q d\mu \quad = \quad \int_\Omega y^\alpha v_1 \left(v_1 y^\alpha + \langle \nabla v_1, \nabla y^\alpha \rangle\right) d\mu = -\frac{1}{2} \int_\Omega |v_1 \nabla y^\alpha|^2 d\mu.
$$

For any positive constant denoted as δ, we deduce

$$
(\zeta_{\alpha+1} - \zeta_1)^{\frac{1}{2}} \int_\Omega |v_1 \nabla y^\alpha|^2 d\mu
$$

$$
\begin{aligned}
&= -2(\zeta_{\alpha+1}-\zeta_1)^{\frac{1}{2}}\int_\Omega \eta^\alpha Q d\mu\\
&\leq \frac{\delta}{2}(\zeta_{\alpha+1}-\zeta_1)\int_\Omega(\eta^\alpha)^2 d\mu+\frac{2}{\delta}\int_\Omega Q^2 d\mu\\
&\leq (2a\delta+\frac{2}{\delta})\int_\Omega Q^2 d\mu - a\delta\int_\Omega |\nabla y^\alpha|^2 v_1\Delta_\psi v_1 d\mu - b\delta\int_\Omega v_1 y^\alpha Q d\mu.
\end{aligned}
$$

Summing over α from 1 to 2, we get

$$
\begin{aligned}
&\sum_{\alpha=1}^{2}(\zeta_{\alpha+1}-\zeta_1)^{\frac{1}{2}}\int_\Omega |v_1\nabla y^\alpha|^2 d\mu\\
&\leq (2a\delta+\frac{2}{\delta})\sum_{\alpha=1}^{2}\int_\Omega Q^2 d\mu - a\delta\sum_{\alpha=1}^{2}\int_\Omega |\nabla y^\alpha|^2 v_1\Delta_\psi v_1 d\mu - b\delta\sum_{\alpha=1}^{2}\int_\Omega v_1 y^\alpha Q d\mu.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\sum_{\alpha=1}^{2}(\zeta_{\alpha+1}-\zeta_1)^{\frac{1}{2}}\int_\Omega |v_1|^2(1+|\mathbf{z}|^2) d\mu\\
&\leq (2a\delta+\frac{2}{\delta})\int_\Omega\left[v_1^2|\mathbf{z}|^2+\sum_{\alpha=1}^{2}2u_1 y^\alpha\langle\nabla y^\alpha,\nabla v_1\rangle+(1+|\mathbf{z}|^2)|\nabla v_1|^2\right]d\mu\\
&-2a\delta\int_\Omega(1+|\mathbf{z}|^2)v_1\Delta_\psi v_1 d\mu - b\delta\int_\Omega\left[v_1^2|\mathbf{z}|^2+\sum_{\alpha=1}^{2}v_1 y^\alpha\langle\nabla y^\alpha,\nabla v_1\rangle\right]d\mu.
\end{aligned}
$$

Similar to (34), (35), and (36), we deduce

$$
\sum_{\alpha=1}^{2}\int_\Omega v_1 y^\alpha\langle\nabla y^\alpha,\nabla v_1\rangle d\mu = -2\int_\Omega v_1^2|\mathbf{z}|^2 d\mu - 1, \tag{41}
$$

$$
-a\int_\Omega(1+|\mathbf{z}|^2)v_1\Delta_\psi v_1 d\mu \leq \sqrt{a}(1+\Gamma)\zeta_1^{\frac{1}{2}}, \tag{42}
$$

and for $b=0$,

$$
\int_\Omega |\nabla v_1|^2 d\mu \leq \frac{1}{\sqrt{a}}\zeta_1^{\frac{1}{2}}, \tag{43}
$$

and for $b\neq 0$,

$$
\int_\Omega |\nabla v_1|^2 d\mu \leq \frac{1}{b}\zeta_1. \tag{44}
$$

Using above equations for $b=0$, we can write

$$
\begin{aligned}
&(1+\gamma)\sum_{\alpha=1}^{2}(\zeta_{\alpha+1}-\zeta_1)^{\frac{1}{2}}\\
&\leq \sum_{\alpha=1}^{2}(\zeta_{\alpha+1}-\zeta_1)^{\frac{1}{2}}\int_\Omega |v_1|^2(1+|\mathbf{z}|^2) d\mu\\
&\leq (2a\delta+\frac{2}{\delta})\left[-3\gamma-2+\frac{1+\Gamma}{\sqrt{a}}\zeta_1^{\frac{1}{2}}\right]+2\sqrt{a}\delta(1+\Gamma)\zeta_1^{\frac{1}{2}}
\end{aligned}
$$

$$= \delta\left(-6a\gamma - 2a + 4\sqrt{a}(1+\Gamma)\zeta_1^{\frac{1}{2}}\right) + \frac{1}{\delta}\left(-6\gamma - 4 + \frac{2(1+\Gamma)}{\sqrt{a}}\zeta_1^{\frac{1}{2}}\right).$$

Taking

$$\delta = \left\{\frac{-3\gamma - 2 + \frac{(1+\Gamma)}{\sqrt{a}}\zeta_1^{\frac{1}{2}}}{-3a\gamma - a + 2\sqrt{a}(1+\Gamma)\zeta_1^{\frac{1}{2}}}\right\}^{\frac{1}{2}}$$

we infer (6). Also, for $b \neq 0$, we arrive at

$$\begin{aligned} & (1+\gamma)\sum_{\alpha=1}^{2}(\zeta_{\alpha+1} - \zeta_1)^{\frac{1}{2}} \\ \leq\ & (2a\delta + \frac{2}{\delta})\left[-3\gamma - 2 + \frac{1+\Gamma}{b}\zeta_1\right] + 2\sqrt{a}\delta(1+\Gamma)\zeta_1^{\frac{1}{2}} + b\delta(1+\Gamma) \\ & = \delta\left(-6a\gamma - 4a + \frac{2a(1+\Gamma)}{b}\zeta_1 + 2\sqrt{a}(1+\Gamma)\zeta_1^{\frac{1}{2}} + b(1+\Gamma)\right) \\ & + \frac{1}{\delta}\left(-6\gamma - 4 + \frac{2(1+\Gamma)}{b}\zeta_1\right). \end{aligned}$$

Taking

$$\delta = \left\{\frac{-6\gamma - 4 + \frac{2(1+\gamma)}{b}\zeta_1}{-6a\gamma - 4a + \frac{2a(1+\gamma)}{b}\zeta_1 + 2\sqrt{a}(1+\Gamma)\zeta_1^{\frac{1}{2}} + b(1+\Gamma)}\right\}^{\frac{1}{2}}$$

we find (7). □

Remark 2.3. *Notice that if $a = 1$ and $b = 0$ then Theorem 1.2 reduces to theorem 1.2 of [6].*

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Author's contributions

All authors contributed equally in the preparation of this manuscript.

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