Filomat 39:25 (2025), 8817–8826
https://doi.org/10.2298/FIL2525817K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Conformal gradient Ricci soliton on twisted product manifolds

Byung Hak Kim[a], Jin Hyuk Choi[b,*], Sang Deok Lee[c]

[a]*Department of Applied Mathematics and Institute of Natural Sciences, Kyung Hee University, Yongin 17104, Republic of Korea*
[b]*Humanitas college, Kyung Hee University, Yongin 17104, Republic of Korea*
[c]*Department of Mathematics, Dankook University, Cheonan 31116, Republic of Korea*

Abstract. The concept of a conformal Ricci soliton generalizes that of a Ricci soliton. Notably the conformal Ricci soliton is closely related to the conformal Ricci flow. This paper aims to characterize Riemannian manifolds that admit a conformal gradient Ricci soliton. Furthermore, we present methods for constructing such manifolds in the context of product manifolds and warped product manifolds.

1. Introduction

Since R. S. Hamilton[11] introduced the concepts of Ricci flow and Yamabe flow, numerous studies have been published on those topics. In 2004, A. E. Fischer [10] proposed the conformal Ricci flow as a modification of the classical Ricci flow, replacing the unit volume constraint with a scalar curvature constraint. These resulting equations are called the conformal Ricci flow equations, as conformal geometry plays a crucial role in constraining scalar curvature. Furthermore, the equations can be interpreted as the sum of a conformal vector field and a Ricci flow equation. On an m-dimensional Riemannian manifold (M, g) the conformal Ricci flow equation takes the form

$$\frac{\partial \tilde{g}}{\partial t} + 2(\tilde{S} + \frac{g}{m}) = -pg, \qquad \tilde{r}(g) = -1, \tag{1}$$

where $\tilde{S}$ and $\tilde{r}(g)$ are the Ricci curvature and the scalar curvature of the manifold (M, g) respectively, and p is a scalar non-dynamical field. The term-pg acts as a constraint force to preserve the scalar curvature condition [5]. The conformal Ricci flow equations are analogous to the Navier-Stokes equation of fluid mechanics, and because of this analogy, the time-dependent scalar field p is referred to as a conformal pressure [2].

In 2015, N. Basu and A. Bhattacharyya [2] introduced the conformal Ricci soliton equation given by

$$\mathfrak{L}_X g + 2\tilde{S} = [2\tilde{\lambda} - (p + \frac{2}{m})]\tilde{g}, \tag{2}$$

2020 *Mathematics Subject Classification*. Primary 53C15; Secondary 53C25.
Keywords. conformal gradient Ricci soliton.
Received: 13 May 2025; Accepted: 21 July 2025
Communicated by Mića S. Stanković
* Corresponding author: Jin Hyuk Choi
Email addresses: bhkim@khu.ac.kr (Byung Hak Kim), jinhchoi@khu.ac.kr (Jin Hyuk Choi), sdlee@dankook.ac.kr (Sang Deok Lee)
ORCID iD: https://orcid.org/0000-0002-6724-4734 (Jin Hyuk Choi)

where $\tilde{\lambda}$ is a constant and $\mathfrak{L}_X$ denotes the Lie derivative in the direction X. If $\tilde{\lambda}$ is a smooth function, a conformal Ricci soliton is called a conformal almost Ricci soliton [7,8,9,14,17].

In [8], S. Dey studied conformal Ricci soliton and almost conformal Ricci soliton within the framework of paracontact manifolds. In addition, this research method could be applied to the study of odd-dimensional manifolds with an almost contact structure. In [13], authors studied isometries of almost Ricci-Yamabe solitons and explored the conditions under which a given manifold that admits an almost Ricci-Yamabe soliton will be isometric with Euclidean space or Euclidean sphere. They provided non-trivial examples to verify the main theorems.

The equation (2) is the generalization of the classical Ricci soliton equation and simultaneously satisfies the conformal Ricci flow equation. A conformal Ricci soliton [2,7,8,9] is called a conformal gradient Ricci soliton if the associated vector field X is a gradient of some smooth function h defined on a manifold. In this case, the conformal gradient Ricci soliton is given by

$$2\tilde{\nabla}\tilde{\nabla}h + 2\tilde{S} = [2\lambda - (p + \frac{2}{m})]\tilde{g}, \tag{3}$$

where $\tilde{\nabla}$ is the Riemannian connection on M. In this case, M is said to admit conformal gradient Ricci soliton with (h, λ, p). Recently, P. Zhang, Y. Li, S. Roy, S. Dey and A. Bhattacharyya[18] studied the geometrical structure in a perfect fluid space time with a torse-forming vector field in connection with the conformal Ricci-Yamabe metric and the conformal η -Ricci-Yamabe metric. S. Dey [8,9] and S. Uddin [9] studied conformal Ricci soliton and almost conformal Ricci soliton within the framework of paracontact manifolds and Kenmotsu manifolds. In 2022, U. C. De, A. Sardar and K. De [6] studied and classified 3-dimensional Riemannian manifolds endowed with a special type of vector field when the Riemannian metrics are Ricci-Yamabe solitons and gradient Ricci-Yamabe solitons respectively. Moreover they constructed an example related to their main theorem. This paper consist of five sections. The first section introduces the historical background of this paper's topic and research results , and necessary concepts.

In the second section, we study the geometric charecterization of conformal gradient Ricci soliton in the product space. And we present a method to construct a model space with a conformal gradient Ricci soliton in a product space in Theorems 2.2 and 2.3. In sections 3 and 4, we investigate the conformal gradient Ricci soliton in the warped product manifold and also obtain Theorem 3.3 that how we can construct the model space in the warped product manifold $M = L \times_{ce^t} F$. In the last section 5, we are to characterize the twisted manifold with conformal gradient Ricci soliton.

2. Conformal gradient Ricci soliton in the product manifold

Let $M = B \times F$ be the product manifold of an n-dimensional Riemannian manifold (B, g) and a q-dimensional Riemannian manifold $(F, \bar{g})$. If $M = B \times F$ admits a conformal gradient Ricci soliton with $(\tilde{h}, \tilde{\lambda}, p)$, then the following equations holds:

$$\begin{aligned} \nabla_b \tilde{h}_a + S_{ba} &= \tilde{[}2\tilde{\lambda} - (p + \tfrac{2}{n+q})]g_{ba}, \\ \partial_b \tilde{h}_x &= 0, \\ \bar{\nabla}_y \tilde{h}_x + \bar{S}_{yx} &= \tilde{K}\bar{g}_{yx}, \\ \tilde{r} &= r + \bar{r}, \end{aligned} \tag{4}$$

where ∇ and $\bar{\nabla}$ are the Levi-Civita connections for g and $\bar{g}$ respectively, S and $\bar{S}$ are Ricci curvatures on B and F respectively. Additional, $\tilde{r}$, r and $\bar{r}$ are the scalar curvatures on M, B and F respectively. The ranges of the indices $a, b, ...,$ run from 1 to n, and $x, y, ...$ from $n + 1$ to $n + q$. Moreover $\tilde{h}_a = \frac{\partial \tilde{h}}{\partial u_a}$, $\tilde{h}_x = \frac{\partial \tilde{h}}{\partial \bar{u}_x}$ for the coordinate neighborhoods $\{\tilde{U} = (U, \bar{U}); \tilde{U}^h = (u^a, \bar{u}^x)\}$ in $B \times F$.

Since $\partial_b \tilde{h}_x = 0$, it follows from the second equation of (4) that $\tilde{h}$ can be expressed as the sum $k + l$, where k is a smooth function on B and l is a smooth function on F. Thus, the first and third equations of (4) reduces to

$$\nabla_b k_a + S_{ba} = \tilde{K} g_{ba}, \bar{\nabla}_y l_x + \bar{S}_{yx} = \tilde{K} \bar{g}_{yx}. \tag{5}$$

where we have set $\tilde{K} = 2\lambda - (p + \frac{2}{n+q})$. Equation (5) shows that $\tilde{K}$ is constant and consequently, both B and F become gradient Ricci solitons with $(k, \tilde{K})$ and $(l, \tilde{K})$ respectively. This leads to the following.

Theorem 2.1. *If $M = B \times F$ admits a conformal gradient Ricci soliton with $(\tilde{h}, \tilde{\lambda}, p)$, then both B and F are gradient Ricci solitons with $(k, \tilde{K})$ and $(l, \tilde{K})$ respectively.*

Conversely, suppose that B and F are gradient Ricci solitons with (k, ρ_1) and (l, ρ_2). If we set $\tilde{h} = k + l$, $\rho = \rho_1 = \rho_2$, $\tilde{\lambda} = c$(=constant) and $p = \frac{2}{n+q} + \rho - 2c$, then $B \times F$ admits a conformal gradient Ricci soliton with $(\tilde{h}, \lambda, p)$; that is, $M = B \times F$ admits a conformal gradient Ricci soliton with $(\tilde{h}, \lambda, p)$. Thus, we have

Theorem 2.2. *If B and F are gradient Ricci solitons with (k, ρ) and (l, ρ), then $M = B \times F$ admits a conformal gradient Ricci soliton with $(\tilde{h} = k + l, \tilde{\lambda} = c, p = \frac{2}{n+q} + \rho - 2c)$, where c is an arbitrary constant.*

Therefore Theorem 2.2 provides a method to construct model spaces of manifolds that admit conformal gradient Ricci solitons.

In [15], the authors established the following:

Theorem 2.3. *If B and F are gradient Ricci solitons for (h_1, λ) and (h_2, μ), then $M = B \times F$ is a gradient Ricci soliton with $(\tilde{h} = h_1 + h_2, \tilde{\rho} = \lambda = \mu)$.*

3. Conformal gradient Ricci soliton in the warped product manifold $R^n \times_f F$

Let $M = R^n \times_f F$ be a warped product manifold admitting a conformal gradient Ricci soliton $(\tilde{h}, \tilde{\lambda}, p)$. Then the following equations hold:

$$\begin{aligned} 2\nabla_b \tilde{h}_a + 2S_{ba} - \tfrac{2q}{f}(\nabla_b f_a) &= A g_{ba}, \\ 2\partial_b \tilde{h}_x - \tfrac{2f_b}{f}\tilde{h}_x &= 0, \\ 2\bar{\nabla}_y \tilde{h}_x + 2ff^c \tilde{h}_c \bar{g}_{xy} + 2\bar{S}_{yx} - 2f(\triangle f)\bar{g}_{yx} - 2(q-1)\|f_e\|^2 \bar{g}_{yx} &= A f^2 \bar{g}_{yx}, \end{aligned} \tag{6}$$

where $A = 2\lambda - (p + \frac{2}{2n+1})$, $f_b = \frac{\partial f}{\partial u_b}$, and $\triangle f$ is a Laplacian of f with respect to g.

From (6), we obtain the following Theorem

Theorem 3.1. *Let $M = R^n \times_f F$ admit a conformal gradient Ricci soliton with $(\tilde{h}, \tilde{\lambda}, p)$ and assume $q > 2$. Then:*
(a) If $\tilde{h}_x = 0$ for all $x = n+1, ..., n+q$, then F is an Einstein manifold.
(b) If $\tilde{h}_a = 0$ for all $a = 1, 2, ..., n$, then M is either an Einstein manifold or a product manifold. Moreover, F is Einstein.

Proof. (a) From the first equation of (6), it follows that $\tilde{h}$ depends only on R^n. From the third equation of (6), it follows that F must be an Einstein manifold because $\bar{S} = K\bar{g}_{yx}$, where K is constant on F by the second Bianchi identity.
(b) The second equation of (6) shows that either f or $\tilde{h}$ is constant, implying that M is Einsein or a product manifold. Moreover, we see that F becomes an Einstein manifold from the third equation of (6).

If $\tilde{h} = k + l$, then $f_b l_x = 0$ from the second equation of (6). Hence either f is constant, making M a product making, or F is Einstein.

Theorem 3.2. *Let the warped product manifold $M = R^n \times_f F$ admits a conformal gradient Ricci soliton with $(\tilde{h}, \tilde{\lambda}, p)$ and $q > 2$. If $\tilde{h} = k + l$ for some functions k on B and l on F respectively, then either F is an Einstein manifold or M is a product manifold.*

We next investigate how to construct the model spaces of the manifold with a conformal gradient Ricci soliton on the warped product manifold $R \times_f F$ for the Einstein manifold F, referring to Theorems 3.1 and 3.2.

Theorem 3.3. *Let F be an Einstein manifold and if we set $\tilde{h}(t) = \frac{t^2}{2}$ and $f(t) = cost$, then the warped product manifold $M = R \times_f F$ admits a conformal gradient Ricci soliton. Furthermore, the following relations hold: $\tilde{\lambda} - \frac{p}{2} - \frac{1}{n+1} = 1$ and $\frac{\bar{r}}{q} - \frac{tsin2t}{2} - (q-1)sin^2 t = 0$, where $\bar{r}$ is the scalar curvature of F, and $\bar{S} = \frac{\bar{r}}{q}\bar{g}$.*

Proof. To construct a model space where the warped product manifold $M = R \times_f F$ admits conformal gradient Ricci soliton with $(\tilde{h}, \tilde{\lambda}, p)$, the quantities $(\tilde{h}, \tilde{\lambda}, p)$ must satisfy the following equations:

$$\begin{aligned} \tilde{h}_{11} &= \tilde{\lambda} - \frac{p}{2} - \frac{1}{n+1}, \\ \partial_1 \tilde{h}_x - \frac{f_1}{f}\tilde{h}_x &= 0, \\ \bar{\nabla}_y \tilde{h}_x + f f^1 \tilde{h}_1 \bar{g}_{xy} + \bar{S}_{yx} + f(\triangle f)\bar{g}_{yx} - (q-1)\|f_1\|^2 \bar{g}_{yx} &= (\tilde{\lambda} - \frac{p}{2} - \frac{1}{n+1}) f^2 \bar{g}_{yx}. \end{aligned} \tag{7}$$

Then, to satisfy (7), the following must be satisfied:

$$\tilde{\lambda} - \frac{p}{2} - \frac{1}{n+1} = 1, \ \frac{\bar{r}}{q} - \frac{tsin2t}{2} - (q-1)sin^2 t = 0, \tag{8}$$

where we have put $\bar{S} = \frac{\bar{r}}{q}\bar{g}$.

Therefore if the relations in (8) hold, then the warped product manifold $R \times_f F$ becomes a conformal gradient Ricci soliton manifold with $(\tilde{h} = \frac{t^2}{2}, \tilde{\lambda}, p)$ and the warping function $f(t) = cost$. Theorem 3.3 allows us to construct a model space in the warped product manifold admitting a conformal gradient Ricci soliton.

Thus Theorem 3.3 provides a complete model of a conformal gradient Ricci soliton on a warped product manifold with Einstein fiber.

In [4], B.Y.Chen proved

Theorem 3.4. *If M is a m-dimensional Riemannian manifold that admits a nowhere-zero concircular vector field, then M is locally a warped product $I \times_{\phi(s)} F$, where $\phi(s)$ is a nowhere-vanishing function on I and F is a Riemannian $(m-1)$-manifold.*

In [12], the following theorem was proved.

Theorem 3.5. *Let F be a $2n$-dimensional Kaehler manifold and c a nonzero constant. Let $f = ce^t$ be a function on a line L. Then the warped product space $M = L \times_f F$ has an almost contact metric structure (ϕ, ξ, η, g) that satisfies*

$$\begin{aligned} \nabla_X \phi \cdot Y &= -\eta(Y)\phi X - g(X, \phi Y)\xi, \\ \nabla_X \xi &= X - \eta(X)\xi. \end{aligned} \tag{9}$$

In an almost contact Riemannian manifold, if the Ricci tensor S satisfies $S = ag + b\eta \otimes \eta$, where a and b are scalar functions, then it is called an η-Einstein manifold[3,12].

In [12], the author proved

Theorem 3.6. *Let M be an almost contact manifold that satisfies the equation (7) of dimension $2n+1$. If M is η-Einstein and a is constant(or b is constant), then M is an Einstein manifold.*

Let F be a Kaehler manifold and c a nonzero constant. Let $f = ce^t$ be a function on a line L. Then, by Theorem 3.5, the warped product manifold $M = L \times_f F$ admits an almost contact metric structure that satisfies the equation (9).

If M admits gradient Ricci soliton, then $S_{ij} + \nabla_i \eta_j = \rho g_{ij}$ and that

$$S_{ij} = (\rho - 1)g_{ij} + \eta_i \eta_j \tag{10}$$

by the second equation of (9). From the second equation of (9) and the second Bianch's identity, we obtain $S_{ij}\xi^j = (1-2n)\eta_i$. Thus, we see that $\rho = 1 - 2n$, which corresponds to

$$S_{ij} = -2ng_{ij} + \eta_i \eta_j, \tag{11}$$

where $l_x = \frac{\partial l}{\partial \bar{u}_x}$. Hence M becomes η-Einstein. From this fact and Theorems 3.5 and 3.6, we have

Theorem 3.7. *Let F be a Kaehler manifold and c a nonzero constant. Then the warped product space $M = L \times_{ce^t} F$ a gradient Ricci soliton becomes Einstein, where the vector field ξ is in the admitted almost contact structure (ϕ, ξ, η, g) on M.*

If M admits a conformal gradient Ricci soliton, then $[2(\lambda - 1) - (p + \frac{2}{2n+1})]g_{ij} + 2\eta_i\eta_j = 0$. If ξ is in the almost contact structure (ϕ, ξ, η, g), then we get $S_{ij} = [\lambda - 1 + (\frac{p}{2} + n)]g_{ij} + \eta_i \eta_j$, that is M is η-Einstein. From this fact and Theorems 3.5 and 3.6, we obtain the following Theorem

Theorem 3.8. *Let F be a Kaehler manifold and c a nonzero constant. Then the warped product space $M = L \times_{ce^t} F$ with a conformal gradient Ricci soliton becomes Einstein, where the vector field ξ is in the admitted almost contact structure (ϕ, ξ, η, g) on M.*

4. Conformal gradient Ricci soliton in the warped product manifold $B \times_f F$

Let $M = B \times_f F$ be a warped product manifold, where (B, g) is an n-dimensional Riemannian manifold and $(F, \bar{g})$ is a q-dimensional Riemannian manifold. Suppose that $M = B \times_f F$ admits a conformal gradient Ricci soliton with $(\tilde{h}, \tilde{\lambda}, p)$. Then the following equations hold:

$$\begin{aligned} 2\nabla_b \tilde{h}_a + 2(S_{ba} - \tfrac{q}{f}\nabla_b f_a) &= \tilde{K} g_{ab}, \\ \partial_b \tilde{h}_x - \tfrac{f_b}{f}\tilde{h}_x &= 0, \\ 2\bar{\nabla}_y \tilde{h}_x + f f^c \tilde{h}_c \bar{g}_{yx} + 2(\bar{S}yx - f(\triangle f)\bar{g}_{yx} - (q-1)\|f_e\|^2 \bar{g}_{yx}) &= \tilde{K} f^2 \bar{g}_{yx}, \\ \tilde{r} = r + \tfrac{\bar{r}}{f^2} - \tfrac{2q(\triangle f)}{f} - \tfrac{q(q-1)}{f^2}\|f_e\|^2, & \end{aligned} \tag{12}$$

where we have put $\tilde{K} = 2\lambda - (p + \frac{2}{n+q})$.

From (12), we have

Theorem 4.1. *Let the warped product manifold $M = B \times_f F$ admits a conformal gradient Ricci soliton with $(\tilde{h}, \lambda, p)$ and $q > 2$. Then:*
(a) If $\tilde{h}_a = 0$, then $\tilde{K}$ depends only on B. In this case, either M is a product manifold with an Einstein manifold B and F, or both M and F are Einstein.
(b) If $\tilde{h}_x = 0$, then $\tilde{K}$ depends only on B, and F an Einstein manifold.

Proof. (a) since $\tilde{h}_a = 0$ for all a, it follows from the first equation of (12) that $\tilde{K}$ depends only on B and $f_b\tilde{h}_x = 0$. Hence, either f is constant, making B is an Einstein manifold or $\tilde{h}$ is constant, implying that either M is a product manifold of an Einstein manifold and F, or both M and F are Einstein.
(b) If $\tilde{h}_x = 0$ for all x, then we see that $\tilde{K}$ depends only on B from the first equation of (12) and $\bar{S}$ becomes an Einstein manifold from the third equation of (12).

If the potential function $\tilde{h}$ is represented as $\tilde{h} = k + l$ for some functions k on B and l on F respectively, then we observe that $\tilde{K}$ depends only on B from the first equation of (12) and $f_b l_x = 0$. This implies that either f is constant or l is constant.
If f is constant, then M becomes a product manifold and

$$\begin{aligned} 2\nabla_b k_a + 2S_{ba} &= \tilde{K}g_{ba}, \\ 2\bar{\nabla}_y l_x + 2\bar{S}_{yx} &= \tilde{K}f^2\bar{g}_{yx},. \end{aligned} \tag{13}$$

hold according to the equation (12).

Hence we see that $\tilde{K}$ depends only on F from the second equation of (13), which implies that $\tilde{K}$ becomes constant. Therefore M turns into a product manifold of gradient Ricci soliton B with $(k, \tilde{K})$ and the gradient Ricci soliton F with $(l, f^2\tilde{K})$. If l is constant, then we can conclude that F becomes an Einstein manifold base on the second equation of (13). Thus we have

Theorem 4.2. *Let the warped product manifold $M = B \times_f F$ admits a conformal gradient Ricci soliton with $(\tilde{h}, \lambda, p)$ and $q > 2$. Suppose that the potential function $\tilde{h} = k + l$ for some functions k on B and l on F respectively. Then either M is a product manifold of a gradient Ricci soliton B with $(k, \tilde{K})$, and a gradient Ricci soliton F with $(l, \tilde{K})$, or F is an Einstein manifold.*

So far, we have considered the case where the total space M has dimension ≥ 4. We now turn to the case where the dimension $M = 3$. This will allow us to understand the geometric characterizations of each space in all dimensions.

Let us consider the 3-dimensional warped product manifold $M = B \times_f F$ which admits a conformal gradient Ricci soliton with $(\tilde{h}, \lambda, p)$. Since the conformal curvature tensor $C = 0$ in the 3-dimensional Riemannian manifold, we obtain

$$\begin{aligned} \tilde{R}_{kji}{}^h &= (\tilde{S}_{ji}\delta_k{}^h - \tilde{S}_{ki}\delta_j{}^h + \tilde{g}_{ji}\tilde{S}_k{}^h - \tilde{g}_{ki}\tilde{S}_j{}^h) + \tfrac{\tilde{r}}{2}(\tilde{g}_{ki}\delta_j{}^h - \tilde{g}_{ji}\delta_k{}^h), \\ \tilde{S}_{ji} + \tilde{\nabla}_j\tilde{h}_i &= B\tilde{g}_{ji} \\ \tilde{r} &= -\tilde{\Delta}\tilde{h} + 3A, \end{aligned} \tag{14}$$

where we define $A = \lambda - (\frac{p}{2} + \frac{1}{3})$ and $B = 4A - \tilde{\Delta}\tilde{h} - \tilde{r}$. Hence we derive $\tilde{S}_{ji} + \tilde{\nabla}_j\tilde{h}_i = (4A - \tilde{\Delta}\tilde{h} - \tilde{r})g_{ji}$ from the first and second equations of (14). From the second equation of (14), we observe that M becomes an almost gradient Ricci soliton. Moreover, we obtain

$$\begin{aligned} S_{ab} + \nabla_b\tilde{h}_a &= \tfrac{q}{f}\nabla_b f_a + Ag_{ab}, \\ \partial_a\tilde{h}_x &= \tfrac{1}{f}f_a\tilde{h}_x, \\ \bar{S}_{yx} + \bar{\nabla}_y\tilde{h}_x &= (f\triangle f + (q-1)\|f_e\|^2 - ff^a\tilde{h}_a)\bar{g}_{yx}. \end{aligned} \tag{15}$$

If $\tilde{h}_a = 0$, then we see that $f_a\tilde{h}_x = 0$ from the second equation of (15). Since f and $\tilde{h}$ depend only on B and F respectively, either $f_a = 0$ or $\tilde{h}_x = 0$; that is, f is a constant or $\tilde{h}$ is a constant. In the first case, if f is a constant, then M becomes a product manifold. Moreover B becomes an Einstein manifold and $\bar{S} + \bar{\nabla}\bar{\nabla}\tilde{h} = 0$ from the first and third equations of (15), where $\tilde{h}$ depends only on F. In the second case, if $\tilde{h}$ is a constant, F becomes an Einstein manifold by using the third equation of (15).
If $\tilde{h}_x = 0$, then F becomes an Einstein manifold from the third equation of (15). Thus we have

Theorem 4.3. *Let $M = B \times_f F$ be a 3-dimensional warped product manifold that admits a conformal gradient Ricci soliton with $(\tilde{h}, \lambda, p)$. Then:*
(a) If $\tilde{h}_a = 0$, then either f is constant or $\tilde{h}$ is constant. In the former case, M becomes a product manifold of an Einstein manifold B and F with $\bar{S} + \bar{\nabla}\bar{\nabla}\tilde{h} = 0$, where $n \neq 2$. In the latter case, F becomes an Einstein manifold, where $q \neq 2$.

(b) If $\tilde{h}_x = 0$, then F is an Einstein manifold, where $q \neq 2$.

If the potential function $\tilde{h}$ is represented as $\tilde{h} = k + l$ for some functions k on B and l on F respectively, in the 3-dimensional warped product manifold $M = B \times_f F$ with a conformal gradient Ricci soliton with $(\tilde{h}, \lambda, p)$, then we obtain

$$\begin{aligned} S_{ab} + \nabla_b k_a &= \frac{q}{f}\nabla_b f_a + A g_{ab}, \\ f_a l_x &= 0, \\ \bar{S}_{yx} + \bar{\nabla}_y l_x &= (f\triangle f + (q-1)||f_e||^2 - f f^a k_a)\bar{g}_{yx}. \end{aligned} \tag{16}$$

where $l_x = \frac{\partial l}{\partial \bar{u}_x}$.

Hence we see that either $f_a = 0$ or $l_x = 0$ from the second equation of (16) because f and l are depend only on B and F, respectively. In the first case, $f_a = 0$, we see that B becomes (almost) a gradient Ricci soliton and $\bar{S} + \bar{\nabla}\bar{\nabla}l = 0$. In the second case, $l_x = 0$, then we obtains $\bar{S} + \bar{\nabla}\bar{\nabla}l = H\bar{g}_{yx}$, where we have defined $H = f\triangle f + (q-1)||f_e||^2 - ff^a k_a$ and that $\partial_x P = 0$, meaning P is a constant on F. Thus, we have

Theorem 4.4. *Let $M = B \times_f F$ be a 3-dimensional warped product manifold that admits a conformal gradient Ricci soliton with $(\tilde{h}, \lambda, p)$. Suppose $\tilde{h} = k + l$ for some functions k on B and l on F, respectively, then either M becomes a product manifold of (almost) gradient Ricci soliton B with (k, A) and F with $\bar{S} + \bar{\nabla}\bar{\nabla}l = 0$, or F is an Einstein manifold.*

5. Conformal gradient Ricci soliton in the twisted product manifold

Let the $M = B \times_f F$ be a twisted product manifold that admits a conformal gradient Ricci soliton with $(\tilde{h}, \tilde{\lambda}, p)$. Then the following equations hold:

$$\begin{aligned} 2\nabla_b\tilde{h}_a + 2(S_{ab} - \tfrac{q}{f}\nabla_b f_a) &= \tilde{K} g_{ba}, \\ \partial_a\tilde{h}_x - \tfrac{f_a}{f}\tilde{h}_x - (q-1)(\tfrac{1}{f}\partial_a f_x - \tfrac{1}{f^2} f_a f_x) &= 0, \\ 2[\bar{\nabla}_y\tilde{h}_x - \tfrac{1}{f}(f_y\tilde{h}_x + f_x\tilde{h}_y - f^z\tilde{h}_z\bar{g}_{yx}) + ff^c\tilde{h}_c\bar{g}_{yx}] + 2[\bar{S}_{yx} - f(\triangle f)\bar{g}_{yx} - (q-1)||f_e||^2\bar{g}_{yx} & \\ - \tfrac{\bar{\triangle} f}{f}\bar{g}_{yx} - \tfrac{(q-2)}{f}\bar{\nabla}_y f_x + \tfrac{2(q-2)}{f^2} f_y f_x - \tfrac{(q-3)}{f^2}||f_w||^2\bar{g}_{yx}] &= \tilde{K} f^2\bar{g}_{yx}, \end{aligned} \tag{17}$$

where we have defined $\tilde{K} = 2\lambda - (p + \frac{2}{n+q})$, $f_x = \frac{\partial f}{\partial u_x}$, and $\bar{\triangle} f$ is a Laplacian of f with respect to $\bar{g}$.

If the potential function $\tilde{h}$ depends only on B, then $\tilde{h}_x = 0$. Since $\frac{1}{f}\partial_a f_x - \frac{1}{f^2} f_a f_x = \partial_a\partial_x(lnf)$, we derive $(q-1)\partial_a\partial_x(lnf) = 0$ from the second equation in (17). Hence, f can be represented as $f = f^*\bar{f}$ for some functions f^* and $\bar{f}$ on B and F, respectively. From the first equation in (17), we observe that $\tilde{K}$ depends only on B and

$$S_{ab} + \nabla_b\tilde{h}_a = \frac{\tilde{K}}{2} g_{ab} + \frac{q}{f^*}\nabla_b f_a{}^*. \tag{18}$$

Moreover, we find that

$$\bar{S}_{yx} = [f(\triangle f) + (q-1)||f_e||^2 + \frac{\bar{\triangle} f}{f} + \frac{(q-3)}{f^2}||f_w||^2 + \frac{\tilde{K}}{2} f^2 - ff^c\tilde{h}_c]\bar{g}_{yx} + \frac{(q-2)}{\bar{f}}\bar{\nabla}_y\bar{f}_x - \frac{2(q-2)}{\bar{f}^2}\bar{f}_y\bar{f}_x. \tag{19}$$

If we express $f(\triangle f) + (q-1)||f_e||^2 + \frac{\bar{\triangle} f}{f} + \frac{(q-3)}{f^2}||f_w||^2 + \frac{\tilde{K}}{2} f^2 - ff^c\tilde{h}_c = W$, then we see that W depends only on F. Moreover if $q = 2$ and W is a constant, then F becomes an Einstein manifold. Thus, we have

Theorem 5.1. *Suppose that the twisted product manifold $M = B \times_f F$ admits a conformal gradient Ricci soliton with $(\tilde{h}, \lambda, p)$, where $\tilde{h}$ depends only on B. Then:*
(a) The quantity $\tilde{K}$ depends only on B.
(b) the warping function f can be expressed as $f = f^* \bar{f}$ for some function f^* on B and $\bar{f}$ on F, respectively.
(c) the Ricci tensor S on B and $\bar{S}$ on F satisfy the forms given in equations (18) and (19).
(d) if $q = dimF = 2$ and the quantity W is a constant, then F is an Einstein manifold.

If the potential $\tilde{h}$ is a concircular scalar field in the twisted product manifold admitting a conformal gradient Ricci soliton with $(\tilde{h}, \lambda, p)$, then $\tilde{\nabla}_j \tilde{\nabla}_i \tilde{h} = \phi \tilde{g}_{ij}$ for some scalar field ϕ. In this case, we see that $\tilde{S} = A\tilde{g}$, from equation (3), where we have set $A = \lambda - \frac{p}{2} - \frac{1}{n+q} - \phi$. Hence, M becomes an Einstein manifold if $n + q = dimM \geq 3$.

Since $0 = \tilde{\nabla}_a \tilde{h}_x = \partial_a \tilde{h}_x - \frac{f_a \tilde{h}_x}{f}$, we get $0 = -(q-1)(\frac{1}{f}\partial_a f_x - \frac{1}{f^2} f_a f_x) = -(q-1)\partial_a \partial_x (lnf)$ from the second equation of (15). Hence, f can be represented by $f = f^* \bar{f}$ for some functions $f^* and \bar{f}$ on B and F respectively. Thus, we have

Theorem 5.2. *Let $M = B \times_f F$ be a twisted product manifold admitting a conformal gradient Ricci soliton with $(\tilde{h}, \lambda, p)$, where $\tilde{h}$ is a concircular scalar field. Then M becomes Einstein manifold if $dimM \geq 3$,and f can be represented by $f = f^* \bar{f}$ for some functions f^* and $\bar{f}$ on B and F respectively.*

If the mixed component of Ricci curvature $\tilde{S}(A, X) = 0$, or if the manifold M is conformally flat in a twisted product manifold M, then M becomes a warped manifold. The warping function f can be represented as $f = f^* \bar{f}$ for some functions f^* and $\bar{f}$ on B and F, respectively (see [1],[16]), where A and X are vector fields on B and F, respectively. In [16], the authors refer to M as mixed Ricci- flat if $\tilde{S}(A, X) = 0$.
If $f = f^* \bar{f}$, then we obtain $f \partial_a f_x - f_a f_x = 0$. Hence, if $\tilde{h} = k + l$ and $f = f^* \bar{f}$, then the equation (17) becomes

$$\begin{aligned} 2\nabla_b k_a + 2(S_{ab} - \tfrac{q}{f^*}\nabla_b f^*_a) &= \tilde{K} g_{ba}, \\ f^*_a l_x &= 0, \\ \bar{\nabla}_y l_x - \tfrac{1}{\bar{f}}(\bar{f}_y l_x + \bar{f}_x l_y) + \bar{S}_{yx} + \tfrac{2(q-2)}{f^2} f_y f_x - \tfrac{(q-2)}{f}\bar{\nabla}_y f_x & \\ +[\tfrac{\bar{f}^z}{\bar{f}} l_z + f^* f^{*d} \bar{f}^2 k_d - f(\triangle f) - \tfrac{\bar{\triangle} f}{f} - (q-1)\|f_e\|^2 - \tfrac{(q-3)}{f^2}\|f_w\|^2 - \tfrac{\tilde{K}}{2} f^2]\bar{g}_{yx} &= 0 \end{aligned} \tag{20}$$

From the first equation of (20), we see that $\tilde{K}$ depends only on B, and from the second equation of (13), we obtain either $f^*_a = 0$ or $l_x = 0$. If the case of $f^*_a = 0$, we see that f depends only F and B becomes an almost gradient Ricci soliton with $(k, \frac{\tilde{K}}{2})$ from the first equation of (20). In the second case of $l_x = 0$, the third equation of (20) becomes

$$\bar{S}_{yx} - \frac{(q-2)}{\bar{f}}\bar{\nabla}_y \bar{f}_x + \frac{2(q-2)}{\bar{f}^2}\bar{f}_y \bar{f}_x = \frac{1}{2}[\tilde{K} f^2 + \frac{2(q-3)}{\bar{f}^2}\|\bar{f}_w\|^2 + \frac{2\bar{\triangle}\bar{f}}{\bar{f}} + 2f(\triangle f) + 2(q-1)\|f_e\|^2 - 2f f^c k_c]\bar{g}_{yx} \tag{21}$$

Since $\bar{\nabla}_y \bar{\nabla}_x (\frac{1}{\bar{f}}) = \frac{1}{\bar{f}^3}(2\bar{f}_x \bar{f}_y - \bar{f}\bar{\nabla}_y \bar{f}_x)$, we get

$$\bar{S}_{yx} + (q-2)\bar{f}\bar{\nabla}_y \bar{\nabla}_x (\tfrac{1}{\bar{f}}) + \tfrac{1}{2}[\tilde{K} f^2 + \tfrac{2(q-3)}{f^2}\|\bar{f}_w\|^2 + \tfrac{2\bar{\triangle}\bar{f}}{\bar{f}} + 2f(\triangle f) + 2(q-1)\|f_e\|^2 - 2f f^c k_c]\bar{g}_{yx} = 0.$$

Thus we have

Theorem 5.3. *Let the twisted product manifold $M = B \times_f F$ admits a conformal gradient Ricci soliton with $(\tilde{h}, \lambda, p)$. Suppose M is either conformally flat or mixed Ricci-flat and the potential function $\tilde{h}$ is given by $\tilde{h} = k + l$ for some functions k on B and l on F respectively, then:*
(a) The quantity $\tilde{K}$ depends only on B.

(b) Either B is an almost gradient Ricci soliton with $(k, \frac{\tilde{K}}{2})$, or $\bar{S}$ on F satifies the relation $\bar{S} + \bar{f}\bar{\nabla}\bar{\nabla}(\frac{q-2}{\bar{f}}) + A\bar{g} = 0$, where A is defined as $A = \tilde{K}f^2 + \frac{2(q-3)}{f^2}||\bar{f}_w||^2 + \frac{2\bar{\triangle}\bar{f}}{\bar{f}} + 2f(\triangle f) + 2(q-1)||f_e||^2 - 2ff^c k_c$. We see that the quantity A depends only on F. In this case, if $q = dimF = 2$ and A is constant on F, then F is an Einstein manifold.

So far, we have considered the case the dimension $M \geq 4$. We now turn to the case where the dimension of M is equal to 3. This will enable us to know the geometric characterizations of each space in all dimensions. Let us consider the 3-dimensional twisted product manifold $M = B \times_f F$ that admits a conformal gradient Ricci soliton with $(\tilde{h}, \lambda, p)$. Since the conformal curvature tensor C vanishes identically in the 3-dimensional Riemannian manifold, we obtain

$$\begin{aligned} \tilde{R}_{kji}{}^h &= (\tilde{S}_{ji}\delta_k{}^h - \tilde{S}_{ki}\delta_j{}^h + \tilde{g}_{ji}\tilde{S}_k{}^h - \tilde{g}_{ki}\tilde{S}_j{}^h) + \tfrac{\tilde{r}}{2}(\tilde{g}_{ki}\delta_j{}^h - \tilde{g}_{ji}\delta_k{}^h), \\ \tilde{S}_{ji} + \tilde{\nabla}_j\tilde{h}_i &= U\tilde{g}_{ji}, \\ \tilde{r} &= -\tilde{\Delta}\tilde{h} + 3A, \end{aligned} \tag{22}$$

where we have set $A = \lambda - (\frac{p}{2} + \frac{1}{3})$, $U = 4A - \tilde{\Delta}\tilde{h} - \tilde{r}$, and $\tilde{\Delta}\tilde{h}$ is a Laplacian of $\tilde{h}$ with respect to $\tilde{g}$. From the second and third equations of (22), we see that M becomes an almost gradient Ricci soliton and $U = A$, respectively. Moreover, we obtain

$$\begin{aligned} S_{ab} + \nabla_b\tilde{h}_a &= \tfrac{q}{f}\nabla_b f_a + Ag_{ab}, \\ -(q-1)(\tfrac{1}{f}\partial_a f_x - \tfrac{1}{f^2}f_a f_x) + \partial_b\tilde{h}_x - \tfrac{f_b}{f}\tilde{h}_x &= 0, \\ \bar{S}_{yx} + \bar{\nabla}_y\tilde{h}_x - \tfrac{1}{f}f_y\tilde{h}_x - \tfrac{1}{f}f_x\tilde{h}_y - \tfrac{q-2}{f}\bar{\nabla}_y f_x + \tfrac{2(q-2)}{f^2}f_y f_x &= Q\bar{g}_{yx}, \end{aligned} \tag{23}$$

where $Q = Af^2 + f(\triangle f) + \frac{1}{f}(\bar{\Delta}f) + (q-1)||f_e||^2 + \frac{q-3}{f^2}||f_x||^2 - ff^c\tilde{h}_c - \frac{f^z\tilde{h}_z}{f}$.

If the potential function $\tilde{h}$ depends only on F, then $\tilde{h}_a = 0$ and that $S = \frac{q}{f}\nabla\nabla f + Ag$ from the first equation of (23).

If the potential function $\tilde{h}$ depends only on B, then $\tilde{h}_x = 0$. Since $\frac{1}{f}\partial_a f_x - \frac{1}{f^2}f_a f_x = \partial_a\partial_x(lnf)$, we obtain $\partial_a\partial_x(lnf) = 0$ from the second equation of (23) when $q \neq 1$. Therefore, f can be represented as $f = f^*\bar{f}$ for some functions f^* and $\bar{f}$ on B and F, respectively. From the first equation of (23), we see that A depends only on B and

$$S_{ab} + \nabla_b\tilde{h}_a = Ag_{ab} + \frac{q}{f^*}\nabla_b f_a{}^*. \tag{24}$$

Moreover, we obtain

$$\bar{S}_{yx} = [Af^2 + f(\triangle f) + \frac{1}{f}(\bar{\Delta}f) + (q-1)||f_e||^2 + \frac{q-3}{f^2}||f_x||^2 - ff^c\tilde{h}_c - f^z\tilde{h}_z]\bar{g}_{yx} + \frac{(q-2)}{\bar{f}}\bar{\nabla}_y\bar{f}_x - \frac{2(q-2)}{\bar{f}^2}\bar{f}_y\bar{f}_x. \tag{25}$$

If we put $Af^2 + f(\triangle f) + \frac{1}{f}(\bar{\Delta}f) + (q-1)||f_e||^2 + \frac{q-3}{f^2}||f_x||^2 - ff^c\tilde{h}_c - f^z\tilde{h}_z = V$ and if V is a constant on F, then F becomes an Einstein manifold, and that it is a space of constant curvature when $q = 2$. Thus, we have

Theorem 5.4. *Let $M = B \times_f F$be a 3-dimensional twisted product manifold that admits a conformal gradient Ricci soliton with $(\tilde{h}, \lambda, p)$. Then:*

(a) If $\tilde{h}_a = 0$, then the Ricci tensor S on B takes the form $S = \frac{q}{f}\nabla\nabla f + Ag$.

(b) If $\tilde{h}_x = 0$, then the warping function f can be expressed as $f = f^*\bar{f}$ for some functions f^* and $\bar{f}$ on B and F, respectively, when $q \neq 1$. In this case, if a certain function V defined on F is constant, then F becomes an Einstein manifold and is a space of constant curvature when $q = 2$.

References

[1] Y. Agaoka and B. H. Kim, *On conformally flat twisted product manifolds*, Mem. Fac. Integrated Arts and Sci., Hiroshima Univ., Ser IV, **23** (1997), 1-7.

[2] N. Basu and A. Bhattacharyya, *Conformal Ricci soliton in Kenmotsu manifold*, Global Journal of Advanced Research on Classical and Modern Geometries, **4** (2015), 15-21.

[3] D. E. Blair, *Riemannian geometry of contact and symplectic manifolds*, Birkhauser, Boston, 2002

[4] B. Y. Chen, *Some results on concircular vector fields and their applications to Ricci solitons*, Bull. Korean Math. Soc., **52** (2015), 1535-1547.

[5] J. Das, K. Halder and A. Bhattacharyya, *Conformal Yamabe solitons and conformal quasi-Yamabe solitons on hyperbolic Sasakian manifolds*, Ganita, **73** (2023), 75-88.

[6] U. C. De, A. Sardar and K. De, *Ricci-Yamabe solitons and 3-dimensional Riemannian manifolds* , Turkish Journal of Mathematics, **46** (2022) , 1078-1088.

[7] D. Dey and P. Majhi, *Almost Kenmotsu metric as a conformal Ricci soliton, Conformal geometry and dynamics* , An electronic Journal of the American Mathematical society, **23** (2019), 105-116.

[8] S. Dey, *Conformal Ricci soliton and almost conformal Ricci soliton in paracontact geometry*, International Journal of geometric methods in modern physics, **20** (2023), 2350041.

[9] S. Dey and S. Uddin, *Conformal η-Ricci almost solitons of Kenmotsu manifolds* , International Journal of geometric methods in modern physics, **19** (2022), 105-116.

[10] A. E. Fischer, *An introduction to conformal Ricci flow*, Class. Quantum Grav. **21** (2004), 171-218.

[11] R. S. Hamilton , *The Ricci flow on surfaces*, Mathematics and general relativity(Santa Cruz, CA, 1986), 237-262, Contemporary Mathematics, **71**, American Mathematical Society, (1988).

[12] K. Kenmotsu, *A class of almost contact Riemannian manifolds*, Tohoku Math. J., **24** (1972), 93-103.

[13] M. Khatri, C. Zosangzuala and J. P. Singh, *Isometries on almost Ricci-Yamabe solitons*, Arab. J. Math., **12** (2023), 127-138.

[14] B. H. Kim, J. H. Choi and S. D. Lee, *On almost generalized gradient Ricci-Yamabe soliton*, Filomat, **38:11** (2024), 3825-3827.

[15] B. H. Kim, S. D. Lee, J. H. Choi, and Y. O. Lee, *On warped product spaces with a certain Ricci condition*, Bull. Korean Math. Soc., **50** (2013), 1683-1691.

[16] M. F. Lopez, E. G. Rio, D. N. Kupeli and B. Unal, *A curvature condition for a twisted to be a warped product*, Manuscripta Math., **106** (2001), 213-217.

[17] S. Sarkar, *A study of conformal almost Ricci solitons on Kenmotsu manifolds*, International Journal of geometric methods in modern physics, **20** (2023), 1-16.

[18] P. Zhang, Y. Li, S. Roy, S. Dey and A. Bhattacharyya, *Geometrical structure in a perfect fluid space time with conformal Ricci-Yamabe soliton*, Symmetry, **14(3)** (2002), 594.