



Some types hypersurfaces of meta-metallic Riemannian manifolds

Selcen Yüksel Perktaş^{a,*}, Şerife Nur Bozdağ^b, Feyza Esra Erdoğan^b

^aDepartment of Mathematics, Faculty of Arts and Science, Adıyaman University, Adıyaman 02040, Turkey

^bDepartment of Mathematics, Faculty of Science, Ege University, İzmir 35100, Turkey

Abstract. In this paper, we study submanifolds within meta-metallic Riemannian manifolds, focusing specifically on hypersurfaces. We present key properties of the structures induced on these hypersurfaces by the meta-metallic Riemannian structure of the ambient manifold. Additionally, we provide characterizations for both invariant and non-invariant hypersurfaces within this class of manifolds.

1. Introduction

Manifolds are used to solve various problems in natural and engineering sciences. Additionally, they are a popular subject as they contribute to the advancement of these sciences and open up new application areas. Manifolds equipped with differential geometric structures possess rich geometric properties. Various structures in Riemannian (and semi-Riemannian) manifolds, such as almost complex, almost product, almost paracontact, and almost contact structures, provide a fundamental basis for studying the differential and geometric properties of submanifolds.

Hypersurface studies play a crucial role in mathematics and numerous scientific disciplines, serving as a bridge between theoretical geometry and practical applications. In differential geometry, key concepts such as curvature, tangents, and normal vectors provide essential insights into both the intrinsic and extrinsic properties of high-dimensional spaces. Beyond pure mathematics, the study of hypersurfaces has far-reaching applications in physics, engineering, and computer science, offering deeper understanding and innovative solutions to fundamental structural problems.

Manifolds with polynomial structures defined by constant coefficients allow the formulation of many results in classical algebra and geometry through tools such as tensor fields and 1-forms. Yano [13] introduced an f -structure, a generalization of complex and contact manifolds, while Goldberg and Yano [6] further developed this concept, introducing the idea of polynomial structures on manifolds. Crasmareanu

2020 *Mathematics Subject Classification.* Primary 53D15, 53C25; Secondary 58E20, 53C43.

Keywords. Metallic Riemannian manifold, meta-metallic structure, meta-metallic-Chi ratio, invariant hypersurface, non-invariant hypersurface.

Received: 18 April 2025; Revised: 23 June 2025; Accepted: 22 July 2025

Communicated by Mića S. Stanković

Research supported by The Scientific and Technological Research Council of Türkiye (TÜBİTAK) with project number 123F195. For this reason, the authors would like to thank TÜBİTAK.

* Corresponding author: Selcen Yüksel Perktaş

Email addresses: sperktaş@adiyaman.edu.tr (Selcen Yüksel Perktaş), serife.nur.yalcin@ege.edu.tr (Şerife Nur Bozdağ), feyza.esra.erdogan@ege.edu.tr (Feyza Esra Erdoğan)

ORCID iDs: <https://orcid.org/0000-0002-8848-0621> (Selcen Yüksel Perktaş), <https://orcid.org/0000-0002-9651-7834> (Şerife Nur Bozdağ), <https://orcid.org/0000-0003-0568-7510> (Feyza Esra Erdoğan)

and Hretcanu [3] defined the Golden structure and developed the theory of Golden manifolds. Similarly, Özkan et al. introduced silver and bronze structures in [9, 10].

Metallic manifolds are a generalization of Golden manifolds, encompassing broader structural variations. Crasmareanu and Hretcanu [7] defined the metallic structure on Riemannian manifolds using a $(1, 1)$ tensor field J and the structure polynomial $Q(X) = X^2 - pX - qI$. Riemannian manifolds have significant applications in various fields. For instance, Bekiros and Kouloumpou [2] developed the SBDIEM model, which utilizes Riemannian manifolds and stochastic differential equations to model COVID and similar epidemics.

Şahin [11] introduced almost poly Norden manifolds, which include Norden manifolds and Euclidean spaces. Later, Şahin and Şahin [12] defined meta-Golden Riemannian manifolds inspired by the meta-Golden ratio. Following this, Erdoğan et al. [4] studied the hypersurfaces of meta-Golden Riemannian manifolds.

Finally, Erdoğan et al. [5] introduced meta-metallic Riemannian manifolds, inspired by the meta-metallic-chi ratio and metallic manifolds. These manifolds provide a broader framework, encompassing meta-Golden Riemannian manifolds and offering more precise results compared to metallic structures.

This paper consists of four sections. The second section presents the fundamental definitions used throughout the paper. The third section provides essential information on meta-metallic Riemannian manifolds. The fourth section discusses the hypersurfaces of these manifolds in detail, obtaining characterizations for invariant and non-invariant hypersurfaces. Finally, two different examples are provided.

2. Preliminaries

Let p and q be positive integers. The positive solution of equation $x^2 - px - q = 0$ is considered a member of the family of metallic ratios. The solution set is represented as

$$\iota = \frac{p \mp \sqrt{p^2 + 4q}}{2}$$

and these members are also referred to as the (p, q) -metallic numbers. If we substitute $p = q = 1$ into the positive root of the metallic ratio above, we obtain $\dot{\phi} = \frac{1+\sqrt{5}}{2}$, which gives the Golden Ratio. Until 2019, it was claimed that the logarithmic spiral satisfied the Golden Ratio. However, Barlett ([1]) demonstrated that this argument was incorrect and proved that an important class of logarithmic spirals perfectly satisfies the meta-Golden-Chi ratio. Building on this, Şahin and Şahin defined a new class of manifolds using this ratio. From Figure 1 in the article ([12]), the authors obtained

$$\bar{\chi} = \frac{1}{\dot{\phi}} + \frac{1}{\bar{\chi}}$$

which leads to the proposition $\bar{\chi}^2 - \frac{1}{\dot{\phi}}\bar{\chi} - 1 = 0$. Thus, the roots are found as

$$\frac{\frac{1}{\dot{\phi}} \mp \sqrt{4 + \frac{1}{\dot{\phi}^2}}}{2}.$$

The correlation between the meta-Golden Chi ratio $\bar{\chi}$ and continued fractions was established by Hylebrouck in 2014. If we define the positive and negative roots as $\bar{\chi} = \frac{1}{2\dot{\phi}} + \frac{\sqrt{4 + \frac{1}{\dot{\phi}^2}}}{2}$ and $\bar{\bar{\chi}} = \frac{1}{2\dot{\phi}} - \frac{\sqrt{4 + \frac{1}{\dot{\phi}^2}}}{2}$, then

$$\bar{\bar{\chi}} = \frac{1}{\dot{\phi}} - \bar{\chi},$$

$$\dot{\phi}\bar{\chi}^2 = \dot{\phi} + \bar{\chi},$$

and

$$\dot{\phi}\bar{\chi}^2 = \dot{\phi} + \bar{\chi}$$

are obtained, where $\dot{\phi}$ is the Golden Ratio ([8]).

Hretcanu and Crasmareanu [7] obtained that $\check{\mathfrak{J}}$ is a metallic structure, which is a $(1, 1)$ tensor field on a manifold $\mathfrak{M}$, if

$$\check{\mathfrak{J}}^2\mathfrak{X} = p\check{\mathfrak{J}}\mathfrak{X} + q\mathfrak{X} \quad (1)$$

is satisfied for $\mathfrak{X} \in \mathfrak{X}(\mathfrak{M})$ where p, q are positive integers. Hence, let $\check{\mathfrak{g}}$ be the Riemannian metric on $\mathfrak{M}$ and then $(\check{\mathfrak{g}}, \check{\mathfrak{J}})$ is called a metallic Riemannian structure if

$$\check{\mathfrak{g}}(\check{\mathfrak{J}}\mathfrak{X}, \mathfrak{Y}) = \check{\mathfrak{g}}(\mathfrak{X}, \check{\mathfrak{J}}\mathfrak{Y}), \quad (2)$$

for $\mathfrak{X}, \mathfrak{Y} \in \mathfrak{X}(\mathfrak{M})$. Therefore $(\mathfrak{M}, \check{\mathfrak{g}}, \check{\mathfrak{J}})$ is called an almost metallic Riemannian manifold. From (2), we found that

$$\check{\mathfrak{g}}(\check{\mathfrak{J}}\mathfrak{X}, \check{\mathfrak{J}}\mathfrak{Y}) = p\check{\mathfrak{g}}(\mathfrak{X}, \check{\mathfrak{J}}\mathfrak{Y}) + q\check{\mathfrak{g}}(\mathfrak{X}, \mathfrak{Y}). \quad (3)$$

With an approach, similar to the method used to obtain the meta-golden chi ratio in [12], the meta-metallic-chi ratio, which we will use throughout this article, is structured as follows;

$$\dot{\chi} = \frac{q}{\check{\chi}} + \frac{p}{\dot{\chi}},$$

where $\check{\chi}$ denotes the metallic ratio. The roots of this equation are found as

$$\frac{\frac{q}{\check{\chi}} \mp \sqrt{4p + \frac{q^2}{\check{\chi}^2}}}{2}.$$

The correlation between the meta-metallic Chi ratio $\dot{\chi}$ and continued fractions was found in [8].

We define the positive root by

$$\dot{\chi} = \frac{\frac{q}{\check{\chi}} + \sqrt{4p + \frac{q^2}{\check{\chi}^2}}}{2},$$

which is called the silver mean of inverse of metallic mean and the negative root by

$$\ddot{\chi} = \frac{\frac{q}{\check{\chi}} - \sqrt{4p + \frac{q^2}{\check{\chi}^2}}}{2}.$$

3. Meta-Metallic Manifolds

In this section we give basic definitions for a new type of manifold called a meta-metallic manifold which can be considered as a generalization of meta-Golden manifolds, [12].

Definition 3.1. Let $\check{\beta}$ be a $(1, 1)$ tensor field on almost metallic manifold $(\mathfrak{M}, \check{\mathfrak{J}})$ which satisfies

$$\check{\mathfrak{J}}\check{\beta}^2\mathfrak{X} = p\check{\mathfrak{J}}\mathfrak{X} + q\check{\beta}\mathfrak{X}, \quad (4)$$

for any $\mathfrak{X} \in \mathfrak{X}(\mathfrak{M})$, then $\check{\beta}$ is called an almost meta-metallic structure and $(\mathfrak{M}, \check{\mathfrak{J}}, \check{\beta})$ is called an almost meta-metallic manifold, [5].

Theorem 3.2. Let $\check{\beta}$ be a $(1, 1)$ -tensor field on almost metallic manifold $(\mathfrak{M}, \check{\mathfrak{J}})$. In that case, $\check{\beta}$ is almost meta-metallic structure iff

$$\check{\beta}^2 = \check{\mathfrak{J}}\check{\beta} - p\check{\beta} + pI, \quad (5)$$

where I is the identity map, [5].

Definition 3.3. Let $\check{\beta}$ is an almost meta-metallic structure on almost metallic manifold $(\mathfrak{M}, \check{\mathfrak{J}}, \check{\mathfrak{g}})$. If $\check{\beta}$ is compatible with $\check{\mathfrak{g}}$ on $\mathfrak{M}$, namely

$$\check{\mathfrak{g}}(\check{\beta}\mathfrak{X}, \mathfrak{Y}) = \check{\mathfrak{g}}(\mathfrak{X}, \check{\beta}\mathfrak{Y}), \quad (6)$$

or equivalently

$$\check{\mathfrak{g}}(\check{\beta}\mathfrak{X}, \check{\beta}\mathfrak{Y}) = \check{\mathfrak{g}}(\check{\mathfrak{J}}\mathfrak{X}, \check{\beta}\mathfrak{Y}) - p\check{\mathfrak{g}}(\mathfrak{X}, \check{\beta}\mathfrak{Y}) + p\check{\mathfrak{g}}(\mathfrak{X}, \mathfrak{Y}), \quad (7)$$

for any $\mathfrak{X}, \mathfrak{Y} \in \Gamma(T\mathfrak{M})$, then $(\mathfrak{M}, \check{\mathfrak{J}}, \check{\beta}, \check{\mathfrak{g}})$ is called almost meta-metallic Riemannian manifold.

Theorem 3.4. Let $(\mathfrak{M}, \check{\mathfrak{J}}, \check{\beta}, \check{\mathfrak{g}})$ be an almost meta-metallic Riemannian manifold. Then $\check{\beta}$ is integrable if Codazzi-like equation $(\nabla_{\check{\beta}\mathfrak{X}}\check{\beta})\mathfrak{Y} - \check{\beta}(\nabla_{\mathfrak{X}}\check{\beta})\mathfrak{Y} = 0$ is ensured for any $\mathfrak{X}, \mathfrak{Y} \in \Gamma(T\mathfrak{M})$, [5].

Theorem 3.5. Let $(\mathfrak{M}, \check{\mathfrak{J}}, \check{\beta}, \check{\mathfrak{g}})$ be an almost meta-metallic Riemannian manifold. If $\nabla\check{\beta} = 0$ then $\nabla\check{\mathfrak{J}} = 0$, [5].

Note that the Nijenhuis tensor field of $\check{\beta}$ is defined by

$$N_{\check{\beta}}(\mathfrak{X}, \mathfrak{Y}) = \check{\beta}^2[\mathfrak{X}, \mathfrak{Y}] + [\check{\beta}\mathfrak{X}, \check{\beta}\mathfrak{Y}] - \check{\beta}[\mathfrak{X}, \check{\beta}\mathfrak{Y}] - \check{\beta}[\check{\beta}\mathfrak{X}, \mathfrak{Y}],$$

for any $\mathfrak{X}, \mathfrak{Y} \in \Gamma(T\mathfrak{M})$. If the Nijenhuis tensor field $N_{\check{\beta}}$ vanishes then $\check{\beta}$ is integrable. In this case $(\mathfrak{M}, \check{\mathfrak{J}}, \check{\beta}, \check{\mathfrak{g}})$ is called a meta-metallic Riemannian manifold.

Corollary 3.6. Let $(\mathfrak{M}, \check{\mathfrak{J}}, \check{\beta}, \check{\mathfrak{g}})$ be an almost meta-metallic Riemannian manifold. If $\check{\nabla}\check{\beta} = 0$, then the meta-metallic structure is integrable and so $(\mathfrak{M}, \check{\mathfrak{J}}, \check{\beta}, \check{\mathfrak{g}})$ is a meta-metallic Riemannian manifold, [5].

Proposition 3.7. An almost meta-metallic structure $\check{\beta}$ is an isomorphism on $T_p\mathfrak{M}$, for each $p \in \mathfrak{M}$, [5].

Proposition 3.8. Let $(\mathfrak{M}, \check{\mathfrak{J}}, \check{\beta}, \check{\mathfrak{g}})$ be an almost meta-metallic Riemannian manifold. Then

- If κ is the eigenvalue of the metallic structure $\check{\mathfrak{J}}$, then $\check{\chi}$ and $\check{\bar{\chi}}$ are the eigenvalues of the meta-metallic structure $\check{\beta}$.
- If $p - \kappa$ is the eigenvalue of the metallic structure $\check{\mathfrak{J}}$ then

$$G = \frac{\frac{q}{p-\kappa} + \sqrt{4p + \frac{q^2}{(p-\kappa)^2}}}{2}$$

and

$$\check{G} = \frac{\frac{q}{p-\kappa} - \sqrt{4p + \frac{q^2}{(p-\kappa)^2}}}{2}$$

are the eigenvalues of the meta-metallic structure $\check{\beta}$, [5].

4. Hypersurfaces of Meta-Metallic Riemannian Manifolds

Now, let us examine the conditions satisfied by the structures induced on the hypersurface of an almost meta-metallic Riemannian manifold. However, in this study, we analyze the structure induced on the tangent bundle of the hypersurface without considering it as a metallic structure, by taking the induced structures as non-invariant.

Proposition 4.1. *Let $(\check{\mathfrak{M}}, \check{\beta}, \check{\mathfrak{J}}, \check{g})$ be an almost meta-metallic Riemannian manifold, $\mathfrak{M}$ be a hypersurface of $\check{\mathfrak{M}}$ and $\mathfrak{J}$ and β be the structures induced on the tangent bundle of the hypersurface from the metallic and meta-metallic structures of the ambient manifold, respectively. In this case,*

$$\Pi = (\mathfrak{J}, \beta, \mathfrak{U}, \mathfrak{V}, v, u, b, c, g)$$

is the induced structure on $\mathfrak{M}$ by the almost meta-metallic Riemannian structure $(\check{\mathfrak{J}}, \check{\beta}, \check{g})$ satisfies the followings;

1. $\mathfrak{J}^2\mathfrak{X} = p\mathfrak{J}\mathfrak{X} + q\mathfrak{X} - v(\mathfrak{X})\mathfrak{U}$,
2. $v(\mathfrak{J}\mathfrak{X}) = (p - b)v(\mathfrak{X})$,
3. $\mathfrak{J}\mathfrak{U} = (p - b)\mathfrak{U}$,
4. $v(\mathfrak{U}) = pb + q - b^2$,
5. $g(\mathfrak{J}\mathfrak{X}, \mathfrak{Y}) = g(\mathfrak{X}, \mathfrak{J}\mathfrak{Y}) \quad v(\mathfrak{X}) = g(\mathfrak{X}, \mathfrak{U})$,
6. $g(\mathfrak{J}\mathfrak{X}, \mathfrak{J}\mathfrak{Y}) = pg(\mathfrak{J}\mathfrak{X}, \mathfrak{Y}) + qg(\mathfrak{X}, \mathfrak{Y}) - v(\mathfrak{X})v(\mathfrak{Y})$,
7. $\beta^2\mathfrak{X} = \mathfrak{J}\beta\mathfrak{X} - p\beta\mathfrak{X} + p\mathfrak{X} + u(\mathfrak{X})(\mathfrak{U} - \mathfrak{V})$,
8. $v(\beta\mathfrak{X}) = (p - b + c)u(\mathfrak{X}) + u(\beta\mathfrak{X})$,
9. $\beta\mathfrak{V} = \mathfrak{J}\mathfrak{V} - p\mathfrak{V} + c(\mathfrak{U} - \mathfrak{V})$,
10. $u(\mathfrak{V}) = p + c(b - p - c) + v(\mathfrak{V})$,
11. $u(\mathfrak{X}) = g(\mathfrak{X}, \mathfrak{V})$,
12. $g(\beta\mathfrak{X}, \mathfrak{Y}) = g(\mathfrak{X}, \beta\mathfrak{Y})$,
13. $g(\beta\mathfrak{X}, \beta\mathfrak{Y}) = g(\mathfrak{J}\mathfrak{X}, \beta\mathfrak{Y}) - pg(\mathfrak{X}, \beta\mathfrak{Y}) + pg(\mathfrak{X}, \mathfrak{Y}) - u(\mathfrak{X})u(\mathfrak{Y}) + v(\mathfrak{X})u(\mathfrak{Y})$,

where $\mathfrak{X}, \mathfrak{Y} \in \Gamma(T\mathfrak{M})$; $b, c \in C^\infty(\check{\mathfrak{M}}, \mathbb{R})$; $g = \check{g}|_{\mathfrak{M}}$; $\mathfrak{U}, \mathfrak{V} \in \Gamma(T\mathfrak{M})$ and v, u are 1-forms.

Proof. Let $(\check{\mathfrak{M}}, \check{\mathfrak{J}}, \check{\beta}, \check{g})$ be an almost meta-metallic Riemannian manifold and $\mathfrak{M}$ be a hypersurface of $\check{\mathfrak{M}}$. For any $\mathfrak{X} \in \Gamma(T\mathfrak{M})$ and the local unit normal vector field $\mathfrak{N} \in \Gamma(T\mathfrak{M}^\perp)$, we write

$$\check{\mathfrak{J}}\mathfrak{X} = \mathfrak{J}\mathfrak{X} + v(\mathfrak{X})\mathfrak{N}, \tag{8}$$

$$\check{\mathfrak{J}}\mathfrak{N} = \mathfrak{U} + b\mathfrak{N}, \quad b \in C^\infty(\check{\mathfrak{M}}, \mathbb{R}) \tag{9}$$

where $\mathfrak{J}\mathfrak{X} \in \Gamma(T\mathfrak{M})$ and $\mathfrak{U} = (\check{\mathfrak{J}}\mathfrak{N})^\top$. Similarly,

$$\check{\beta}\mathfrak{X} = \beta\mathfrak{X} + u(\mathfrak{X})\mathfrak{N}, \tag{10}$$

$$\check{\beta}\mathfrak{N} = \mathfrak{V} + c\mathfrak{N}, \quad c \in C^\infty(\check{\mathfrak{M}}, \mathbb{R}) \tag{11}$$

where $\beta\mathfrak{X} \in \Gamma(T\mathfrak{M})$ and $\mathfrak{V} = (\check{\beta}\mathfrak{N})^\top$.

From (8),(9) and (2), we have

$$v(\mathfrak{X}) = \check{g}(\check{\mathfrak{J}}\mathfrak{X}, \mathfrak{N}) = \check{g}(\mathfrak{X}, \check{\mathfrak{J}}\mathfrak{N}) = g(\mathfrak{X}, \mathfrak{U}),$$

where g is the induced metric on $\mathcal{M}$ defined by $g = \check{g}|_{\mathcal{M}}$.

By applying $\check{\mathfrak{J}}$ to the both sides of (8), we obtain

$$\begin{aligned}\check{\mathfrak{J}}^2\mathfrak{X} &= \check{\mathfrak{J}}\mathfrak{J}\mathfrak{X} + v(\mathfrak{X})\check{\mathfrak{J}}\mathfrak{N}, \\ p\mathfrak{J}\mathfrak{X} + pv(\mathfrak{X})\mathfrak{N} + q\mathfrak{X} &= \check{\mathfrak{J}}^2\mathfrak{X} + v(\mathfrak{J}\mathfrak{X})\mathfrak{N} + v(\mathfrak{X})\mathfrak{U} + bv(\mathfrak{X})\mathfrak{N}.\end{aligned}$$

By equating the tangential and normal parts of the last equation,

$$\check{\mathfrak{J}}^2\mathfrak{X} = p\mathfrak{J}\mathfrak{X} + q\mathfrak{X} - v(\mathfrak{X})\mathfrak{U}$$

and

$$v(\mathfrak{J}\mathfrak{X}) = (p - b)v(\mathfrak{X})$$

are obtained.

Similarly, by applying $\check{\mathfrak{J}}$ to the both sides of (9),

$$\begin{aligned}\check{\mathfrak{J}}^2\mathfrak{N} &= \check{\mathfrak{J}}\mathfrak{U} + b\check{\mathfrak{J}}\mathfrak{N}, \\ p\check{\mathfrak{J}}\mathfrak{N} + q\mathfrak{N} &= \check{\mathfrak{J}}\mathfrak{U} + v(\mathfrak{U})\mathfrak{N} + b\mathfrak{U} + b^2\mathfrak{N}, \\ p\mathfrak{U} + pb\mathfrak{N} + q\mathfrak{N} &= \check{\mathfrak{J}}\mathfrak{U} + v(\mathfrak{U})\mathfrak{N} + b\mathfrak{U} + b^2\mathfrak{N}\end{aligned}$$

which implies

$$\check{\mathfrak{J}}\mathfrak{U} = (p - b)\mathfrak{U}$$

and

$$v(\mathfrak{U}) = q + pb - b^2.$$

Now by using (2), (3) and (8); we get

$$pg(\mathfrak{J}\mathfrak{X}, \mathfrak{Y}) + qg(\mathfrak{X}, \mathfrak{Y}) = g(\mathfrak{J}\mathfrak{X}, \check{\mathfrak{J}}\mathfrak{Y}) + v(\mathfrak{X})v(\mathfrak{Y}),$$

which implies

$$g(\mathfrak{J}\mathfrak{X}, \check{\mathfrak{J}}\mathfrak{Y}) = pg(\mathfrak{J}\mathfrak{X}, \mathfrak{Y}) + qg(\mathfrak{X}, \mathfrak{Y}) - v(\mathfrak{X})v(\mathfrak{Y}).$$

Also we have

$$\check{g}(\check{\mathfrak{J}}\mathfrak{X}, \mathfrak{Y}) = g(\mathfrak{J}\mathfrak{X}, \mathfrak{Y})$$

and

$$\check{g}(\mathfrak{X}, \check{\mathfrak{J}}\mathfrak{Y}) = g(\mathfrak{X}, \mathfrak{J}\mathfrak{Y})$$

which gives $g(\mathfrak{J}\mathfrak{X}, \mathfrak{Y}) = g(\mathfrak{X}, \mathfrak{J}\mathfrak{Y})$ via (2).

On the other hand by applying $\check{\beta}$ to the both sides of (10),

$$\begin{aligned}\check{\beta}^2\mathfrak{X} &= \check{\beta}\beta\mathfrak{X} + u(\mathfrak{X})\check{\beta}\mathfrak{N}, \\ \check{\mathfrak{J}}\check{\beta}\mathfrak{X} - p\check{\beta}\mathfrak{X} + p\mathfrak{X} &= \beta^2\mathfrak{X} + u(\beta\mathfrak{X})\mathfrak{N} + u(\mathfrak{X})\mathfrak{V} + cu(\mathfrak{X})\mathfrak{N},\end{aligned}$$

which implies

$$\check{\mathfrak{J}}\beta\mathfrak{X} + v(\beta\mathfrak{X})\mathfrak{N} + u(\mathfrak{X})\mathfrak{U} + bu(\mathfrak{X})\mathfrak{N} - p\beta\mathfrak{X} - pu(\mathfrak{X})\mathfrak{N} + p\mathfrak{X} = \beta^2\mathfrak{X} + u(\beta\mathfrak{X})\mathfrak{N} + u(\mathfrak{X})\mathfrak{V} + cu(\mathfrak{X})\mathfrak{N}, \quad (12)$$

via (8) and (9). By equating the tangential and normal components of (12), following equalities are obtained;

$$\begin{aligned}\beta^2\mathfrak{X} &= \check{\mathfrak{J}}\beta\mathfrak{X} - p\beta\mathfrak{X} + p\mathfrak{X} + u(\mathfrak{X})\mathfrak{U} - u(\mathfrak{X})\mathfrak{V}, \\ v(\beta\mathfrak{X}) &= (p - b + c)u(\mathfrak{X}) + u(\beta\mathfrak{X}).\end{aligned}$$

Applying $\check{\beta}$ to the both sides of (11), we get

$$\begin{aligned}\check{\beta}^2\mathfrak{N} &= \check{\beta}\mathfrak{V} + c\check{\beta}\mathfrak{N}, \\ \check{\mathfrak{J}}\check{\beta}\mathfrak{N} - p\check{\beta}\mathfrak{N} + p\mathfrak{N} &= \beta\mathfrak{V} + u(\mathfrak{V})\mathfrak{N} + c\mathfrak{V} + c^2\mathfrak{N}, \\ \check{\mathfrak{J}}\mathfrak{V} + v(\mathfrak{V})\mathfrak{N} + c\mathfrak{U} + bc\mathfrak{N} - p\mathfrak{V} - pc\mathfrak{N} + p\mathfrak{N} &= \beta\mathfrak{V} + u(\mathfrak{V})\mathfrak{N} + c\mathfrak{V} + c^2\mathfrak{N}\end{aligned}$$

which gives

$$\begin{aligned}\beta\mathbb{V} &= \mathfrak{I}\mathbb{V} - p\mathbb{V} + c\mathbb{U} - c\mathbb{V}, \\ u(\mathbb{V}) &= v(\mathbb{V}) + (bc - pc + p - c^2),\end{aligned}$$

by equating the the tangential and normal components.

Moreover, from (6), (10) and (11);

$$\begin{aligned}\check{g}(\beta\mathbb{X} + u(\mathbb{X})\mathbb{N}, \mathbb{Y}) &= \check{g}(\mathbb{X}, \beta\mathbb{Y} + u(\mathbb{Y})\mathbb{N}), \\ g(\beta\mathbb{X}, \mathbb{Y}) &= g(\mathbb{X}, \beta\mathbb{Y})\end{aligned}$$

and

$$\begin{aligned}\check{g}(\beta\mathbb{X} + u(\mathbb{X})\mathbb{N}, \mathbb{N}) &= \check{g}(\mathbb{X}, \mathbb{V} + c\mathbb{N}), \\ u(\mathbb{X}) &= g(\mathbb{X}, \mathbb{V}),\end{aligned}$$

are obtained. Then by using (7), we get

$$\begin{aligned}\check{g}(\check{\beta}\mathbb{X}, \check{\beta}\mathbb{Y}) &= \check{g}(\check{\mathfrak{I}}\mathbb{X}, \check{\beta}\mathbb{Y}) - p\check{g}(\mathbb{X}, \check{\beta}\mathbb{Y}) + p\check{g}(\mathbb{X}, \mathbb{Y}), \\ \check{g}(\beta\mathbb{X} + u(\mathbb{X})\mathbb{N}, \beta\mathbb{Y} + u(\mathbb{Y})\mathbb{N}) &= \check{g}(\mathfrak{I}\mathbb{X} + v(\mathbb{X})\mathbb{N}, \beta\mathbb{Y} + u(\mathbb{Y})\mathbb{N}) \\ &\quad - p\check{g}(\mathbb{X}, \beta\mathbb{Y} + u(\mathbb{Y})\mathbb{N}) + p\check{g}(\mathbb{X}, \mathbb{Y}), \\ g(\beta\mathbb{X}, \beta\mathbb{Y}) + u(\mathbb{X})u(\mathbb{Y}) &= g(\mathfrak{I}\mathbb{X}, \beta\mathbb{Y}) + v(\mathbb{X})u(\mathbb{Y}) - pg(\mathbb{X}, \beta\mathbb{Y}) + pg(\mathbb{X}, \mathbb{Y}),\end{aligned}$$

which gives

$$g(\beta\mathbb{X}, \beta\mathbb{Y}) = g(\mathfrak{I}\mathbb{X}, \beta\mathbb{Y}) - pg(\mathbb{X}, \beta\mathbb{Y}) + pg(\mathbb{X}, \mathbb{Y}) - u(\mathbb{X})u(\mathbb{Y}) + v(\mathbb{X})u(\mathbb{Y}).$$

□

Proposition 4.2. Let $\mathfrak{M}$ be a hypersurface of a meta-metallic Riemannian manifold $(\check{\mathfrak{M}}, \check{\mathfrak{I}}, \check{\beta}, \check{g})$ and $\mathfrak{I}$ and β be the structures induced on the tangent bundle of the hypersurface from the metallic and meta-metallic structures of the ambient manifold, respectively. Then the induced structure

$$\Pi = (\mathfrak{I}, \beta, \mathbb{U}, \mathbb{V}, v, u, b, c, g)$$

on $\mathfrak{M}$ satisfies the followings;

1. $(\nabla_{\mathbb{X}}\beta)\mathbb{Y} = g(A_{\mathbb{N}}\mathbb{X}, \mathbb{Y})\mathbb{V} + u(\mathbb{Y})A_{\mathbb{N}}\mathbb{X},$
2. $(\nabla_{\mathbb{X}}u)\mathbb{Y} = cg(A_{\mathbb{N}}\mathbb{X}, \mathbb{Y}) - g(A_{\mathbb{N}}\mathbb{X}, \beta\mathbb{Y}),$
3. $\nabla_{\mathbb{X}}\mathbb{V} = (c - \beta)A_{\mathbb{N}}\mathbb{X},$
4. $\mathbb{X}(c) = -2g(A_{\mathbb{N}}\mathbb{X}, \mathbb{V}),$
5. $(\nabla_{\mathbb{X}}\mathfrak{I})\mathbb{Y} = g(A_{\mathbb{N}}\mathbb{X}, \mathbb{Y})\mathbb{U} + v(\mathbb{Y})A_{\mathbb{N}}\mathbb{X},$
6. $(\nabla_{\mathbb{X}}v)\mathbb{Y} = bg(A_{\mathbb{N}}\mathbb{X}, \mathbb{Y}) - g(A_{\mathbb{N}}\mathbb{X}, \mathfrak{I}\mathbb{Y}),$
7. $\nabla_{\mathbb{X}}\mathbb{U} = (b - \mathfrak{I})A_{\mathbb{N}}\mathbb{X},$
8. $\mathbb{X}(b) = -2g(A_{\mathbb{N}}\mathbb{X}, \mathbb{U}),$

where $\mathbb{X}, \mathbb{Y} \in \Gamma(T\mathfrak{M})$; $b, c \in C^\infty(\check{\mathfrak{M}}, \mathbb{R})$; $g = \check{g}|_{\mathfrak{M}}$; $\mathbb{U}, \mathbb{V} \in \Gamma(T\mathfrak{M})$ and v, u are 1-forms and ∇ is the induced connection on $\mathfrak{M}$ from $\check{\mathfrak{M}}$.

Proof. For a hypersurface $\mathfrak{M}$ of $(\mathfrak{M}, \check{\beta}, \check{\mathfrak{J}}, \check{g})$, the Gauss and Weingarten formulas are given by

$$\check{\nabla}_{\mathfrak{X}} \mathfrak{Y} = \nabla_{\mathfrak{X}} \mathfrak{Y} + h(\mathfrak{X}, \mathfrak{Y}) \mathbb{N}, \quad \nabla_{\mathfrak{X}} \mathbb{N} = -A_{\mathbb{N}} \mathfrak{X}$$

for any $\mathfrak{X} \in \Gamma(T\mathfrak{M})$ and $\mathbb{N} \in \Gamma(T\mathfrak{M}^\perp)$, respectively. Here h denotes the second fundamental form of $\mathfrak{M}$; $A_{\mathbb{N}}$ is the shape operator in the direction of $\mathbb{N}$ given by $g(A_{\mathbb{N}} \mathfrak{X}, \mathfrak{Y}) = h(\mathfrak{X}, \mathfrak{Y})$ and ∇ is the induced connection on $\mathfrak{M}$.

Now assume that $\mathfrak{M}$ is a hypersurface of a meta-metallic Riemannian manifold $(\mathfrak{M}, \check{\mathfrak{J}}, \check{\beta}, \check{g})$. It is well known that $\check{\nabla} \check{\beta} = 0$, [4]. By using (10) and (11),

$$\check{\nabla}_{\mathfrak{X}} \check{\beta} \mathfrak{Y} = \check{\beta} \check{\nabla}_{\mathfrak{X}} \mathfrak{Y},$$

namely,

$$\nabla_{\mathfrak{X}} \check{\beta} \mathfrak{Y} + h(\mathfrak{X}, \check{\beta} \mathfrak{Y}) \mathbb{N} + \mathfrak{X}(u(\mathfrak{Y})) \mathbb{N} - u(\mathfrak{Y}) A_{\mathbb{N}} \mathfrak{X} = \check{\beta} \nabla_{\mathfrak{X}} \mathfrak{Y} + u(\nabla_{\mathfrak{X}} \mathfrak{Y}) \mathbb{N} + h(\mathfrak{X}, \mathfrak{Y}) \mathbb{W} + ch(\mathfrak{X}, \mathfrak{Y}) \mathbb{N} \quad (13)$$

is obtained, for any $\mathfrak{X}, \mathfrak{Y} \in \Gamma(T\mathfrak{M})$. If the normal and tangential parts of (13) equate to each other which implies that

$$(\nabla_{\mathfrak{X}} u) \mathfrak{Y} = ch(\mathfrak{X}, \mathfrak{Y}) - h(\mathfrak{X}, \check{\beta} \mathfrak{Y}),$$

and

$$(\nabla_{\mathfrak{X}} \check{\beta}) \mathfrak{Y} = g(A_{\mathbb{N}} \mathfrak{X}, \mathfrak{Y}) \mathbb{W} + u(\mathfrak{Y}) A_{\mathbb{N}} \mathfrak{X}.$$

Additionally, since $(\check{\nabla}_{\mathfrak{X}} \check{\beta}) \mathbb{N} = 0$, it follows

$$\begin{aligned} \check{\nabla}_{\mathfrak{X}} \check{\beta} \mathbb{N} &= \check{\beta} \check{\nabla}_{\mathfrak{X}} \mathbb{N}, \\ \nabla_{\mathfrak{X}} \mathbb{W} + h(\mathfrak{X}, \mathbb{W}) \mathbb{N} + \mathfrak{X}(c) \mathbb{N} - c A_{\mathbb{N}} \mathfrak{X} &= -\beta A_{\mathbb{N}} \mathfrak{X} - u(A_{\mathbb{N}} \mathfrak{X}) \mathbb{N}, \end{aligned}$$

via (10), (11) and Gauss-Weingarten formulas. By equating the tangential and normal components of the last equation we obtain

$$\nabla_{\mathfrak{X}} \mathbb{W} = c A_{\mathbb{N}} \mathfrak{X} - \beta A_{\mathbb{N}} \mathfrak{X}$$

and

$$\mathfrak{X}(c) = -u(A_{\mathbb{N}} \mathfrak{X}) - h(\mathfrak{X}, \mathbb{W}).$$

It is well known that if $\check{\nabla} \check{\beta} = 0$ then $\check{\nabla} \check{\mathfrak{J}} = 0$. So from (8) and (9) for a hypersurface $\mathfrak{M}$ of a meta-metallic Riemannian manifold $(\mathfrak{M}, \check{\mathfrak{J}}, \check{\beta}, \check{g})$, we have

$$\check{\nabla}_{\mathfrak{X}} \check{\mathfrak{J}} \mathfrak{Y} = \check{\mathfrak{J}} \check{\nabla}_{\mathfrak{X}} \mathfrak{Y},$$

$$\nabla_{\mathfrak{X}} \check{\mathfrak{J}} \mathfrak{Y} + h(\mathfrak{X}, \check{\mathfrak{J}} \mathfrak{Y}) \mathbb{N} + \mathfrak{X}(v(\mathfrak{Y})) \mathbb{N} - v(\mathfrak{Y}) A_{\mathbb{N}} \mathfrak{X} = \check{\mathfrak{J}} \nabla_{\mathfrak{X}} \mathfrak{Y} + v(\nabla_{\mathfrak{X}} \mathfrak{Y}) \mathbb{N} + h(\mathfrak{X}, \mathfrak{Y}) \mathbb{U} + bh(\mathfrak{X}, \mathfrak{Y}) \mathbb{N} \quad (14)$$

which implies

$$(\nabla_{\mathfrak{X}} \check{\mathfrak{J}}) \mathfrak{Y} = v(\mathfrak{Y}) A_{\mathbb{N}} \mathfrak{X} + h(\mathfrak{X}, \mathfrak{Y}) \mathbb{U}$$

and

$$(\nabla_{\mathfrak{X}} v) \mathfrak{Y} = -h(\mathfrak{X}, \check{\mathfrak{J}} \mathfrak{Y}) + bh(\mathfrak{X}, \mathfrak{Y}),$$

by equating the tangential and normal parts of (14). On the other hand, again by using (8), (9) and $\check{\nabla} \check{\mathfrak{J}} = 0$, then we get

$$\begin{aligned} \check{\nabla}_{\mathfrak{X}} \check{\mathfrak{J}} \mathbb{N} &= \check{\mathfrak{J}} \check{\nabla}_{\mathfrak{X}} \mathbb{N}, \\ \nabla_{\mathfrak{X}} \mathbb{U} + h(\mathfrak{X}, \mathbb{U}) \mathbb{N} + \mathfrak{X}(b) \mathbb{N} - b A_{\mathbb{N}} \mathfrak{X} &= -\check{\mathfrak{J}} A_{\mathbb{N}} \mathfrak{X} - v(A_{\mathbb{N}} \mathfrak{X}) \mathbb{N}, \end{aligned}$$

which gives

$$\nabla_{\mathfrak{X}} \mathbb{U} = -\check{\mathfrak{J}} A_{\mathbb{N}} \mathfrak{X} + b A_{\mathbb{N}} \mathfrak{X},$$

and

$$\mathfrak{X}(b) = -h(\mathfrak{X}, \mathbb{U}) - v(A_{\mathbb{N}} \mathfrak{X}).$$

□

Definition 4.3. Let $\mathfrak{M}$ be a hypersurface of an almost meta-metallic Riemannian manifold $(\mathfrak{M}, \check{\mathfrak{J}}, \check{\beta}, \check{g})$. For any point $p \in \mathfrak{M}$;

i- if $\check{\mathfrak{J}}(T_p\mathfrak{M}) \subset T_p\mathfrak{M}$ and $\check{\beta}(T_p\mathfrak{M}) \subset T_p\mathfrak{M}$ which implies that $\check{\mathfrak{J}}\check{\beta}(T_p\mathfrak{M}) \subset T_p\mathfrak{M}$ then $\mathfrak{M}$ is called an invariant hypersurface,

ii- if $\check{\mathfrak{J}}\check{\beta}(T_p\mathfrak{M}) \not\subset T_p\mathfrak{M}$ then $\mathfrak{M}$ is called a non-invariant hypersurface.

Let $\mathfrak{M}$ be an invariant hypersurface of an almost meta-metallic Riemannian manifold $(\mathfrak{M}, \check{\mathfrak{J}}, \check{\beta}, \check{g})$. In this case, from (8)-(11), we have $v = 0$ (or equivalently $\mathfrak{U} = 0$) and $u = 0$ (or equivalently $\mathfrak{V} = 0$). Then the following theorem is stated.

Theorem 4.4. A hypersurface $\mathfrak{M}$ of an almost meta-metallic Riemannian manifold $(\mathfrak{M}, \check{\mathfrak{J}}, \check{\beta}, \check{g})$ is invariant iff the normal vector field of $\mathfrak{M}$ is the eigenvector of $\check{\mathfrak{J}}$ and $\check{\beta}$ with the eigenvalue b and c , respectively.

Theorem 4.5. Let $\mathfrak{M}$ be a hypersurface of an almost meta-metallic Riemannian manifold $(\mathfrak{M}, \check{\mathfrak{J}}, \check{\beta}, \check{g})$. Then $\mathfrak{M}$ is invariant iff

$$b = \frac{p \pm \sqrt{p^2 + 4q}}{2} \quad (15)$$

and

$$c = \frac{b - p \pm \sqrt{p^2 - 2p(b+2) + b^2}}{2}. \quad (16)$$

Proof. From equalities (4) and (10) of Proposition 4.1; if $\mathfrak{M}$ is an invariant hypersurface namely, $v = 0$ (or equivalently $\mathfrak{U} = 0$) and $u = 0$ (or equivalently $\mathfrak{V} = 0$) then we have

$$pb + q - b^2 = 0$$

and

$$p + c(b - p - c) = 0$$

which imply

$$b = \frac{p \pm \sqrt{p^2 + 4q}}{2}$$

and

$$c = \frac{b - p \pm \sqrt{p^2 - 2p(b+2) + b^2}}{2}.$$

Conversely, assume that b and c are given above. Then from equalities (4) and (10) of Proposition 4.1 we have $v = 0$ and $u = 0$, which show that $\mathfrak{M}$ is an invariant hypersurface. $\square$

Corollary 4.6. Let $\mathfrak{M}$ be an invariant hypersurface of a meta-metallic Riemannian manifold $(\mathfrak{M}, \check{\mathfrak{J}}, \check{\beta}, \check{g})$. Then $\check{\mathfrak{J}}$ and $\check{\beta}$ are parallel with respect to ∇ .

Theorem 4.7. Let $\mathfrak{M}$ be a non-invariant hypersurface of a meta-metallic Riemannian manifold $(\mathfrak{M}, \check{\mathfrak{J}}, \check{\beta}, \check{g})$ with the induced structure $\Pi = (\check{\mathfrak{J}}, \beta, \mathfrak{U}, \mathfrak{V}, v, u, b, c, g)$. Then $\mathfrak{M}$ is totally geodesic iff $\check{\mathfrak{J}}$ and β are parallel on $\mathfrak{M}$ with respect to ∇ .

Proof. From equalities (1) and (5) of Proposition 4.2; the proof is obvious. $\square$

Let $\mathfrak{M}$ be a totally umbilical hypersurface of a meta-metallic Riemannian manifold $(\mathfrak{M}, \mathfrak{J}, \check{\beta}, \check{g})$ with the induced structure $\Pi = (\mathfrak{J}, \beta, \mathbb{U}, \mathbb{V}, v, u, b, c, g)$. Since $A = \lambda I$ then from Proposition 4.2, we have

$$\left. \begin{aligned} (\nabla_{\mathbb{X}}\beta)Y &= \lambda(g(\mathbb{X}, Y)\mathbb{V} + u(Y)\mathbb{X}), \\ (\nabla_{\mathbb{X}}u)Y &= \lambda(cg(\mathbb{X}, Y) - g(\mathbb{X}, \beta Y)), \\ \nabla_{\mathbb{X}}\mathbb{V} &= \lambda(c - \beta)\mathbb{X}, \\ \mathbb{X}(c) &= -2\lambda g(\mathbb{X}, \mathbb{V}), \\ (\nabla_{\mathbb{X}}\mathfrak{J})Y &= \lambda(g(\mathbb{X}, Y)\mathbb{U} + \mathbb{V}(Y)\mathbb{X}), \\ (\nabla_{\mathbb{X}}v)Y &= \lambda(bg(\mathbb{X}, Y) - g(\mathbb{X}, \mathfrak{J}Y)), \\ \nabla_{\mathbb{X}}\mathbb{U} &= \lambda(b - \mathfrak{J})\mathbb{X}, \\ \mathbb{X}(b) &= -2\lambda g(\mathbb{X}, \mathbb{U}), \end{aligned} \right\} \quad (17)$$

for all $\mathbb{X}, Y \in \Gamma(T\mathfrak{M})$.

Theorem 4.8. Let $\mathfrak{M}$ be a totally umbilical invariant hypersurface of a meta-metallic Riemannian manifold with the induced structure $\Pi = (\mathfrak{J}, \beta, \mathbb{U}, \mathbb{V}, v, u, b, c, g)$. Then $\mathfrak{J} = bI$ and $\beta = cI$, where b and c are constant functions.

Proof. From the third, fourth and the last two equations in (17), we complete the proof.

Conversely, suppose that M is a hypersurface of a meta-metallic Riemannian manifold such that $\mathfrak{J} = bI$ and $\beta = cI$. Then from (17) one can easily see that $\nabla_{\mathbb{X}}\mathbb{V} = 0$ and $\nabla_{\mathbb{X}}\mathbb{U} = 0$. $\square$

Example 4.9. Let $\mathbb{E}^5$ be a Euclidean space with an almost metallic structure $\check{\mathfrak{J}}$ given by

$$\begin{aligned} \check{\mathfrak{J}} : \mathbb{E}^5 &\rightarrow \mathbb{E}^5 \\ (x_1, x_2, x_3, y_1, y_2) &\rightarrow (\check{\epsilon}x_1, \check{\epsilon}x_2, \check{\epsilon}x_3, (p - \check{\epsilon})y_1, (p - \check{\epsilon})y_2). \end{aligned}$$

Now, we define a $(1, 1)$ tensor field $\check{\beta}$ on $(\mathbb{E}^5, \check{\mathfrak{J}})$ by

$$\begin{aligned} \check{\beta} : \mathbb{E}^5 &\rightarrow \mathbb{E}^5 \\ (x_1, x_2, x_3, y_1, y_2) &\rightarrow (\check{\chi}x_1, \check{\chi}x_2, \check{\chi}x_3, -\check{\chi}y_1, -\check{\chi}y_2), \end{aligned}$$

where $\check{\chi} = \frac{\check{\epsilon} + \sqrt{\check{\epsilon}^2 + 4p}}{2}$.

It is easy to see that $\check{\beta}$ is a meta-metallic structure on $\mathbb{E}^5$ and so $(\mathbb{E}^5, \check{\mathfrak{J}}, \check{\beta}, \langle, \rangle)$ is an almost meta-metallic Riemannian manifold, where $\langle, \rangle$ is the usual Euclidean metric on $\mathbb{E}^5$.

Now, we consider a hypersurface $\mathfrak{M}$ of $\mathbb{E}^5$ given by $x_1 = x_2$. Then $T\mathfrak{M}$ is spanned by

$$\mathbb{Z}_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}, \quad \mathbb{Z}_2 = \frac{\partial}{\partial x_3}, \quad \mathbb{Z}_3 = \frac{\partial}{\partial y_1}, \quad \mathbb{Z}_4 = \frac{\partial}{\partial y_2}.$$

In this case it is easy to see that $\check{\mathfrak{J}}(T\mathfrak{M}) \subset T\mathfrak{M}$ and $\check{\beta}(T\mathfrak{M}) \subset T\mathfrak{M}$ which implies that $\mathfrak{M}$ is an invariant hypersurface of $\mathbb{E}^5$.

Example 4.10. Let $\mathbb{R}^{n+k}$ be the $(n+k)$ -dimensional real number space with a coordinate system $(x_1, \dots, x_n, x_{n+1}, \dots, x_{n+k})$. We define

$$\check{\mathfrak{J}}(x_1, \dots, x_n, x_{n+1}, \dots, x_{n+k}) = (\check{\epsilon}x_1, \dots, \check{\epsilon}x_n, (p - \check{\epsilon})x_{n+1}, \dots, (p - \check{\epsilon})x_{n+k})$$

and

$$\tilde{\beta}(x_1, \dots, x_n, x_{n+1}, \dots, x_{n+k}) = (\dot{\chi}x_1, \dot{\chi}x_2, \dots, \dot{\chi}x_n, -\check{\chi}x_{n+1}, -\check{\chi}x_{n+2}, \dots, -\check{\chi}x_{n+k})$$

where κ denotes the metallic ratio,

$$\dot{\chi} = \frac{\frac{q}{\kappa} + \sqrt{4p + \frac{q^2}{\kappa^2}}}{2}$$

and

$$\check{\chi} = \frac{\kappa \mp \sqrt{\kappa^2 + 4p}}{2}.$$

Since we write

$$\tilde{\mathfrak{J}}\tilde{\beta}^2(x_1, \dots, x_n, x_{n+1}, \dots, x_{n+k}) = (\dot{\chi}^2\kappa x_1, \dots, \dot{\chi}^2\kappa x_n, \check{\chi}^2(p - \kappa)x_{n+1}, \dots, \check{\chi}^2(p - \kappa)x_{n+k}),$$

then it is easy to see that

$$\tilde{\mathfrak{J}}\tilde{\beta}^2 = p\tilde{\mathfrak{J}} + q\tilde{\beta},$$

which implies $(\mathbb{R}^{n+k}, \tilde{\mathfrak{J}}, \tilde{\beta})$ is an almost meta-metallic manifold. Moreover, the usual product $\langle, \rangle$ on $\mathbb{R}^{n+k}$ satisfies (2) (or equivalently (3)) then $(\mathbb{R}^{n+k}, \tilde{\mathfrak{J}}, \tilde{\beta}, \langle, \rangle)$ is an almost meta-metallic Riemannian manifold.

Consider the hypersurface $S^{n+k-1}(r)$ of $\mathbb{R}^{n+k}$ which is given by

$$S^{n+k-1}(r) = \left\{ (x_1, \dots, x_n, x_{n+1}, \dots, x_{n+k}) : \sum_{i=1}^n x_i^2 + \sum_{j=n+1}^{n+k} x_j^2 = r^2 \right\} \subset \mathbb{R}^{n+k},$$

and the normal vector field of $S^{n+k-1}(r)$ at any point $(x_1, \dots, x_n, x_{n+1}, \dots, x_{n+k}) \in S^{n+k-1}(r)$ is defined by

$$\mathbb{N} = \frac{1}{r}(x_1, \dots, x_n, x_{n+1}, \dots, x_{n+k}). \quad (18)$$

Then there exists a tangent vector $(\mathbb{X}_1, \dots, \mathbb{X}_n, \mathbb{X}_{n+1}, \dots, \mathbb{X}_{n+k})$ on hypersphere for every point $p = (x_1, \dots, x_n, x_{n+1}, \dots, x_{n+k}) \in S^{n+k-1}(r)$ iff

$$\sum_{i=1}^n x_i \mathbb{X}_i + \sum_{j=n+1}^{n+k} x_j \mathbb{X}_j = 0. \quad (19)$$

Then by using (8) and (9) we write

$$\begin{aligned} \tilde{\mathfrak{J}}(\mathbb{X}_i, \mathbb{X}_j) &= \mathfrak{J}(\mathbb{X}_i, \mathbb{X}_j) + v(\mathbb{X}_i, \mathbb{X}_j)\mathbb{N}, \\ \tilde{\mathfrak{J}}\mathbb{N} &= \mathbb{U} + b\mathbb{N}, \end{aligned} \quad (20)$$

where $(\mathbb{X}_i, \mathbb{X}_j) = (\mathbb{X}_1, \dots, \mathbb{X}_n, \mathbb{X}_{n+1}, \dots, \mathbb{X}_{n+k}) \in T_p S^{n+k-1}(r)$. Since

$$\tilde{\mathfrak{J}}\mathbb{N} = \frac{1}{r}(\kappa x_1, \dots, \kappa x_n, (p - \kappa)x_{n+1}, \dots, (p - \kappa)x_{n+k}) \quad (21)$$

and $b = \langle \tilde{\mathfrak{J}}\mathbb{N}, \mathbb{N} \rangle$, then we obtain

$$b = \frac{1}{r^2}(\kappa \sum_{i=1}^n x_i^2 + (p - \kappa) \sum_{j=n+1}^{n+k} x_j^2). \quad (22)$$

Using $\mathbb{U} = \tilde{\mathfrak{J}}\mathbb{N} - b\mathbb{N}$ with the last equation above, we get

$$\mathbb{U} = \frac{2\kappa - p}{r^3} \left(\left(\sum_{j=n+1}^{n+k} x_j^2 \right) x_1, \dots, \left(\sum_{j=n+1}^{n+k} x_j^2 \right) x_n, - \left(\sum_{i=1}^n x_i^2 \right) x_{n+1}, \dots, - \left(\sum_{i=1}^n x_i^2 \right) x_{n+k} \right), \quad (23)$$

which implies that

$$v(\mathbb{X}_i, \mathbb{X}_j) = \langle (\mathbb{X}_i, \mathbb{X}_j), \mathbb{U} \rangle = \frac{2\kappa - p}{r^3} \left(\left(\sum_{j=n+1}^{n+k} x_j^2 \right) \left(\sum_{i=1}^n x_i \mathbb{X}_i \right) - \left(\sum_{i=1}^n x_i^2 \right) \left(\sum_{j=n+1}^{n+k} x_j \mathbb{X}_j \right) \right).$$

If we put $\sum_{i=1}^n x_i \mathbb{X}_i = - \sum_{j=n+1}^{n+k} x_j \mathbb{X}_j = \theta$, then the last equation can be written as

$$v(\mathbb{X}_i, \mathbb{X}_j) = \frac{2\kappa - p}{r} \theta. \quad (24)$$

By using (20) and (24) we get

$$\mathfrak{J}(\mathbb{X}_i, \mathbb{X}_j) = (\kappa \mathbb{X}_i - \frac{2\kappa - p}{r} \theta x_i, (p - \kappa) \mathbb{X}_j - \frac{2\kappa - p}{r} \theta x_j). \quad (25)$$

Furthermore by using (10) and (11) we write

$$\begin{aligned} \tilde{\beta}(\mathbb{X}_i, \mathbb{X}_j) &= \beta(\mathbb{X}_i, \mathbb{X}_j) + u(\mathbb{X}_i, \mathbb{X}_j)\mathbb{N}, \\ \tilde{\beta}\mathbb{N} &= \mathbb{V} + c\mathbb{N}, \end{aligned} \quad (26)$$

since

$$\tilde{\beta}\mathbb{N} = \frac{1}{r} (\dot{\chi} x_1, \dots, \dot{\chi} x_n, -\check{\chi} x_{n+1}, \dots, -\check{\chi} x_{n+k}) \quad (27)$$

and $c = \langle \tilde{\beta}\mathbb{N}, \mathbb{N} \rangle$, we calculate

$$c = \frac{1}{r^2} (\dot{\chi} \left(\sum_{i=1}^n x_i^2 \right) - \check{\chi} \left(\sum_{j=n+1}^{n+k} x_j^2 \right)). \quad (28)$$

By using $\mathbb{V} = \tilde{\beta}\mathbb{N} - c\mathbb{N}$, we get

$$\mathbb{V} = \frac{\dot{\chi} + \check{\chi}}{r^3} \left(\left(\sum_{j=n+1}^{n+k} x_j^2 \right) x_1, \dots, \left(\sum_{j=n+1}^{n+k} x_j^2 \right) x_n, - \left(\sum_{i=1}^n x_i^2 \right) x_{n+1}, \dots, - \left(\sum_{i=1}^n x_i^2 \right) x_{n+k} \right), \quad (29)$$

which implies that

$$u(\mathbb{X}_i, \mathbb{X}_j) = \langle (\mathbb{X}_i, \mathbb{X}_j), \mathbb{V} \rangle = \frac{\dot{\chi} + \check{\chi}}{r^3} \left(\left(\sum_{j=n+1}^{n+k} x_j^2 \right) \left(\sum_{i=1}^n x_i \mathbb{X}_i \right) - \left(\sum_{i=1}^n x_i^2 \right) \left(\sum_{j=n+1}^{n+k} x_j \mathbb{X}_j \right) \right).$$

Then we get

$$u(\mathbb{X}_i, \mathbb{X}_j) = \frac{\dot{\chi} + \check{\chi}}{r} \theta. \quad (30)$$

By using (26) and (30), we obtain

$$\beta(\mathbb{X}_i, \mathbb{X}_j) = (\dot{\chi} \mathbb{X}_i - \frac{\dot{\chi} + \check{\chi}}{r^2} \theta x_i, -\check{\chi} \mathbb{X}_j - \frac{\dot{\chi} + \check{\chi}}{r^2} \theta x_j). \quad (31)$$

Hence $S^{n+k-1}(r)$ is a non-invariant hypersurface of the almost meta-metallic Riemannian manifold $(\mathbb{R}^{n+k}, \tilde{\mathfrak{J}}, \tilde{\beta}, \langle, \rangle)$ endowed with the induced structure $(\mathfrak{J}, \beta, \mathbb{U}, \mathbb{V}, v, u, b, c)$ given by (23 - 25) and (29 - 31).

Example 4.11. Let $\mathbb{E}^5$ be a Euclidean space with an almost metallic structure $\check{\mathfrak{J}}$ given by

$$\check{\mathfrak{J}} : \mathbb{E}^5 \rightarrow \mathbb{E}^5$$

$$(x_i, y_i, z) \rightarrow \check{\mathfrak{J}}(x_i, y_i, z) = \left(\frac{p}{2}x_i + \frac{\sqrt{p^2 + 4q}}{2}y_i, \quad \frac{p}{2}y_i + \frac{\sqrt{p^2 + 4q}}{2}x_i, \quad (p - \mathfrak{k})z \right),$$

where $\mathfrak{k}$ is metallic ratio and (x_i, y_i, z) is a coordinate system on $\mathbb{E}^5$ for $i=1,2$. Now, we define a $(1, 1)$ tensor field $\check{\beta}$ on $(\mathbb{E}^5, \check{\mathfrak{J}})$ by

$$\check{\beta} : \mathbb{E}^5 \rightarrow \mathbb{E}^5$$

$$(x_i, y_i, z) \rightarrow \check{\beta}(x_i, y_i, z) = \left(-\frac{\check{\chi} + \check{\chi}}{2}x_i + \frac{\check{\chi} - \check{\chi}}{2}y_i, \quad \frac{\check{\chi} - \check{\chi}}{2}x_i - \frac{\check{\chi} + \check{\chi}}{2}y_i, \quad -\check{\chi}z \right),$$

where $\check{\chi} = \frac{\mathfrak{k} \mp \sqrt{\mathfrak{k}^2 + 4p}}{2}$ and $\check{\chi} = \frac{-(p - \mathfrak{k}) \mp \sqrt{(p - \mathfrak{k})^2 + 4p}}{2}$.

It is easy to see that $\check{\beta}$ is a meta-metallic structure on $\mathbb{E}^5$ and so $(\mathbb{E}^5, \check{\mathfrak{J}}, \check{\beta}, \langle, \rangle)$ is an almost meta-metallic Riemannian manifold, where $\langle, \rangle$ is the usual Euclidean metric on $\mathbb{E}^5$.

Now, we consider a hypersurface $\mathfrak{M}$ of $\mathbb{E}^5$ given by $y_1 = y_2^2$. Then $T\mathfrak{M}$ is spanned by

$$\mathbb{Z}_1 = \frac{\partial}{\partial x_1}, \quad \mathbb{Z}_2 = \frac{\partial}{\partial x_2}, \quad \mathbb{Z}_3 = 2y_2 \frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2}, \quad \mathbb{Z}_4 = \frac{\partial}{\partial z}$$

and $T\mathfrak{M}^\perp$ is spanned by

$$N = \frac{\partial}{\partial y_1} - 2y_2 \frac{\partial}{\partial y_2}.$$

In this case it is easy to see that $\check{\mathfrak{J}}\check{\beta}(T\mathfrak{M}) \not\subset T\mathfrak{M}$ which implies that $\mathfrak{M}$ is a non-invariant hypersurface of $\mathbb{E}^5$.

References

- [1] C. Barlett, Nautilus spirals and the meta-Golden ratio Chi, *Nexus Netw. J.* **21** (2019), 641-656.
- [2] S. Bekiros and D. Kouloumpou, SBDiEM: A new mathematical model of infectious disease dynamics, *Chaos, Solitons, Fractals* **136** (2019), 109828.
- [3] M. Crasmareanu and C.E. Hretcanu, Golden differential geometry, *Chaos Solitons Fractals* **38** (2008), 1229-12387.
- [4] F.E. Erdoğan, C. Yıldırım and Ş.N. Bozdağ, Structures induced on hypersurfaces of meta-Golden Riemannian manifolds, *Filomat* **38(27)** (2024), 9593–9606.
- [5] F.E. Erdoğan, S. Yüksel Perktas and Ş.N. Bozdağ, Meta-metallic Riemannian manifolds, *Filomat* **38(1)** (2024), 315-323.
- [6] S. I. Goldberg and K. Yano, Polynomial structures on manifolds, *Kodai Math. Sem. Rep.* **22** (1970), 199-218.
- [7] C.E. Hretcanu and M. Crasmareanu, Metallic structures on Riemannian manifolds, *Revista de la Union Matematica Argentina* **54(2)** (2013), 15-27.
- [8] D. Huylebrouck, The meta-Golden ratio Chi, *Proceedings of Bridges 2014: Mathematics, Music, Art, Architecture, Culture*, (2014), 151-158.
- [9] M. Özkan and B. Peltek, A New Structure on Manifolds: Silver Structure, *International Electronic Journal of Geometry* **9(2)** (2016), 59-69.
- [10] M. Özkan and S. Doğan, Almost bronze structures on differentiable manifolds, *Journal of Mathematics* **1** (2022), 6940387.
- [11] B. Şahin, Almost poly-Norden manifolds, *Int. J. Maps Math.* **1** (2018), 68-79.
- [12] F. Şahin and B. Şahin, Meta-Golden Riemannian manifolds, *Math. Meth. Appl. Sci.* **45(16)** (2022), 10491-10501.
- [13] K. Yano, On structure defined by a tensor field f of type $(1, 1)$ satisfying $f^3 + f = 0$, *Tensor, NS*, **14** (1963) 99-109.