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Analytical approach for fractional differential hybrid inclusions involving Ψ-Hilfer fractional derivative

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Abstract. Our approach involved constructing appropriate operator spaces and verifying the conditions necessary for Dhage's theorem to hold. We demonstrated that our specific Ψ-Hilfer hybrid inclusion problem meets these criteria, thus guaranteeing the existence of at least one solution. To illustrate and validate our theoretical findings, we presented a concrete example showcasing the application of our theoretical framework, which highlights the practical relevance and effectiveness of our approach in solving fractional differential inclusions. This example serves not only as a proof of concept but also as a practical demonstration of the theoretical results, affirming the robustness and applicability of the Ψ-Hilfer hybrid fractional differential inclusion framework.

1. Introduction

Fractional differential equations have recently become a focal point of research due to their capacity to model complex phenomena across various scientific and engineering disciplines. Unlike traditional differential equations, fractional differential equations incorporate derivatives of non-integer order, allowing them to capture a broader range of dynamics and behaviors. This characteristic makes them particularly valuable in representing real-world issues in fields such as physics, mechanics, biology, chemistry, and control theory. For instance, these equations can describe anomalous diffusion processes, complex materials, and dynamic systems with memory effects. Comprehensive discussions on their applications and theoretical developments can be found in[6, 12, 15, 19–21, 26, 29, 30]. Moreover, the study of fractional differential equations and inclusions has been extensively explored by researchers, highlighting their versatility and significance in modeling diverse systems[2–4, 17, 22–24].
Hybrid systems are crucial in embedded control systems, especially when dealing with the physical environment. Many researchers have been diving into developing the theoretical aspects and methods for solving hybrid fractional differential equations using various fixed-point techniques, [1, 8–11, 28, 31].
In traditional fractional calculus, the primary focus has been on derivatives and integrals of non-integer order, described by operators such as the Riemann-Liouville and Caputo derivatives. However, these

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conventional fractional derivatives, while powerful, often lack the adaptability required for certain complex phenomena. The Hilfer fractional derivative addresses this limitation by introducing an additional parameter β , alongside the fractional order α.
This parameter β effectively modulates the fractional integration component, enabling a more nuanced control over the memory and history-dependent effects in the system being studied.
As the study and application of the Hilfer fractional derivative continue to expand, its ability to provide detailed insights and solutions to complex problems underscores its significance in both theoretical research and practical applications. Its dual-parameter framework offers a robust and adaptable tool for advancing our understanding of systems where traditional approaches may fall short. For more comprehensive understanding of the Hilfer and ψ-Hilfer fractional derivatives, we direct readers to the detailed discussions presented in references [5, 25, 28].
Hilal and Kajouni [16] discussed the following boundary value problems for hybrid differential equation with fractional order involving Caputo differential operators of order $0 < \alpha < 1$,

$$\begin{cases} D^{\alpha}\Big(\dfrac{x(t)}{f(t,x(t))}\Big) = g(t,x(t)), \quad t \in J = [0,T] \\ a\frac{x(0)}{f(0,x(0))} + b\frac{x(T)}{f(T,x(T))} = c, \end{cases}$$

where $f \in C\Big(J \times \mathbb{R}, \mathbb{R} \setminus \{0\}\Big)$, $g : [a,b] \times \mathbb{R} \longrightarrow \mathbb{R}$ and a,b,c are real constants with $a + b \neq 0$.
Inspired by the previous work addressing fractional hybrid differential equations, in this examination, we integrate their concepts to examine the potential of a solution for the following problem.

$$\begin{cases} {}^{H}D^{\alpha,\beta;\Psi}\Big({}^{H}D^{p,q;\Psi}\Big(\dfrac{x(t)}{f(t,x(t))}\Big)\Big) \in G(t,x(t)), \quad t \in \mathbb{J} = [a,b] \\ x(a) = 0 \quad , \qquad x(b) = \displaystyle\sum_{i=1}^{n} \omega_i I^{\sigma_i,\Psi} x(\eta_i) + \sum_{j=1}^{m} \nu_j x(\tau_j). \end{cases} \tag{1}$$

With ${}^{H}D^{\alpha,\beta;\Psi}$ and ${}^{H}D^{p,q;\Psi}$are the Ψ-Hilfer fractional derivative of order α , p $0 < \alpha \leq 1$, $0 < p \leq 1$, and parameter β, q , $0 \leq q \leq 1$, $0 \leq \beta \leq 1$ respectively, $\omega_i \in \mathbb{R}$, $i = 1, ..., n$, $a < \eta_1 < ... < \eta_n < b$, $\nu_j \in \mathbb{R}$, $j = 1, ..., m$, $a < \tau_1 < ... < \tau_m < b$, $f \in C\Big(\mathbb{J} \times \mathbb{R}, \mathbb{R} \setminus \{0\}\Big)$ and $G : \mathbb{J} \times \mathbb{R} \longrightarrow \mathcal{P}(\mathbb{R})$ is a multivalued map, where $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$.

In this research, the innovative aspect lies in exploring a fresh and complex form of a complicated kind of fractional derivative [28] known as Ψ-Hilfer derivative, and this is achieved by taking into account different values of the parameter α and β and function Ψ.

The paper is set up like this: the second part, covers several terms, concepts, and lemmas associated with multivalued analysis and fractional calculus that are crucial for our research. Then, in section 3, we discuss how to use Dhage's fixed point theorem[14] to demonstrate the availability of solutions to the multivalued problem in section 3. Finally, in section 4, we provide a practical example to demonstrate the key results we've discussed.

2. Preliminaries

2.1. Fractional Calculus

Some concepts, lemmas, and annotations that are important to our study will be laid forth in this section.

Let $C(\mathbb{J}, \mathbb{R})$ denote the Banach space of all continuous functions from $\mathbb{J}$ into $\mathbb{R}$ with the norm defined by $\|f\| = sup_{t\in\mathbb{J}}\{|f(t)|\}$. We denote by $AC^n(\mathbb{J}, \mathbb{R})$ the n-times absolutely continuous functions given by

$$AC^n((\mathbb{J}, \mathbb{R}) = \Big\{f : \mathbb{J} \longrightarrow \mathbb{R} : f^{(n-1)} \in AC(\mathbb{J}, \mathbb{R})\Big\}. \tag{2}$$

Definition 2.1. *[20]*
Let $\alpha > 0$, f is an integrable function and Ψ is an increasing function on $\mathbb{J}$ with $\Psi'(y)$ is a continuous derivate for all $y \in \mathbb{J}$. The fractional integral in the sense of Ψ Riemann-Liouville of order α of f is determined by :

$$I_{a^+}^{\alpha;\Psi} f(y) = \frac{1}{\Gamma(\alpha)} \int_a^y \Psi'(y)(\Psi(y) - \Psi(s))^{\alpha-1} f(s)ds \tag{3}$$

Definition 2.2. *[20]*
For $y \in \mathbb{J}$, and $n \in \mathbb{N}$.The Ψ-Riemann-Liouville derivative of order α is determined accordingly, whether α is a strictly positive real, $f : \mathbb{J} \longrightarrow \mathbb{R}$ is an integrable function, and $\Psi \in C^n((\mathbb{J}, \mathbb{R})$ an increasing function with $\Psi'(t) \neq 0$.

$$D_{a^+}^{\alpha;\Psi} f(y) = \left(\frac{1}{\Psi'(y)}\frac{d}{dy}\right)^n I_{a^+}^{n-\alpha;\Psi} f(y) \tag{4}$$

$$= \frac{1}{\Gamma(n-\alpha)}\left(\frac{1}{\Psi'(y)}\frac{d}{dy}\right)^n \int_a^y \Psi'(y)(\Psi(y) - \Psi(s))^{n-\alpha-1} f(s)ds \tag{5}$$

with $n-1 < \alpha < n, n = [\alpha]+1$, and $[\alpha]$ symbolizes the real number's integer part α.

Definition 2.3. *[20]*
Let $f, \Psi \in C^n(\mathbb{J}, \mathbb{R})$ be two functions with Ψ is increasing and $\Psi'(y) \neq 0$ for all $y \in \mathbb{J}$.For a function of order α and type $0 \leq \beta \leq 1$,The Ψ-Hilfer fractional derivative of f , is determined by

$$\begin{aligned} {}^H D_{a^+}^{\alpha,\beta;\Psi} f(y) &= I_{a^+}^{\beta(n-\alpha);\Psi}\left(\frac{1}{\Psi'(y)}\frac{d}{dy}\right)^n I_{a^+}^{(1-\beta)(n-\alpha);\Psi} f(y) \\ &= I_{a^+}^{\gamma-\alpha;\Psi} D_{a^+}^{\gamma;\Psi} f(y), \end{aligned} \tag{6}$$

with $n-1 < \alpha < n, n = [\alpha]+1$, and $[\alpha]$ symbolizes the real number's integer part α, with $\gamma = \alpha + \beta(n-\alpha)$.

Lemma 2.4. *[20]*
If $\alpha, \beta > 0$. The semi group's subsequent attribute is as follows, provided by

$$I_{a^+}^{\alpha;\Psi} I_{a^+}^{\beta;\Psi} f(y) = I_{a^+}^{\alpha+\beta;\Psi} f(y), y > a. \tag{7}$$

Proposition 2.5. *[7, 20]*
Let $\alpha \geq 0$, $\varsigma > 0$ and $\varrho \in \mathbb{J}$

(I) $I_{a^+}^{\alpha;\Psi}(\Psi(\varsigma) - \Psi(a))^{\varrho-1} = \dfrac{\Gamma(\varrho)}{\Gamma(\alpha+\varrho)}(\Psi(\varsigma) - \Psi(a))^{\alpha+\varrho-1}$.

(II) ${}^H D_{a^+}^{\alpha;\Psi}(\Psi(\varsigma) - \Psi(a))^{\varsigma-1} = \dfrac{\Gamma(\varrho)}{\Gamma(\alpha-\varrho)}(\Psi(\varrho) - \Psi(a))^{\alpha-\varrho-1}$,
$n-1 < \alpha < n$, and $\varsigma > n$.

Lemma 2.6. *[20]*
If $f \in C^n([a,b], \mathbb{R})$, $n-1 < \alpha < n$, $\quad 0 \leq \beta \leq 1$ and $\gamma = \alpha + \beta(n-\alpha)$, then

$$I_{a^+}^{\alpha;\Psi}({}^H D_{a^+}^{\alpha,\beta;\psi} f)(t) = f(t) - \sum_{k=1}^{n} \frac{f_{\Psi}^{[n-k]} I_{a^+}^{(1-\beta)(n-\alpha);\psi} f(a)}{\Gamma(\gamma-k+1)}(\Psi(t) - \Psi(a))^{\gamma-k}, \tag{8}$$

for all $t \in \mathbb{J}$, where $f_{\Psi}^{[n-k]} := \left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^{n-k} f(t)$.

Lemma 2.7. *Let $a \geq 0$, $0 < \alpha \leq 1$, $0 \leq \beta \leq 1$, $0 < p \leq 1$, $0 \leq q \leq 1$, $\gamma = \alpha + \beta - \alpha\beta$, and $h \in C(\mathbb{J}, \mathbb{R})$. Thus, the function x is a solution to the boundary value problem that follows:*

$$\begin{cases} {}^{H}D^{\alpha,\beta;\Psi}\Big({}^{H}D^{p,q;\Psi}\Big(\dfrac{x(t)}{f(t,x(t))}\Big)\Big) = h(t), \quad t \in \mathbb{J} = [a,b] \\ x(a) = 0 \quad , \qquad x(b) = \displaystyle\sum_{i=1}^{n} \omega_i I^{\sigma_i,\Psi} x(\eta_i) + \sum_{j=1}^{m} \nu_j x(\tau_j). \end{cases} \tag{9}$$

if and only if

$$x(t) = f(t,x(t))\Bigg[I^{\alpha+\beta;\Psi}h(t) + \frac{(\Psi(t)-\Psi(a))^{\gamma+p-1}}{\Lambda\Gamma(\gamma+p)}\Big(I^{\alpha+p;\Psi}f(b,x(b))h(b) - \sum_{j=1}^{m}\nu_j I^{\alpha+p;\Psi}f(\tau_j,x(\tau_j))h(\tau_j) - \sum_{i=1}^{n}\omega_i I^{\alpha+p+\sigma_i;\Psi}f(\eta_i,x(\eta_i))h(\eta_i)\Big)\Bigg], \tag{10}$$

where

$$\Lambda = \sum_{i=1}^{n}\omega_i.f(\eta_i,x(\eta_i)).\frac{(\Psi(\eta_i)-\Psi(a))^{\gamma+p+\sigma_i-1}}{\Gamma(\gamma+p+\sigma_i)} + \sum_{j=1}^{m}\nu_j.f(\tau_j,x(\tau_j)).\frac{(\Psi(\tau_j)-\Psi(a))^{\gamma+p-1}}{\Gamma(\gamma+p)} - f(b,x(b)).\frac{(\Psi(b)-\Psi(a))^{\gamma+p-1}}{\Gamma(\gamma+p)} \neq 0 \tag{11}$$

Proof. The problem can be expressed as

$$I^{\gamma-\alpha;\Psi}\ {}^{H}D^{\gamma;\Psi}\Big({}^{H}D^{p,q;\Psi}\Big(\frac{x(t)}{f(t,x(t))}\Big)\Big) = h(t) \tag{12}$$

When the Ψ-Riemann-Liouville fractional integral of order α is applied to each side of (12),employing Lemma 2.6, we arrive at

$${}^{H}D^{p,q;\Psi}\Big(\frac{x(t)}{f(t,x(t))}\Big) = I^{\alpha;\Psi}h(t) + \frac{c_0}{\Gamma(\gamma)}((\Psi(t)-\Psi(a))^{\gamma-1} \tag{13}$$

where c_0 is a constant and $\gamma = \alpha + \beta - \alpha\beta$. Then we will apply the Ψ-Riemann-Liouville fractional integral of order p to each side,we will have as result by using Lemma 2.6

$$\frac{x(t)}{f(t,x(t))} = I^{\alpha+p;\Psi}h(t) + \frac{c_0}{\Gamma(\gamma+p)}((\Psi(t)-\Psi(a))^{\gamma+p-1} + \frac{c_1}{\Gamma(\mu)}((\Psi(t)-\Psi(a))^{\mu-1} \tag{14}$$

with c_1 being a constant and $\mu = p + q - pq$.With the boundary condition $x(a) = 0$ in (12) , we attain that $c_1 = 0$, then we get

$$x(t) = f(t,x(t))\Bigg[I^{\alpha+p;\Psi}h(t) + \frac{c_0}{\Gamma(\gamma+p)}((\Psi(t)-\Psi(a))^{\gamma+p-1}\Bigg], \tag{15}$$

By making use of the boundary condition $x(b) = \sum_{i=1}^{n}\omega_i I^{\sigma_i,\Psi}x(\eta_i) + \sum_{j=1}^{m}\nu_j x(\tau_j)$, in (13) we find

$$c_0 = \frac{1}{\Lambda}\Bigg[f(b,x(b))I^{\alpha+p;\Psi}h(b) - \sum_{i=1}^{n}\omega_i(f(\eta_i,x(\eta_i))I^{\alpha+p+\sigma_i;\Psi}h(\eta_i) - \sum_{j=1}^{m}\nu_j f(\tau_j,x(\tau_j))I^{\alpha+p;\Psi}h(\tau_j)\Bigg] \tag{16}$$

Inserting the value of c_0 in (14) in a manner that result in the solution expressed in (9). The converse is demonstrated through direct computation. □

2.2. Multi-valued Analysis

For a normed space $(X, \|.\|)$, we define :
$\mathcal{P}(X) = \{Y \subset X : Y \neq \emptyset\}$;
$\mathcal{P}_{cl}(X) = \{Y \subset X : Y \quad is \quad closed \quad \}$;
$\mathcal{P}_{cp}(X) = \{Y \subset X : Y \quad is \quad compact \quad \}$.
$\mathcal{P}_{cv}(X) = \{Y \subset X : Y \quad is \quad convex \quad \}$.
$\mathcal{P}_{cp,cv}(X) = \{Y \subset X : Y \quad is \quad compact \quad and \quad convex\}$.

Definition 2.8. *A multivalued map $G : [a, b] \times \mathbb{R} \longrightarrow \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if :*

(i) $t \longrightarrow G(t, x)$ *is measurable for each* $x \in \mathbb{R}$.

(ii) $x \longrightarrow G(t, x)$ *is upper semicontinuous for almost all* $t \in \mathbb{J}$.
Thus, a Carathéodory function G is called $\mathbb{L}^1$*-Carathéodory if :*

(iii) *There exists a function* $h \in \mathbb{L}^1([a, b]; \mathbb{R}^+)$ *such that*

$$\|G(t, x)\| = sup\{|z| : z \in G(t, x)\} \leq h(t),$$

for all $x \in \mathbb{R}$ *and for a.e.* $t \in \mathbb{J}$.

Dhage's fixed point theorem[14], detailed below plays an important role in the Principal result

Theorem 2.9. *Let X be a Banach algebra and let* $\mathcal{R} : X \longrightarrow X$ *be a single valued and* $\mathcal{T} : X \longrightarrow \mathcal{P}_{cl,cv}(X)$ *be a multi-valued operator satisfying:*

1- $\mathcal{R}$ *is single-valued Lipschitz with a Lipschitz constant l.*

2- $\mathcal{T}$ *being a compact and upper semi-continuous.*

3- $2Ml < 1$, *with* $M = \|\mathcal{T}(X)\|$.
Then one of the following holds:

(i) *The operator inclusion* $x \in \mathcal{R}x\mathcal{T}x$ *has a solution*

(ii) $\Delta = \{x \in X : \theta x \in \mathcal{R}x\mathcal{T}x), \theta > 1\}$ *is unbounded.*

3. Main Result

Throughout this section, we deal with the existence of solution for the problem (1) using Dhage fixed point theorem. The following notation is used to make the calculations in the proof clearer and avoid long expressions :

$$\begin{aligned}\Phi = \frac{(\Psi(b) - \Psi(a))^{\alpha+\beta}}{\Gamma(\alpha + \beta + 1)} + \frac{(\Psi(b) - \Psi(a))^{\gamma+p-1}}{|\Lambda|\Gamma(\gamma + p)} \Bigg[\widehat{f} \frac{(\Psi(b) - \Psi(a))^{\alpha+p}}{\Gamma(\alpha + p)} \\ + \widehat{f} \sum_{j=1}^{m} |\nu_j| \frac{(\Psi(\tau_j) - \Psi(a))^{\alpha+p}}{\Gamma(\alpha + p + 1)} \quad + \widehat{f} \sum_{i=1}^{n} |\omega_i| \frac{(\Psi(\eta_i) - \Psi(a))^{\alpha+p+\sigma_i}}{\Gamma(\alpha + p + \sigma_i + 1)} \Bigg]\end{aligned} \tag{17}$$

where $\widehat{f} = max(|f(b, x(b))|; |f(\tau_j, x(\tau_j))|; |f(\eta_i, x(\eta_i))|)$, $i = 1, ..., n$, $j = 1, ..., m$

Definition 3.1. *A function $x \in C(\mathbb{J}, \mathbb{R})$ called a solution of problem (1) if, $x(a) = 0$, $x(b) = \sum_{i=1}^{n} \omega_i I^{\sigma_i,\Psi} x(\eta_i) + \sum_{j=1}^{m} \nu_j x(\tau_j)$. and a function exist $z \in \mathbb{L}^1(\mathbb{J}, \mathbb{R})$ with $z \in G(t, x(t))$ a.e, on $\mathbb{J}$,so that*

$$x(t) = f(t, x(t))\Big[I^{\alpha+\beta;\Psi} z(t) + \frac{(\Psi(t) - \Psi(a))^{\gamma+p-1}}{\Lambda\Gamma(\gamma + p)}\Big(f(b, x(b))I^{\alpha+p;\Psi} z(b) - \sum_{j=1}^{m} \nu_j f(\tau_j, x(\tau_j))I^{\alpha+p;\Psi} z(\tau_j)$$
$$- \sum_{i=1}^{n} \omega_i (f(\eta_i, x(\eta_i))I^{\alpha+p+\sigma_i;\Psi} z(\eta_i)\Big)\Big]. \tag{18}$$

For each $x \in C(\mathbb{J}, \mathbb{R})$, define the set of selections of G by

$$\mathcal{S}_{G,x} := \Big\{z \in \mathbb{L}^1(\mathbb{J}, \mathbb{R}) : z(t) \in G(t, x(t)) \; for \; a.e \; t \in \mathbb{J}\Big\}$$

Lemma 3.2. *([14]) Let Θ be a linear continuous mapping from $L^1(\mathbb{J}, X)$ to $C(\mathbb{J}, X)$. Then the operator :*

$$\Theta \circ \mathcal{S}_{G,x} : C(\mathbb{J}, X) \longrightarrow \mathcal{P}_{bd,cl}\Big(C(\mathbb{J}, X)\Big)$$

is a closed graph operator in $C(\mathbb{J}, X) \times C(\mathbb{J}, X)$

The existence of this result bases itself on the Dhage fixed point theory.[14].

Theorem 3.3. *Suppose that:*

(H1). The function $f : \mathbb{J} \times \mathbb{R} \longrightarrow \mathbb{R} \setminus \{0\}$ is continuous and φ is a bounded function ,with the condition that $\varphi(t) > 0$, for almost $t \in \mathbb{J}$ and for almost $x, y \in \mathbb{R}$.

$$|f(t, x(t)) - f(t, y(t))| \leq \varphi(t)|x(t) - y(t)|,$$

(H2). $G : \mathbb{J} \times \mathbb{R} \longrightarrow \mathcal{P}(\mathbb{R})$ is $\mathbb{L}^1$-Carathéodory and possesses nonempty convex values, and for every fixed $x \in C(\mathbb{J}, \mathbb{R})$ the set :

$$\mathcal{S}_{G,x} = \{z \in \mathbb{L}^1(\mathbb{J}, X) : z(t) \in G(t, x(t)); \quad t \in \mathbb{J}\}$$

is nonempty and convex.

(H3). $|G(t, x)| := sup\{|z| : z \in G(t, x)\} \leq p(t)\psi(|x|)$ for all $t \in \mathbb{J}$ and all $x \in C(\mathbb{J}, X)$, where $p \in \mathbb{L}^1(\mathbb{J}, \mathbb{R}^+)$ and $\psi : \mathbb{R}^+ \longrightarrow [0, +\infty)$ is continuous, bounded and nondecreasing function.

(H4) $\|\varphi\|\|p\|\psi(\|x\|)\Phi < \frac{1}{2}$.(where Φ is given by (17))

Consequently,the problem (1) possesses one or more solutions on $\mathbb{J}$.

Proof. Let us introduce the multivalued map $\Delta : C(\mathbb{J}, \mathbb{R}) \longrightarrow \mathcal{P}(C(\mathbb{J}, \mathbb{R}))$, for the purpose of transforming problem (1) into a fixed point problem, we introduce Δ by:

$$\Delta x := \left\{ h' \in C(\mathbb{J}, \mathbb{R}) : h'(t) = \begin{cases} f(t, x(t))\Big[I^{\alpha+\beta;\Psi} z(t) + \dfrac{(\Psi(t) - \Psi(a))^{\gamma+p-1}}{\Lambda\Gamma(\gamma + p)} \\ \times\Big(I^{\alpha+p;\Psi} f(b, x(b))z(b) \\ -\displaystyle\sum_{j=1}^{m} \nu_j I^{\alpha+p;\Psi} f(\tau_j, x(\tau_j))z(\tau_j) \\ -\displaystyle\sum_{i=1}^{n} \omega_i I^{\alpha+p+\sigma_i;\Psi} f(\eta_i, x(\eta_i))z(\eta_i)\Big)\Big] \\ t \in \mathbb{J}, z \in \mathcal{S}_{G,x} \end{cases} \right\}.$$

In what follows we define two operators, $\mathcal{R} : C(\mathbb{J}, \mathbb{R}) \longrightarrow C(\mathbb{J}, \mathbb{R})$ by

$$\mathcal{R}x(t) = f(t, x(t)),$$

and, $\mathcal{T} : C(\mathbb{J}, \mathbb{R}) \longrightarrow \mathcal{P}(C(\mathbb{J}, \mathbb{R}))$ by

$$\mathcal{T}(x) = \Big\{S \in C(J, \mathbb{R}) : S(t) = I^{\alpha+\beta;\Psi} z(t) + \frac{\Psi(t) - \Psi(a))^{\gamma+p-1}}{\Lambda\Gamma(\gamma + p)} \times \Big(I^{\alpha+p;\Psi} f(b, x(b))z(b) - \sum_{j=1}^{m} \nu_j I^{\alpha+p;\Psi} f(\tau_j, x(\tau_j))z(\tau_j) - \sum_{i=1}^{n} \omega_i I^{\alpha+p+\sigma_i;\Psi} f(\eta_i, x(\eta_i))z(\eta_i)\Big); z \in \mathcal{S}_{G,x}\Big\}.$$

Then the operator Δ can be written as $\Delta x = \mathcal{R}x\mathcal{T}x$. Next, We will demonstrate that the operators $\mathcal{R}$ and $\mathcal{T}$ matches the specifications of the theorem 2.9, the proof will be given through a series of steps:
Step 1 : $\mathcal{R}$ is a Lipschitz on $C(\mathbb{J}, \mathbb{R})$.
Let $x, y \in C(\mathbb{J}, \mathbb{R})$, for all $t \in \mathbb{J}$, and by $(H1)$ we have

$$|\mathcal{R}x(t) - \mathcal{R}y(t)| \leq |f(t, x(t)) - f(t, y(t))| \leq |\varphi(t)||x(t) - y(t)|.$$

It follows

$$\|\mathcal{R}x - \mathcal{R}y\| \leq \|\varphi\|\|x - y\|.$$

Step 2 : $\mathcal{T}x$ is convex for every $x \in C(\mathbb{J}, \mathbb{R})$.
If $S_1, S_2 \in \mathcal{T}x$, therefore there exist $z_1, z_2 \in \mathcal{S}_{G,x}$ such that for every $t \in, \mathbb{J}$ we get:

$$S_l(t) = I^{\alpha+\beta;\Psi} z_l(t) + \frac{(\Psi(t) - \Psi(a))^{\gamma+p-1}}{\Lambda\Gamma(\gamma + p)} \times \Big(I^{\alpha+p;\Psi} f(b, x(b))z_l(b) - \sum_{j=1}^{m} \nu_j I^{\alpha+p;\Psi} f(\tau_j, x(\tau_j))z_l(\tau_j) - \sum_{i=1}^{n} \omega_i I^{\alpha+p+\sigma_i;\Psi} f(\eta_i, x(\eta_i))z_l(\eta_i)\Big),$$

For $l = 1, 2$. Let $0 \leq \lambda \leq 1$ consequently, for every $t \in, \mathbb{J}$, we get

$$\begin{aligned}\lambda S_1(t) - (1 - \lambda)S_2(t) = {} & I^{\alpha+\beta;\Psi}\Big(\lambda z_1(t) - (1 - \lambda)z_2(t)\Big) + \frac{(\Psi(t) - \Psi(a))^{\gamma+p-1}}{\Lambda\Gamma(\gamma + p)} \\ & \times \Big(I^{\alpha+p;\Psi} f(b, x(b))\Big(\lambda z_1(b) - (1 - \lambda)z_2(b)\Big) \\ & - \sum_{j=1}^{m} \nu_j I^{\alpha+p;\Psi} f(\tau_j, x(\tau_j))\Big(\lambda z_1(\tau_j) - (1 - \lambda)z_2(\tau_j)\Big) \\ & - \sum_{i=1}^{n} \omega_i I^{\alpha+p+\sigma_i;\Psi} f(\eta_i, x(\eta_i))\Big(\lambda z_1(\eta_i) - (1 - \lambda)z_2(\eta_i)\Big)\Big).\end{aligned}$$

Using the convexity property of $\mathcal{S}_{G,x}$,we get $\lambda S_1 + (1 - \lambda)S_2 \in \mathcal{T}x$, therefore $\mathcal{T}x$ is convex for every $x \in C(\mathbb{J}, \mathbb{R})$
Step 3 : $\mathcal{T}$ is bounded
Indeed, for $\rho > 0$, let $Q_\rho = \{x \in C(J, \mathbb{R}) : \|x\| \leq \rho\}$ denote the bounded ball in $C(\mathbb{J}, \mathbb{R})$, Hence, for every $S \in \mathcal{T}(x)$ and $x \in Q_\rho$ there exists $z \in \mathcal{S}_{G,x}$,in which :

$$S(t) = I^{\alpha+\beta;\Psi} z(t) + \frac{(\Psi(t) - \Psi(a))^{\gamma+p-1}}{\Lambda\Gamma(\gamma + p)}\Big(I^{\alpha+p;\Psi} f(b, x(b))z(b) - \sum_{j=1}^{m} \nu_j I^{\alpha+p;\Psi} f(\tau_j, x(\tau_j))z(\tau_j) - \sum_{i=1}^{n} \omega_i I^{\alpha+p+\sigma_i;\Psi} f(\eta_i, x(\eta_i))z(\eta_i)\Big),$$

then for every $t \in \mathbb{J}$ we have

$$\begin{aligned}
\|S\| &\leq \sup_{t\in J}\Big| I^{\alpha+\beta;\Psi}z(t) + \frac{(\Psi(t)-\Psi(a))^{\gamma+p-1}}{\Lambda\Gamma(\gamma+p)}\Big(I^{\alpha;\psi}f(b,x(b))z(b) \\
&\quad - \sum_{j=1}^{m}\nu_j I^{\alpha+p;\Psi}f(\tau_j,x(\tau_j))z(\tau_j) - \sum_{i=1}^{n}\omega_i I^{\alpha+p+\sigma_i;\Psi}f(\eta_i,x(\eta_i))z(\eta_i)\Big)\Big|, \\
&\leq \|p\|\psi(\|x\|)\Bigg\{\frac{(\Psi(b)-\Psi(a))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{(\Psi(b)-\Psi(a))^{\gamma+p-1}}{|\Lambda|\Gamma(\gamma+p)} \\
&\quad \times\Bigg[\widehat{f}\frac{(\Psi(b)-\Psi(a))^{\alpha+p}}{\Gamma(\alpha+p+1)} + \widehat{f}\sum_{j=1}^{m}|\nu_j|\frac{(\Psi(\tau_j)-\Psi(a))^{\alpha+p}}{\Gamma(\alpha+p+1)} + \\
&\quad \widehat{f}\sum_{i=1}^{n}|\omega_i|\frac{(\Psi(\eta_i)-\Psi(a))^{\alpha+p+\sigma_i}}{\Gamma(\alpha+p+\sigma_i+1)}\Bigg]\Bigg\} \\
&\leq \|p\|\psi(\|x\|)\Phi, \\
&\leq \|p\|\psi(\rho)\Phi,
\end{aligned}$$

then

$$\|S\| \leq \|p\|\psi(\rho)\Phi,$$

where Φ is given by (17). Finaly $\mathcal{T}$ is bounded.

Step 4 : $\mathcal{T}$ maps bounded set into equicontinuous sets of $C(J,\mathbb{R})$.

Let $\zeta_1,\zeta_2 \in J; \zeta_1 < \zeta_2$, and $x \in \mathcal{Q}_\rho$, where $\mathcal{Q}_\rho$ as above then for each $x \in \mathcal{Q}_\rho$ and $S \in \mathcal{T}x$, there exist $z \in \mathcal{S}_{G,x}$ then we have :

$$\begin{aligned}
\Big|S(\zeta_2)-S(\zeta_1)\Big| &\leq \frac{1}{\Gamma(\alpha+\beta)}\Big|\int_a^{\zeta_1}\Psi'(s)\Big((\Psi(\zeta_2)-\Psi(s))^{\alpha+\beta-1} - (\Psi(\zeta_1)-\Psi(s))^{\alpha+\beta-1}\Big) \\
&\quad z(s)ds + \int_{\zeta_1}^{\zeta_2}\Psi'(s)(\Psi(\tau_2)-\Psi(s))^{\alpha+\beta-1}z(s)ds\Big| \\
&\quad + \frac{|(\Psi(\tau_2)-\Psi(a))^{\gamma+p-1} - (\Psi(\tau_1)-\Psi(a))^{\gamma+p-1}|}{|\Lambda|\Gamma(\gamma+p)} \\
&\quad \times\Bigg[\|z(s)\|\widehat{f}\frac{(\Psi(b)-\Psi(a))^{\alpha+p}}{\Gamma(\alpha+p+1)} + \|z(s)\|\widehat{f}\sum_{i=1}^{n}|\omega_i|\frac{(\Psi(\eta_i)-\Psi(a))^{\alpha+p+\sigma_i}}{\Gamma(\alpha+p+\sigma_i+1)} \\
&\quad + \|z(s)\|\widehat{f}\sum_{j=1}^{m}|\nu_i|\frac{(\Psi(\tau_j)-\Psi(a))^{\alpha+p}}{\Gamma(\alpha+p+1)}\Bigg] \\
&\leq \frac{\|p\|\psi(\rho)}{\Gamma(\alpha+\beta)}\Big|\int_a^{\zeta_1}\Psi'(s)\Big((\Psi(\tau_2)-\Psi(s))^{\alpha+\beta-1} - (\Psi(\tau_1)-\Psi(s))^{\alpha+\beta-1}\Big) \\
&\quad + \int_{\zeta_1}^{\zeta_2}\Psi'(t)(\psi(\zeta_2)-\psi(s))^{\alpha+\beta-1}\Big| \\
&\quad + \frac{\|p\|\psi(\rho)|(\Psi(\zeta_2)-\Psi(a))^{\gamma+p-1} - (\Psi(\zeta_1)-\Psi(a))^{\gamma+p-1}|}{|\Lambda|\Gamma(\gamma+p)} \\
&\quad \times\Bigg[\widehat{f}\frac{(\Psi(b)-\Psi(a))^{\alpha+p}}{\Gamma(\alpha+p+1)} + \widehat{f}\sum_{j=1}^{m}|\nu_i|\frac{(\Psi(\tau_j)-\Psi(a))^{\alpha+p}}{\Gamma(\alpha+p+1)} + \\
&\quad \widehat{f}\sum_{i=1}^{n}|\omega_i|\frac{(\Psi(\eta_i)-\Psi(a))^{\alpha+p+\sigma_i}}{\Gamma(\alpha+p+\sigma_i+1)}\Bigg]
\end{aligned}$$

As $\zeta_2 \longrightarrow \zeta_1$ The fact that the right-hand side of the inequality approaches zero, implies that $\mathcal{T}x$ exhibits equicontinuity. By using Arzelà-Ascoli's theorem, we deduce that $\mathcal{T}$is compact, therfore $\mathcal{T}$ is completely continuous.
Next, To establish that the multivalued operator $\mathcal{T}$ is characterized by upper semicontinuity, we should demonstrate that $\mathcal{T}$ has a closed graph.
Step 5 : $\mathcal{T}$ has a closed graph.
Let $x_n \longrightarrow x_*$, $S_n \in \mathcal{T}(x_n)$ and $S_n \longrightarrow S_*$, we shall demonstrate that $S_* \in \mathcal{T}(x_*)$.
For $S_n \in \mathcal{T}(x_n)$, there exists $z_n \in \mathcal{S}_{G,x_n}$ such that for each $t \in \mathbb{J}$,

$$S_n(t) = I^{\alpha+\beta;\Psi} z_n(t) + \frac{(\Psi(t)-\Psi(a))^{\gamma+p-1}}{\Lambda\Gamma(\gamma+p)} \Bigg(I^{\alpha+p;\Psi} f(b, x_n(b)) z_n(b) - \sum_{j=1}^{m} \nu_j I^{\alpha+p;\Psi} f(\tau_j, x(\tau_j)) z_n(\tau_j) - \sum_{i=1}^{n} \omega_i I^{\alpha+p+\sigma_i;\Psi} f(\eta_i, x_n(\eta_i)) z_n(\eta_i)\Bigg).$$

Our task is to demonstrate that $z_* \in \mathcal{S}_{G,x_*}$ exist, in which for every $t \in J$, we determine

$$S_*(t) = I^{\alpha+\beta;\Psi} z_*(t) + \frac{(\Psi(t)-\Psi(a))^{\gamma+p-1}}{\Lambda\Gamma(\gamma+p)} \Bigg(I^{\alpha+p;\Psi} f(b, x_n(b)) z_*(b) - \sum_{j=1}^{m} \nu_j I^{\alpha+p;\Psi} f(\tau_j, x(\tau_j)) z_*(\tau_j) - \sum_{i=1}^{n} \omega_i I^{\alpha+p+\sigma_i;\Psi} f(\eta_i, x_n(\eta_i)) z_*(\eta_i)\Bigg).$$

we have that

$$\begin{aligned}
\|S_n(t) - S_*(t)\| = \|&I^{\alpha+\beta;\Psi}(z_n(t) - z_*(t)) \\
&+ \frac{(\psi(t)-\Psi(a))^{\gamma-1}}{\Lambda\Gamma(\gamma)}\Big(I^{\alpha+p;\Psi} f(b, x(b))(z_n(b) - z_*(b)) \\
&- \sum_{i=1}^{n} \omega_i I^{\alpha+p+\sigma_j;\Psi} f(\eta_i, x(\eta_i))(z_n(\eta_i) - z_*(\eta_i)) \\
&- \sum_{j=1}^{m} \nu_j I^{\alpha+p;\Psi} f(\tau_j, x(\tau_j))(z_n(\tau_j) - z_*(\tau_j))\Big)\| \to 0 \\
&\text{as } n \longrightarrow \infty.
\end{aligned}$$

We introduce the operator as it is described below:

$$\Theta : \mathbb{L}^1(J, \mathbb{R}) \longrightarrow C(\mathbb{J}, \mathbb{R})$$
$$z \longrightarrow \Theta(z)(t).$$

With

$$\Theta(z)(t) = I^{\alpha+\beta;\Psi} z(t) + \frac{(\Psi(t)-\Psi(a))^{\gamma+p-1}}{\Lambda\Gamma(\gamma+p)}\Bigg(I^{\alpha+p;\Psi} f(b, x_n(b)) z(b) - \sum_{j=1}^{m} \nu_j I^{\alpha+p;\Psi} f(\tau_j, x(\tau_j)) z(\tau_j) - \sum_{i=1}^{n} \omega_i I^{\alpha+p+\sigma_i;\Psi} f(\eta_i, x(\eta_i)) z(\eta_i)\Bigg). \tag{19}$$

From Lemma 3.2, the operator $\Theta \circ \mathcal{S}_{G,x}$ possesses the closed graph property.
So, we have $S_n \in \Theta\big(\mathcal{S}_{G,x_n}\big)$, since $x_n \longrightarrow x_*$,and $S_n \longrightarrow S_*$, it follows the existence of $z_* \in \mathcal{S}_{G,x_*}$ in which for

every $t \in \mathbb{J}$:

$$S_*(t) = I^{\alpha+\beta;\Psi} z_*(t) + \frac{(\Psi(t)-\Psi(a))^{\gamma+p-1}}{\Lambda\Gamma(\gamma+p)}\Big(I^{\alpha+p;\Psi} f(b, x_*(b)) z_*(b) - \sum_{j=1}^{m} \nu_j I^{\alpha+p;\Psi} f(\tau_j, x_*(\tau_j)) z_*(\tau_j) - \sum_{i=1}^{n} \omega_i I^{\alpha+p+\sigma_i;\Psi} f(\eta_i, x_*(\eta_i)) z_*(\eta_i)\Big). \tag{20}$$

As a result we deduce that the multivalued operator $\mathcal{T}$ has the properties of compactness and upper semicontinuity.

Step 6 :

We show that $2Ml < 1$.

From Step 3, we get

$$M := \|\mathcal{T}(C(\mathbb{J}, \mathbb{R}))\| = sup\{|\mathcal{T}(x)| : x \in C(\mathbb{J}, \mathbb{R})\} \leq \|p\|\psi(\|x\|)\Phi,$$

it follows by using $(H4)$:

$$\|\varphi\|\|p\|\psi(\|x\|)\Phi < \frac{1}{2},$$

this implies that $2Ml < 1$, where $l = \|\varphi\|$.

Therefore, every condition specified in theorem 2.9 is satisfied., to finish the proof it remains to prove the boundedness of the set $\Omega = \{x \in C(\mathbb{J}, \mathbb{R}) : \theta x \in \mathcal{R}x\mathcal{T}x,\ \theta > 1\}$.

Step 7:

$\Omega = \{x \in X : \theta x \in \mathcal{R}x\mathcal{T}x),\ \theta > 1\}$ is bounded.

If $x \in \Omega$,it follows that $\theta x \in \mathcal{R}x\mathcal{T}x$ for some $\theta > 1$, consequently, a specific function exist $v \in \mathcal{S}_{G,x}$ in which

$$x(t) = \theta^{-1} f(t, x(t))\Big[I^{\alpha+\beta;\Psi} v(t) + \frac{(\Psi(t)-\Psi(a))^{\gamma+p-1}}{\Lambda\Gamma(\gamma+p)}\Big(I^{\alpha+\beta;\Psi} f(b, x(b)) v(b) - \sum_{j=1}^{m} \nu_j I^{\alpha+p;\Psi} f(\tau_j, x(\tau_j)) v(\tau_j) - \sum_{i=1}^{n} \omega_i I^{\alpha+p+\sigma_i;\Psi} f(\eta_i, x(\eta_i)) v(\eta_i)\Big)\Big].$$

Let us put $\mathfrak{f} = \sup_{t\in J} |f(t, 0)| > 0$, and for every $t \in \mathbb{J}$ we establish

$$\begin{aligned}
|x(t)| &\leq \theta^{-1}\Big|f(t, x(t)\Big|\Big|I^{\alpha;\Psi} v(t) + \frac{(\Psi(t)-\Psi(a))^{\gamma-1}}{\Lambda\Gamma(\gamma)}\Big(f(b, x(b)) I^{\alpha;\Psi} v(b) \\
&\quad - \sum_{i=1}^{n} \omega_i f(\eta_i, x(\eta_i)) I^{\alpha;\Psi} v(\eta_i) - \sum_{j=1}^{m} \nu_j f(\tau_j, x(\tau_j)) I^{\alpha+p;\Psi} v(\tau_j)\Big)\Big|, \\
&\leq \Big(\|\varphi\|\|x\| + \mathfrak{f}\Big)\|p\|\psi(\|x\|)\Big\{\frac{(\Psi(b)-\Psi(a))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{(\Psi(b)-\Psi(a))^{\gamma+p-1}}{|\Lambda|\Gamma(\gamma+p)} \\
&\quad \times \Big[\widehat{f}\frac{(\Psi(b)-\Psi(a))^{\alpha+p}}{\Gamma(\alpha+p+1)} + \widehat{f}\sum_{i=1}^{n} |\omega_i| \frac{(\Psi(\eta_i)-\Psi(a))^{\alpha+p+\sigma_j+1}}{\Gamma(\alpha+p+\sigma_j+1)} \\
&\quad + \widehat{f}\sum_{j=1}^{m} |\nu_j| \frac{(\Psi(\tau_j)-\Psi(a))^{\alpha+p}}{\Gamma(\alpha+p+1)}\Big]\Big\} \\
&\leq \Big(\|\varphi\|\|x\| + \mathfrak{f}\Big)\|p\|\psi(\|x\|)\Phi, \\
&\leq \|\varphi\|\|x\|\|p\|\psi(\|x\|)\Phi + \mathfrak{f}\|p\|\psi(\|x\|)\Phi,
\end{aligned}$$

this implies

$$\|x\| \leq \frac{\mathfrak{f}\|p\|\psi(\|x\|)}{1 - \|\varphi\|\|p\|\psi(\|x\|)\Phi}.$$

We conclude that Ω is bounded in $C(\mathbb{J},\mathbb{R})$. If it is not bounded, dividing the above inequality by $\delta = \|x\|$ such that $\delta \longrightarrow +\infty$, then we get

$$1 \leq \frac{\mathfrak{f}\|p\|\psi(\delta)}{\delta(1-\|\varphi\|\|p\|\psi(\delta)\Phi)},$$

by $(H3)$, using the fact that ψ is bounded then there exists $\mathcal{L} > 0$ in which $\psi(\delta) \leq \mathcal{L}$, we get by letting $\delta \longrightarrow +\infty$

$$1 \leq \lim_{\delta\to+\infty} \frac{\mathfrak{f}\|p\|\mathcal{L}}{\delta(1-\|\varphi\|\|p\|\mathcal{L}\Phi)} = 0,$$

Since this leads to a contradiction, it follows that the set Ω is bounded in $C(\mathbb{J},\mathbb{R})$, According to Theorem 2.9, we can then conclude that the operator Δ has at least one fixed point which serves as the solution to problem(1) on $\mathbb{J}$. □

4. Example

Review the following problem

$$\begin{cases} {}^{H}D^{\frac{2}{5},\frac{2}{3};\frac{e^t}{6}}\left({}^{H}D^{\frac{1}{3},\frac{4}{3};\frac{e^t}{6}}\dfrac{x(t)}{sinx(t)+3}\right) \in \left[\dfrac{|x(t)|^3}{2(|x(t)|^3+3)} + \dfrac{t}{4}; \dfrac{|sinx(t)|}{4(|sinx(t)|+5)} + \dfrac{t}{2}\right], \\ 0 \leq t \leq 1 \\ x(0) = 0 \quad , \qquad x(1) = \frac{7}{8} I^{\frac{1}{3},\frac{e^t}{6}}.x(\frac{1}{5}) + \frac{1}{8}.I^{\frac{1}{2},\frac{e^t}{6}} x(\frac{1}{4}) + \frac{1}{3}x(\frac{2}{5}) + \frac{1}{2}x(\frac{2}{3}) \end{cases}$$

where $\alpha = \dfrac{2}{5}$, $\beta = \dfrac{2}{3}$, $p = \frac{1}{3}$,$q = \frac{3}{4}$, $a = 0$, $b = 1$, $\mathbb{J} := [0,1]$, $n = 2$, $\omega_1 = \dfrac{7}{8}$, $\omega_2 = \dfrac{1}{8}$, $\nu_1 = \frac{1}{3}$, $\nu_2 = \frac{1}{2}$, $\eta_1 = \dfrac{1}{6}$, $\eta_2 = \dfrac{1}{2}$,$\sigma_1 = \frac{1}{3}$,$\sigma_2 = \frac{1}{2}$,$\tau_1 = \frac{2}{5}$,$\tau_2 = \frac{2}{3}$ with and $\Psi(t) = \dfrac{e^t}{6}$.

Set, $G : \mathbb{J}\times\mathbb{R} \longrightarrow \mathcal{P}(\mathbb{R})$, is a multivalued map defined by

$$(t,x) \longrightarrow G(t,x) = \left[\frac{|x(t)|^3}{2(|x(t)|^3+3)} + \frac{t}{4}; \frac{|sinx(t)|}{4(|sinx(t)|+5)} + \frac{t}{2}\right].$$

For $z \in G(t,x)$ we have

$$|z| \leq max\left\{\frac{|x(t)|^3}{2(|x(t)|^3+3)} + \frac{t}{4}; \frac{|sinx(t)|}{4(|sinx(t)|+5)} + \frac{t}{2}\right\} \leq \frac{3}{4}.$$

Thus

$$\|G(t,x)\| = sup\{|z| : z \in G(t,x)\} \leq \frac{3}{4} = p(t)\Psi(\|x\|),\ for\ all\ x \in \mathbb{R}.$$

Where $p(t) = 1$, $\Psi(\|x\|) = \dfrac{3}{4}$.

we got $f(t,x) = sinx + 5$. Consequently, for any $x, y \in \mathbb{R}$, we have

$$|f(t,x) - f(t,y)| = |sinx - siny| < |x-y| = \varphi(t)|x-y|,$$

where $\varphi(t) = 1$ for all $t \in \mathbb{J}$.

With the provided information, we can calculate.$\gamma = \alpha+\beta-\alpha\beta = \frac{12}{15}$, $|\Lambda| \simeq 0.293$, $\Phi = 0.5429696$.

Then

$$\|\varphi\|\|p\|\psi(\|x\|)\Phi = 1\times 1\times\frac{3}{4}\times 0.5429696 = 0.4072272 < \frac{1}{2}$$

With all conditions of Theorem 3.3 satisfied, it can be concluded that problem has at least one solution defined on $\mathbb{J}$.

5. Conclusion

The results of this study contribute to the broader field of fractional differential inclusions by addressing a specific and sophisticated type of problem namely, the Ψ-Hilfer hybrid inclusion. By bridging theory and practice, we have not only confirmed the feasibility of solutions under certain conditions but also paved the way for further research in this domain. Future work could explore more diverse types of fractional differential inclusions, extend the results to more complex boundary conditions, or develop more advanced numerical techniques to solve such problem.
In summary, the application of Dhage's fixed point theorem has proven to be a robust approach for solving the Ψ-Hilfer hybrid fractional differential inclusion problem. The combination of theoretical analysis and numerical examples demonstrates the efficacy of our approach and provides a solid foundation for future research in this intricate field of study.

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Data availability

The data used to support the finding of this study are available from the corresponding author upon request.

Conflicts of interest

The authors declare that they have no conflict of interest.

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