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Well-posedness and exponential stability for a shear beam model with a second sound

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Abstract. In this manuscript, we consider a beam model called the Shear beam model (no rotary inertia) with a second sound. First, we establish well-posedness findings by applying the Faedo-Galerkin method, and by constructing an appropriate Lyapunov functional, we show exponential decay findings for the solutions of the system. Furthermore, our obtained findings are not related to any relationship between the system parameters.

1. Introduction

In 1921, Timoshenko [17] introduced the classical system, which is made up of two hyperbolic equations given by

$$\begin{cases} \rho_1\varphi_{\tau\tau} - \kappa\left(\varphi_x + \psi\right)_x = 0 \text{ in } (0, l_0) \times (0, \infty), \\ \rho_2\psi_{\tau\tau} - b\psi_{xx} + \kappa\left(\varphi_x + \psi\right) = 0 \text{ in } (0, l_0) \times (0, \infty), \end{cases}$$

such that the functions ψ and φ represent the rotational angle of the filament of the beam and the transverse displacement, respectively, ρ_1, ρ_2, b and κ are fixed positive physical constants. For almost a century, numerous researchers have spent a lot of time and effort studying this model. In [14] Said-Houari and Laskri considered the Timoshenko system

$$\begin{cases} \rho_1\varphi_{\tau\tau} - \kappa\left(\varphi_x + \psi\right)_x = 0, \ (x, \tau) \in (0, 1) \times (0, \infty), \\ \rho_2\psi_{\tau\tau} - b\psi_{xx} + \kappa\left(\varphi_x + \psi\right) + \mu_1\psi_\tau + \mu_2\psi_\tau\left(x, \tau - r\right) = 0, \ (x, \tau) \in (0, 1) \times (0, \infty), \end{cases} \tag{1}$$

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such that μ_1, μ_2 are positive constants and $r > 0$ represents the time delay. The initial data $(\varphi_0, \varphi_1, \psi_0, \psi_1, f_0)$ belongs to an appropriate functional space. This system is equipped with the next boundary and initial conditions

$$\begin{cases} \varphi(0,\tau) = \varphi(1,\tau) = \psi(0,\tau) = \psi(1,\tau) = 0, \ \tau \in (0,\infty), \\ \varphi(x,0) = \varphi_0(x), \ \varphi_\tau(x,0) = \varphi_1(x), \ \psi(x,0) = \psi_0(x), \ x \in (0,1), \\ \psi_\tau(x,0) = \psi_1(x), \ \psi_\tau(x,\tau - r) = f_0(x,\tau - r), \ (x,\tau) \in (0,1) \times [0,r]. \end{cases}$$

Under a suitable assumption on the weights of the two feedbacks, the authors showed by applying the semigroup method the well-posedness of the system and also demonstrated that (1) is exponentially stable for the equal-speed wave propagation case. After the usual Timoshenko equations, the Shear model is the first set of coupled equations for modeling wave propagation in beams. The Shear model, in essence, considers the action of Shear distortion (but without rotary inertia) on the Euler-Bernoulli model, which results in the coupled equations given by

$$\begin{cases} \rho_1\varphi_{\tau\tau} - \kappa(\varphi_x + \psi)_x = 0 \text{ in } (0,l_0) \times (0,\infty), \\ -b\psi_{xx} + \kappa(\varphi_x + \psi) = 0 \text{ in } (0,l_0) \times (0,\infty), \end{cases}$$

where the functions ψ, ψ_x, φ and $\kappa(\varphi_x + \psi)$ represent, respectively, the angle of rotation due to the bending moment, the dimensionless moment, the dimensionless displacement and the dimensionless Shear. This system is equipped with the next initial and Dirichlet-Neumann boundary conditions

$$\begin{cases} \varphi(0,\tau) = \varphi(l_0,\tau) = \psi_x(0,\tau) = \psi_x(l_0,\tau) = 0, \ \tau \in (0,\infty), \\ \varphi(x,0) = \varphi_0(x), \ \varphi_\tau(x,0) = \varphi_1(x), \ \psi(x,0) = \psi_0(x), \ x \in (0,l_0). \end{cases}$$

In [1], Júnior et al. considered a damped Shear beam model given by

$$\begin{cases} \rho_1\varphi_{\tau\tau} - \kappa(\varphi_x + \psi)_x + \mu\varphi_\tau = 0 \text{ in } (0,l_0) \times (0,\infty), \\ -b\psi_{xx} + \kappa(\varphi_x + \psi) = 0 \text{ in } (0,l_0) \times (0,\infty). \end{cases} \tag{2}$$

This system is equipped with the next initial and boundary conditions

$$\begin{cases} \varphi(0,\tau) = \varphi(l_0,\tau) = \psi_x(0,\tau) = \psi_x(l_0,\tau) = 0, \ \tau \in (0,\infty), \\ \varphi(x,0) = \varphi_0(x), \ \varphi_\tau(x,0) = \varphi_1(x), \ \psi(x,0) = \psi_0(x), \ x \in (0,l_0). \end{cases} \tag{3}$$

The authors, by applying the Faedo-Galerkin method, showed the existence and uniqueness of weak and strong solutions to (2)–(3). Moreover, through the multiplier techniques and the energy method, they proved that the energy $\mathbb{E}(\tau)$ of (2)–(3) decays exponentially, regardless of any relationship between the system coefficients. In [12], Ramos et al., based on Júnior et al. [1], considered a Shear beam model given by

$$\begin{cases} \rho_1\varphi_{\tau\tau} - \kappa(\varphi_x + \psi)_x = 0 \text{ in } (0,l_0) \times (0,\infty), \\ -b\psi_{xx} + \kappa(\varphi_x + \psi) + \gamma\psi_\tau = 0 \text{ in } (0,l_0) \times (0,\infty). \end{cases} \tag{4}$$

This system is equipped with the next initial and boundary conditions

$$\begin{cases} \varphi(0,\tau) = \varphi(l_0,\tau) = \psi_x(0,\tau) = \psi_x(l_0,\tau) = 0, \ \tau > 0, \\ \varphi(x,0) = \varphi_0(x), \ \varphi_\tau(x,0) = \varphi_1(x), \ \psi(x,0) = \psi_0(x), \ 0 < x < l_0. \end{cases} \tag{5}$$

The authors by using semigroup techniques established that the system is non-exponentially stable. In addition, the semigroup theory was used to achieve the system's well-posedness (4)–(5). In [5], Ben Moussa et al. considered a Shear beam system with thermal dissipation given by

$$\begin{cases} \rho\varphi_{\tau\tau} - \kappa(\varphi_x + \psi)_x + \mu\theta_x = 0 \text{ in } (0,l_0) \times (0,\infty), \\ -b\psi_{xx} + \kappa(\varphi_x + \psi) = 0 \text{ in } (0,l_0) \times (0,\infty), \\ c\theta_\tau - \delta\theta_{xx} + \mu\varphi_{x\tau} = 0 \text{ in } (0,l_0) \times (0,\infty), \end{cases}$$

where θ and $c > 0$ represent, respectively, the difference in temperature from the configuration value T_0 and a physical constant that characterizes the heat conductivity of the material. Furthermore, this system is equipped with the next initial and boundary conditions

$$\begin{cases} \varphi(0,\tau) = \varphi(l_0,\tau) = \psi(0,\tau) = \psi(l_0,\tau) = \theta_x(0,\tau) = \theta_x(l_0,\tau) = 0, \ \tau \in (0,\infty), \\ \varphi(x,0) = \varphi_0(x), \ \varphi_\tau(x,0) = \varphi_1(x), \ \psi(x,0) = \psi_0(x), \ \theta(x,0) = \theta_0(x), \ x \in [0,l_0]. \end{cases}$$

The authors investigated a thermoelastic Shear beam model with thermal dissipation. They demonstrated, by using the Faedo-Galerkin method, the well-posedness of the problem and the exponential stability by using the multiplier technique. In addition, they display some numerical experiments to clearly show the theoretical results. Recently, Ayadi et al. [3] considered the following coupled Timoshenko system

$$\begin{cases} \rho_1\varphi_{\tau\tau} - \kappa(\varphi_x + \psi)_x = 0 \text{ in } (0,1)\times(0,\infty), \\ \rho_2\psi_{\tau\tau} - b\psi_{xx} + \kappa(\varphi_x + \psi) + \delta\theta_x + \alpha(\tau)h(\psi_\tau) = 0 \text{ in } (0,1)\times(0,\infty), \\ \rho_3\theta_\tau + q_x + \delta\psi_{x\tau} = 0 \text{ in } (0,1)\times(0,\infty), \\ \varsigma q_\tau + \beta q + \theta_x = 0 \text{ in } (0,1)\times(0,\infty). \end{cases} \tag{6}$$

The authors discussed the solutions' regularity and well-posedness using semi-group theory. Furthermore, for a large class of relaxation functions, they established an explicit and general decay result that is dependent on a stability number μ, which is defined as follows

$$\mu = \left(\varsigma - \frac{\rho_1}{\kappa\rho_3}\right)\left(\frac{\rho_2}{b} - \frac{\rho_1}{\kappa}\right) - \frac{\varsigma\delta^2\rho_1}{b\kappa\rho_3}.$$

This number μ is important in establishing the asymptotic behavior of the energy associated with the system (6). In [13], Ramos et al. considered a one-dimensional piezoelectric beam system with magnetic effect and thermal dissipation given by

$$\begin{cases} \rho v_{\tau\tau} - \alpha v_{xx} + \gamma\beta p_{xx} + \hbar\theta_x = 0 \text{ in } (0,l_0)\times(0,\infty), \\ \mu p_{\tau\tau} - \beta p_{xx} + \gamma\beta v_{xx} = 0 \text{ in } (0,l_0)\times(0,\infty), \\ c\theta_\tau + q_x + \hbar v_{x\tau} = 0 \text{ in } (0,l_0)\times(0,\infty), \\ \varsigma q_\tau + q + k\theta_x = 0 \text{ in } (0,l_0)\times(0,\infty). \end{cases}$$

This system is subjected to the next boundary and initial conditions

$$\begin{cases} v(x,0) = v_0(x), \ v_\tau(x,0) = v_1(x), \ x \in (0,l_0), \\ p(x,0) = p_0(x), \ p_\tau(x,0) = p_1(x), \ x \in (0,l_0), \\ \theta(x,0) = \theta_0(x), \ q(x,0) = q_0(x), \ x \in (0,l_0), \\ v(0,\tau) = \alpha v_x(l_0,\tau) - \gamma\beta p_x(l_0,\tau) = 0, \ \tau \in (0,\infty), \\ p(0,\tau) = \gamma v_x(l_0,\tau) - p_x(l_0,\tau) = 0, \ \tau \in (0,\infty), \\ q(0,\tau) = \theta(l_0,\tau) = 0, \ \tau \in (0,\infty), \end{cases}$$

where ρ, α, γ, μ, β, $\hbar$, c, ς and k are positive constants. The authors showed the existence and uniqueness of system solutions by applying semigroup theory. Also, by using the energy method with the multiplier techniques, they established the system's exponential stability. This result is independent of any relationship between the coefficients. Informed by the above works, in this manuscript we consider the problem

$$\begin{cases} \rho_1\varphi_{\tau\tau} - \kappa(\varphi_x + \psi)_x + \delta\theta_x + \mu\varphi_\tau = 0 \text{ in } (0,l_0)\times(0,\infty), \\ -b\psi_{xx} + \kappa(\varphi_x + \psi) = 0 \text{ in } (0,l_0)\times(0,\infty), \\ c\theta_\tau + q_x + \delta\varphi_{x\tau} = 0 \text{ in } (0,l_0)\times(0,\infty), \\ \varsigma q_\tau + \beta q + \theta_x = 0 \text{ in } (0,l_0)\times(0,\infty), \end{cases} \tag{7}$$

with the next initial and boundary conditions

$$\begin{cases} \varphi(0,\tau) = \varphi(l_0,\tau) = \psi_x(0,\tau) = \psi_x(l_0,\tau) = q(0,\tau) = \theta(l_0,\tau) = 0, \ \tau \in (0,\infty), \\ (\varphi,\varphi_\tau,\psi,\theta,q)(x,0) = (\varphi_0,\varphi_1,\psi_0,\theta_0,q_0)(x), \ x \in (0,l_0). \end{cases}$$

This paper's goal is to investigate the well-posedness and asymptotic behavior of the solution of (7) with a second sound. We establish the exponential decay. Furthermore, our results are not related to any relationship between the system's parameters. The role of the second sound term and its effect on the asymptotic behavior of the solution appear in many works for various types of problems. To learn more about this term, we recommend that readers consult the following sources [2, 7, 8, 10] in the case of Timoshenko, porous-elastic systems, Bresse systems, and thermoelastic Laminated beams (see also [9, 11, 16] and reference therein). Because the boundary conditions on ψ are of Newmann type, we present a few transformation that enables application of the Poincaré inequality to ψ. According to the second equation in (7), we get

$$-b\int_0^{l_0}\psi_{xx}dx+\kappa\int_0^{l_0}(\varphi_x+\psi)\,dx=0. \tag{8}$$

So, by solving (8), we obtain

$$\int_0^{l_0}\psi(x,\tau)\,dx=0,\ \forall\tau\geq 0. \tag{9}$$

Outline of the manuscript. This manuscript is organized as follows. In Section 2, we apply the Faedo-Galerkin method to establish the well-posedness of (7). Next, in Section 3, we demonstrate that the system (7) is exponentially stable by constructing a suitable Lyapunov functional.

2. The well-posedness of the problem

By applying the Faedo-Galerkin method in this section, we demonstrate the existence of a weak solution for (7). To achieve this, we use the Sobolev space $H_0^1(0,l_0)$ and the Lebesgue space $L^2(0,l_0)$, with their usual scalar products and norms. Let us define the space $\mathcal{H}$ as follows:

$$\mathcal{H}=H_0^1(0,l_0)\times L^2(0,l_0)\times H_*^1(0,l_0)\times L^2(0,l_0)\times L^2(0,l_0),$$

where $H_*^1(0,l_0)=H^1(0,l_0)\cap L_*^2(0,l_0)$ such that

$$L_*^2(0,l_0)=\left\{f\in L^2(0,l_0):\int_0^{l_0}f(x)\,dx=0\right\},$$

and

$$H_*^2(0,l_0)=\left\{f\in H^2(0,l_0):f_x(0)=f_x(l_0)=0\right\}.$$

We now define the following spaces as follows:

$$H_\alpha^1(0,l_0)=\left\{f\in H^1(0,l_0):f(l_0)=0\right\}\text{ and }H_\varrho^1(0,l_0)=\left\{f\in H^1(0,l_0):f(0)=0\right\}.$$

Definition 2.1. *Let $(\varphi_0,\varphi_1,\psi_0,\theta_0,q_0)\in\mathcal{H}$. We will say that (φ,ψ,θ,q) or $U=(\varphi,\varphi_\tau,\psi,\theta,q)$ is a weak solution of (7), if $U\in C([0,T_0];\mathcal{H})$ and satisfies*

$$\begin{cases}\rho_1\int_0^{l_0}\varphi_{\tau\tau}u\,dx+\kappa\int_0^{l_0}(\varphi_x+\psi)\,u_x\,dx+\delta\int_0^{l_0}\theta_x u\,dx+\mu\int_0^{l_0}\varphi_\tau u\,dx=0,\ \forall u\in H_0^1(0,l_0),\\ b\int_0^{l_0}\psi_x w_x\,dx+\kappa\int_0^{l_0}(\varphi_x+\psi)\,w\,dx=0,\ \forall w\in H_*^1(0,l_0),\\ c\int_0^{l_0}\theta_\tau v\,dx+\int_0^{l_0}q_x v\,dx+\delta\int_0^{l_0}\varphi_{\tau x}v\,dx=0,\ \forall v\in L^2(0,l_0),\\ \varsigma\int_0^{l_0}q_\tau z\,dx+\beta\int_0^{l_0}qz\,dx+\int_0^{l_0}\theta_x z\,dx=0,\ \forall z\in L^2(0,l_0),\end{cases} \tag{10}$$

for a.e. $\tau\in(0,T_0)$ and

$$(\varphi,\varphi_\tau,\psi,\theta,q)(0)=(\varphi_0,\varphi_1,\psi_0,\theta_0,q_0). \tag{11}$$

Theorem 2.2. *If the initial data* $(\varphi_0, \varphi_1, \psi_0, \theta_0, q_0) \in \mathcal{H}$*, then (7) admits a weak solution satisfying*

$$\begin{cases} \varphi \in L^\infty(0, T_0; H_0^1(0, l_0)),\ \varphi_\tau \in L^\infty(0, T_0; L^2(0, l_0)),\ \psi \in L^\infty(0, T_0; H_*^1(0, l_0)), \\ \theta \in L^\infty(0, T_0; L^2(0, l_0)),\ q \in L^\infty(0, T_0; L^2(0, l_0)). \end{cases}$$

Proof. The proof is provided by applying the Faedo-Galerkin method. Let us describe the method. We divide into four steps. In Step 1, solutions of the approximate problem. In Step 2, give energy estimates for approximate solutions. In Step 3, passage to limits. In Step 4, justify the initial conditions.

Step 1. Approximate problem. Let $\{u_j\}_{j=1}^\infty$, $\{v_j\}_{j=1}^\infty$, $\{w_j\}_{j=1}^\infty$ and $\{z_j\}_{j=1}^\infty$ be orthonormal bases in $H^2(0, l_0) \cap H_0^1(0, l_0)$, $H_*^2(0, l_0) \cap H_*^1(0, l_0)$, $H_\alpha^1(0, l_0)$ and $H_\varrho^1(0, l_0)$, respectively, which are all of them orthonormal in $L^2(0, l_0)$, and the both bases $\{u_j\}_{j=1}^\infty$, $\{v_j\}_{j=1}^\infty$ constituted by the eigenfunctions of $-\partial_{xx}(.)$ associated, respectively, to the eigenvalues $\left\{\lambda_j'\right\}$ and $\left\{\lambda_j''\right\}$, that are

$$-\partial_{xx} u_j = \lambda_j' u_j \text{ and } -\partial_{xx} v_j = \lambda_j'' v_j,\ 1 \le j \le n.$$

Now, for every integer $n \in \mathbb{N}$, we define the finite-dimensional subspaces by

$$(U_n, V_n, W_n, Z_n) = (span\{u_1, u_2, ..., u_n\}, span\{v_1, v_2, ..., v_n\}, span\{w_1, w_2, ..., w_n\}, span\{z_1, z_2, ..., z_n\}).$$

So, we will find an approximate solution

$$(\varphi^n, \psi^n, \theta^n, q^n)(x, \tau) = \left(\sum_{j=1}^n f_j^n(\tau) u_j(x), \sum_{j=1}^n g_j^n(\tau) v_j(x), \sum_{j=1}^n h_j^n(\tau) w_j(x), \sum_{j=1}^n L_j^n(\tau) z_j(x) \right),$$

for the following approximate problem

$$\begin{cases} \rho_1(\varphi_{\tau\tau}^n, u) + \kappa(\varphi_x^n + \psi^n, u_x) + \delta(\theta_x^n, u) + \mu(\varphi_\tau^n, u) = 0,\ \forall u \in U_n, \\ b(\psi_x^n, v_x) + \kappa(\varphi_x^n + \psi^n, v) = 0,\ \forall v \in V_n, \\ c(\theta_\tau^n, w) + (q_x^n, w) + \delta(\varphi_{\tau x}^n, w) = 0,\ \forall w \in W_n, \\ \varsigma(q_\tau^n, z) + \beta(q^n, z) + (\theta_x^n, z) = 0,\ \forall z \in Z_n, \end{cases} \tag{12}$$

with the initial conditions

$$(\varphi^n, \varphi_\tau^n, \psi^n, \theta^n, q^n)(0) = \left(\varphi_0^n, \varphi_1^n, \psi_0^n, \theta_0^n, q_0^n\right), \tag{13}$$

and

$$\left(\varphi_0^n, \varphi_1^n, \psi_0^n, \theta_0^n, q_0^n\right) \to (\varphi_0, \varphi_1, \psi_0, \theta_0, q_0) \text{ strongly in } \mathcal{H}. \tag{14}$$

Substituting φ^n, ψ^n, θ^n, q^n into (12) and taking $u = u_j$, $v = v_j$, $w = w_j$ and $z = z_j$, for $j = 1, ..., n$, we get the linear ordinary differential system shown below

$$\begin{cases} \rho_1 f_{j\tau\tau}^n + \lambda_j' \kappa f_j^n + \kappa \sum\limits_{k=1}^n \left\langle v_k, u_{jx} \right\rangle g_k^n - \delta \sum\limits_{k=1}^n \left\langle w_k, u_{jx} \right\rangle h_k^n + \mu f_{j\tau}^n = 0, \\ \left(b\lambda_j'' + \kappa\right) g_j^n + \kappa \sum\limits_{k=1}^n \left\langle u_{kx}, v_j \right\rangle f_k^n = 0, \\ c h_{j\tau}^n + \sum\limits_{k=1}^n \left\langle z_{kx}, w_j \right\rangle L_k^n - \delta \sum\limits_{k=1}^n \left\langle u_k, w_{jx} \right\rangle f_{k\tau}^n = 0, \\ \varsigma L_{j\tau}^n + \beta L_j^n - \sum\limits_{k=1}^n \left\langle w_k, z_{jx} \right\rangle h_k^n = 0, \end{cases} \tag{15}$$

with initial conditions

$$\begin{cases} f_j^n(0) = \left(\varphi_0^n, u_j\right),\ f_{j\tau}^n(0) = \left(\varphi_1^n, u_j\right),\ g_j^n(0) = \left(\psi_0^n, v_j\right), \\ h_j^n(0) = \left(\theta_0^n, w_j\right),\ L_j^n(0) = \left(q_0^n, z_j\right),\ j = 1, ..., n. \end{cases}$$

The application of the basic ODE theory yields the existence of a unique local solution $U_n = (\varphi^n, \varphi^n_\tau, \psi^n, \theta^n, q^n)$ for (15) in a maximal interval $[0, \tau_n)$ with $0 < \tau_n \le T_0$ for every $n \in \mathbb{N}$.

Step 2. Energy estimates for approximate solutions. Replacing u by φ^n_τ, $v = \psi^n_\tau$, $w = \theta^n$ and $z = q^n$ in (12), we obtain

$$\begin{cases} \rho_1 \int_0^{l_0} \varphi^n_{\tau\tau}\varphi^n_\tau dx + \kappa \int_0^{l_0} (\varphi^n_x + \psi^n)\, \varphi^n_{\tau x} dx + \delta \int_0^{l_0} \theta^n_x \varphi^n_\tau dx + \mu \int_0^{l_0} \left|\varphi^n_\tau\right|^2 dx = 0, \\ b \int_0^{l_0} \psi^n_x \psi^n_{x\tau} dx + \kappa \int_0^{l_0} (\varphi^n_x + \psi^n)\, \psi^n_\tau dx = 0, \\ c \int_0^{l_0} \theta^n_\tau \theta^n dx + \int_0^{l_0} q^n_x \theta^n dx + \delta \int_0^{l_0} \varphi^n_{\tau x} \theta^n dx = 0, \\ \varsigma \int_0^{l_0} q^n_\tau q^n dx + \beta \int_0^{l_0} \left|q^n\right|^2 dx + \int_0^{l_0} \theta^n_x q^n dx = 0. \end{cases}$$

By performing some calculations, we obtain

$$\frac{d}{d\tau}\mathbb{E}^n(\tau) + \mu \int_0^{l_0} \left|\varphi^n_\tau\right|^2 dx + \beta \int_0^{l_0} \left|q^n\right|^2 dx = 0, \tag{16}$$

where

$$\mathbb{E}^n(\tau) := \frac{\rho_1}{2}\int_0^{l_0} \left|\varphi^n_\tau\right|^2 dx + \frac{b}{2}\int_0^{l_0} \left|\psi^n_x\right|^2 dx + \frac{\kappa}{2}\int_0^{l_0} \left|\varphi^n_x + \psi^n\right|^2 dx + \frac{c}{2}\int_0^{l_0} \left|\theta^n\right|^2 dx + \frac{\varsigma}{2}\int_0^{l_0} \left|q^n\right|^2 dx.$$

Then, according to (16), for all $\tau \in [0, T_0]$, $n \in \mathbb{N}$, we find

$$\mathbb{E}^n(\tau) + \mu \int_0^\tau \int_0^{l_0} \left|\varphi^n_\tau\right|^2 dxds + \beta \int_0^\tau \int_0^{l_0} \left|q^n\right|^2 dxds = \mathbb{E}^n(0) \le \sigma_1, \tag{17}$$

such that $\sigma_1 > 0$ depends on the initial data. Hence, approximate solutions are defined for the whole range $[0, T_0]$.

Step 3. Passage to the limit. According to (17) and the definition of $\mathbb{E}^n(\tau)$, we conclude that

$$\begin{cases} \{\varphi^n\} \text{ is bounded in } L^\infty\left(0, T_0; H^1_0(0, l_0)\right), \\ \{\varphi^n_\tau\} \text{ is bounded in } L^\infty\left(0, T_0; L^2(0, l_0)\right), \\ \{\psi^n\} \text{ is bounded in } L^\infty\left(0, T_0; H^1_*(0, l_0)\right), \\ \{\theta^n\} \text{ is bounded in } L^\infty\left(0, T_0; L^2(0, l_0)\right), \\ \{q^n\} \text{ is bounded in } L^\infty\left(0, T_0; L^2(0, l_0)\right). \end{cases} \tag{18}$$

Then we can extract a subsequence of $\{\varphi^n\}, \{\psi^n\}, \{\theta^n\}$ and $\{q^n\}$ still denoted by $\{\varphi^n\}, \{\psi^n\}, \{\theta^n\}$ and $\{q^n\}$ such that

$$\begin{cases} \varphi^n \overset{*}{\rightharpoonup} \varphi \text{ in } L^\infty\left(0, T_0; H^1_0(0, l_0)\right), \\ \varphi^n_\tau \overset{*}{\rightharpoonup} \varphi_\tau \text{ in } L^\infty\left(0, T_0; L^2(0, l_0)\right), \\ \psi^n \overset{*}{\rightharpoonup} \psi \text{ in } L^\infty\left(0, T_0; H^1_*(0, l_0)\right), \\ \theta^n \overset{*}{\rightharpoonup} \theta \text{ in } L^\infty\left(0, T_0; L^2(0, l_0)\right), \\ q^n \overset{*}{\rightharpoonup} q \text{ in } L^\infty\left(0, T_0; L^2(0, l_0)\right). \end{cases} \tag{19}$$

Using the weak star convergence (19), we obtain for every $j \le n$,

$$\begin{cases} \kappa \int_0^{l_0} (\varphi^n_x + \psi^n)\, u_{jx} dx + \delta \int_0^{l_0} \theta^n_x u_j dx + \mu \int_0^{l_0} \varphi^n_\tau u_j dx \\ \to \kappa \int_0^{l_0} (\varphi_x + \psi)\, u_{jx} dx + \delta \int_0^{l_0} \theta_x u_j dx + \mu \int_0^{l_0} \varphi_\tau u_j dx \text{ in } L^\infty(0, T_0) \\ b \int_0^{l_0} \psi^n_x v_{jx} dx + \kappa \int_0^{l_0} (\varphi^n_x + \psi^n)\, v_j dx \\ \to b \int_0^{l_0} \psi_x v_{jx} dx + \kappa \int_0^{l_0} (\varphi_x + \psi)\, v_j dx \text{ in } L^\infty(0, T_0) \\ \int_0^{l_0} q^n_x w_j dx + \delta \int_0^{l_0} \varphi^n_{\tau x} w_j dx \to \int_0^{l_0} q_x w_j dx + \delta \int_0^{l_0} \varphi_{\tau x} w_j dx, \text{ in } L^\infty(0, T_0) \\ \beta \int_0^{l_0} q^n z_j dx + \int_0^{l_0} \theta^n_x z_j dx \to \beta \int_0^{l_0} q z_j dx + \int_0^{l_0} \theta_x z_j dx \text{ in } L^\infty(0, T_0). \end{cases} \tag{20}$$

Note that if $u_n \to u$ weakly star in $L^\infty(0,T_0)$, then

$$\int_0^{T_0} \frac{du_n}{d\tau} g(\tau)\, d\tau = -\int_0^{T_0} u_n g'(\tau)\, d\tau \to -\int_0^{T_0} u g'(\tau)\, d\tau = \int_0^{T_0} \frac{du}{d\tau} g(\tau)\, d\tau,\ \forall g \in C_0^1(0,T_0).$$

So, $\frac{du_n}{d\tau} \to \frac{du}{d\tau}$ in $\mathfrak{D}'(0,T_0)$. Using this information to $\int_0^{l_0} \varphi_{\tau\tau}^n u_j dx$, $\int_0^{l_0} \theta_\tau^n w_j dx$ and $\int_0^{l_0} q_\tau^n z_j dx$, we get

$$\begin{cases} \rho_1 \int_0^{l_0} \varphi_{\tau\tau}^n u_j dx \to \rho_1 \int_0^{l_0} \varphi_{\tau\tau} u_j dx \text{ in } \mathfrak{D}'(0,T_0), \\ c \int_0^{l_0} \theta_\tau^n w_j dx \to c \int_0^{l_0} \theta_\tau w_j dx \text{ in } \mathfrak{D}'(0,T_0), \\ \varsigma \int_0^{l_0} q_\tau^n z_j dx \to \varsigma \int_0^{l_0} q_\tau z_j dx \text{ in } \mathfrak{D}'(0,T_0). \end{cases} \tag{21}$$

Passing to the limit in (12), using (20)–(21), we get for all $j \geq 1$,

$$\begin{cases} \rho_1 \int_0^{l_0} \varphi_{\tau\tau} u_j dx + \kappa \int_0^{l_0} (\varphi_x + \psi) u_{jx} dx + \delta \int_0^{l_0} \theta_x u_j dx + \mu \int_0^{l_0} \varphi_\tau u_j dx = 0, \\ b \int_0^{l_0} \psi_x v_{jx} dx + \kappa \int_0^{l_0} (\varphi_x + \psi) v_j dx = 0, \\ c \int_0^{l_0} \theta_\tau w_j dx + \int_0^{l_0} q_x w_j dx + \delta \int_0^{l_0} \varphi_{\tau x} w_j dx = 0, \\ \varsigma \int_0^{l_0} q_\tau z_j dx + \beta \int_0^{l_0} q z_j dx + \int_0^{l_0} \theta_x z_j dx = 0. \end{cases}$$

Since $\{u_j\}_{j=1}^\infty$, $\{v_j\}_{j=1}^\infty$, $\{w_j\}_{j=1}^\infty$ and $\{z_j\}_{j=1}^\infty$ are Hilbert bases in $H_0^1(0,l_0)$, $H_*^1(0,l_0)$, $L^2(0,l_0)$ and $L^2(0,l_0)$, respectively, (10) follows immediately.

Step 4. Initial data. By applying the Aubin-Lions-Simon theorem (Theorem **II.5.16,** [6]), because

$$\begin{cases} \text{The embedding of } H_0^1(0,l_0) \text{ in } L^2(0,l_0) \text{ is compact,} \\ \text{The embedding of } L^2(0,l_0) \text{ in } L^2(0,l_0) \text{ is continuous.} \end{cases}$$

Then, we get the embedding of $\mathbb{E}_{\infty,\infty}$ in $C(0,T_0;L^2(0,l_0))$ is compact where

$$\mathbb{E}_{\infty,\infty} = \left\{ \varphi^n \in L^\infty\left(0,T_0;H_0^1(0,l_0)\right),\ \varphi_\tau^n = \frac{d\varphi^n}{d\tau} \in L^\infty\left(0,T_0;L^2(0,l_0)\right) \right\},$$

by (18), we get $\{\varphi^n\}$ bounded in $\mathbb{E}_{\infty,\infty}$, then there exist $\{\varphi^n\}$ subsequence of $\{\varphi^n\}$

$$\varphi^n \overset{n\to\infty}{\longrightarrow} \varphi \text{ strongly in } C(0,T_0;L^2(0,l_0)).$$

Therefore,

$$\varphi(0) = \varphi_0.$$

Let $\eta(\tau)$ be a smooth function in τ with $\eta \in C^\infty([0,T_0])$ such that $\eta(0) = 1$ and $\eta(T_0) = 0$. Next, by integration the third equation of (10) with respect to τ over $(0,T_0)$ and using integration by part, then taking $v = \eta(\tau)\pi(x)$, we obtain for every $\pi \in H_0^1(0,l_0)$,

$$c \int_0^{T_0} \int_0^{l_0} \theta_\tau \eta(\tau)\pi(x) dx d\tau - \int_0^{T_0} \int_0^{l_0} q\eta(\tau)\pi_x(x) dx d\tau - \delta \int_0^{T_0} \int_0^{l_0} \varphi_\tau \eta(\tau)\pi_x(x) dx d\tau = 0.$$

The integration of the first term by part yields

$$\int_0^{T_0} \int_0^{l_0} \theta_\tau \eta(\tau)\pi(x) dx d\tau = -\int_0^{l_0} \theta(x,0)\pi(x) dx - \int_0^{T_0} \int_0^{l_0} \theta \eta_\tau(\tau)\pi(x) dx d\tau.$$

Thus,

$$c\int_0^{l_0}\theta(0)\,\pi(x)dx = -c\int_0^{T_0}\int_0^{l_0}\theta(x,\tau)\,\eta_\tau(\tau)\pi(x)dxd\tau - \int_0^{T_0}\int_0^{l_0} q\eta(\tau)\pi_x(x)dxd\tau$$
$$-\delta\int_0^{T_0}\int_0^{l_0}\varphi_\tau\eta(\tau)\pi_x(x)dxd\tau, \tag{22}$$

for all $\pi \in H_0^1(0,l_0)$. By using a similar technique with the third equation of (12) and by exploiting the initial conditions (13), we obtain

$$c\int_0^{l_0}\theta_0^n\pi(x)dx = -c\int_0^{T_0}\int_0^{l_0}\theta^n(x,\tau)\,\eta_\tau(\tau)\pi(x)dxd\tau - \int_0^{T_0}\int_0^{l_0} q^n\eta(\tau)\pi_x(x)dxd\tau$$
$$-\delta\int_0^{T_0}\int_0^{l_0}\varphi_\tau^n\eta(\tau)\pi_x(x)dxd\tau. \tag{23}$$

By recalling (14) and (19) and passing to the limit in (23), we get

$$c\int_0^{l_0}\theta_0\pi(x)dx = -c\int_0^{T_0}\int_0^{l_0}\theta(x,\tau)\,\eta_\tau(\tau)\pi(x)dxd\tau - \int_0^{T_0}\int_0^{l_0} q\eta(\tau)\pi_x(x)dxd\tau$$
$$-\delta\int_0^{T_0}\int_0^{l_0}\varphi_\tau\eta(\tau)\pi_x(x)dxd\tau, \tag{24}$$

for all $\pi \in H_0^1(0,l_0)$. The comparison of (22) and (24) leads to

$$\theta(0) = \theta_0.$$

Similarly, we get

$$(\varphi_\tau,\psi,q)(0) = (\varphi_1,\psi_0,q_0).$$

As a result, (φ,ψ,θ,q) is weak solution of (7). Using (10) and similar technique to obtain (16), we have

$$\frac{d}{d\tau}\mathbb{E}(\tau) = -\mu\int_0^{l_0}\left|\varphi_\tau\right|^2dx - \beta\int_0^{l_0}\left|q\right|^2dx \le 0, \tag{25}$$

where

$$\mathbb{E}(\tau) := \frac{\rho_1}{2}\int_0^{l_0}\left|\varphi_\tau\right|^2dx + \frac{b}{2}\int_0^{l_0}\left|\psi_x\right|^2dx + \frac{\kappa}{2}\int_0^{l_0}\left|\varphi_x+\psi\right|^2dx + \frac{c}{2}\int_0^{l_0}|\theta|^2dx + \frac{\varsigma}{2}\int_0^{l_0}\left|q\right|^2dx.$$

Then, from (25), we can deduce that for every $\tau \in [0,T_0]$, $n \in \mathbb{N}$,

$$\mathbb{E}(\tau) \le \mathbb{E}(0) \le \sigma_2, \tag{26}$$

where $\sigma_2 > 0$ depends on the initial data. As a result, the solution of (7) can be applied to the entire interval $[0,T_0]$. So,

$$\sup_{\tau\in[0,T_0]}\mathbb{E}(\tau) \le \mathbb{E}(0) \le \sigma_2.$$

Hence,

$$\left\|\varphi\right\|_{L^\infty(0,T_0;H_0^1(0,l_0))} + \left\|\psi\right\|_{L^\infty(0,T_0;H_*^1(0,l_0))} + \left\|\varphi_\tau\right\|_{L^\infty(0,T_0;L^2(0,l_0))} + \|\theta\|_{L^\infty(0,T_0;L^2(0,l_0))} + \left\|q\right\|_{L^\infty(0,T_0;L^2(0,l_0))} \le \sigma_2.$$

Consequently, we obtain

$$\begin{cases} \varphi \in L^\infty(0,T_0;H_0^1(0,l_0)),\ \varphi_\tau \in L^\infty(0,T_0;L^2(0,l_0)),\ \psi \in L^\infty(0,T_0;H_*^1(0,l_0)), \\ \theta \in L^\infty(0,T_0;L^2(0,l_0)),\ q \in L^\infty(0,T_0;L^2(0,l_0)). \end{cases}$$

which completes the proof of the Theorem (2.2). □

3. Exponential stability

In this section, we use the multiplier technique to demonstrate the energy of the system's solution is exponentially stable. So, we need the next lemmas.

Lemma 3.1. *Let $U = (\varphi, \varphi_\tau, \psi, \theta, q)$ be the solution of (7). Then, the energy functional, defined by*

$$\mathbb{E}(\tau) = \frac{1}{2}\int_0^{l_0} \left[\rho_1\varphi_\tau^2 + b\psi_x^2 + \kappa\left(\varphi_x + \psi\right)^2 + c\theta^2 + \varsigma q^2\right] dx, \tag{27}$$

satisfies

$$\mathbb{E}'(\tau) = -\beta\int_0^{l_0} q^2 dx - \mu\int_0^{l_0} \varphi_\tau^2 dx \leq 0. \tag{28}$$

Proof. We multiply $(7)_1$, $(7)_2$, $(7)_3$, $(7)_4$ by φ_τ, ψ_τ, θ, q respectively, and integrate over $(0, l_0)$, through the integration by parts and the boundary conditions, we arrive at (28). □

Lemma 3.2. *Let $(\varphi, \psi, \theta, q)$ be a solution of (7). Then, the functional*

$$I_1(\tau) = \rho_1\int_0^{l_0} \varphi_\tau\varphi dx,\ \tau \geq 0,$$

satisfies

$$\begin{aligned} I_1'(\tau) \leq &-\left(b - \varepsilon_1\left(2\mu C_P^2 + 2\delta C_P\right)\right)\int_0^{l_0}\psi_x^2 dx - \left(\kappa - \varepsilon_1\left(2\delta + 2\mu C_P\right)\right)\int_0^{l_0}\left(\varphi_x + \psi\right)^2 dx \\ &+ \left(\rho_1 + \frac{\mu}{4\varepsilon_1}\right)\int_0^{l_0}\varphi_\tau^2 dx + \frac{\delta}{4\varepsilon_1}\int_0^{l_0}\theta^2 dx. \end{aligned}$$

Choosing $\varepsilon_1 \leq \min\left(\frac{b}{4\mu C_P^2 + 4\delta C_P}, \frac{\kappa}{4\delta + 4\mu C_P}\right)$, *we get*

$$I_1'(\tau) \leq -\frac{b}{2}\int_0^{l_0}\psi_x^2 dx - \frac{\kappa}{2}\int_0^{l_0}\left(\varphi_x + \psi\right)^2 dx + \left(\rho_1 + \frac{\mu}{4\varepsilon_1}\right)\int_0^{l_0}\varphi_\tau^2 dx + \frac{\delta}{4\varepsilon_1}\int_0^{l_0}\theta^2 dx. \tag{29}$$

Proof. By differentiating $I_1(\tau)$, exploiting $(7)_1$, $(7)_2$, applying the integration by parts and through the boundary conditions, we arrive at

$$\begin{aligned} I_1'(\tau) &= -b\int_0^{l_0}\psi_x^2 dx - \kappa\int_0^{l_0}\left(\varphi_x + \psi\right)^2 dx + \rho_1\int_0^{l_0}\varphi_\tau^2 dx - \delta\int_0^{l_0}\varphi\theta_x dx - \mu\int_0^{l_0}\varphi_\tau\varphi dx \\ &= -b\int_0^{l_0}\psi_x^2 dx - \kappa\int_0^{l_0}\left(\varphi_x + \psi\right)^2 dx + \rho_1\int_0^{l_0}\varphi_\tau^2 dx + \delta\int_0^{l_0}\varphi_x\theta dx - \mu\int_0^{l_0}\varphi_\tau\varphi dx. \end{aligned} \tag{30}$$

By applying the Young and Poincaré inequalities, we obtain

$$\begin{aligned} \delta\int_0^{l_0}\varphi_x\theta dx &\leq \delta\varepsilon_1\int_0^{l_0}\varphi_x^2 dx + \frac{\delta}{4\varepsilon_1}\int_0^{l_0}\theta^2 dx \\ &\leq 2\delta\varepsilon_1\int_0^{l_0}\left(\varphi_x + \psi\right)^2 dx + 2\delta C_P\varepsilon_1\int_0^{l_0}\psi_x^2 dx + \frac{\delta}{4\varepsilon_1}\int_0^{l_0}\theta^2 dx, \end{aligned} \tag{31}$$

and

$$-\mu\int_0^{l_0}\varphi\varphi_\tau dx \le \mu\varepsilon_1 C_P\int_0^{l_0}\varphi_x^2dx+\frac{\mu}{4\varepsilon_1}\int_0^{l_0}\varphi_\tau^2dx$$
$$\le 2\mu\varepsilon_1 C_P\int_0^{l_0}(\varphi_x+\psi)^2dx+2\mu\varepsilon_1 C_P^2\int_0^{l_0}\psi_x^2dx+\frac{\mu}{4\varepsilon_1}\int_0^{l_0}\varphi_\tau^2dx. \tag{32}$$

We get (29) by substituting (31) and (32) in (30). □

Lemma 3.3. *Let (φ,ψ,θ,q) be a solution of (7). Then, the functional*

$$I_2(\tau)=-\varsigma\int_0^{l_0}q\int_0^x\theta(y)\,dydx,\ \tau\ge 0,$$

satisfies

$$I_2'(\tau)\le-(1-\beta l_0\varepsilon_3)\int_0^{l_0}\theta^2dx+\left(\frac{\beta l_0}{4\varepsilon_3}+\frac{\varsigma}{c}+\frac{\varsigma\delta}{c}\varepsilon_4\right)\int_0^{l_0}q^2dx+\frac{\varsigma\delta}{4c\varepsilon_4}\int_0^{l_0}\varphi_\tau^2dx. \tag{33}$$

Proof. By differentiating $I_2(\tau)$, from $(7)_3$, $(7)_4$ and through the boundary conditions with integration by parts, we arrive at

$$\begin{aligned}I_2'(\tau)&=-\varsigma\int_0^{l_0}q_\tau\int_0^x\theta(y)\,dydx-\varsigma\int_0^{l_0}q\int_0^x\theta_\tau(y)\,dydx\\&=\beta\int_0^{l_0}q\int_0^x\theta(y)\,dydx+\int_0^{l_0}\theta_x\int_0^x\theta(y)\,dydx+\frac{\varsigma}{c}\int_0^{l_0}q\int_0^x q_ydydx+\frac{\varsigma\delta}{c}\int_0^{l_0}q\int_0^x\varphi_{y\tau}dydx\\&=\beta\int_0^{l_0}q\int_0^x\theta(y)\,dydx+\int_0^{l_0}\theta_x\int_0^x\theta(y)\,dydx+\frac{\varsigma}{c}\int_0^{l_0}q^2dx+\frac{\varsigma\delta}{c}\int_0^{l_0}q\varphi_\tau dx\\&=\beta\int_0^{l_0}q\int_0^x\theta(y)\,dydx-\int_0^{l_0}\theta^2dx+\frac{\varsigma}{c}\int_0^{l_0}q^2dx+\frac{\varsigma\delta}{c}\int_0^{l_0}q\varphi_\tau dx.\end{aligned} \tag{34}$$

By applying the Young inequality, we obtain

$$\beta\int_0^{l_0}q\int_0^x\theta(y)\,dydx\le\beta l_0\varepsilon_3\int_0^{l_0}\theta^2dx+\frac{\beta l_0}{4\varepsilon_3}\int_0^{l_0}q^2dx, \tag{35}$$

and

$$\frac{\varsigma\delta}{c}\int_0^{l_0}q\varphi_\tau dx\le\frac{\varsigma\delta}{c}\varepsilon_4\int_0^{l_0}q^2dx+\frac{\varsigma\delta}{4c\varepsilon_4}\int_0^{l_0}\varphi_\tau^2dx. \tag{36}$$

Substituting (36) and (35) in (34), we get (33). □

Now, we give the next definition of the Lyapunov functional

$$\mathbb{L}(\tau):=N_0\mathbb{E}(\tau)+I_1(\tau)+N_1I_2(\tau), \tag{37}$$

such that N_0 and N_1 are positive constants.

Lemma 3.4. *Let (φ,ψ,θ,q) be the solution of (7). Then, there exist two positive constants κ_1 and κ_2 such that the Lyapunov functional (37) satisfies*

$$\kappa_1\mathbb{E}(\tau)\le\mathbb{L}(\tau)\le\kappa_2\mathbb{E}(\tau),\ \forall\tau\ge 0, \tag{38}$$

and

$$\mathbb{L}'(\tau)\le-\beta_1\mathbb{E}(\tau),\ \forall\tau\ge 0. \tag{39}$$

Proof. Through (37), we have

$$|\mathbb{L}(\tau) - N_0\mathbb{E}(\tau)| \leq \rho_1 \int_0^{l_0} \left|\varphi_\tau\varphi\right| dx + N_1\varsigma \int_0^{l_0} \left|q\right| \int_0^x \left|\theta(y)\right| dydx.$$

Using $\varphi_x^2 \leq 2(\varphi_x + \psi)^2 + 2\psi^2$ and applying the inequalities of Poincaré, Young and Cauchy-Schwarz, we arrive at

$$|\mathbb{L}(\tau) - N_0\mathbb{E}(\tau)| \leq C_1N_1\mathbb{E}(\tau),$$

which yields

$$(N_0 - C_1N_1)\,\mathbb{E}(\tau) \leq \mathbb{L}(\tau) \leq (N_0 + C_1N_1)\,\mathbb{E}(\tau),$$

where $C_1 = l_0 \max\left\{1, \frac{\varsigma}{c}\right\}$, $\max\left\{\frac{1}{N_1}, \frac{2\rho_1 C_P^2}{bN_1}, \frac{2\rho_1 C_P}{\kappa N_1}\right\} \leq l_0$ and N_1 is large enough such that $N_1 \geq 1$ with C_P is the Poincaré constant. By selecting N_0 sufficiently large such that $N_0 > C_1N_1$ we find (38). From now on, we will show that (39) holds. By utilizing (28), (29), and (33), we get to

$$\begin{aligned}\mathbb{L}'(\tau) \leq &-\left(N_0\beta - N_1\left(\frac{\beta l_0}{4\varepsilon_3} + \frac{\varsigma}{c} + \frac{\varsigma\delta}{c}\varepsilon_4\right)\right)\int_0^{l_0} q^2dx - \left(N_0\mu - \left(\rho_1 + \frac{\mu}{4\varepsilon_1}\right) - N_1\frac{\varsigma\delta}{4c\varepsilon_4}\right)\int_0^{l_0}\varphi_\tau^2dx \\ &- \frac{b}{2}\int_0^{l_0}\psi_x^2dx - \frac{\kappa}{2}\int_0^{l_0}(\varphi_x + \psi)^2dx - \left(N_1(1 - \beta l_0\varepsilon_3) - \frac{\delta}{4\varepsilon_1}\right)\int_0^{l_0}\theta^2dx.\end{aligned}$$

By setting $\varepsilon_3 = \dfrac{1}{2\beta l_0}$,

$$\begin{aligned}\mathbb{L}'(\tau) \leq &-\left(N_0\beta - N_1\left(\frac{\beta^2 l_0^2}{2} + \frac{\varsigma}{c} + \frac{\varsigma\delta}{c}\varepsilon_4\right)\right)\int_0^{l_0} q^2dx - \left(N_0\mu - \left(\rho_1 + \frac{\mu}{4\varepsilon_1}\right) - N_1\frac{\varsigma\delta}{4c\varepsilon_4}\right)\int_0^{l_0}\varphi_\tau^2dx \\ &- \frac{b}{2}\int_0^{l_0}\psi_x^2dx - \frac{\kappa}{2}\int_0^{l_0}(\varphi_x + \psi)^2dx - \left(\frac{N_1}{2} - \frac{\delta}{4\varepsilon_1}\right)\int_0^{l_0}\theta^2dx.\end{aligned}$$

We now choose the following parameters appropriately.

First, we select N_1 large enough such that $N_1 \geq 1$ and

$$\delta_0 = \frac{N_1}{2} - \frac{\delta}{4\varepsilon_1} > 0.$$

Next, we pick N_0 large enough so that $N_0 > C_1N_1$,

$$\delta_1 = N_0\beta - N_1\left(\frac{\beta^2 l_0^2}{2} + \frac{\varsigma}{c} + \frac{\varsigma\delta}{c}\varepsilon_4\right) > 0 \text{ and } \delta_2 = N_0\mu - \left(\rho_1 + \frac{\mu}{4\varepsilon_1}\right) - N_1\frac{\varsigma\delta}{4c\varepsilon_4} > 0.$$

So, we end up with

$$\mathbb{L}'(\tau) \leq -\delta_1\int_0^{l_0} q^2dx - \delta_2\int_0^{l_0}\varphi_\tau^2dx - \frac{b}{2}\int_0^{l_0}\psi_x^2dx - \frac{\kappa}{2}\int_0^{l_0}(\varphi_x + \psi)^2dx - \delta_0\int_0^{l_0}\theta^2dx.$$

Moreover,

$$\mathbb{L}'(\tau) \leq -\varpi\left[\int_0^{l_0} q^2dx + \int_0^{l_0}\varphi_\tau^2dx + \int_0^{l_0}\psi_x^2dx + \int_0^{l_0}(\varphi_x + \psi)^2dx + \int_0^{l_0}\theta^2dx\right], \tag{40}$$

where $\varpi = \min\left(\delta_0, \delta_1, \delta_2, \frac{b}{2}, \frac{\kappa}{2}\right) > 0$. We have it on the other hand,

$$\mathbb{E}(\tau) \leq C_2 \left[\int_0^{l_0} q^2 dx + \int_0^{l_0} \varphi_\tau^2 dx + \int_0^{l_0} \psi_x^2 dx + \int_0^{l_0} (\varphi_x + \psi)^2 dx + \int_0^{l_0} \theta^2 dx \right],$$

which implies that

$$-\left[\int_0^{l_0} q^2 dx + \int_0^{l_0} \varphi_\tau^2 dx + \int_0^{l_0} \psi_x^2 dx + \int_0^{l_0} (\varphi_x + \psi)^2 dx + \int_0^{l_0} \theta^2 dx \right] \leq -C_3 \mathbb{E}(\tau). \tag{41}$$

The combination of (40) and (41) gives (39). □

In what follows, we estimate the system's energy (7) using the equivalence relation (38) and the estimation (39). Now, we can specify and demonstrate the following stability result.

Theorem 3.5. *Let $(\varphi, \psi, \theta, q)$ be the solution of (7). Then, the solution $(\varphi, \psi, \theta, q)$ decays exponentially, i.e., there are two positive constants, λ_1 and λ_2 such that*

$$\mathbb{E}(\tau) \leq \lambda_2 e^{-\lambda_1 \tau}, \ \forall \tau \geq 0. \tag{42}$$

Proof. By exploiting the estimation (39), we get to

$$\mathbb{L}'(\tau) \leq -\beta_1 \mathbb{E}(\tau), \ \forall \tau \geq 0.$$

Using $\mathbb{L} \sim \mathbb{E}$, we deduce that

$$\mathbb{L}'(\tau) \leq -\lambda_1 \mathbb{L}(\tau), \ \forall \tau \geq 0, \tag{43}$$

with $\lambda_1 = \beta_1/\kappa_2 > 0$. By performing integration of (43), we get

$$\mathbb{L}(\tau) \leq \mathbb{L}(0) e^{-\lambda_1 \tau}, \ \forall \tau \geq 0. \tag{44}$$

By exploiting $\mathbb{L} \sim \mathbb{E}$ and (44), there exist a constant $\lambda_2 = \mathbb{L}(0)/\kappa_1 > 0$ such that (42) holds. The proof is complete. □

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