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Novel classes on generating functions for the products of Gaussian Pell Padovan and Gaussian Perrin numbers

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Abstract. In this work, we prove a new theorem by using properties related to symmetric functions. All of the conclusions drawn in this work are based on this theorem. We introduce new generating functions for the product of Gaussian Pell and Padovan numbers, Gaussian Perrin numbers with (p, q)-numbers, and the product of earlier Gaussian numbers with bivariate polynomials.

1. Introduction and main results

Many modern sciences depend in their research and theories on sequences. However, they are employed in the fields of mathematics and physics, one such sequence is the Fibonacci sequence, which is regarded as one of the most well-known sequences and has been examined by several scholars for many decades due to its significance. It has been used extensively in many different fields and sciences (physics, biology, computer science, engineering, mathematics, etc.) for many centuries by numerous researchers and scientists. Many authors have made various generalizations about these numbers [5]. Some have preserved the recursive relationship while changing the initial conditions, while others have generalized this sequence by keeping the initial conditions after making a small adjustment to the recursive relationship.

In this study, we are interested in sequences of numbers defined by third-order recurrences relations such as Gaussian Pell Padovan and Gaussian Perrin which are defined respectively by [11, 23]

$$\begin{cases} GR_n = 2GR_{n-2} + GR_{n-3}, \quad n \geq 3, \\ GR_0 = 1 - i; GR_1 = 1 + i; GR_2 = 1 + i, \end{cases}$$

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and

$$\begin{cases} Gr_n = Gr_{n-2} + Gr_{n-3}, & n \geq 3, \\ Gr_0 = -1 + 3i; Gr_1 = 3; Gr_2 = 2i. \end{cases}$$

There are many studies on these two kinds of sequences. For examples, Kartal in [11] defined and studied the Gaussian Padovan and Gaussian Perrin numbers, Bhoi and Ray in [3] explored Perrin numbers expressed as sums of two base representations. Moreover, Zerroug in [25] worked on Gaussian Padovan, Gaussian Pell-Padovan numbers, and new generating functions for certain numbers and polynomials.

The (p,q)-Pell and (p,q)-Pell Lucas numbers were recently defined by Gulec and Taskara in [9]. They demonstrated the matrix sequences and examined their properties. Regarding the (p,q)-Fibonacci numbers, Suvarnamani and Tatong in [22] explored several findings utilizing the well-known Binet's formula. Moreover, Suvarnamani in [21] derived useful characteristics of the (p,q)-Lucas numbers and introduced innovative identities for the (p,q)-Fibonacci numbers using matrix techniques in a follow-up study. In our work, we focus on the application of certain (p,q)-numbers, such as (p,q)-Fibonacci, (p,q)-Lucas and (p,q)-Pell Lucas numbers, as defined respectively by

$$\begin{cases} F_{p,q,n} = pF_{p,q,n-1} + qF_{p,q,n-2}, & n \geq 2 \\ F_{p,q,0} = 0; F_{p,q,1} = 1 \end{cases},$$

$$\begin{cases} L_{p,q,n} = pL_{p,q,n-1} + qL_{p,q,n-2}, & n \geq 2 \\ L_{p,q,0} = 2; L_{p,q,1} = p \end{cases},$$

and

$$\begin{cases} Q_{p,q,n} = 2pQ_{p,q,n-1} + qQ_{p,q,n-2}, & n \geq 2 \\ Q_{p,q,0} = 2; Q_{p,q,1} = 2p \end{cases}.$$

On the other hand, Alves in [2] introduced the notion of bivariate Mersenne polynomials, then in [15] the bivariate Mersenne Lucas polynomials are defined by the same recurrence as bivariate Mersenne polynomials but with different initial terms which are defined by

$$\begin{cases} m_n(x,y) = 3ym_{n-1}(x,y) - 2xm_{n-2}(x,y), & n \geq 2 \\ m_0(x,y) = 2; m_1(x,y) = 3y \end{cases}. \tag{1}$$

Additionally, Catalani defined the bivariate Fibonacci and bivariate Lucas polynomials in [8] by

$$\begin{cases} F_n(x,y) = xF_{n-1}(x,y) + yF_{n-2}(x,y), & n \geq 2 \\ F_0(x,y) = 0; F_1(x,y) = 1 \end{cases}, \tag{2}$$

$$\begin{cases} L_n(x,y) = xL_{n-1}(x,y) + yL_{n-2}(x,y), & n \geq 2 \\ L_0(x,y) = 2, \quad L_1(x,y) = x \end{cases}. \tag{3}$$

In [24], Zorcelik and Uygun introduced sequences called bivariate Jacobsthal and bivariate Jacobsthal Lucas polynomial sequences as follows

$$\begin{cases} J_n(x,y) = xyJ_{n-1}(x,y) + 2yJ_{n-2}(x,y), & n \geq 2 \\ J_0(x,y) = 0, \quad J_1(x,y) = 1 \end{cases}, \tag{4}$$

$$\begin{cases} j_n(x,y) = xyj_{n-1}(x,y) + 2yj_{n-2}(x,y), & n \geq 2 \\ j_0(x,y) = 2, \quad j_1(x,y) = xy \end{cases}. \tag{5}$$

Lastly, in [20], H.Serpil and A.Zeynep defined the bivariate Pell polynomials as follows

$$\begin{cases} P_n(x,y) = 2xyP_{n-1}(x,y) + yP_{n-2}(x,y), & n \geq 2 \\ P_0(x,y) = 0, \quad P_1(x,y) = 1 \end{cases}. \tag{6}$$

Remark 1.1. *If we set y = 1 in Eqs.* (1)-(6)*, we obtain the recurrence relations of Mersenne Lucas, Fibonacci, Lucas, Jacobsthal, Jacobsthal Lucas and Pell polynomials, respectively.*

Remark 1.2. *If we put x=y = 1 in Eqs.* (1)-(6)*, we get the recurrence relations of Mersenne Lucas, Fibonacci, Lucas, Jaocobsthal, Jacobsthal Lucas and Pell numbers, respectively.*

In this part, we present the preliminary tools and notions necessary for understanding the following section, we introduce some definitions of the symmetric functions that are needed in this part and will be utilized throughout the paper.

Definition 1.3. *[1] Let A and E be two alphabets. Then $S_n(A-E)$ is defined by the following form*

$$\sum_{n=0}^{\infty} S_n(A-E)t^n = \frac{\prod\limits_{e\in E}(1-et)}{\prod\limits_{a\in A}(1-at)}, \tag{7}$$

with the condition $S_n(A-E) = 0$ for $n < 0$.

Corollary 1.4. *Taking $A = \{0\}$ in* (7) *gives*

$$\sum_{n=0}^{\infty} S_n(-E)t^n = \prod_{e\in E}(1-et). \tag{8}$$

Thus, we obtain

$$\sum_{n=0}^{\infty} S_n(A-E)t^n = \left(\sum_{n=0}^{\infty} S_n(A)t^n\right)\times\left(\sum_{n=0}^{\infty} S_n(-E)t^n\right).$$

Remark 1.5. *If $A = E$, so*

$$\sum_{n=0}^{\infty} S_n(A)t^n = \frac{1}{\sum\limits_{n=0}^{\infty} S_n(-A)t^n}.$$

Definition 1.6. *[18] Let n be a positive integer and $A = \{a_1, a_2\}$ is a set of given variables. Then, the n^{th} symmetric function $S_n(a_1 + a_2)$ is defined by*

$$S_n(A) = S_n(a_1+a_2) = \frac{a_1^{n+1} - a_2^{n+1}}{a_1 - a_2},$$

with

$$\begin{aligned} S_0(A) &= S_0(a_1+a_2) = 1, \\ S_1(A) &= S_1(a_1+a_2) = a_1 + a_2, \\ S_2(A) &= S_2(a_1+a_2) = a_1^2 + a_1a_2 + a_2^2, \\ &\vdots \end{aligned}$$

Definition 1.7. *[7] The symmetrizing operator $\delta^k_{a_1a_2}$ is defined by*

$$\delta^k_{a_1a_2} f(a_1) = \frac{a_1^k f(a_1) - a_2^k f(a_2)}{a_1 - a_2}, \quad \text{for all } k \in \mathbb{N}_0.$$

Gaussian Pell Padovan numbers	$GR_n = (1-i)S_n(E) + (1+i)S_{n-1}(E) + (-1+3i)S_{n-2}(E)$
Gaussian Perrin numbers	$Gr_n = (-1+3i)S_n(E) + 3S_{n-1}(E) + (1-i)S_{n-2}(E)$
(p,q)-Fibonacci numbers	$F_{p,q,n} = S_{n-1}(a_1 + [-a_2])$
(p,q)-Pell Lucas numbers	$Q_{p,q,n} = 2S_n(a_1 + [-a_2] - 2pS_{n-1}(a_1 + [-a_2])$
(p,q)-Lucas numbers	$L_{p,q,n} = 2S_n(a_1 + [-a_2] - pS_{n-1}(a_1 + [-a_2])$
bivariate Pell polynomials	$P_n(x,y) = S_{n-1}(a_1 + [-a_2])$
bivariate Mersenne Lucas polynomials	$m_n(x,y) = 2S_n(a_1 + [-a_2]) - 3yS_{n-1}(a_1 + [-a_2])$
bivariate Fibonacci polynomials	$F_n(x,y) = S_{n-1}(a_1 + [-a_2])$
bivariate Lucas polynomials	$L_n(x,y) = 2S_n(a_1 + [-a_2]) - xS_{n-1}(a_1 + [-a_2])$
bivariate Jacobsthal polynomials	$S_n(a_1 + [-a_2])$
bivariate Jacobsthal-Lucas polynomials	$j_n(x,y) = 2S_n(a_1 + [-a_2]) - xyS_{n-1}(a_1 + [-a_2])$

Proposition 1.8. *[14, 17, 19] Let's consider $n \in \mathbb{N}$, the symmetric functions of Gaussian Pell Padovan, Gaussian Perrin, (p,q)-Fibonacci, (p,q)-Pell Lucas, (p,q)-Lucas numbers, bivariate Pell, bivariate Mersenne Lucas, bivariate Fibonacci and Lucas polynomials, bivariate Jacobsthal and Jacobsthal Lucas polynomials are respectively given by*

In this paper, we proved a new main theorem using the operator $\delta_{a_1a_2}^{k-l+1}$. From this theorem, we obtained novel generating functions of numbers and polynomials.

Theorem 1.9. *Let $E = \{e_1, e_2, e_3\}$ and $A = \{a_1, a_2\}$ be two alphabets that we have*

$$\sum_{n=0}^{\infty} S_n(E)\, S_{n+k-l}(A)\, z^n =$$

$$\frac{S_{k-l}(A) + S_1(-E)a_1a_2S_{k-l-1}(A)z + S_2(-E)a_1^2a_2^2S_{k-l-2}(A)z^2 - a_1^{k-l+1}a_2^{k-l+1}z^{k-l+2}\sum_{n=0}^{\infty} S_{n+k-l+2}(-E)\, S_n(A)\, z^n}{\left(\sum_{n=0}^{\infty} S_n(-E)\, a_1^n z^n\right)\left(\sum_{n=0}^{\infty} S_n(-E)\, a_2^n z^n\right)}, \quad (9)$$

for all $n, k \in \mathbb{N}_0$ and $k, l \in \{0, 1, 2, 3, 4, 5\}$.

The rest of the paper is organized as follows: The proof of the theorem is provided in Section 2. In section 3, we derive new generating functions for the products of Gaussian numbers with (p,q)-Fibonacci, (p,q)-Lucas, (p,q)-Pell Lucas numbers, bivariate Mersenne-Lucas polynomials, bivariate Fibonacci and bivariate Lucas polynomials, bivariate Pell polynomials, and bivariate Jacobsthal and Jacobsthal-Lucas polynomials. In the final section, Section 4, we present an application in signal processing.

2. Proof of Theorem

Applying the operator $\delta_{a_1a_2}^{k-l+1}$ to the series $f(a_1z) = \sum_{n=0}^{\infty} S_n(E)\, a_1^n z^n$, we get

$$\begin{aligned}
\delta_{a_1a_2}^{k-l+1} f(a_1z) &= \frac{a_1^{k-l+1}\sum_{n=0}^{\infty} S_n(E)\, a_1^n z^n - a_2^{k-l+1}\sum_{n=0}^{\infty} S_n(E)\, a_2^n z^n}{a_1 - a_2} \\
&= \sum_{n=0}^{\infty} S_n(E)\left(\frac{a_1^{n+k-l+1} - a_2^{n+k-l+1}}{a_1 - a_2}\right) z^n \qquad (10) \\
&= \sum_{n=0}^{\infty} S_n(E)\, S_{n+k-l}(A) z^n.
\end{aligned}$$

Alternatively, by using the operation $\delta_{a_1a_2}^{k-l+1}$ to the series $f(a_1z) = \dfrac{1}{\sum\limits_{n=0}^{\infty} S_n(-E)\,a_1^n z^n}$, we get

$$\begin{aligned}
\delta_{a_1a_2}^{k-l+1} f(e_1z) &= \frac{\dfrac{a_1^{k-l+1}}{\sum\limits_{n=0}^{\infty} S_n(-E)\,a_1^n z^n} - \dfrac{a_2^{k-l+1}}{\sum\limits_{n=0}^{\infty} S_n(-E)\,a_2^n z^n}}{a_1 - a_2} \\
&= \frac{\sum\limits_{n=0}^{\infty} S_n(-E)\,a_1^n a_2^n \dfrac{a_1^{k-l+1-n} - a_2^{k-l+1-n}}{a_1 - a_2} z^n}{\left(\sum\limits_{n=0}^{\infty} S_n(-E)\,a_1^n z^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-E)\,a_2^n z^n\right)}.
\end{aligned}$$

Equivalently,

$$\begin{aligned}
\delta_{a_1a_2}^{k-l+1} f(a_1z) &= \frac{\sum\limits_{n=0}^{\infty} S_n(-E)\,a_1^n a_2^n S_{k-l-n}(A)\,z^n}{\left(\sum\limits_{n=0}^{\infty} S_n(-E)\,a_1^n z^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-E)\,a_2^n z^n\right)} \\
&= \frac{\sum\limits_{n=0}^{k-l} S_n(-E)\,a_1^n a_2^n S_{k-l-n}(A)\,z^n + \sum\limits_{n=k-l+2}^{\infty} S_n(-E)\,a_1^n a_2^n S_{k-l-n}(A)\,z^n}{\left(\sum\limits_{n=0}^{\infty} S_n(-E)\,a_1^n z^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-E)\,a_2^n z^n\right)} \\
&= \frac{\begin{array}{c} S_{k-l}(A) + S_1(-E)a_1a_2S_{k-l-1}(A)z + S_2(-E)a_1^2a_2^2S_{k-l-2}(A)z^2 \\ - \sum\limits_{n=k-l+2}^{\infty} S_n(-E)\,a_1^{k-l+1}a_2^{k-l+1}\left(\dfrac{a_1^{n-k+l-1} - a_2^{n-k+l-1}}{a_1 - a_2}\right) z^n \end{array}}{\left(\sum\limits_{n=0}^{\infty} S_n(-E)\,a_1^n z^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-E)\,a_2^n z^n\right)} \\
&= \frac{\begin{array}{c} S_{k-l}(A) + S_1(-E)a_1a_2S_{k-l-1}(A)z + S_2(-E)a_1^2a_2^2S_{k-l-2}(A)z^2 \\ -a_1^{k-l+1}a_2^{k-l+1} \sum\limits_{n=k-l+2}^{\infty} S_n(-E).S_{n-k+l-2}(A)z^n \end{array}}{\left(\sum\limits_{n=0}^{\infty} S_n(-E)\,a_1^n z^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-E)\,a_2^n z^n\right)},
\end{aligned}$$

which also gives

$$\delta_{a_1a_2}^{k-l+1} f(a_1z) = \frac{\begin{array}{c} S_{k-l}(A) + S_1(-E)a_1a_2S_{k-l-1}(A)z + S_2(-E)a_1^2a_2^2S_{k-l-2}(A)z^2 \\ -a_1^{k-l+1}a_2^{k-l+1}z^{k-l+2} \sum\limits_{n=0}^{\infty} S_{n-k+l+2}(-E)\,S_n(A)\,z^n \end{array}}{\left(\sum\limits_{n=0}^{\infty} S_n(-E)\,a_1^n z^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-E)\,a_2^n z^n\right)}.$$

3. Lemmas

In this section, according to Theorem 1.9, we provide some lemmas.

- For $A = \{a_1, a_2\}$, $E = \{e_1, e_2, e_3\}$ and $k, l \in \{0, 1, 2, 3, 4, 5\}$ in Theorem 1.9, we deduce the following lemmas.

Lemma 3.1. *[12] Considering that $E = \{e_1, e_2, e_3\}$ and $A = \{a_1, a_2\}$ are two alphabets, we are given*

$$\sum_{n=0}^{\infty} S_n(E)S_n(A)z^n = \frac{1 - a_1a_2S_2(-E)z^2 - a_1a_2(a_1+a_2)S_3(-E)z^3}{\left(\sum\limits_{n=0}^{\infty} S_n(-E)a_1^nz^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-E)a_2^nz^n\right)}. \tag{11}$$

Corollary 3.2. *Relationship* (11) *allows us to get*

$$\sum_{n=0}^{\infty} S_{n-1}(E)S_{n-1}(A)z^n = \frac{z - a_1a_2S_2(-E)z^3 - a_1a_2(a_1+a_2)S_3(-E)z^4}{\left(\sum\limits_{n=0}^{\infty} S_n(-E)a_1^nz^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-E)a_2^nz^n\right)}. \tag{12}$$

Lemma 3.3. *[12] Provided that $E = \{e_1, e_2, e_3\}$ and $A = \{a_1, a_2\}$ are two alphabets, we are given*

$$\sum_{n=0}^{\infty} S_n(E)S_{n-1}(A)z^n = \frac{-S_1(-E)z - (a_1+a_2)S_2(-E)z^2 - ((a_1+a_2)^2 - a_1a_2)S_3(-E)z^3}{\left(\sum\limits_{n=0}^{\infty} S_n(-E)a_1^nz^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-E)a_2^nz^n\right)}. \tag{13}$$

Using the relationship (13), we obtain

$$\sum_{n=0}^{\infty} S_{n-1}(E)S_{n-2}(A)z^n = \frac{-S_1(-E)z^2 - (a_1+a_2)S_2(-E)z^3 - ((a_1+a_2)^2 - a_1a_2)S_3(-E)z^4}{\left(\sum\limits_{n=0}^{\infty} S_n(-E)a_1^nz^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-E)a_2^nz^n\right)}. \tag{14}$$

Lemma 3.4. *[12] Assuming that $E = \{e_1, e_2, e_3\}$ and $A = \{a_1, a_2\}$ are two alphabets, we obtain*

$$\sum_{n=0}^{\infty} S_n(E)S_{n+1}(A)z^n = \frac{(a_1+a_2) + S_1(E)a_1a_2z - a_1^2a_2^2S_3(-E)z^2}{\left(\sum\limits_{n=0}^{\infty} S_n(-E)a_1^nz^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-E)a_2^nz^n\right)}. \tag{15}$$

According to relationship (15), we obtain

$$\sum_{n=0}^{\infty} S_{n-1}(E)S_n(A)z^n = \frac{(a_1+a_2)z + S_1(-E)a_1a_2z^2 - a_1^2a_2^2S_3(-E)z^3}{\left(\sum\limits_{n=0}^{\infty} S_n(-E)a_1^nz^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-E)a_2^nz^n\right)}. \tag{16}$$

According to relationship (16), we obtain

$$\sum_{n=0}^{\infty} S_{n-2}(E)S_{n-1}(A)z^n = \frac{(a_1+a_2)z^2 + S_1(-E)a_1a_2z^3 - a_1^2a_2^2S_3(-E)z^4}{\left(\sum\limits_{n=0}^{\infty} S_n(-E)a_1^nz^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-E)a_2^nz^n\right)}. \tag{17}$$

Lemma 3.5. *[12] Assuming that $E = \{e_1, e_2, e_3\}$ and $A = \{a_1, a_2\}$, are two alphabets, then we are given*

$$\sum_{n=0}^{\infty} S_n(E)S_{n+2}(A)z^n = \frac{[(a_1+a_2)^2 - a_1a_2] + S_1(-E)a_1a_2(a_1+a_2)z + a_1^2a_2^2S_2(-E)z^2}{\left(\sum\limits_{n=0}^{\infty} S_n(-E)a_1^nz^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-E)a_2^nz^n\right)}. \tag{18}$$

Using the relationship (18), we get

$$\sum_{n=0}^{\infty} S_{n-2}(E)S_n(A)z^n = \frac{[(a_1+a_2)^2 - a_1a_2]z^2 + a_1a_2(a_1+a_2)S_1(-E)z^3 + a_1^2a_2^2S_2(-E)z^4}{\left(\sum\limits_{n=0}^{\infty} S_n(-E)a_1^nz^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-E)a_2^nz^n\right)}. \tag{19}$$

- For the case where $A = \{a_1, -a_2\}$ and $E = \{e_1, e_2, e_3\}$, by replacing a_2 with $(-a_2)$ in equations (11), (12), (13), (14), (16), (17) and (19), we have

$$\sum_{n=0}^{\infty} S_n(E)S_n(a_1 + [-a_2])z^n = \frac{1 + a_1a_2S_2(-E)z^2 + a_1a_2(a_1 - a_2)S_3(-E)z^3}{\prod_{e\in E}(1 - ea_1z)\prod_{e\in E}(1 + ea_2z)}, \tag{20}$$

$$\sum_{n=0}^{\infty} S_{n-1}(E)S_{n-1}(a_1 + [-a_2])z^n = \frac{z + a_1a_2S_2(-E)z^3 + a_1a_2(a_1 - a_2)S_3(-E)z^4}{\prod_{e\in E}(1 - ea_1z)\prod_{e\in E}(1 + ea_2z)}, \tag{21}$$

$$\sum_{n=0}^{\infty} S_n(E)S_{n-1}(a_1 + [-a_2])z^n = \frac{-S_1(E)z - (a_1 - a_2)S_2(-E)z^2 - ((a_1 - a_2)^2 + a_1a_2)S_3(-E)z^3}{\prod_{e\in E}(1 - ea_1z)\prod_{e\in E}(1 + ea_2z)}, \tag{22}$$

$$\sum_{n=0}^{\infty} S_{n-1}(E)S_{n-2}(a_1 + [-a_2])z^n = \frac{-S_1(E)z^2 - (a_1 - a_2)S_2(-E)z^3 - ((a_1 - a_2)^2 + a_1a_2)S_3(-E)z^4}{\prod_{e\in E}(1 - ea_1z)\prod_{e\in E}(1 + ea_2z)}, \tag{23}$$

$$\sum_{n=0}^{\infty} S_{n-1}(E)S_n(a_1 + [-a_2])z^n = \frac{(a_1 - a_2)z - S_1(-E)a_1a_2z^2 - a_1^2a_2^2S_3(-E)z^4}{\prod_{e\in E}(1 - ea_1z)\prod_{e\in E}(1 + ea_2z)}, \tag{24}$$

$$\sum_{n=0}^{\infty} S_{n-2}(E)S_{n-1}(a_1 + [-a_2])z^n = \frac{(a_1 - a_2)z^2 - S_1(-E)a_1a_2z^3 + a_1^2a_2^2S_3(-E)z^5}{\prod_{e\in E}(1 - ea_1z)\prod_{e\in E}(1 + ea_2z)}, \tag{25}$$

$$\sum_{n=0}^{\infty} S_{n-2}(E)S_n(a_1 + [-a_2])z^n = \frac{[(a_1 - a_2)^2 + a_1a_2]z^2 - a_1a_2(a_1 - a_2)S_1(E)z^3 + a_1^2a_2^2S_2(-E)z^3}{\prod_{e\in E}(1 - ea_1z)\prod_{e\in E}(1 + ea_2z)}, \tag{26}$$

$$\begin{aligned}\prod_{e\in E}(1 - ea_1z)\prod_{e\in E}(1 + ea_2z) \quad = \quad & 1 + (a_1 - a_2)S_1(-E)z + [(a_1 - a_2)^2S_2(-E) - a_1a_2(S_1^2(-E) - 2S_2(-E))]z^2 \\ & +[(a_1 - a_2)^3S_3(-E) - a_1a_2(a_1 - a_2)(S_1(-E)S_2(-E) - 3S_3(-E))]z^3 \\ & -[a_1a_2(a_1 - a_2)^2S_3(-E)S_1(-E) - a_1^2a_2^2(S_2^2(-E) - 2S_3(-E)S_1(-E))]z^4 \\ & +a_1^2a_2^2(a_1 - a_2)S_3(-E)S_2(-E)z^5a_1^3a_2^3S_3^2(-E)z^6.\end{aligned}$$

4. Generating functions of the products of Gaussians numbers with (p, q)-numbers

In this section, we derive new generating functions of the products of Gaussians Pell Padovan and Gaussian Perrin numbers with (p, q)-Fibonacci, (p, q)-Lucas and (p, q)-Pell-Lucas.

Theorem 4.1. *The novel generating function of the product of (p, q)-Lucas numbers and Gaussian Pell Padovan numbers for $n \in \mathbb{N}$ is provided by*

$$\sum_{n=0}^{\infty} GR_nL_{p,q,n}t^n = \frac{G_1}{D_1}, \tag{27}$$

where

$$G_1 = 2(1-i) + p(1+i)t + [(-3p^2 - 6p) + (5p^2 + 10q)i]t^2 + [(-p^3 - pq + (p^3 + 5pq)i]t^3 + [(p^2q + 6q^2) + (p^2q - 10pq)i]t^4 - pq^2(-1+3i)t^5,$$

and

$$D_1 = 1 - (2p^2 + 4q)t^2 - (p^3 + 3pq)t^3 + 4q^2t^4 + 2pq^2t^5 - q^3t^6.$$

Proof. By referred to [6] and [14], we have

$$GR_n = (1-i)S_n(E) + (1+i)S_{n-1}(E) + (-1+3i)S_{n-2}(E) \text{ and } L_{p,q,n} = 2S_n(a_1 + [-a_2] - pS_{n-1}(a_1 + [-a_2]).$$

We see that

$$\begin{aligned}
\sum_{n=0}^{\infty} GR_n L_{p,q,n} t^n &= \sum_{n=0}^{\infty} ((1-i)S_n(E) + (1+i)S_{n-1}(E) + (-1+3i)S_{n-2}(E))\,(2S_n(a_1 + [-a_2] - pS_{n-1}(a_1 + [-a_2]))t^n \\
&= 2(1-i)\sum_{n=0}^{\infty} S_n(E)S_n(a_1 + [-a_2])t^n - p(1-i)\sum_{n=0}^{\infty} S_n(E)S_{n-1}(a_1 + [-a_2])t^n \\
&\quad + 2(1+i)\sum_{n=0}^{\infty} S_{n-1}(E)S_n(a_1 + [-a_2])t^n - p(1+i)\sum_{n=0}^{\infty} S_{n-1}(E)S_{n-1}(a_1 + [-a_2])t^n \\
&\quad + 2(-1+3i)\sum_{n=0}^{\infty} S_{n-2}(E)S_n(a_1 + [-a_2])t^n - p(-1+3i)\sum_{n=0}^{\infty} S_{n-2}(E)S_{n-1}(a_1 + [-a_2])t^n,
\end{aligned}$$

we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} GR_n L_{p,q,n} t^n &= \frac{2(1-i)(1 - 2qt^2 - pqt^4)}{D_1} - \frac{p(1-i)(2pt^2 + (p^2+q)t^3)}{D_1} \\
&\quad + \frac{2(1+i)(pt + q^2t^4)}{D_1} - \frac{p(1+i)(t - 2qt^3 - pqt^4)}{D_1} \\
&\quad + \frac{2(-1+3i)((p^2+q)t^2 - 2q^2t^4)}{D_1} - \frac{p(-1+3i)(pt^2 + q^2t^5)}{D_1}.
\end{aligned}$$

So, the proof is completed. □

Theorem 4.2. *The novel generating function of the product of* (p,q)*-Lucas numbers and Gaussian Perrin numbers for* $n \in \mathbb{N}$ *is supplied by*

$$\sum_{n=0}^{\infty} Gr_n L_{p,q,n} t^n = \frac{G_2}{D_2}, \tag{28}$$

where

$$G_2 = -2 + 6i + 3pt + [2q + (-6q - 2p^2)i]t^2 + [(p^3 + 2p^2 + 6pq + 2q) + (-3p^3 - 2p^2 - 9pq - 2q)i]t^3 + [(4q^2 + 3p^2q) + 2q^2i]t^4 - pq^2(1-i)t^5,$$

and

$$D_2 = 1 - (p^2 + 2q)t^2 - (p^3 + 3pq)t^3 + q^2t^4 + pq^2t^5 - q^3t^6.$$

Proof. The same technique described in Theorem 4.1 can be applied to prove the result. □

Theorem 4.3. *For $n \in \mathbb{N}$, when Gaussian Pell Padovan numbers are multiplied by (p,q)-Fibonacci numbers, the new generating function is*

$$\sum_{n=0}^{\infty} GR_n F_{p,q,n} t^n = \frac{G_3}{D_1}, \tag{29}$$

with

$$G_3 = (1+i)t + p(1+i)t^2 + [(p^2-q) + (-p^2-3q)i]t^3 - pq(1+i)t^4 + q^2(-1+3i)t^5.$$

Proof. We see that

$$\begin{aligned}\sum_{n=0}^{\infty} GR_n F_{p,q,n} t^n &= \sum_{n=0}^{\infty} ((1-i)S_n(E) + (1+i)S_{n-1}(E) + (-1+3i)S_{n-2}(E))\,(S_{n-1}(a_1+[-a_2])t^n \\ &= (1-i)\sum_{n=0}^{\infty} S_n(E)S_{n-1}(a_1+[-a_2])t^n + (1+i)\sum_{n=0}^{\infty} S_{n-1}(E)S_{n-1}(a_1+[-a_2])t^n \\ &\quad +(-1+3i)\sum_{n=0}^{\infty} S_{n-2}(E)S_{n-1}(a_1+[-a_2])t^n;\end{aligned}$$

we obtain

$$\sum_{n=0}^{\infty} GR_n F_{p,q,n} t^n = \frac{(1-i)(2pt^2+(p^2+q)t^3)}{D_1} + \frac{(1+i)(t-2qt^3-pqt^4)}{D_1} + \frac{(-1+3i)(pt^2+q^2t^5)}{D_1}.$$

So, the proof is completed. □

Theorem 4.4. *The novel generating function of the product of (p,q)-Fibonacci numbers and Gaussian Perrin numbers for $n \in \mathbb{N}$ is provided by*

$$\sum_{n=0}^{\infty} Gr_n F_{p,q,n} t^n = \frac{G_4}{D_2}, \tag{30}$$

with

$$G_4 = 3t + 2pit^2 + [(p^2-2q) + (-3p^2-3q)i]t^3 - 3pqt^4 + q^2(1-i)t^5.$$

Proof. The same technique described in Theorem 4.3 can be applied to prove the result. □

Theorem 4.5. *The novel generating function of the product of (p,q)-Pell Lucas numbers and Gaussian Pell Padovan numbers for $n \in \mathbb{N}$ is provided by*

$$\sum_{n=0}^{\infty} GR_n Q_{p,q,n} t^n = \frac{G_5}{D_3}, \tag{31}$$

where

$$\begin{aligned}G_5 &= 2(1-i) + 2p(1+i)t + [(-4p^2-6q-8p) + i(12p^2+10q+8p)]t^2 + [(-8p^3-2pq) + i(8p^3+10pq)]t^3 \\ &\quad +[(4p^2q+6q^2) + i(4p^2q-10q^2)]t^4 - 2pq^2(-1+3i)t^5,\end{aligned}$$

and

$$D_3 = 1 - (p^2+4q)t^2 - (8p^3+6pq)t^3 + 4q^2t^4 + 4pq^2t^5 - q^3t^6.$$

Proof. We see that

$$\begin{aligned}
\sum_{n=0}^{\infty} GR_n Q_{p,q,n} t^n &= \sum_{n=0}^{\infty}\left(\begin{array}{c} ((1-i)(S_n(E)+(1+i)S_{n-1}(E)+(-1+3i)S_{n-2}(E)) \\ \times(2S_n(a_1+[-a_2])-2pS_{n-1}(a_1+[-a_2])) \end{array}\right)t^n \\
&= 2(1-i)\sum_{n=0}^{\infty} S_n(E)S_n(a_1+[-a_2])t^n - 2p(1-i)\sum_{n=0}^{\infty} S_n(E)S_{n-1}(a_1+[-a_2])t^n \\
&\quad +2(1+i)\sum_{n=0}^{\infty} S_{n-1}(E)S_n(a_1+[-a_2])t^n - 2p(1+i)\sum_{n=0}^{\infty} S_{n-1}(E)S_{n-1}(a_1+[-a_2])t^n \\
&\quad +2(-1+3i)\sum_{n=0}^{\infty} S_{n-2}(E)S_n(a_1+[-a_2])t^n - 2p(-1+3i)\sum_{n=0}^{\infty} S_{n-2}(E)S_{n-1}(a_1+[-a_2])t^n;
\end{aligned}$$

we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} GR_n Q_{p,q,n} t^n &= \frac{2(1-i)(1-2qt^2-2pqt^4)}{D_3} - \frac{2p(1-i)(4pt^2+(4p^2+q)t^3)}{D_3} \\
&\quad +\frac{(2+2i)(2pt+q^2t^4)}{D_3} - \frac{2p(1+i)(t-2qt^3-2pqt^4)}{D_3} \\
&\quad +\frac{2(-1+3i)((4p^2+q)t^2-2q^2t^4)}{D_3} - \frac{2p(-1+3i)(2pt^2+q^2t^5)}{D_3}.
\end{aligned}$$

So, the proof is completed. □

Theorem 4.6. *The novel generating function of the product of (p, q)-Pell Lucas numbers and Gaussian Perrin numbers for $n \in \mathbb{N}$ is provided by*

$$\sum_{n=0}^{\infty} Gr_n Q_{p,q,n} t^n = \frac{G_6}{D_4}, \tag{32}$$

with

$$\begin{aligned}
G_6 &= (-2+6i)+6pt+[(8p^2+4q)+i(-16p^2-8q)]t^2+[(8p^3+12pq)+i(-24p^3-18pq)]t^3 \\
&\quad +[(12p^2q+4q^2)+i(2q^2)]t^4-2pq^2(1-i)t^5,
\end{aligned}$$

and

$$D_4 = 1-(4p^2+2q)t^2-(8p^3+6pq)t^3+q^2t^4+2pq^2t^5-q^3t^6.$$

Proof. The method outlined in Theorem 4.5 may also be utilized to demonstrate this result. □

5. A new generating functions of the products of Gaussian numbers with bivariate polynomials

In this section, we derive new generating functions for the products of Gaussians Pell Padovan and Gaussian Perrin numbers with bivariate Fibonacci, bivariate Mersenne Lucas, bivariate Pell, bivariate Jacobsthal, bivariate Jacobsthal Lucas, Lucas polynomials.

Theorem 5.1. *The new ordinary generating function of $(GR_n, F_n(x, y))$ for every natural n is determined to be*

$$\sum_{n=0}^{\infty} GR_n F_n(x,y) t^n = \frac{G_7}{D_5}, \tag{33}$$

where

$$G_7 = (1+i)t + x(1+i)t^2 + ((x^2-y) - i(x^2+3y))t^3 - xy(1+i)t^4 + y^2(-1+3i)t^5,$$

and

$$D_5 = 1 - (2x^2+4y)t^2 - (x^3+3xy)t^3 + 4y^2t^4 + 2xy^2t^5 - y^3t^6.$$

Proof. We see that

$$\begin{aligned}\sum_{n=0}^{\infty} GR_nF_n(x,y)t^n &= \sum_{n=0}^{\infty}[(1-i)S_n(E) + (1+i)S_{n-1}(E) + (-1+3i)S_{n-2}(E] \times S_{n-1}(a_1+[-a_2])t^n \\ &= (1-i)\sum_{n=0}^{\infty} S_n(E)S_{n-1}(a_1+[-a_2])t^n + (1+i)\sum_{n=0}^{\infty} S_{n-1}(E)S_{n-1}(a_1+[-a_2])t^n \\ &\quad +(-1+3i)\sum_{n=0}^{\infty} S_{n-2}(E)S_{n-1}(a_1+[-a_2])t^n;\end{aligned}$$

we obtain

$$\sum_{n=0}^{\infty} GR_nF_n(x,y)t^n = \frac{(1-i)(2xt^2+(x^2+y)t^3}{D_5} + \frac{(1+i)(t-2yt^3-xyt^4)}{D_5} + \frac{(-1+3i)(xt^2+y^2t^5)}{D_5}.$$

So, the proof is completed. □

Theorem 5.2. *The new ordinary generating function of* $(Gr_n, F_n(x,y))$ *for every natural n is determined to be*

$$\sum_{n=0}^{\infty} Gr_nF_n(x,y)t^n = \frac{G_8}{D_6}, \tag{34}$$

such that

$$G_8 = 3t + 2xit^2 + ((-x^2-4y) + i(3x^2+3y))t^3 - 3xyt^4 + (1-i)y^2t^5,$$

and

$$D_6 = 1 - (x^2+2y)t^2 - (x^3+3xy)t^3 + y^2t^4 + xy^2t^5 - y^3t^6.$$

Proof. The same technique described in Theorem 5.1 can be applied to prove the result. □

Theorem 5.3. *For* $n \in \mathbb{N}$, *the new generating function of the product of Gaussian Pell Padovan numbers with bivariate Mersenne Lucas polynomials is given by*

$$\sum_{n=0}^{\infty} GR_nm_n(x,y)t^n = \frac{G_9}{D_6}, \tag{35}$$

such that

$$\begin{aligned}G_9 &= 2(1-i) + 3y(1+i)t + \left[(12x-27y^2) + (45y^2-20x)i\right]t^2 + (6xy - 27y^3 + i(27y^3-30xy))t^3 \\ &\quad + \left[(24x^2-18xy^2) + (-18xy^2-40x^2)i\right]t^4 - 12x^2y(-1+3i)t^5.\end{aligned}$$

Proof. We see that

$$\begin{aligned}
\sum_{n=0}^{\infty} GR_n m_n(x,y)t^n &= \sum_{n=0}^{\infty}[(1-i)S_n(E)+(1+i)S_{n-1}(E)+(-1+3i)S_{n-2}(E)\times(2m)t^n \\
&= 2(1-i)\sum_{n=0}^{\infty} S_n(E)S_n(a_1+[-a_2])t^n - 3y(1-i)\sum_{n=0}^{\infty} S_n(E)S_{n-1}(a_1+[-a_2])t^n \\
&\quad +2(1+i)\sum_{n=0}^{\infty} S_{n-1}(E)S_n(a_1+[-a_2])t^n - 3y(1+i)\sum_{n=0}^{\infty} S_{n-1}(E)S_{n-1}(a_1+[-a_2])t^n \\
&\quad +2(-1+3i)\sum_{n=0}^{\infty} S_{n-2}(E)S_n(a_1+[-a_2])t^n - 3y(-1+3i)\sum_{n=0}^{\infty} S_{n-2}(E)S_{n-1}(a_1+[-a_2])t^n;
\end{aligned}$$

we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} GR_n m_n(x,y)t^n &= \frac{(2-2i)(1+4xt^2+6xyt^3)}{D_6} - \frac{3y(1-i)(6yt^2+(9y^2-2x)t^3)}{D_6} \\
&\quad + \frac{(2+2i)(3yt+4x^2t^4)}{D_6} - \frac{3y(1+i)(t+4xt^3+6xyt^4)}{D_6} \\
&\quad + \frac{(-2+6i)((9y^2-2x)t^2-8x^2t^4)}{D_6} - \frac{3y(-1+3i)(3yt^2+4x^2t^5)}{D_6}.
\end{aligned}$$

After bringing to a common denominator and simplifying, we obtain

$$\sum_{n=0}^{\infty} GR_n m_n(x,y)t^n = \frac{G_9}{D_6}.$$

The proof is completed. □

Theorem 5.4. *The new ordinary generating function of $(Gr_n, m_n(x,y))$ for every natural n is determined to be*

$$\sum_{n=0}^{\infty} Gr_n m_n(x,y)t^n = \frac{G_{10}}{D_6}, \tag{36}$$

where

$$\begin{aligned}
G_{10} &= 2(-1+3i)+9yt+\left[(-8x+18y^2)+(-36y^2+16x)i\right]t^2+\left[-36xy+27y^3+i(-81y^3+54xy)\right]t^3 \\
&\quad +\left[(16x^2-54xy^2)+8x^2i\right]t^4-12x^2y(1-i)t^5.
\end{aligned}$$

Proof. This result can be derived by adopting the same strategy as in Theorem 5.3. □

Theorem 5.5. *The new ordinary generating function of $(GR_n, L_n(x,y))$ for every natural n is determined to be*

$$\sum_{n=0}^{\infty} GR_n L_n(x,y)t^n = \frac{G_{11}}{D_6}, \tag{37}$$

with

$$\begin{aligned}
G_{11} &= 2(1-i)+x(1+i)t+[(-3x^2-6y)+i(5x^2+10y)]t^2+[(-x^3-xy)+i(x^3+3xy)]t^3 \\
&\quad +[(6y^2+x^2y)+i(x^2y-10y^2)]t^4-xy^2(-1+3i)t^5.
\end{aligned}$$

Proof. We see that

$$\begin{aligned}\sum_{n=0}^{\infty} GR_nL_n(x,y)t^n &= \sum_{n=0}^{\infty}[(1-i)S_n(E)+(1+i)S_{n-1}(E)+(-1+3i)S_{n-2}(E)]\\ &\quad\times[2S_n(a_1+[-a_2])-xS_{n-1}(a_1+[-a_2])]t^n\\ &= 2(1-i)\sum_{n=0}^{\infty} S_n(E)S_n(a_1+[-a_2])t^n - x(1-i)\sum_{n=0}^{\infty} S_n(E)S_{n-1}(a_1+[-a_2])t^n\\ &\quad+2(1+i)\sum_{n=0}^{\infty} S_{n-1}(E)S_n(a_1+[-a_2])t^n - x(1+i)\sum_{n=0}^{\infty} S_{n-1}(E)S_{n-1}(a_1+[-a_2])t^n\\ &\quad+2(-1+3i)\sum_{n=0}^{\infty} S_{n-2}(E)S_n(a_1+[-a_2])t^n - x(-1+3i)\sum_{n=0}^{\infty} S_{n-2}(E)S_{n-1}(a_1+[-a_2])t^n;\end{aligned}$$

we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} GR_nL_n(x,y)t^n &= \frac{2(1-i)(1-2yt^2-xyt^3)}{D_6} - \frac{x(1-i)(2xt^2+(x^2+y)t^3)}{D_6}\\ &\quad+\frac{2(1+i)(xt+y^2t^4)}{D_6} - \frac{x(1+i)(t-2yt^3-xyt^4)}{D_6}\\ &\quad+\frac{2(-1+3i)((x^2+y)t^2-2y^2t^4)}{D_6} - \frac{x(-1+3i)(xt^2+y^2t^5)}{D_6}.\end{aligned}$$

So, the proof is completed. □

Theorem 5.6. *Let n denote a natural number, the new generating function of the product of Gaussian Perrin numbers with bivariate Lucas polynomials is given by*

$$\sum_{n=0}^{\infty} Gr_nL_n(x,y)t^n = \frac{G_{12}}{D_6}, \tag{38}$$

where

$$\begin{aligned}G_{12} &= 2(-1+3i)+3xt+\left[(2x^24y)+i(-4x^2-7y)\right]t^2+\left[(x^3+6xy)+i(-3x^3-9xy)\right]t^3\\ &\quad+\left[(3x^2y+4y^2)+i(2y^2)\right]t^4-xy^2(1-i)t^5.\end{aligned}$$

Proof. The method outlined in Theorem 5.5 may also be utilized to demonstrate this result. □

Theorem 5.7. *Let n denote a natural number, the novel ordinary generating function of $(GR_n j_n(x;y))$ is given as*

$$\sum_{n=0}^{\infty} GR_n j_n(x,y)t^n = \frac{G_{13}}{D_7}, \tag{39}$$

where

$$\begin{aligned}G_{13} &= 2(1-i)+xy(1+i)t+((-12y-3x^2y^2)+i(20y+5x^2y^2))t^2+[(-x^3y^3-2xy^2))\\ &\quad+i(10xy^2+x^3+y^3)]t^3+((8y^2+2x^2y^3)+i(8y^2+2x^2y^3))t^4-4xy^3(-1+3i)t^5,\end{aligned}$$

and

$$D_7 = 1-(2x^2y^2+8y)t^2-(x^3y^3+6xy^2)t^3+16y^2t^4+8xy^3t^5-8y^3t^6.$$

Proof. We see that

$$\begin{aligned}\sum_{n=0}^{\infty} GR_n j_n(x,y)t^n &= \sum_{n=0}^{\infty}[(1-i)S_n(E)+(1+i)S_{n-1}(E)+(-1+3i)S_{n-2}(E)]\\ &\quad\times[2S_n(a_1+[-a_2])-xyS_{n-1}(a_1+[-a_2])t^n\\ &= 2(1-i)\sum_{n=0}^{\infty} S_n(E)S_n(a_1+[-a_2])t^n - xy(1-i)\sum_{n=0}^{\infty} S_n(E)S_{n-1}(a_1+[-a_2])t^n\\ &\quad+2(1+i)\sum_{n=0}^{\infty} S_{n-1}(E)S_n(a_1+[-a_2])t^n - xy(1+i)\sum_{n=0}^{\infty} S_{n-1}(E)S_{n-1}(a_1+[-a_2])t^n\\ &\quad+2(-1+3i)\sum_{n=0}^{\infty} S_{n-2}(E)S_n(a_1+[-a_2])t^n - xy(-1+3i)\sum_{n=0}^{\infty} S_{n-2}(E)S_{n-1}(a_1+[-a_2])t^n,\end{aligned}$$

we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} GR_n j_n(x,y)t^n &= \frac{2(1-i)(1-4yt^2-2xy^2t^3)}{D_7} - \frac{xy(1-i)(2xyt^2+(x^2y^2+2y)t^3)}{D_7}\\ &\quad+\frac{2(1+i)(xyt+4y^2t^4)}{D_7} - \frac{xy(1+i)(t-4yt^3-2xy^2t^4)}{D_7}\\ &\quad+\frac{2(-1+3i)((x^2y^2+2y)t^2-8y^2t^4)}{D_7} - \frac{xy(-1+3i)(xyt^2+4y^2t^5)}{D_7}.\end{aligned}$$

So, the proof is completed. □

Theorem 5.8. *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Perrin numbers with bivariate Jacobsthal Lucas polynomials is given by:*

$$\sum_{n=0}^{\infty} Gr_n j_n(x,y)t^n = \frac{G_{14}}{D_7}, \tag{40}$$

with

$$\begin{aligned}G_{14} &= 2(-1+3i)+3xyt+((8y-2x^2y^2)+i(-16y-4x^2y^2))t^2+[(x^3y^3+12xy^2)+i(-3x^3y^3-18xy^2)]t^3\\ &\quad+(16y^2+6x^2y^3+8y^2i)t^4-4xy^3(1-i)t^5.\end{aligned}$$

Proof. This result can be proven using the same technique as in Theorem 5.7. □

Theorem 5.9. *Considering $n \in \mathbb{N}$, the new generating function that emerges from multiplying Jacobsthal polynomials by Gaussian Pell Padovan numbers is*

$$\sum_{n=0}^{\infty} GR_n J_n(x,y)t^n = \frac{G_{15}}{D_7}, \tag{41}$$

with

$$G_{15} = (1+i)t+xy(1+i)t^2+[(x^2y^2-2y))+i(-x^2y^2-6y))]t^3-2xy^2(1+i)t^4+4y^2(-1+3i)t^5.$$

Proof. We see that

$$\begin{aligned}\sum_{n=0}^{\infty} GR_n J_n(x,y)t^n &= \sum_{n=0}^{\infty}[(1-i)S_n(E)+(1+i)S_{n-1}(E)+(-1+3i)S_{n-2}(E)]\times S_{n-1}(a_1+[-a_2]t^n \\ &= (1-i)\sum_{n=0}^{\infty} S_n(E)S_{n-1}(a_1+[-a_2])t^n+(1+i)\sum_{n=0}^{\infty} S_{n-1}(E)S_{n-1}(a_1+[-a_2])t^n \\ &\quad +(-1+3i)\sum_{n=0}^{\infty} S_{n-2}(E)S_{n-1}(a_1+[-a_2])t^n;\end{aligned}$$

we obtain

$$\sum_{n=0}^{\infty} GR_n J_n(x,y)t^n = \frac{(1-i)(2xyt^2+(x^2y^2+2y)t^3)}{D_7}+\frac{(1+i)(t-4yt^3-2xy^2t^4)}{D_7}+\frac{(-1+3i)(xyt^2+4y^2t^5)}{D_7}.$$

That is what we want to happen. □

Theorem 5.10. *Let's consider $n \in \mathbb{N}$, the new generating function that results from multiplying Gaussian Perrin numbers by Jacobsthal polynomials is*

$$\sum_{n=0}^{\infty} Gr_n J_n(x,y)t^n = \frac{G_{16}}{D_8}, \tag{42}$$

where

$$G_{16} = 3t+2xyit^2+[(-x^2y^2-8y)+i(3x^2y^2+6y)]t^3-6xy^2t^4+4y^2(1-i)t^5,$$

and

$$D_8 = 1-(x^2y^2+4y)t^2-(x^3y^3+6xy^2)t^3+4y^2t^4+4y^3xt^5-8y^3t^6.$$

Proof. This result can be proven using the same technique as in Theorem 5.9. □

6. Application in Signal Processing: Noise Reduction Example

Signal processing is a field that involves the analysis, modification, and synthesis of signals, typically to improve the quality or clarity of information being transmitted [10, 13]. One common task in signal processing is *noise reduction*, which aims to minimize or eliminate unwanted interference from a signal, thereby enhancing the original information [4].

The Gaussian Pell Padovan and (p,q)-Lucas sequences, due to their complex structure and recurrence properties, can be applied in signal processing tasks such as noise reduction, pattern recognition, and signal modulation. In this work, we demonstrate the application of the Gaussian Pell Padovan sequence in an adaptive filtering technique for noise reduction.

6.1. Gaussian Pell Padovan Sequence for Noise Reduction

The Gaussian Pell Padovan sequence can be used to simulate structured noise patterns. This sequence, defined by the recurrence relation:

$$GR_n = 2GR_{n-2}+GR_{n-3}, \text{ with initial terms } GR_0 = 1-i,\ GR_1 = 1+i,\ GR_2 = 1+i,$$

produces complex values that offer unique characteristics suitable for signal processing. To simplify, we use only the real parts of the sequence for our noise pattern. Calculating the first few terms, we get:

- $GR_0 = 1 - i$,
- $GR_1 = 1 + i$,
- $GR_2 = 1 + i$,
- $GR_3 = 3 + i$,
- $GR_4 = 3 + 3i$,
- $GR_5 = 7 + 3i$,
- $GR_6 = 9 + 7i$.

Thus, the real parts form the following noise pattern:

$$\text{Noise Pattern} = [1, 1, 1, 3, 3, 7, 9].$$

6.2. Simulating a Noisy Signal

To illustrate the noise reduction process, we construct a sine wave signal and add the Gaussian Pell Padovan noise pattern to simulate interference.

- **Original Signal**: A sine wave sampled at 10 points over the interval $[0, 3\pi/2]$, represented as $S = \sin(x)$.
- **Noisy Signal**: The original signal with added noise, $S_{\text{noisy}} = S + \text{Noise Pattern}$.

The signals are as follows:

$$S = [0, 0.5, 0.87, 1, 0.87, 0.5, 0, -0.5, -0.87, -1],$$

$$S_{\text{noisy}} = [1, 1.5, 1.87, 4, 3.87, 7.5, 9, 2.5, 3.13, 8].$$

6.3. Noise Reduction Using Adaptive Filtering

To reduce the noise, we apply a simple adaptive filter by subtracting the noise pattern from the noisy signal:

$$S_{\text{filtered}} = S_{\text{noisy}} - \text{Noise Pattern}.$$

This filtering step allows us to approximate the original signal by removing the structured noise. The results are illustrated in the following plots:

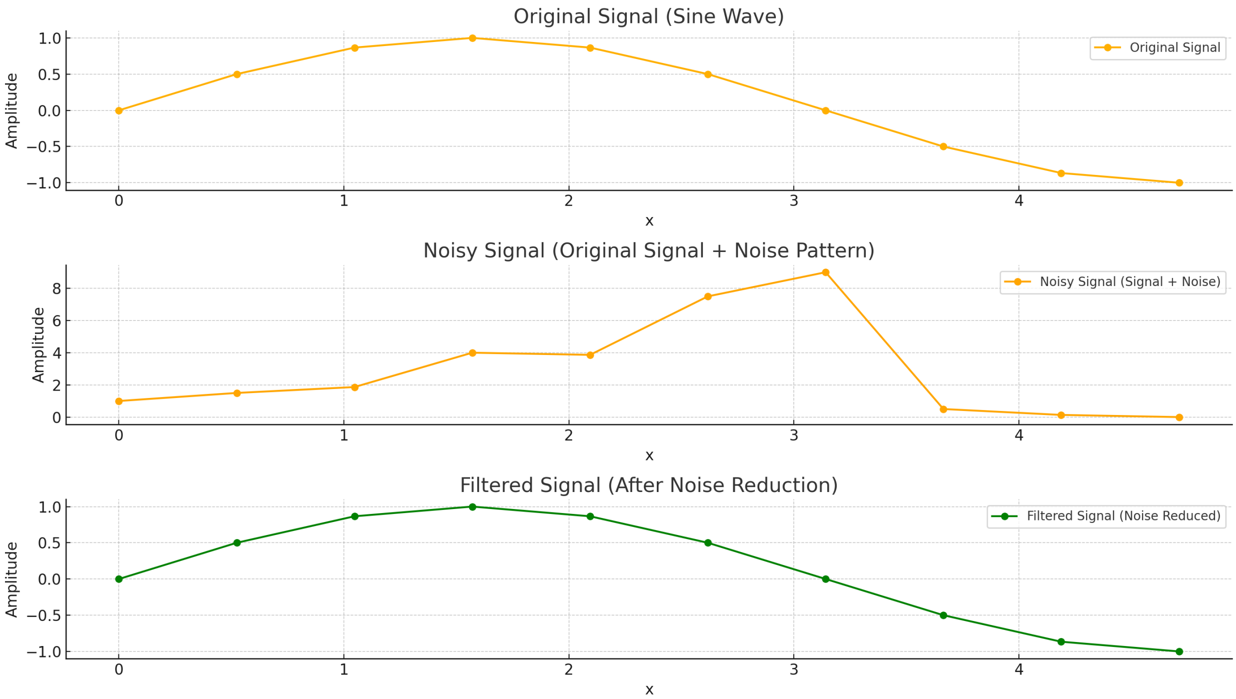

Figure 1: Original Signal: A clean sine wave representing the signal without interference.

This example demonstrates the effectiveness of Gaussian Pell Padovan sequences in signal processing for noise reduction. By leveraging the unique structure of these sequences, it is possible to design adaptive filters that can effectively cancel structured noise, enhancing the clarity of the original signal. This application highlights the potential of Gaussian sequences for practical tasks in signal processing.

7. Conclusion

In conclusion, in this study, we propose a novel theorem. By using this theorem, we were able through this paper to find generating function for the products of some Gaussians numbers of the third order, (p, q)-numbers and bivariate polynomials of the second order.

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References

[1] A. Abderrezzak, *Généralisation de la transformation d'Euler d'une série formelle*, Adv. Math. **103** (1994), 180–195.
[2] F. R. V. Alves, *Bivariate Mersenne polynomials and matrices*, Notes Number Theory Discrete Matt., **26** (2020), 83–95.
[3] K. Bhoi, P. K. RAY, *Perrin numbers expressible as sums of two base brepdigits*, Rendiconti dell'Istituto di Matematica dell'Universita di Trieste, **53** (2021), 1–11.
[4] S. Boll, *Suppression of acoustic noise in speech using spectral subtraction*, IEEE Transactions on Acoustics, Speech, and Signal Processing, **27** (1979), 113–120.
[5] S. Boughaba, A. Boussayoud, *On Some Identities And Generating Function of Both K-jacobsthal Numbers and Symmetric Functions in Several Variables*, Konuralp Journal of Mathematics., **7** (2019), 235–242.

[6] A. Boussayoud, A. Abderrezzak, *Complete homogeneous symmetric functions and Hadamard product*, Ars Comb., **144** (2019), 81-90.

[7] A. Boussayoud, M. Kerada, A. Abderrezzak, *A Generalization of Some Orthogonal Polynomials*, Springer Proc. Math. Stat. **41** (2013), 229–235.

[8] M. Catalani, *Some formulae for bivariate Fibonacci and Lucas polynomials*, arXive: math/0406323v155, (2004).

[9] H. H. Gulec, N. Teskara. *On the Pell and Pell-Lucas sequences and their matrix representations*, Applied Mathematics Letters. **5** (2012), 1554–1559.

[10] S. Haykin, *Adaptive Filter Theory*, 4th Ed., Upper Saddle River, Prentice-Hall, (2002).

[11] M.Y. Kartal, *Gaussian Padovan and Gaussian Perrin Numbers and Properties of Them*, Asian-European Journal of Mathematics **12** (2019), 2040014.

[12] H. Merzouk, A. Boussayoud, M. Chelgham, *Generating Functions of Generalized Tribonacci and Tricobstal polynomials*, Montes Taurs J. Pure App. Math. **2** (2020), 7–37.

[13] A. V. Oppenheim and R. W. Schafer, *Discrete-Time Signal Processing*, Prentice-Hall, (1999).

[14] N. Saba, A. Boussayoud, *New theorem on symmetric functions and their applications on some (p,q)-numbers*, J. Appl.Math. Inform., **40** (2022), 243–257.

[15] N. Saba, A. Boussayoud, *On the bivariate Mersenne Lucas polynomials and their properties*, Chaos, Solitons and Fractals., **146** (2021), 110899.

[16] N. Saba, A. Boussayoud, M. Kerada, *Generating functions of even and odd Gaussian numbers and polynomials*, J. Sci.Arts., **21.1** (2021), 125–144.

[17] N. Saba, A. Boussayoud, A. Abderrezzak, *Complete homogeneous symmetric functions of third and second-order linear recurrence sequences*, Electron. J. Math. Analysis Appl., **9** (2021), 226–242.

[18] N. Saba, A. Boussayoud, M. Chelgham, *Symmetric and generating functions for some generalized polynomials*, Malaya J.Mat., **8** (2020), 1756–1765.

[19] N. Saba, A. Boussayoud, *Complete homogeneous symmetric functions of Gauss Fibonacci polynomials and bivariate Pell polynomials*, Open J. Math. Sci., **4** (2020), 179–185.

[20] H. Serpil , A.Zeynep, *On sum formulae for bivariate Pell polynomials*, Far East Journal of Applied Mathematics., **41** (2010).

[21] A. Suvarnamani, Some Properties of (p, q)- Lucas Number, Kyungpook Mathematical Journal, **56** (2016), 367–370.

[22] A. Suvarnamani, M. Tatong, *Some Properties of* (p, q)*-Fibonacci Numbers*, Science and Technology RMUTT Journal. **5** (2015), 17–21.

[23] D. Tasci, *Gaussian Padovan and Gaussian Pell-Padovan sequences*, Commun. Fac. Sci. Univ.Ank. Ser. A1 Math. Stat. **67** (2018), 82–88.

[24] S. Uygun, A. Zorçelik, *Bivariate Jacobsthal and bivariate Jacobsthal-Lucas matrix polynomials*, J. Math. Comput. Sci., **8** (2018), 331–344.

[25] H. Zerroug, A. Boussayoud, B. Aloui, M. Kerada, *Gaussian Padovan, Gaussian Pell-Padovan numbers and new generating functions with some numbers and polynomials*, Journal of Information and Optimization Sciences, **43** (2022), 745–760.