



The maximum forcing numbers of quadriculated tori

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Abstract. Klein and Randić (1985) proposed the concept of forcing number, which has an application in chemical resonance theory. Let G be a graph with a perfect matching M . The forcing number of M is the smallest cardinality of a subset of M that is contained only in M . The maximum forcing number of G is the maximum value of forcing numbers over all perfect matchings of G . Kleinerman (2006) obtained that the maximum forcing number of $2m \times 2n$ quadriculated torus is mn . By improving Kleinerman's approach, we obtain the maximum forcing numbers of all 4-regular quadriculated graphs on torus except one class.

1. Introduction

Let G be a graph with a perfect matching M . A subset $S \subseteq M$ is called a *forcing set* of M if it is contained in no other perfect matchings of G . The smallest cardinality of a forcing set of M is called the *forcing number* of M , denoted by $f(G, M)$. The *minimum* and *maximum forcing number* of G , denoted by $f(G)$ and $F(G)$, are respectively defined as the minimum and maximum values of $f(G, M)$ over all perfect matchings M of G .

The concept of the forcing number of a perfect matching was first introduced by Klein and Randić [4, 11] in 1985 when they studied the molecular resonance structures, which was called “innate degree of freedom” in chemical literatures. It was turned out that the perfect matchings with the maximum forcing number contribute more to the stability of molecule [12]. Afshani, Hatami and Mahmoodian [2] pointed out that the computational complexity of the maximum forcing number of a graph is still an open problem. Xu, Bian and Zhang [15] obtained that maximum forcing numbers of hexagonal systems are equal to the resonant numbers. The same result also holds for polyominoes [8, 16] and BN-fullerene graphs [13]. Abeledo and Atkinson [1] had already obtained that resonant numbers of 2-connected plane bipartite graphs can be computed in polynomial time. Thus, the maximum forcing numbers of such three classes of graphs can be solved in polynomial time.

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The Cartesian product of graphs G and H , denoted by $G \square H$, is the graph with vertex set

$$V(G) \square V(H) = \{(g, h) \mid g \in V(G), h \in V(H)\},$$

and two vertices (g_1, h_1) and (g_2, h_2) are adjacent if and only if

$$g_1 = g_2 \text{ and } h_1 h_2 \in E(H) \text{ or } g_1 g_2 \in E(G) \text{ and } h_1 = h_2.$$

The maximum forcing numbers of Cartesian product of some special graphs, such as paths and cycles, have been obtained. Let P_n and C_n denote a path and a cycle with n vertices, respectively. Pachter and Kim [10], Lam and Pachter [6] obtained that $F(P_{2n} \square P_{2n}) = n^2$ using different methods. In general, Afshani et al. [2] proved that $F(P_m \square P_n) = \lfloor \frac{m}{2} \rfloor \cdot \lfloor \frac{n}{2} \rfloor$ for even mn . Besides, they [2] obtained that $F(P_{2m} \square C_{2n}) = mn$ and $F(P_{2m+1} \square C_{2n}) = mn + 1$, and asked such a question: what is the maximum forcing number of a non-bipartite cylinder $P_{2m} \square C_{2n+1}$? Jiang and Zhang [3] solved this problem and obtained that $F(P_{2m} \square C_{2n+1}) = m(n + 1)$. By a method of marking independent sets, Kleinerman [5] obtained that $F(C_{2m} \square C_{2n}) = mn$. Obviously, $C_{2m} \square C_{2n}$ is a special type of 4-regular quadrilaterated graphs on torus.

As early as 1991, Thomassen [14] classified all 4-regular quadrilaterated graphs on torus (abbreviated to “quadrilaterated tori”) into two classes, which were reduced into one class by Li [7]. For $n \geq 1$ and $m \geq 2$, a quadrilaterated torus $T(n, m, r)$ is obtained from an $n \times m$ chessboard (n rows, each consists of m squares) by sticking the left and right sides together and then identifying the top and bottom sides with a torsion of r squares where $1 \leq r \leq m$ (see Fig. 1). Obviously, $T(n, m, m)$ is isomorphic to $C_n \square C_m$. Based on the parity of the three parameters, quadrilaterated tori with perfect matchings can be divided into six classes $T(2n, 2m, 2r)$, $T(2n, 2m, 2r - 1)$, $T(2n + 1, 2m, 2r)$, $T(2n + 1, 2m, 2r - 1)$, $T(2n, 2m + 1, 2r)$ and $T(2n, 2m + 1, 2r - 1)$.

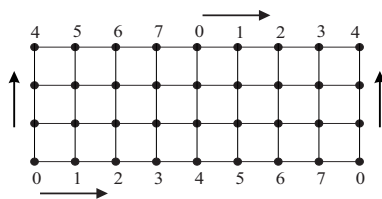


Fig. 1: Quadrilaterated torus $T(3, 8, 4)$.

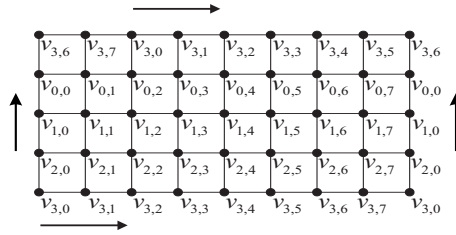
In this paper, we obtain a simple expression for the maximum forcing numbers of all quadrilaterated tori except for $T(2n + 1, 2m, 2r - 1)$. In Section 2, we give some notations and terminologies, and prove some crucial lemmas. In Section 3, we prove that $F(T(2n, 2m + 1, t)) = n(m + 1)$ for $1 \leq t \leq 2m + 1$ by choosing a fixed independent set. In Section 4, we obtain that $F(T(2n, 2m, r)) = mn + 1$ if $(r, 2m) = 2$, and $F(T(2n, 2m, r)) = mn$ otherwise, where $(r, 2m)$ represents the greatest common factor of r and $2m$, and $1 \leq r \leq 2m$. In Section 5, by another representation of the quadrilaterated torus, we obtain the maximum forcing number of $T(2n + 1, 2m, 2r)$ for $1 \leq r \leq m$.

2. Preliminaries

In this section, we give some notations and terminologies, and prove some important lemmas.

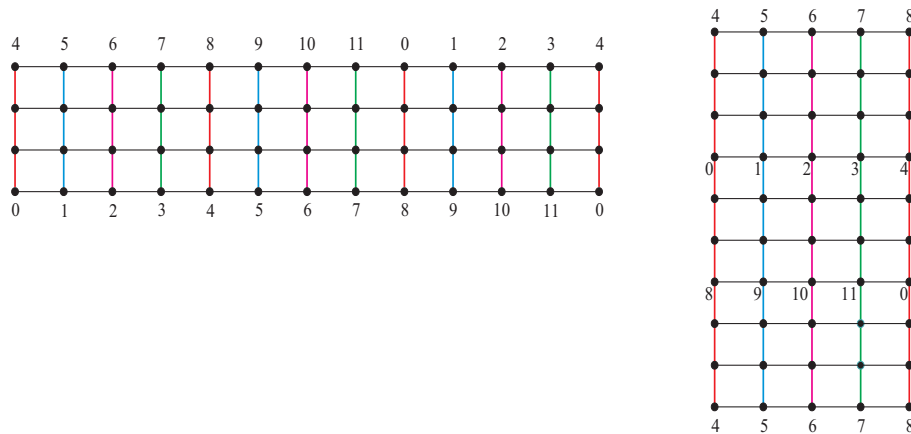
Let $T(n, m, r)$ be a quadrilaterated tori. According to positions of vertices in the chessboard, we label the vertices of $T(n, m, r)$ as $\{v_{i,j} \mid i \in \mathbb{Z}_n, j \in \mathbb{Z}_m\}$ (see Fig. 2), where $\mathbb{Z}_m := \{0, 1, \dots, m - 1\}$. Hence $v_{i,0}$ is adjacent to $v_{i,m-1}$ for $i \in \mathbb{Z}_n$, and $v_{0,j}$ is adjacent to $v_{n-1,m-r+j}$ for $j \in \mathbb{Z}_m$.

For $j \in \mathbb{Z}_m$, let $v_{0,j}v_{1,j} \cdots v_{n-1,j}$ be a path called j -column, and $v_{0,j}$ and $v_{n-1,j}$ are initial and terminal of j -column. For convenience, we call j -column a column for $j \in \mathbb{Z}_m$. If initial v_{0,j_2} of j_2 -column is adjacent to terminal v_{n-1,j_1} of j_1 -column, that is, $j_2 \equiv j_1 + r \pmod{m}$, then j_2 -column is the successor of j_1 -column. Let $j_0, j_1, \dots, j_{g-1}$ -columns be pairwise different such that j_{k+1} -column is the successor of j_k -column for each $k \in \mathbb{Z}_g$. Then these g columns form a cycle, called a I -cycle. In [9], we had proved the following lemma.

Fig. 2: Labels of the vertices in $T(4, 8, 2)$.

Lemma 2.1. [9] $T(n, m, r)$ has (r, m) I-cycles and each I-cycle contains $\frac{m}{(r,m)}$ columns. Moreover, any consecutive (r, m) columns lie on different I-cycles.

Intuitively, we call $v_{i,j}v_{i,j+1}$ a *horizontal edge* and $v_{i,j}v_{i+1,j}$ a *vertical edge* for $i \in \mathbb{Z}_n$ and $j \in \mathbb{Z}_m$. Obviously, all vertical edges form (r, m) I-cycles, and all horizontal edges form n II-cycles (consisting of all vertices and edges on a row). Preserving the horizontal and vertical edges, we can obtain another representation of this quadriculated tori, denoted by $T^*(n, m, r)$, in which all vertices of a I-cycle of $T(n, m, r)$ lie on a column and all vertices of a II-cycle of $T(n, m, r)$ are divided into different rows (see Fig. 3). Therefore, I-cycles (resp. II-cycles) in $T(n, m, r)$ corresponds to II-cycles (resp. I-cycles) in $T^*(n, m, r)$.

Fig. 3: Quadriculated tori $T(3, 12, 8)$ (left) and $T(4, 9, 3) = T^*(3, 12, 8)$ (right).

For a non-empty subset $S \subseteq V(G)$, the *subgraph induced by S* , denoted by $G[S]$, is a graph whose vertex set is S and edge set consists of those edges of G that have both end vertices in S . The induced subgraph $G[V(G) \setminus S]$ is denoted by $G - S$. For an edge subset $F \subseteq E(G)$, we use $V(F)$ to denote the set of all end vertices of edges in F .

For $i \in \mathbb{Z}_n$, the subgraph of $T(n, m, r)$ induced by all vertices of any consecutive two rows

$$\{v_{i,0}, v_{i,1}, \dots, v_{i,m-1}\} \cup \{v_{i+1,0}, v_{i+1,1}, \dots, v_{i+1,m-1}\}$$

is denoted by $R_{i,i+1}$. Then $R_{i,i+1}$ contains a subgraph isomorphic to $C_m \square P_2$. Particularly, $R_{i,i+1}$ is isomorphic to $C_m \square P_2$ for $n \geq 2$ where $i \in \mathbb{Z}_n$. Relabeling the vertices of $T(n, m, r)$ according to I-cycle, we can obtain the following lemma. For details, see Section 2 of ref. [9].

Lemma 2.2. [9] For $n \geq 1$, $m \geq 2$ and $1 \leq r \leq m$, $T^*(n, m, r) = T((r, m), \frac{mn}{(r,m)}, (\frac{m}{(r,m)} - k)n)$, where $0 \leq k \leq \frac{m}{(r,m)} - 1$ is an integer satisfying the equation $(r, m) \equiv rk \pmod{m}$. Furthermore, $T^{**}(n, m, r) = T(n, m, r)$.

Let G be a graph with a perfect matching M . We give an independent set T of G called *marked vertices* of G . Define $M_T = \{e \in M \mid e \text{ has an end vertex in } T\}$. Then $M_T \subseteq M$ and $|M_T| = |T|$. A cycle of G is *M -alternating* if its edges appear alternately in M and off M .

Lemma 2.3. Let G be a graph with a perfect matching M . If the union of all paths of length 2 whose initial and terminal lie in T contains no M -alternating cycles, then $f(G, M) \leq |M| - |T|$.

Proof. We prove that $G[V(M_T)]$ contains no M -alternating cycles. Suppose to the contrary that $G[V(M_T)]$ contains an M -alternating cycle C . Then C is also an M_T -alternating cycle. Since T is an independent set, half vertices of C are marked, and marked and unmarked vertices appear alternately. Thus, C can be viewed as the union of paths of length two whose initial and terminal lie in T , which is a contradiction.

Since $G[V(M_T)]$ contains no M -alternating cycles, $G[V(M_T)]$ has a unique perfect matching. Thus, $M \setminus M_T$ is a forcing set of M , and $f(G, M) \leq |M \setminus M_T| = |M| - |T|$. $\square$

For convenience, “the union of all paths of length 2 whose initial and terminal are marked vertices” is defined as “marked subgraph”.

Next we give the concept of 2×2 -polyomino, which is a kind of general “marked subgraph”. A *polyomino* is a finite connected subgraph in the infinite plane square grid in which every interior face is surrounded by a square and every edge belongs to at least one square. A 2×2 -polyomino is also a polyomino which is obtained by replacing each square in a polyomino by a 2×2 chessboard (see Fig. 4).

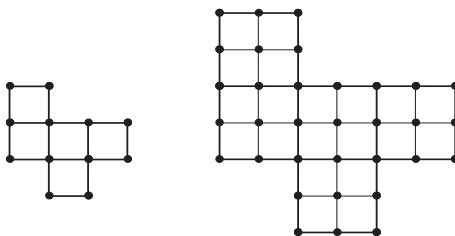


Fig. 4: A polyomino and its corresponding 2×2 -polyomino.

An *interior vertex* of a plane graph is a vertex which is not on the boundary of the unbounded face. For a polyomino, an interior vertex means a vertex of degree 4. By the proof of Theorem 3.2 in [3], Jiang and Zhang obtained the following result.

Lemma 2.4. [3] A 2×2 -polyomino has an odd number of interior vertices.

3. The maximum forcing number of $T(2n, 2m + 1, r)$ for $1 \leq r \leq 2m + 1$

In this section, we will obtain the maximum forcing number of $T(2n, 2m + 1, r)$ by the method of marking independent sets for $1 \leq r \leq 2m + 1$.

For $T(2n, m, r)$, we define some subsets of vertices and edges. For $i \in Z_n$, let

$$X_i = \{v_{i,2k} | k \in Z_{\lfloor \frac{m}{2} \rfloor}\} \text{ and } Y_i = \{v_{i,2k+1} | k \in Z_{\lfloor \frac{m}{2} \rfloor}\}.$$

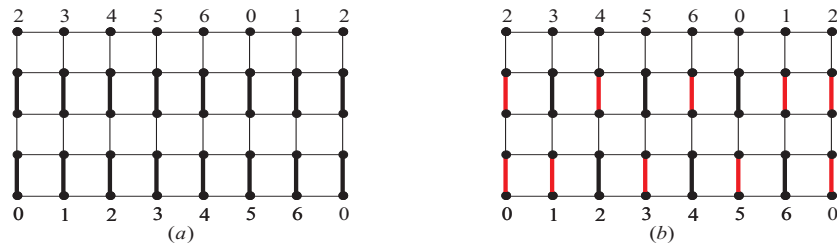
For $j \in Z_m$, let $W_j = \{v_{2k,j}v_{2k+1,j} | k \in Z_n\}$,

$$W_j^1 = \{v_{4k+2,j}v_{4k+3,j} | k \in Z_{\lfloor \frac{n}{2} \rfloor}\} \text{ and } W_j^2 = \{v_{4k,j}v_{4k+1,j} | k \in Z_{\lfloor \frac{n+1}{2} \rfloor}\}$$

be two subsets of W_j .

Theorem 3.1. For $n, m \geq 1$ and $1 \leq r \leq 2m + 1$, $F(T(2n, 2m + 1, r)) = (m + 1)n$.

Proof. Let $M_1 = W_0 \cup W_1 \cup \dots \cup W_{2m}$ be a perfect matching of $T(2n, 2m + 1, r)$ (see Fig. 5(a)). We will prove that $f(T(2n, 2m + 1, r), M_1) = (m + 1)n$. For $i \in Z_n$, since $R_{2i,2i+1}$ contains a subgraph isomorphic to $C_{2m+1} \square P_2$, any forcing set of $M_1 \cap E(R_{2i,2i+1})$ has size at least $m + 1$. Thus, M_1 has the forcing number at least $n(m + 1)$. Let $S = W_0 \cup W_1^1 \cup W_2^2 \cup W_3^1 \cup W_4^2 \cup \dots \cup W_{2m-1}^1 \cup W_{2m}^2$ be a subset of M_1 shown as red lines in Fig. 5(b), so

Fig. 5: The perfect matching M_1 of $T(4,7,5)$, and a forcing set of M_1 shown in red lines.

that exactly $m + 1$ edges of $R_{2i,2i+1}$ are chosen to belong to S . Obviously, S is a forcing set of M_1 with size $n(m + 1)$. Hence, we obtain that $f(T(2n, 2m + 1, r), M_1) = n(m + 1)$.

For any perfect matching M of $T(2n, 2m + 1, r)$, we will choose an independent set T of size mn such that “marked subgraph” contains no M -alternating cycles. By Lemma 2.3, we have

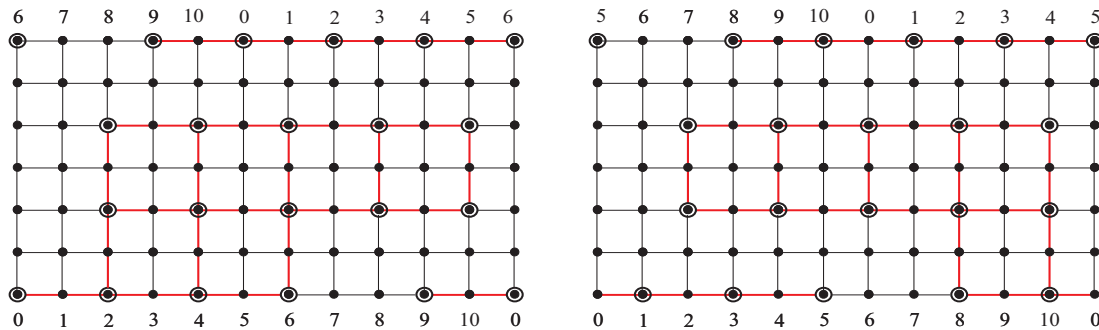
$$f(T(2n, 2m + 1, r), M) \leq |M| - |T| = (2m + 1)n - mn = (m + 1)n.$$

By the arbitrariness of M , we have $F(T(2n, 2m + 1, r)) \leq (m + 1)n$.

To achieve this goal, we will take m appropriate vertices on $1, 3, \dots, 2n-1$ rows. Let $X'_i = (X_i - \{v_{i,0}\}) \cup \{v_{i,2m}\}$ for $i \in \mathbb{Z}_{2n-1}$ and

$$X^* = \{v_{2n-1,2m+1-r}\} \cup \{v_{2n-1,2m+1-r+j} | j = 3, 5, \dots, 2m-1\}.$$

Take marked vertices $T = X'_1 \cup X'_3 \cup \dots \cup X'_{2n-3} \cup X^*$ shown as Fig. 6, where marked vertices are represented by circles.

Fig. 6: Marked vertices of $T(6,11,5)$ and $T(6,11,6)$.

From left to right, we choose 1'st, 4'th, 6'th, $\dots$, $(2m)$ 'th vertices in the first row and 3'th, 5'th, $\dots$, $(2m+1)$ 'th vertices in the third row as marked vertices. Hence, all edges incident with $v_{0,j}$ are not contained in “marked subgraph” for $0 \leq j \leq 2m$. Thus such $2m+1$ vertices are not contained in “marked subgraph”, and “marked subgraph” is a plane graph. The “marked subgraph” formed by all paths of length two whose initial and terminal are in $X'_1 \cup X'_3 \cup \dots \cup X'_{2n-3}$ is a 2×2 -polyomino corresponding to a $(n-2) \times (m-1)$ chessboard, and the “marked subgraph” formed by all paths of length two whose initial and terminal are in $X'_{2n-3} \cup X^*$ is a 2×2 -polyomino corresponding to some $1 \times t$ ($0 \leq t \leq m-1$) chessboard attaching a path. Thus, “marked subgraph” is a 2×2 -polyomino attaching a path.

Suppose to the contrary that C is an M -alternating cycle contained in “marked subgraph”. Then $\text{Int}[C]$ (the subgraph of $T(2n, 2m + 1, r)$ induced by the vertices of C and its interior) is a 2×2 -polyomino. By Lemma 2.4, $\text{Int}[C]$ has an odd number of interior vertices, which contradicts that C is M -alternating. Thus, “marked subgraph” contains no M -alternating cycles. $\square$

4. The maximum forcing number of $T(2n, 2m, r)$ for $1 \leq r \leq 2m$

In this section, we are to obtain the maximum forcing number of $T(2n, 2m, r)$ for $1 \leq r \leq 2m$.

In the proof of Theorem 3.1, we fix mn marked vertices to prove that “marked subgraph” contains no M -alternating cycles for any perfect matching M of $T(2n, 2m+1, r)$, where $1 \leq r \leq 2m+1$. But for a perfect matching M of $T(2n, 2m, r)$, “marked subgraph” contains an M -alternating cycle no matter which sets with size mn we mark. For the case that each II-cycle is not M -alternating, we can prove the following result.

Lemma 4.1. For $n, m \geq 2$ and $1 \leq r \leq 2m$, assume that M is a perfect matching of $T(2n, 2m, r)$ and each II-cycle is not M -alternating. Then we can mark mn vertices so that “marked subgraph” contains no M -alternating cycles.

Proof. First we choose an independent set T of $T(2n, 2m, r)$ with size mn as marked vertices. If n is odd, then take

$$T = \{Y_{4k+1} | k = 0, 1, 2, \dots, \frac{n-1}{2}\} \cup \{X_{4k+3} | k = 0, 1, 2, \dots, \frac{n-3}{2}\}.$$

Otherwise, take

$$T = \{Y_{4k+1} | k = 0, 1, 2, \dots, \frac{n-2}{2}\} \cup \{X_{4k+3} | k = 0, 1, 2, \dots, \frac{n-2}{2}\}.$$

See two examples in Fig. 7, where marked vertices are represented by cycles. If r is odd (resp. even), then

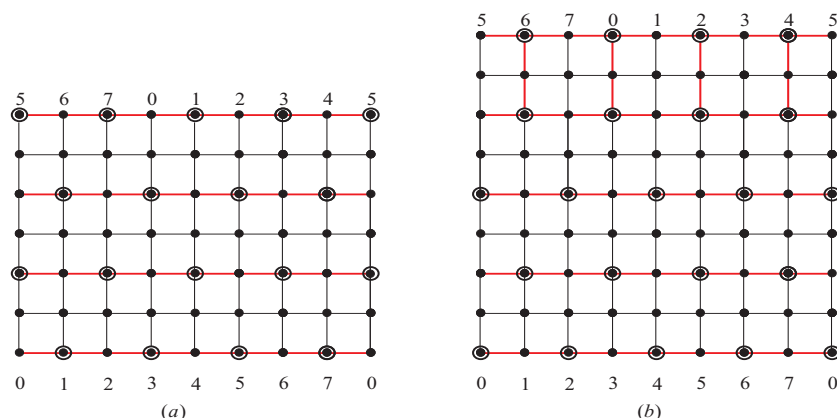


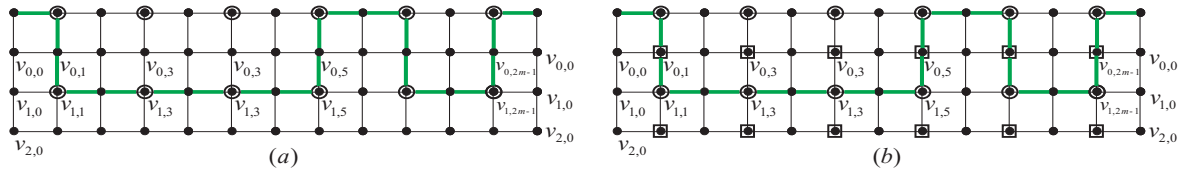
Fig. 7: Marked vertices and “marked subgraph” of $T(6, 8, 3)$ and $T(8, 8, 3)$.

marked vertices on the first and last rows are located at different (resp. same) columns. For the case that r and n have the same parity, “marked subgraph” consists of n II-cycles. By the assumption, each II-cycle is not M -alternating. Thus, “marked subgraph” contains no M -alternating cycles, and T is the marked vertices we require. It suffices to consider the case that r and n have different parity.

In the sequel, we only prove the lemma for the case that r is odd and n is even, and the proof is similar for the other case. Now marked vertices on the first and third rows are located at the same columns. Thus “marked subgraph” consists of m paths of length two $\{v_{2n-1, 2m-r+j}v_{0,j} | j = 1, 3, \dots, 2m-1\}$ and n II-cycles shown as red lines in Fig. 7(b).

By the assumption, each II-cycle is not M -alternating. Hence, each M -alternating cycle (if exists) of “marked subgraph” is contained in the subgraph induced by all vertices of the first three rows, and contains at least two vertices on the second row. By Lemma 2.4, an M -alternating cycle cannot form the boundary of a 2×2 -polyomino which corresponds to a $1 \times l$ chessboard for $1 \leq l \leq m-1$. Therefore, any M -alternating cycle of “marked subgraph” has the following form: it starts with a II-cycle in the first row and moves to the third row and backs at specified intervals shown as green lines in Fig. 8(a). Notice that each such cycle contains exactly $2m$ horizontal edges, divided in some way between the two rows.

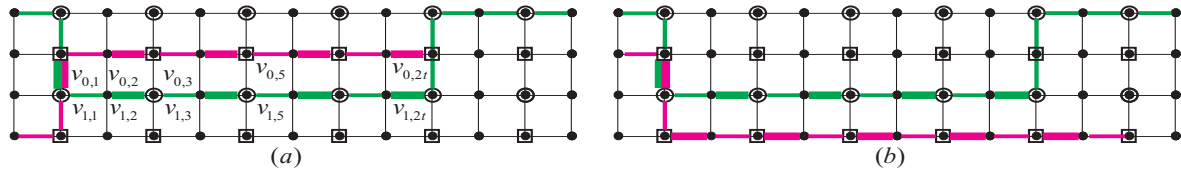
Translating the marked vertices down by one row shown as Fig. 8(b), where new marked vertices are represented by squares, we also have an M -alternating cycle lying on the subgraph induced by the vertices

Fig. 8: M -alternating cycle of "marked subgraph".

of the second, third and fourth rows (otherwise, new marked vertices we obtained is what we want). We will demonstrate that the new M -alternating cycle has more horizontal edges in the bottom (i.e., the fourth) row than the first one does. Consider the set of horizontal edges in the bottom row of the first M -alternating cycle, which is partitioned into subsets naturally by proximity: there is a set of horizontal edges, then a cross-over, then perhaps a cross-back, then another set of horizontal edges, and so forth. Consider one of these sets, say $\{v_{1,1}v_{1,2}, v_{1,2}v_{1,3}, \dots, v_{1,2t}v_{1,2t+1}\}$ shown as green lines on the third row of Fig. 9(a), where $t \geq 1$. By the form of M -alternating cycles, edges of $\{v_{1,1}v_{0,1}, v_{0,1}v_{2n-1,2m-r+1}\}$ and $\{v_{1,2t+1}v_{0,2t+1}, v_{0,2t+1}v_{2n-1,2m-r+2t+1}\}$ are contained in the first M -alternating cycle. It suffices to prove that the set of edges

$$\{v_{2,0}v_{2,1}, v_{2,1}v_{2,2}, v_{2,2}v_{2,3}, \dots, v_{2,2t}v_{2,2t+1}\} \text{ or } \{v_{2,1}v_{2,2}, v_{2,2}v_{2,3}, \dots, v_{2,2t}v_{2,2t+1}, v_{2,2t+1}v_{2,2t+2}\}$$

is contained in the bottom row of the new M -alternating cycle.

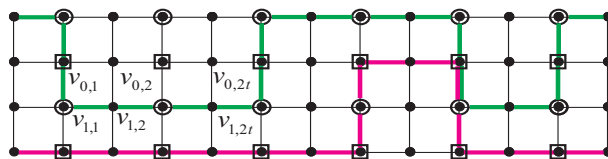
Fig. 9: Part of the two M -alternating cycles lying in corresponding "marked subgraphs".

Since all horizontal edges of the first M -alternating cycle lie on the first and third rows, and these of the new M -alternating cycle lie on the second and fourth rows, only vertical edges of two M -alternating cycles in $\{v_{0,2k+1}v_{1,2k+1} | k = 0, 1, \dots, m-1\}$ may be intersected. If $v_{0,1}v_{1,1}$ belongs to the new M -alternating cycle, then $v_{0,1}v_{1,1} \in M$, and $v_{1,1}v_{2,1}$ is contained in the new M -alternating cycle. We claim that $v_{0,0}v_{0,1}$ is contained in the new M -alternating cycle. Otherwise, $v_{0,1}v_{0,2}$ and $v_{0,2}v_{0,3} \in M$ are contained in the new M -alternating cycle. Since $v_{1,2}v_{1,3} \in M$, $v_{0,3}v_{1,3}$ does not lie on the new M -alternating cycle. Hence the path $v_{0,1}v_{0,2}v_{0,3} \dots v_{0,2t}v_{0,2t+1}$ lies on the new M -alternating cycle (see Fig. 9(a)). Note that $v_{0,2t}v_{0,2t+1} \in M$, which contradicts that $v_{2n-1,2m-r+2t+1}v_{0,2t+1}$ and $v_{0,2t+1}v_{1,2t+1}$ belong to the first M -alternating cycle. Now we prove the claim. Thus, $v_{0,0}v_{0,1}$ and $v_{1,1}v_{2,1}$ lie on the new M -alternating cycle (see Fig. 9(b)). Since $v_{1,1}v_{1,2}v_{1,3} \dots v_{1,2t}v_{1,2t+1}$ is on the first M -alternating cycle, we can obtain that the path $v_{2,1}v_{2,2}v_{2,3} \dots v_{2,2t}v_{2,2t+1}v_{2,2t+2}$ lies on the second M -alternating cycle by a simple argument. If $v_{0,2t+1}v_{1,2t+1}$ belongs to the new M -alternating cycle, then, by a similar argument, we can obtain that

$$v_{0,2t+2}v_{0,2t+1}v_{1,2t+1}v_{2,2t+1}v_{2,2t} \dots v_{2,2}v_{2,1}v_{2,0}$$

lies on the second M -alternating cycle. If neither $v_{0,1}v_{1,1}$ nor $v_{0,2t+1}v_{1,2t+1}$ belongs to the new M -alternating cycle (see Fig. 10), then, by the form of M -alternating cycles, such two M -alternating cycles have no common edges in this area, and the result holds naturally. This means that all horizontal edges in the bottom row of the first M -alternating cycle give rise to abutting horizontal edges in the bottom row of the second one. Because the intersected vertical edges cannot overlap, there is at least one more horizontal edge in the bottom row of the second M -alternating cycle.

Each time we translate the marked vertices down by one row, we obtain an abutting M -alternating cycle which contains more horizontal edges in the bottom row than the first one does. Since any M -alternating

Fig. 10: Part of the two M -alternating cycles lying in corresponding “marked subgraphs”.

cycle contains no more than $2m$ horizontal edges on its bottom row, there is a placement of marked vertices such that “marked subgraph” contains no M -alternating cycles. $\square$

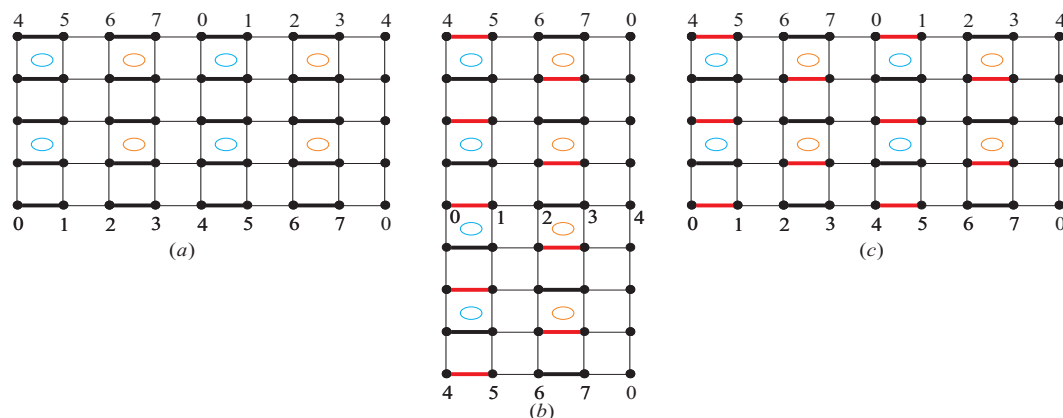
4.1. The maximum forcing number of $T(2n, 2m, 2r)$ for $1 \leq r \leq m$

By Lemma 2.1, $T(n, m, r)$ contains (r, m) I-cycles, and each I-cycle contains $\frac{mn}{(r, m)}$ vertices. For $(r, m) \geq 2$ and $j \in Z_{(r, m)}$, the subgraph induced by all vertices of the two I-cycles containing j -column and $(j + 1)$ -column contains a subgraph isomorphic to $C_{\frac{mn}{(r, m)}} \square P_2$, denoted by $C_{j, j+1}$. Particularly, $C_{j, j+1}$ is isomorphic to $C_{\frac{mn}{(r, m)}} \square P_2$ for $(r, m) \geq 3$ where $j \in Z_{(r, m)}$.

Theorem 4.2. For $n, m \geq 2$ and $1 \leq r \leq m$, we have

$$F(T(2n, 2m, 2r)) = \begin{cases} mn + 1, & \text{if } (r, m) = 1; \\ mn, & \text{otherwise.} \end{cases}$$

Proof. First we prove the case that $(r, m) \neq 1$. Let $M_1 = E_0 \cup E_2 \cup \dots \cup E_{2m-2}$ be a perfect matching of $T(2n, 2m, 2r)$ shown as Fig. 11(a), where $E_j = \{v_{i,j}v_{i,j+1} | i \in Z_{2n}\}$. Then $C_{2j, 2j+1}$ contains a subgraph isomorphic to $C_{\frac{2mn}{(r, m)}} \square P_2$ for $j \in Z_{(r, m)}$ and contains $\frac{mn}{(r, m)}$ disjoint M_1 -alternating cycles. Hence, $T(2n, 2m, 2r)$ contains mn disjoint M_1 -alternating cycles and $f(T(2n, 2m, 2r), M_1) \geq mn$. Form a forcing set of size mn so that half horizontal edges of $C_{2j, 2j+1}$ are chosen for $j \in Z_{(r, m)}$. Precisely, from top to bottom we choose 1'th, 3'th, $\dots$, $(\frac{2mn}{(r, m)} - 1)$ 'th horizontal edges of $C_{4j, 4j+1}$ for $j \in \lceil \frac{(r, m)}{2} \rceil$ and 2'th, 4'th, $\dots$, $\frac{2mn}{(r, m)}$ 'th horizontal edges of $C_{4j+2, 4j+3}$ for $j \in \lfloor \frac{(r, m)}{2} \rfloor$ (red lines of $T^*(2n, 2m, 2r)$ in Fig. 11(b) and that of $T(2n, 2m, 2r)$ in Fig. 11(c) form a forcing set). Hence, $f(T(2n, 2m, 2r), M_1) = mn$.

Fig. 11: The perfect matching M_1 of $T(4, 8, 4)$, where red lines form a forcing set of M_1 .

Let M be any perfect matching of $T(2n, 2m, 2r)$. It suffices to prove that

$$f(T(2n, 2m, 2r), M) \leq mn.$$

If none of II-cycles is M -alternating, then we can mark mn vertices so that “marked subgraph” contains no M -alternating cycles by Lemma 4.1. Otherwise, there is an M -alternating II-cycle. Then each I-cycle is not M -alternating. By Lemma 2.2, $T(2n, 2m, 2r)$ has another representation

$$T^*(2n, 2m, 2r) = T(2(r, m), \frac{2nm}{(r, m)}, 2n(\frac{m}{(r, m)} - k)),$$

in which each II-cycle is not M -alternating. By Lemma 4.1, we can mark mn vertices so that “marked subgraph” contains no M -alternating cycles. By Lemma 2.3,

$$f(T(2n, 2m, 2r), M) = f(T^*(2n, 2m, 2r), M) \leq |M| - |T| = mn.$$

By the arbitrariness of M , we have $F(T(2n, 2m, 2r)) \leq mn$.

Next we prove the case that $(r, m) = 1$. By Lemma 2.1, $T(2n, 2m, 2r)$ has exactly two I-cycles. Let $M_1 = E_0 \cup E_2 \cup \dots \cup E_{2m-2}$ be a perfect matching of $T(2n, 2m, 2r)$ shown as bold lines in Fig. 12(a). Since

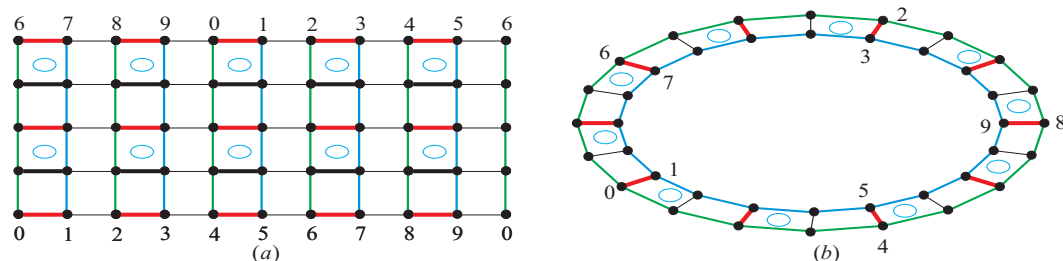


Fig. 12: The perfect matching M_1 of $T(4, 10, 4)$, and red lines cannot form a forcing set of M_1 .

$C_{0,1}$ contains a subgraph isomorphic to $C_{2nm} \square P_2$, $T(2n, 2m, 2r)$ contains mn disjoint M_1 -alternating cycles. Since a forcing set of M_1 contains at least one edge from each M_1 -alternating cycle, any forcing set of M_1 has size at least mn . To find a forcing set of size mn , we need to choose one of the horizontal edges in any two consecutive ones of $C_{0,1}$. In $C_{0,1}$, starting with the two consecutive edges $v_{0,0}v_{0,1}$ and $v_{1,0}v_{1,1}$, in which the latter are chosen, we choose a set of horizontal edges with size mn shown as red lines in Fig. 12(b), where each E_{2j} for $j \in Z_m$ has n edges $\{v_{2i+1,2j}v_{2i+1,2j+1} | i \in Z_n\}$ being chosen. But the chosen mn edges cannot form a forcing set of M_1 for there are still n II-cycles being not intersected with such mn edges (see red lines in Fig. 12(a)). Hence, $f(T(2n, 2m, 2r), M_1) \geq mn + 1$. It's easy to find a forcing set of size $mn + 1$. Thus $f(T(2n, 2m, 2r), M_1) = mn + 1$.

For any perfect matching M of $T(2n, 2m, 2r)$, we are to prove that

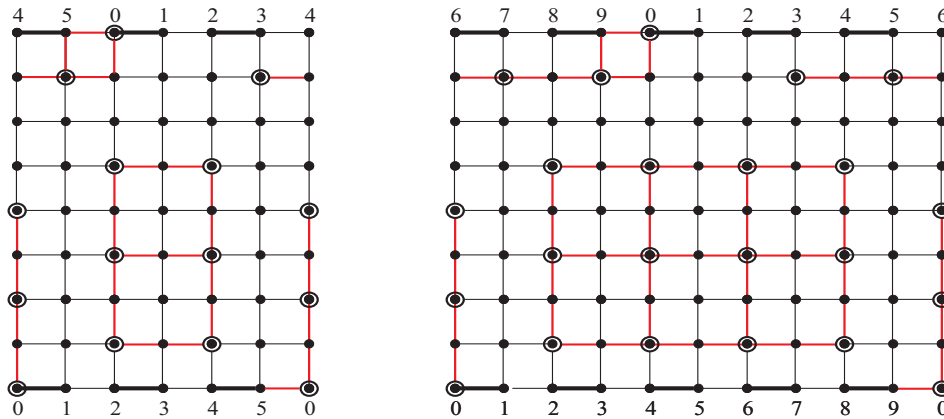
$$f(T(2n, 2m, 2r), M) \leq mn + 1.$$

By Lemma 2.3, it suffices to prove that we can mark at least $mn - 1$ vertices in $T(2n, 2m, 2r)$ such that “marked subgraph” contains no M -alternating cycles. If each II-cycle is not M -alternating, then we can mark mn vertices so that “marked subgraph” contains no M -alternating cycles by Lemma 4.1. Otherwise, assume that $v_{2n-1,0}v_{2n-1,1} \dots v_{2n-1,2m-1}v_{2n-1,0}$ is an M -alternating cycle, and $\{v_{2n-1,2j}v_{2n-1,2j+1} | j \in Z_m\} \subseteq M$. Let

$$X_* = \{v_{0,1}, v_{0,3}, \dots, v_{0,2r-1}, v_{0,2r+3}, v_{0,2r+5}, \dots, v_{0,2m-1}\} \text{ and } Y_* = \{v_{3,0}, v_{5,0}, \dots, v_{2n-1,0}\}.$$

Take $T = Y_* \cup X_* \cup X'_2 \cup X'_4 \cup \dots \cup X'_{2n-2}$ as marked vertices shown as Fig. 13, where $X'_i = X_i - \{v_{i,0}\}$ for $i \in Z_{2n}$, and marked vertices are represented by cycles. Then all vertices on the third row don't lie on the “marked subgraph”, and “marked subgraph” is a plane graph shown as red lines in Fig. 13.

The “marked subgraph” formed by all paths of length two whose initial and terminal are in $X'_2 \cup X'_4 \cup \dots \cup X'_{2n-2}$ is a 2×2 -polyomino corresponding to a $(n-2) \times (m-2)$ chessboard. Noting that both $v_{2n-1,0}$ and $v_{0,2r-1}$ are marked vertices, $v_{2n-1,0}v_{2n-1,2m-1}v_{0,2r-1}v_{0,2r}v_{2n-1,0}$ is contained in “marked subgraph”, and the “marked subgraph” formed by all paths of length two whose initial and terminal are in $X_* \cup Y_*$ is a cycle of length 4

Fig. 13: Marked vertices and “marked subgraph” of $T(8, 6, 2)$ and $T(8, 10, 4)$.

attaching a path on $2m - 3$ vertices and a path on $2n - 3$ vertices. Furthermore, “marked subgraph” consists of a 2×2 -polyomino corresponding to a $(n - 2) \times (m - 2)$ chessboard and a 4-cycle attaching a path on $2m - 2$ vertices and a path on $2n - 3$ vertices. Since $v_{2n-1,0}v_{2n-1,1} \in M$, such 4-cycle $v_{2n-1,0}v_{2n-1,2m-1}v_{0,2r-1}v_{0,2r}v_{2n-1,0}$ is not M -alternating. By Lemma 2.4, a 2×2 -polyomino contains no M -alternating cycles. Thus, “marked subgraph” contains no M -alternating cycles.

By Lemma 2.3, $M \setminus E_T$ is a forcing set of M and

$$f(T(2n, 2m, 2r), M) \leq |M| - |T| \leq 2mn - (mn - 1) = mn + 1.$$

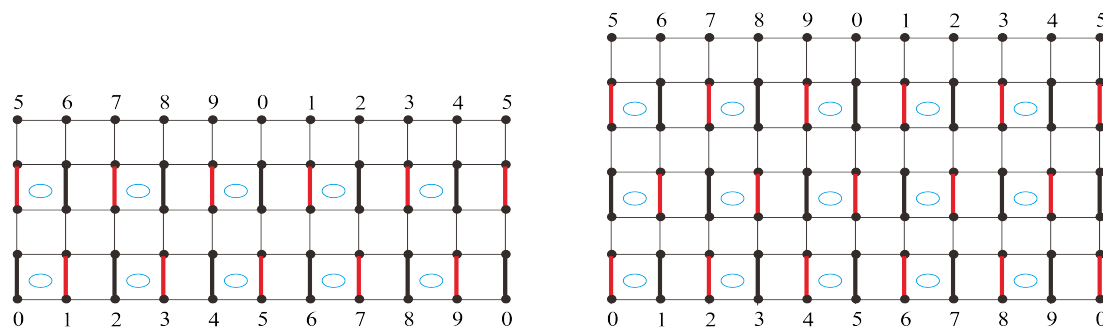
By the arbitrariness of M , we have $F(T(2n, 2m, 2r)) \leq nm + 1$. $\square$

4.2. The maximum forcing number of $T(2n, 2m, 2r - 1)$ for $1 \leq r \leq m$

Next we will obtain the maximum forcing number of $T(2n, 2m, 2r - 1)$ for $1 \leq r \leq m$.

Theorem 4.3. For $n \geq 1, m \geq 2$ and $1 \leq r \leq m$, $F(T(2n, 2m, 2r - 1)) = mn$.

Proof. Let $M_1 = W_0 \cup W_1 \cup \cdots \cup W_{2m-1}$ be a perfect matching of $T(2n, 2m, 2r - 1)$. Since $R_{2i, 2i+1}$ contains a subgraph isomorphic to $C_{2m} \square P_2$, it contains m disjoint M_1 -alternating cycles for $i \in \mathbb{Z}_n$. Thus, any forcing set of M_1 has size at least mn . Clearly, $W_0^2 \cup W_1^1 \cup W_2^2 \cup \cdots \cup W_{2m-2}^1 \cup W_{2m-1}^1$ shown as red lines in Fig. 14 is a forcing set of M_1 with size mn . Hence, we obtain that $f(T(2n, 2m, 2r - 1), M_1) = mn$.

Fig. 14: Perfect matchings M_1 of $T(4, 10, 5)$ and $T(6, 10, 5)$, where red lines form a forcing set.

Let M be any perfect matching of $T(2n, 2m, 2r - 1)$, we are to prove that

$$f(T(2n, 2m, 2r - 1), M) \leq mn.$$

It suffices to mark mn vertices of $T(2n, 2m, 2r - 1)$ such that “marked subgraph” contains no M -alternating cycles. If we have done, then by Lemma 2.3, we have

$$f(T(2n, 2m, 2r - 1), M) \leq |M| - mn = mn.$$

By the arbitrariness of M , we have $F(T(2n, 2m, 2r - 1)) \leq mn$.

For $n \geq 2$, we only suffice to prove the case that there is a Π -cycle is M -alternating by Lemma 4.1. For $n = 1$, n and $2r - 1$ are of the same parity, by the proof of Lemma 4.1, we also need to prove the same case as $n \geq 2$. Without loss of generality, we suppose that $v_{2n-1,0}v_{2n-1,1} \cdots v_{2n-1,2m-1}v_{2n-1,0}$ is an M -alternating Π -cycle, and $\{v_{2n-1,2j}v_{2n-1,2j+1} | j \in Z_m\} \subseteq M$. Let $T = Y_* \cup X'_0 \cup X'_2 \cup \cdots \cup X'_{2n-2}$ (see Fig. 15) as marked vertices which are represented by cycles, where

$$Y_* = \{v_{2n-1,2m-2r+1}, v_{1,0}, v_{3,0}, \dots, v_{2n-3,0}\} \text{ and } X'_i = X_i - \{v_{i,0}\} \text{ for } i \in Z_{2n}.$$

Then T is of size mn . Since any vertices of Y_* and that of X'_{2i} belong to no same rows for $i \in Z_n$, any vertices of $\{v_{i,1}, v_{i,2m-1} | i \in Z_{2n}\}$ are not contained in “marked subgraph”. Furthermore, any vertices of $\{v_{2n-1,2m-2r+1+j} | j = 2, 3, \dots, 2m-2\}$ are not contained in “marked subgraph”. Thus, “marked subgraph” is a plane graph shown as red lines in Fig. 15. The “marked subgraph” formed by all paths of length two whose initial and terminal are in $X'_0 \cup X'_2 \cup X'_4 \cup \cdots \cup X'_{2n-2}$ is a 2×2 -polyomino corresponding to a $(n-1) \times (m-2)$ chessboard, which contains no M -alternating cycles by Lemma 2.4.

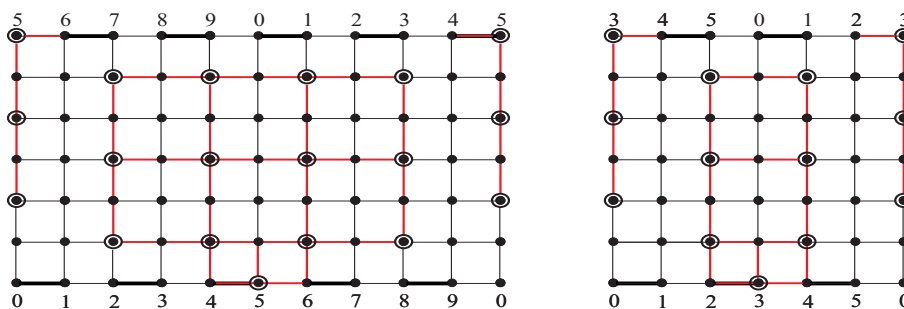


Fig. 15: Marked vertices and “marked subgraph” of $T(6, 10, 5)$ and $T(6, 6, 3)$.

Since $v_{2n-1,2m-2r+1}$, $v_{2n-2,2m-2r}$ and $v_{2n-2,2m-2r+2}$ are marked vertices, four paths of length two $v_{2n-2,2m-2r}v_{2n-1,2m-2r}v_{2n-1,2m-2r+1}$, $v_{2n-2,2m-2r}v_{2n-2,2m-2r+1}v_{2n-1,2m-2r+1}$, $v_{2n-2,2m-2r+1}v_{2n-2,2m-2r+2}v_{2n-1,2m-2r+2}$ and $v_{2n-2,2m-2r+1}v_{2n-1,2m-2r+1}v_{2n-1,2m-2r+2}$ are contained in “marked subgraph”. Let C be an M -alternating cycle of “marked subgraph”. Then C contains the vertex $v_{2n-1,2m-2r+1}$. Since C is M -alternating, it also contains three edges $v_{2n-1,2m-2r}v_{2n-2,2m-2r}$, $v_{2n-1,2m-2r}v_{2n-1,2m-2r+1}$ and $v_{2n-1,2m-2r+1}v_{2n-2,2m-2r+1}$, and such four vertices $v_{2n-1,2m-2r}$, $v_{2n-1,2m-2r+1}$, $v_{2n-2,2m-2r}$ and $v_{2n-2,2m-2r+1}$ are on the boundary of $\text{Int}[C]$. Next, we prove that C contains exactly such four vertices. If C contains at least six vertices, then $\text{Int}[C]$ and $\text{Int}[C] - \{v_{2n-1,2m-2r}, v_{2n-1,2m-2r+1}\}$ have the same number of interior vertices. Since $\text{Int}[C] - \{v_{2n-1,2m-2r}, v_{2n-1,2m-2r+1}\}$ is a 2×2 -polyomino, it has an odd number of interior vertices by Lemma 2.4. Thus, $\text{Int}[C]$ has an odd number of interior vertices, which contradicts that C is M -alternating. Thus

$$C = v_{2n-1,2m-2r}v_{2n-1,2m-2r+1}v_{2n-2,2m-2r+1}v_{2n-2,2m-2r}v_{2n-1,2m-2r}.$$

If $v_{2n-2,2m-2r}v_{2n-2,2m-2r+1} \notin M$, then C is not M -alternating. Hence none of cycles in “marked subgraph” is M -alternating. So we assume that $v_{2n-2,2m-2r}v_{2n-2,2m-2r+1} \in M$. Translating marked vertices right by two columns, by a similar argument, we suffice to consider the case that $v_{2n-2,2m-2r+2}v_{2n-2,2m-2r+3} \in M$. Proceeding like this, it suffices to consider the case that M has the same matching form on the last $2n$ rows, i.e., $\{v_{i,2j}v_{i,2j+1} | j \in Z_m\} \subseteq M$ for $0 \leq i \leq 2n-1$. Since the torsion is $2r-1$, M has different matching form on the first two rows. By the previous argument, we have done. $\square$

5. Discussion of the maximum forcing number of $T(2n+1, 2m, r)$ for $1 \leq r \leq 2m$

By Theorems 3.1 and 4.3, we obtain the maximum forcing number of $T(2n+1, 2m, 2r)$ for $1 \leq r \leq m$.

Theorem 5.1. For $n \geq 1$, $m \geq 2$ and $1 \leq r \leq m$, we have

$$F(T(2n+1, 2m, 2r)) = \begin{cases} \frac{m(2n+1)+(r,m)}{2}, & \text{if } \frac{m}{(r,m)} \text{ is odd;} \\ \frac{m(2n+1)}{2}, & \text{otherwise.} \end{cases}$$

Proof. By Lemma 2.2, $T(2n+1, 2m, 2r)$ has another representation

$$T^*(2n+1, 2m, 2r) = T(2(r, m), \frac{m(2n+1)}{(r, m)}, (2n+1)(\frac{m}{(r, m)} - k))$$

where $0 \leq k \leq \frac{m}{(r, m)} - 1$ satisfies the equation $(2r, 2m) \equiv 2rk \pmod{2m}$.

If $\frac{m}{(r, m)}$ is even, then $2rk - (2r, 2m) = 2mp$ for some non-negative integer p . That is, $rk - (r, m) = mp$. Thus $\frac{r}{(r, m)}k = \frac{m}{(r, m)}p + 1$. Since $\frac{m}{(r, m)}$ is even and $\frac{m}{(r, m)}p + 1$ is odd, we obtain that k is an odd number. Hence $\frac{m}{(r, m)} - k$ and $(2n+1)(\frac{m}{(r, m)} - k)$ are also odd numbers. Let $n' = (r, m)$, $m' = \frac{m(2n+1)}{2(r, m)}$ and $2r' - 1 = (2n+1)(\frac{m}{(r, m)} - k)$. Then $T^*(2n+1, 2m, 2r) = T(2n', 2m', 2r' - 1)$. Since $0 \leq k \leq \frac{m}{(r, m)} - 1$, we have $2n+1 \leq 2r' - 1 \leq (2n+1)\frac{m}{(r, m)} = 2m'$. Thus $n+1 \leq r' < m'$. By Theorem 4.3, we have

$$F(T(2n+1, 2m, 2r)) = F(T(2n', 2m', 2r' - 1)) = m'n' = \frac{m(2n+1)}{2}.$$

If $\frac{m}{(r, m)}$ is odd, then $2(r, m)$ is even, $\frac{m(2n+1)}{(r, m)}$ is odd. Let $n' = (r, m)$, $2m' + 1 = \frac{m(2n+1)}{(r, m)}$ and $r' = (2n+1)(\frac{m}{(r, m)} - k)$. Since $0 \leq k \leq \frac{m}{(r, m)} - 1$, we have $2n+1 \leq r' \leq (2n+1)\frac{m}{(r, m)} = 2m' + 1$. By Theorem 3.1, we have

$$F(T(2n+1, 2m, 2r)) = F(T(2n', 2m' + 1, r')) = (m' + 1)n' = \frac{m(2n+1) + (r, m)}{2}.$$

Now we finish the proof. $\square$

For $T(2n+1, 2m, 2r-1)$, we have not been able to obtain a general expression for the maximum forcing number for $1 \leq r \leq m$. Therefore, computing the maximum forcing number of $T(2n+1, 2m, 2r-1)$ is an open problem.

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