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On a weighted N-Laplacian problem with double exponential nonlinearities

Brahim Dridi[a], **Rached Jaidane**[b,*]

[a]*Mathematics Department, El Manar Preparatory Institute for Engineering Studies, University of Tunis El Manar, Tunis 2092, Tunisia*
[b]*Mathematics Department, Faculty of Sciences of Tunis, University of Tunis El Manar, Tunisia*

Abstract. In this paper, we study the nonlinear elliptic problem

$$-\mathrm{div}(w(x)|\nabla u|^{N-2}\nabla u) = f(x,u), \quad u \in W_0^{1,N}(B,w) \qquad (P)$$

where B is the unit ball of $\mathbb{R}^{\mathbb{N}}$, $N \geq 2$ and $w(x) = \Big(\log \frac{e}{|x|}\Big)^{N-1}$ is the singular logarithm weight with the limiting exponent $N-1$ in the Trudinger-Moser embedding. We consider the problem (P) when the nonlinearity is sub-critical and critical with respect to a maximal growth of double exponent type and we prove the existence of positive solution by using Mountain Pass theorem without the Ambrosetti-Rabionowitz condition. When the nonlinearity is critical, we prove that the associated energy satisfies the Palais-Smale condition only to a given limit level and we prove that the min-max level is less than this limit.

1. Introduction and Main results

In this paper, we consider the following elliptic nonlinear problem:

$$\begin{cases} L_{N,w} = -\mathrm{div}(w(x)|\nabla u|^{N-2}\nabla u) &= f(x,u) \quad \text{in} \quad B \\ u &> 0 \quad \text{in} \quad B \\ u &= 0 \quad \text{on} \quad \partial B, \end{cases} \qquad (1)$$

where $B = B(0,1)$ is the unit open ball in $\mathbb{R}^N$, $N > 2$, $f(x,t)$ is a radial function with respect to x and the weight $w(x)$ is given by

$$w(x) = \Big(\log \frac{e}{|x|}\Big)^{N-1}. \qquad (2)$$

Since 1970, when Moser gives the famous result about the Trudinger-Moser inequality many applications take place as in conformal deformation theory on manifolds, the study of the prescribed Gauss curvature

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* Corresponding author: Rached Jaidane
Email addresses: dridibr@gmail.com (Brahim Dridi), rachedjaidane@gmail.com (Rached Jaidane)
ORCID iDs: https://orcid.org/0000-0001-5863-029X (Brahim Dridi), https://orcid.org/0000-0001-7241-6847 (Rached Jaidane)

and mean field equations.
After that, a logarithmic Trudinger-Moser inequality was used in crucial way in [23] to study the Liouville equation of the form

$$\begin{cases} -\Delta u &= \lambda \frac{e^u}{\int_\Omega e^u} \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{3}$$

where Ω is an open domain of $\mathbb{R}^N$, $N \geq 2$ and λ is a positive parameter.
The equation (3) has a long history and has been derived in the study of multiple condensate solution in the Chern-Simons-Higgs theory [27, 28] and also, it appeared in the study of Euler Flow [7, 8, 15, 19].
Later, the Trudinger-Moser inequality was improved to a weighted inequalities [2, 10, 11, 14]. The influence of the weight in the Sobolev norm was studied as the compact embedding in [20].
When the weight is of logarithmic type, Calanchi and Ruf [12] extend the Trudinger-Moser inequality and give some applications when $N = 2$ and for prescribed nonlinearities. After that, Calanchi et al. [13] considered a more general nonlinearities and proved the existence of radial solutions.
In this paper, we investigate the case $N > 2$ and use Trudinger-Moser inequality to study and prove the existence of solutions to the problem (1) without use of the Ambrosetti-Rabionowitz condition.

Let $\Omega \subset \mathbb{R}^N$, $N > 2$ be a bounded domain and $w \in L^1(\Omega)$ be a nonnegative function, the weighted Sobolev space is defined as

$$W_0^{1,N}(\Omega, w) = cl\{u \in C_0^\infty(\Omega) / \int_B |\nabla u|^N w(x)dx < \infty\}.$$

We will restrict our attention to radial functions and then consider the subspace

$$\mathbf{E} = W_{0,rad}^{1,N}(B, w) = cl\{u \in C_{0,rad}^\infty(B) / \int_B |\nabla u|^N w(x)dx < \infty\} \tag{4}$$

endowed with the norm

$$\|u\| = \Big(\int_B |\nabla u|^N w(x)dx\Big)^{\frac{1}{N}}.$$

The choice of the weight and the space $W_{0,rad}^{1,N}(B, w)$ are motivated by the following double exponential inequalities.

Theorem 1.1. *[11] Let w given by (2), then*

$$\int_B exp(e^{|u|^{\frac{N}{N-1}}})dx < +\infty, \quad \forall \; u \in W_{0,rad}^{1,N}(B, w), \tag{5}$$

and

$$\sup_{\substack{u \in W_{0,rad}^{1,N}(B,w) \\ \|u\|_w \leq 1}} \int_B exp(\beta e^{\omega_{N-1}^{\frac{1}{N-1}} |u|^{\frac{N}{N-1}}})dx < +\infty \quad \Leftrightarrow \quad \beta \leq N. \tag{6}$$

where ω_{N-1} is the area of the unit sphere S^{N-1} in $\mathbb{R}^N$.

Let N' be the Hölder conjugate of N that is $N' = \frac{N}{N-1}\cdot$ In view of inequality (6), we say that f has subcritical growth at $+\infty$ if

$$\lim_{s \to +\infty} \frac{|f(x, s)|}{\exp(Ne^{\alpha s^{N'}})} = 0, \quad \text{for all} \;\; \alpha > 0 \tag{7}$$

and f has critical growth at $+\infty$ if there exists some $\alpha_0 > 0$ such that

$$\lim_{s\to+\infty}\frac{|f(x,s)|}{\exp(Ne^{\alpha s^{N'}})}=0,\ \ \forall\,\alpha>\alpha_0 \quad\text{and}\quad \lim_{s\to+\infty}\frac{|f(x,s)|}{\exp(Ne^{\alpha s^{N'}})}=+\infty,\ \ \forall\,\alpha<\alpha_0. \tag{8}$$

As mentioned in [13] for $N=2$, if there exist $\alpha_0>0$ and $\beta_0>0$ such that

$$\lim_{s\to+\infty}\frac{|f(x,s)|}{\exp(\beta_0 e^{\alpha s^{N'}})}=0,\ \ \forall\,\alpha>\alpha_0 \quad\text{and}\quad \lim_{s\to+\infty}\frac{|f(x,s)|}{\exp(\beta_0 e^{\alpha s^{N'}})}=+\infty,\ \ \forall\,\alpha<\alpha_0,$$

then for all $\beta>0$, we have

$$\lim_{s\to+\infty}\frac{|f(x,s)|}{\exp(\beta e^{\alpha s^{N'}})}=0,\ \ \forall\,\alpha>\alpha_0 \quad\text{and}\quad \lim_{s\to+\infty}\frac{|f(x,s)|}{\exp(\beta e^{\alpha s^{N'}})}=+\infty,\ \ \forall\,\alpha<\alpha_0$$

and so in (8), we choose $\beta=N$ by convenience.

In this paper, we consider the problem (1) with subcritical and critical growth nonlinearities $f(x,t)$. Furthermore, we suppose that $f(x,t)$ satisfies the following hypothesis:

(H1) $f: B\times\mathbb{R}\to\mathbb{R}$ is continuous, positive, radial in x, and $f(x,t)=0$ for $t\le 0$

(H2) There exist $t_0>0$ and $M>0$ such that for all $t>t_0$ and for all $x\in B$ we have

$$0<F(x,t)\le Mf(x,t),$$

where

$$F(x,t)=\int_0^t f(x,s)ds.$$

(H3) $0<F(x,t)\le \dfrac{1}{N}f(x,t)t,\quad \forall t>0, \forall x\in B.$

Before announcing our first result, for the weight function $w(x)$ given by (2), we denote

$$\lambda_1=\inf_{\substack{u\in W^{1,N}_{0,rad}(B,w)\\ u\neq 0}}\frac{\int_B|\nabla u|^N w(x)dx}{\int_B |u|^N dx}.$$

the first eigenvalue of $(L_{N,w}, W^{1,N}_{0,rad}(B,w))$. It is well known that λ_1 is isolated simple positive eigenvalue and has a positive bounded associated eigenfunction, [17]. We will prove the following results.

Theorem 1.2. *Let $f(x,t)$ be a function that has a subcritical growth at $+\infty$ and satisfies $(H1)$, $(H2)$ and $(H3)$. In addition, suppose that $f(x,t)$ verifies the condition*

$$(H4)\qquad \limsup_{t\to 0}\frac{NF(x,t)}{t^N}<\lambda_1\quad \textit{uniformly in}\ \ x,$$

then problem (1) has a non trivial radial solution.

For a critical growth nonlinearity, the following result holds.

Theorem 1.3. *Assume that $f(x,t)$ has a critical growth at $+\infty$ for some α_0 and satisfies the conditions $(H1)$, $(H2)$, $(H3)$ and $(H4)$. If in addition $f(x,t)$ satisfies the asymptotic condition*

$$(H5)\qquad \lim_{t\to\infty}\frac{f(x,t)t}{\exp(Ne^{\alpha_0 t^{N'}})}\ge\gamma_0\quad \textit{uniformly in}\ \ x,\ \ \textit{with}\quad \gamma_0>\frac{1}{\alpha_0^{N-1}e^N},$$

then the problem (1) has a nontrivial solution.

We give an example of such non-linearity f. Let $f(t) = F'(t)$ with $F(t) = \frac{t^{N+1}}{N+1} + t^{\tau} \exp(Ne\alpha_0 t^{N'})$, with $\tau > N+1$. A simple calculation shows that f verifies the conditions $(H1)$, $(H2)$, $(H3)$,$(H4)$ and $(H5)$.

Remark 1.1 *The authors in [16, 30] proved that there is a non-trivial solution to this problem using the Mountain Pass Theorem . The authors used the following condition:*

$(F3)$ *There is a constant $\theta > N$ such that for all $x \in B$ and $t > 0$,*

$$0 < \theta F(x,t) \leq t f(x,t),$$

which is weaker than our hypothesis $(H3)$.

Zhang added in [30] another assumption $(F6)$, *namely:*

(F6) $\dfrac{f(t)}{t}$ *is increasing for all $t > 0$*

to obtain a ground state solution.

The geometric requirements of the mountain Pass Theorem follow from the assumptions on the nonlinear reaction term f but the difficulty is in the proof of the compactness condition. We will prove that when f has subcritical growth, the functional J satisfies the compactness condition as required in the Ambrosetti-Rabionowitz Theorem [3], but in the critical growth case, the compactness is lost and we follows the schemas of [13, 18] and find a logarithmic concentrating sequence (Moser sequence) to avoid this non-compactness level.

We point out that the special case $N = 2$, i.e the following problem

$$\begin{cases} L_{2,w} := -\operatorname{div}(w(x)\nabla u) & = & f(x,u) & \text{in} & B \\ u & > & 0 & \text{in} & B \\ u & = & 0 & \text{on} & \partial B, \end{cases}$$

was studied in [13].

Also, the problems (1) without weight *(w=const.)* has been extensively studied by several authors, see [1, 14, 21, 24, 29] for example and references therein.

Finally, the problem (1) is important and has several applications as in non-Newtonian fluids, reaction diffusion problem, turbulent flows in porous media and image treatment [4, 5, 25, 26]. In this work, the constant C may change from line to another and sometimes we index the constants in order to show how they changes.

2. Preliminaries and Variational formulation

Let B the unit ball in $\mathbb{R}^N$, $N > 2$ and throughout this paper we denote

$$\|u\|_p = \Big(\int_B |u|^p dx\Big)^{\frac{1}{p}}$$

the standard norm in the Lebesgue space $L^p(B)$, for $1 \leq p < \infty$.

We set $\mathbf{E} = W^{1,N}_{0,rad}(B,w)$ equipped with norm

$$\|u\| = \Big(\int_B |\nabla u|^N w(x)\, dx\Big)^{\frac{1}{N}}$$

where $w(x)$ is given by

$$w(x) = \Big(\log\frac{e}{|x|}\Big)^{N-1}.$$

We will consider the following definition of solutions.

Definition 2.1. *We say that a function $u \in \mathbf{E}$ is a solution of the problem (1) if*

$$\int_B |\nabla u|^{N-2}\nabla u.\nabla\varphi\, w(x)dx = \int_B f(x,u)\varphi dx, \;\; \forall\, \varphi \in \mathbf{E}. \tag{9}$$

Let $J : \mathbf{E} \to \mathbb{R}$ be the functional given by

$$J(u) = \frac{1}{N}\int_B |\nabla u|^N w(x)dx - \int_B F(x,u)dx, \tag{10}$$

where

$$F(x,t) = \int_0^t f(x,s)ds.$$

The functional J is well defined and of class C^1 since there exist $a,\ C > 0$ positive constants and there exists $t_1 > 1$ such for that

$$|f(x,t)| \le C\exp(Ne^{a\, t^{N'}}), \qquad \forall |t| > t_1, \tag{11}$$

whenever the nonlinearity $f(x,t)$ is critical or subcritical at $+\infty$.
In such cases, in order to prove the existence of nontrivial solution to the problem (1), we will prove the existence of nonzero critical point of the functional J by using the following theorem introduced by Ambrosetti and Rabionowitz in [3] (Mountain Pass Theorem).

Definition 2.2. *Let (u_n) be a sequence in a Banach space E and $J \in C^1(E,\mathbb{R})$ and let $c \in \mathbb{R}$. We say that the sequence (u_n) is a Palais-Smale sequence at level c (or $(PS)_c$ sequence) for the functional J if*

$$J(u_n) \to c \;\; in \;\; \mathbb{R}, \;\; as \;\; n \to +\infty$$

and

$$J'(u_n) \to 0 \;\; in \;\; E', \;\; as \;\; n \to +\infty.$$

We say that the functional J satisfies the Palais-Smale condition $(PS)_c$ at the level c if every $(PS)_c$ sequence (u_n) is relatively compact in E.

Theorem 2.3. *[3] Let E be a Banach space and $J : E \to \mathbb{R}$ a C^1 functional satisfying $J(0) = 0$. Suppose that*

(i) There exist $\rho,\ \beta > 0$ such that $\forall u \in \partial B(0,\rho), J(u) \ge \beta$;
(ii) There exists $x_1 \in E$ such that $\|x_1\| > \rho$ and $J(x_1) < 0$;
(iii) J satisfies the Palais-Smale condition (PS), that is for all sequence (u_n) in E satisfying

$$J(u_n) \to d \quad as \;\; n \to +\infty \tag{12}$$

for some $d \in \mathbb{R}$ and

$$\|J'(u_n)\|_* \to 0 \quad as \;\; n \to +\infty, \tag{13}$$

the sequence (u_n) is relatively compact.

Then, J has a critical point u and the critical value $c = J(u)$ verifies

$$c := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J(\gamma(t))$$

where $\Gamma := \{\gamma \in C([0,1],X) \;\; such\ that \;\; \gamma(0) = 0 \;\; and \;\; \gamma(1) = x_1\}$ and $c \ge \beta$.

Remark 2.4. *In [16, 30] the authors used the following Theorem without the Palais-Smale condition.*

Theorem 2.5. *[3] Let E be a Banach space and $I : E \to \mathbb{R}$ a C^1 functional satisfying $I(0) = 0$. Suppose that there exist ρ, $\bar{\beta_0} > 0$ and $e \in E$ with $\|e\| > \rho$ such that*

$$\inf_{\|u\|=\rho} I(u) \geq \bar{\beta_0} \ \ and \ \ J(e) \leq 0.$$

Then there is a sequence $(u_n) \subset E$ such

$$I(u_n) \to \bar{c} \ \ and \ \ I'(u_n) \to 0,$$

where

$$\bar{c} := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) \geq \bar{\beta_0}$$

and

$$\Gamma := \{\gamma \in C([0, 1], E) \ \ such\ that \ \ \gamma(0) = 0 \ \ and \ \ \gamma(1) = e\}.$$

The number $\bar{c}$ is called mountain pass level or minimax level of the functional I.

Before starting the proof of the geometric properties for the functional J, we recall the following radial Lemma introduced in [11].

Lemma 2.6. *[11] Let u be a radially symmetric function in $C^1_0(B)$. Then, we have*

$$|u(x)| \leq \frac{1}{\omega_{N-1}^{\frac{1}{N}}} \log^{\frac{1}{N'}}(\log(\frac{e}{|x|}))\|u\|,$$

where ω_{N-1} is the area of the unit sphere $S_{N-1} \in \mathbb{R}^N$.

Now, since the function $\log(\log(\frac{e}{|x|}))$ is in $W^{1,N}(B)$ and the embedding $W^{1,N}(B) \hookrightarrow L^q(B)$ is continuous for all $q \geq 1$, there exists a constant $C > 0$ such that $\|u\|_{qN'} \leq c\|u\|$, for all $u \in \mathbf{E}$.
In the next Lemma, we prove that the functional J satisfies the first geometric property.

Lemma 2.7. *Suppose that $(H1) - (H4)$ hold. Then, there exist ρ, $\beta > 0$ such that $J(u) \geq \beta$ for all $u \in \mathbf{E}$ with $\|u\| = \rho$.*

Proof. It follows from the hypothesis $(H4)$ that there exists $t_0 > 0$ and there exists $\varepsilon \in (0, 1)$ such that

$$F(x,t) \leq \frac{1}{N}\lambda_1(1 - \varepsilon_0)|t|^N, \qquad \text{for } \ |t| < t_0. \tag{14}$$

From $(H3)$ and (11) and for all $q > N$, there exist a constant $C > 0$ such that

$$F(x,t) \leq C|t|^q exp(Ne^{a\ t^{N'}}), \ \ \forall\ |t| > t_1. \tag{15}$$

and so

$$F(x,t) \leq \frac{1}{N}\lambda_1(1 - \varepsilon_0)|t|^N + C|t|^q exp(Ne^{a\ t^{N'}}), \qquad \text{for } \ t \in \mathbb{R}. \tag{16}$$

Since

$$J(u) = \frac{1}{N}\|u\|^N - \int_B F(x,u)dx,$$

we get

$$J(u) \geq \frac{1}{N}\|u\|^N - \frac{1}{N}\lambda_1(1 - \varepsilon_0)|t|^N - C\int_B |u|^q exp(e^{a\ u^{N'}})\,dx.$$

But $\lambda_1\|u\|_N^N \leq \|u\|^N$ and from the Hölder inequality, we obtain

$$J(u) \geq \frac{\varepsilon_0}{N}\|u\|^N - C(\int_B exp(Ne^{a\ |u|^{N'}})dx)^{\frac{1}{N}}\|u\|^q_{qN'} \tag{17}$$

From the Theorem 1.1, if we choose $u \in \mathbf{E}$ such that

$$a\|u\|^{N'} \leq \omega_{N-1}^{\frac{1}{N-1}}, \tag{18}$$

we get

$$\int_B \exp(Ne^{a\ |u|^{N'}})dx = \int_B \exp(Ne^{a\ \|u\|^{N'}(\frac{|u|}{\|u\|})^{N'}})dx < +\infty.$$

On the other hand $\|u\|_{N'q} \leq C\|u\|$ (Lemma 2.6), so

$$J(u) \geq \frac{\varepsilon_0}{N}\|u\|^N - C\|u\|^q,$$

for all $u \in \mathbf{E}$ satisfying (18). Since $N < q$, we can choose $\rho = \|u\|$ small enough such that then there exists $\beta > 0$ small such that $J(u) \geq \beta > 0$. □

By the following Lemma, we prove the second geometric property for the functional J.

Lemma 2.8. *Suppose that* $(H1)$ *and* $(H2)$ *hold. Let* φ_1 *be a normalized eigenfunction associated to* λ_1 *in* $\mathbf{E}$. *Then,* $J(t\varphi_1) \to -\infty$, *as* $t \to +\infty$.

Proof. It follows from the condition $(H2)$ that

$$f(x,t) = \frac{\partial}{\partial t}F(x,t) \geq \frac{1}{M}F(x,t)$$

for all $t \geq t_0$. So

$$F(x,t) \geq C\, e^{\frac{t}{M}},\ \ \forall\ \ t \geq t_0. \tag{19}$$

It follows that, there exist $b > \lambda_1$ and $C > 0$ such that $F(x,t) \geq \frac{b}{N}t^N + C$ for all $t > 0$.

$$J(t\varphi_1) \leq \frac{t^N}{N}\|\varphi_1\|^N - \frac{b}{N}t^N\|\varphi_1\|_N^N - C|B|,$$

where $|B| = mes(B) = Vol(B)$. Then, from the definition of λ_1, we get

$$J(t\varphi_1) \leq t^N\frac{\lambda_1 - b}{N}\|\varphi_1\|_N^N < 0,\ \ \forall t > 0.$$

So, the Lemma 2.8 follows. □

3. Proof of Theorem 1.2

In the Theorem 1.2, we suppose that the function $f(x,t)$ is subcritical, that is satisfies the condition (7) for all $\alpha_0 > 0$. In order to prove that the functional J satisfies the (PS) condition, we use the same idea as in lions' Lemma [22].

Lemma 3.1. *Let* $\{u_k\}_k$ *be a sequence in* $\mathbf{E}$. *Suppose that* $\|u_k\| = 1$, $u_k \rightharpoonup u$ *weakly in* $\mathbf{E}$, $\nabla u_k \to \nabla u$, $u_k(x) \to u(x)$ *a.e* $x \in B$ *and* $u \not\equiv 0$. *Then*

$$\sup_k \int_B \exp\left(Ne^{p\omega_{N-1}^{\frac{1}{N-1}}|u_k|^{N'}}\right)dx < +\infty$$

for all $1 < p < U$ *where* U *is given by:*

$$U = \begin{cases} (1 - \|u\|^N)^{\frac{-1}{N-1}} & \text{if } \|u\| < 1 \\ +\infty & \text{if } \|u\| = 1. \end{cases}$$

Proof. For $a,\ b \in \mathbb{R},\ \ q > 1$. If q' its conjugate i.e. $\frac{1}{q} + \frac{1}{q'} = 1$ we have, by young inequality, that

$$e^{a+b} \leq \frac{1}{q}e^{qa} + \frac{1}{q'}e^{q'b}$$

and so,

$$exp(Ne^{a+b}) \leq exp(\frac{N}{q}e^{qa} + \frac{N}{q'}e^{q'b}).$$

Therefore,

$$exp(Ne^{a+b}) \leq \frac{1}{q}exp(Ne^{qa}) + \frac{1}{q'}exp(Ne^{q'b}). \tag{20}$$

Also, we have

$$(1+a)^q \leq (1+\varepsilon)a^q + (1 - \frac{1}{(1+\varepsilon)^{\frac{1}{q-1}}})^{1-q}, \ \ \forall a \geq 0,\ \ \forall \varepsilon > 0\ \ \forall q > 1. \tag{21}$$

So, we get

$$\begin{array}{rcl} |u_k|^{N'} & = & |u_k - u + u|^{N'} \\ & \leq & (|u_k - u| + u|)^{N'} \\ & \leq & (1+\varepsilon)|u_k - u|^{N'} + \Big(1 - \frac{1}{(1+\varepsilon)^{N-1}}\Big)^{\frac{-1}{N-1}}|u|^{N'}. \end{array}$$

which implies that

$$\begin{aligned} \int_B \exp\Big(Ne^{p\omega_{N-1}^{\frac{1}{N-1}}|u_k|^{N'}}\Big)dx \leq \frac{1}{q}\int_B & \exp(Ne^{pq\omega_{N-1}^{\frac{1}{N-1}}(1+\varepsilon)|u_k-u|^{N'}})dx \\ &+ \frac{1}{q'}\int_B \exp(Ne^{pq'\omega_{N-1}^{\frac{1}{N-1}}(1-\frac{1}{(1+\varepsilon)^{N-1}})^{\frac{-1}{N-1}})|u|^{N'}})dx, \end{aligned}$$

for any $p > 1$.
From (5), the last integral is finite. To complete the proof we have to prove that for every p such that $1 < p < U$,

$$\sup_k \int_B \exp(Ne^{pq\omega_{N-1}^{\frac{1}{N-1}}(1+\varepsilon)|u_k-u|^{N'}})dx < +\infty, \tag{22}$$

for some $\varepsilon > 0$ and $q > 1$.
In the following, we suppose that $\|u\| < 1$, and in the case of $\|u\| = 1$, the proof is similar.
When

$$\|u\| < 1$$

and

$$p < \frac{1}{(1 - \|u\|^N)^{\frac{1}{N-1}}},$$

there exists $\nu > 0$ such that

$$p(1 - \|u\|^N)^{\frac{1}{N-1}}(1+\nu) < 1.$$

On the other hand, By Brezis-Lieb's lemma [6] we have

$$\|u_k - u\|^N = \|u_k\|^N - \|u\|^N + o(1) \ \text{ where } \ o(1) \to 0 \ \text{ as } \ k \to +\infty. \tag{23}$$

Then,

$$\|u_k - u\|^N = 1 - \|u\|^N + o(1),$$

and so,

$$\lim_{k\to+\infty} \|u_k - u\|^N = 1 - \|u\|^N$$

that is,

$$\lim_{k\to+\infty} \|u_k - u\|^{N'} = (1 - \|u\|^N)^{\frac{1}{N-1}}.$$

Therefore, for every $\varepsilon > 0$, there exists $k_\varepsilon \geq 1$ such that

$$\|u_k - u\|^{N'} \leq (1+\varepsilon)(1 - \|u\|^N)^{\frac{1}{N-1}}, \quad \forall \ k \geq k_\varepsilon.$$

If we take $q = 1 + \varepsilon$ with $\varepsilon = \sqrt[3]{1+\nu} - 1$, then $\forall k \geq k_\varepsilon$, we have

$$pq(1+\varepsilon)\|u_k - u\|^{N'} \leq 1.$$

Consequently,

$$\begin{aligned}
\int_B \exp(Ne^{pq\omega_{N-1}^{\frac{1}{N-1}}(1+\varepsilon)|u_k-u|^{N'}})dx &\leq \int_B \exp(Ne^{(1+\varepsilon)pq\omega_{N-1}^{\frac{1}{N-1}}(\frac{|u_k-u|}{\|u_k-u\|})^{N'}\|u_k-u\|^{N'}})dx \\
&\leq \int_B \exp(Ne^{\omega_{N-1}^{\frac{1}{N-1}}(\frac{|u_k-u|}{\|u_k-u\|})^{N'}})dx \\
&\leq \sup_{\|u\|\leq 1} \int_B \exp(Ne^{\omega_{N-1}^{\frac{1}{N-1}}|u|^{N'}})dx < +\infty
\end{aligned}$$

Now, (22) follows from (6). We complete the proof. □

The next result assures the existence of critical point for the functional J when the nonlinearity is subcritical or critical.

Proposition 3.2. *Suppose that* $(H1)$, $(H2)$ *and* $(H3)$ *hold and the function* $f(x,t)$ *satisfies the condition (8) for* $\alpha_0 > 0$. *Then the functional* J *satisfies the Palais-Smale condition* $(PS)_c$ *for any*

$$c < \frac{\omega_{N-1}}{N\alpha_0^{N-1}},$$

where ω_{N-1} *is the area of the unit sphere* S_{N-1} *in* $\mathbb{R}^N$.

Proof. Consider a $(PS)_c$ sequence in $\mathbf{E}$, for some $c \in \mathbb{R}$, that is

$$J(u_n) = \frac{1}{N}\|u_n\|^N - \int_B F(x,u_n)dx \to c, \quad n \to +\infty \tag{24}$$

and

$$|\langle J'(u_n),\varphi\rangle| = \Big|\int_B w(x)|\nabla u_n|^{N-2}\nabla u_n.\nabla\varphi dx - \int_B f(x,u_n)\varphi dx\Big| \leq \varepsilon_n\|\varphi\|, \tag{25}$$

for all $\varphi \in \mathbf{E}$, where $\varepsilon_n \to 0$, when $n \to +\infty$.
Also, inspired by [13], it follows from $(H2)$ that for all $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that

$$F(x,t) \leq \varepsilon t f(x,t), \quad \text{for all } |t| > t_\varepsilon \text{ and uniformly in } x \in B, \tag{26}$$

and so, by (24), for all $\varepsilon > 0$ there exists a constant $C > 0$

$$\frac{1}{N}\|u_n\|^N \leq C + \int_B F(x,u_n)dx,$$

hence
$$\frac{1}{N}\|u_n\|^N \leq C + \int_{|u_n|\leq t_\varepsilon} F(x,u_n)dx + \varepsilon\int_B f(x,u_n)u_n dx$$
and so, from (25), we get
$$\frac{1}{N}\|u_n\|^N \leq C_1 + \varepsilon\varepsilon_n\|u_n\| + \varepsilon\|u_n\|^N,$$
for some constant $C_1 > 0$. Since
$$(\frac{1}{N} - \varepsilon)\|u_n\|^N \leq C_1 + \varepsilon\varepsilon_n\|u_n\|, \tag{27}$$
we deduce that the sequence (u_n) is bounded in $\mathbf{E}$. As consequence, there exists $u \in \mathbf{E}$ such that, up to subsequence, $u_n \rightharpoonup u$ weakly in $\mathbf{E}$, $u_n \to u$ strongly in $L^q(B)$, for all $q \geq 1$ and $u_n(x) \to u(x)$ a.e. in B and so $\nabla u_n \to \nabla u$ as in [1]. Furthermore, we have from (24) and (25), that
$$0 < \int_B f(x,u_n)u_n \leq C, \tag{28}$$
and
$$0 < \int_B F(x,u_n) \leq C. \tag{29}$$
Since by Lemma 2.1 in [18], we have
$$f(x,u_n) \to f(x,u) \text{ in } L^1(B) \text{ as } n \to +\infty, \tag{30}$$
then, it follows from $(H2)$ and the generalized Lebesgue dominated convergence theorem that
$$F(x,u_n) \to F(x,u) \text{ in } L^1(B) \text{ as } n \to +\infty. \tag{31}$$
So,
$$\lim_{n\to+\infty}\|u_n\|^N = N(c + \int_B F(x,u)dx). \tag{32}$$
and using (24), we have
$$\lim_{n\to+\infty}\int_B f(x,u_n)u_n dx = N(c + \int_B F(x,u)dx). \tag{33}$$
By the condition $(H3)$,
$$\lim_{n\to+\infty} N\int_B F(x,u_n)dx \leq N(c + \int_B F(x,u)dx)$$
and so $c \geq 0$. Also, it follows from (24) and (24), that u is a weak solution of the problem (1) and if we take $\varphi = u$ as a test function, we get
$$\int_B |\nabla u|^{N2}\, w(x)dx = \int_B f(x,u)udx,$$
so $J(u) \geq 0$ and .
We will finish the proof by considering three cases for the level c.

Case 1. $c = 0$. In this case
$$0 \leq J(u) \leq \liminf_{n\to+\infty} J(u_n) = 0.$$

So,

$$J(u) = 0$$

and then

$$\lim_{n\to+\infty} \frac{1}{N}\|u_n\|^N = \int_B F(x,u)dx = \frac{1}{N}\|u\|^N.$$

Case 2. $c > 0$ and $u = 0$. We prove that this case cannot happen.
From (46) and (48) with $v = u_n$ we have

$$\lim_{n\to+\infty} \|u_n\|^N = Nc \ \text{ and } \ \lim_{n\to+\infty} \int_B f(x,u_n)u_n dx = Nc$$

and by (48), we have also

$$\Big|\|u_n\|^N - \int_B f(x,u_n)u_n dx\Big| \le C\varepsilon_n.$$

First we claim that there exists $q > 1$ such that

$$\int_B |f(x,u_n)|^q dx \le C \tag{34}$$

so

$$\|u_n\|^N \le C\varepsilon_n + \Big(\int_B |f(x,u_n)|^q\Big)^{\frac{1}{q}} dx \Big(\int_B |u_n|^{q'}\Big)^{\frac{1}{q'}}$$

where q' is the conjugate of q. Since (u_n) converge to $u = 0$ in $L^{q'}(B)$,

$$\lim_{n\to+\infty} \|u_n\|^N = 0$$

and this is a contradiction with $c > 0$.

For the proof of the claim, since f has subcritical or critical growth, for every $\varepsilon > 0$ and $q > 1$ there exists $t_\varepsilon > 0$ and $C > 0$ such that for all $|t| \ge t_\varepsilon$, we have

$$|f(x,t)|^q \le C\exp(Ne^{\alpha_0(\varepsilon+1)t^{N'}}). \tag{35}$$

Consequently,

$$\begin{aligned}\int_B |f(x,u_n)|^q dx &= \int_{\{|u_n|\le t_\varepsilon} |f(x,u_n)|^q dx + \int_{\{|u_n|>t_\varepsilon\}} |f(x,u_n)|^q dx \\ &\le \omega_{N-1} \max_{B\times[-t_\varepsilon,t_\varepsilon]} |f(x,t)|^q + C\int_B \exp\Big(Ne^{\alpha_0(\varepsilon+1)|u_n|^{N'}}\Big)dx.\end{aligned}$$

Since $Nc < \frac{\omega_{N-1}}{\alpha_0^{N-1}}$, there exists $\eta \in (0, \frac{1}{N})$ such that $Nc = (1-N\eta)\dfrac{\omega_{N-1}}{\alpha_0^{N-1}}$. On the other hand, $\|u_n\|^{N'} \to (Nc)^{\frac{1}{N-1}}$, so there exists $n_\eta > 0$ such that for all $n \ge n_\eta$, we get $\|u_n\|^{N'} \le (1-\eta)\frac{\omega_{N-1}^{\frac{1}{N-1}}}{\alpha_0}$. Therefore,

$$\alpha_0(1+\varepsilon)\Big(\frac{|u_n|}{\|u_n\|}\Big)^{N'}\|u_n\|^{N'} \le (1+\varepsilon)(1-\eta)\omega_{N-1}^{\frac{1}{N-1}}.$$

We choose $\varepsilon > 0$ small enough to get

$$\alpha_0(1+\varepsilon)\|u_n\|^{N'} \le \omega_{N-1}^{\frac{1}{N-1}},$$

therefore the second integral is uniformly bounded in view of (5), and the claim is proved.

Case 3. $c > 0$ and $u \neq 0$. In this case, we claim that $J(u) = c$ and therefore we get

$$\lim_{n\to+\infty} \|u_n\|^N = N\Big(c + \int_B F(x,u)dx\Big) = N\Big(J(u) + \int_B F(x,u)dx\Big) = \|u\|^N.$$

Now, in order to prove the claim we remark that

$$J(u) \leq \frac{1}{N} \liminf_{n+\to\infty} \|u_n\|^N - \int_B F(x,u)dx = c.$$

Suppose that $J(u) < c$. We have

$$\|u\|^{N'} < (N\Big(c + \int_B F(x,u)dx\Big)\Big)^{\frac{1}{N-1}}. \tag{36}$$

Set

$$v_n = \frac{u_n}{\|u_n\|}$$

and

$$v = \frac{u}{(N\Big(c + \int_B F(x,u)dx\Big))^{\frac{1}{N}}}.$$

We have $\|v_n\| = 1$, $v_n \rightharpoonup v$ in E, $v \not\equiv 0$ and $\|v\| < 1$. So, by Lemma 4.1, we get

$$\sup_n \int_B \exp\Big(Ne^{p\omega_{N-1}^{\frac{1}{N-1}} |v_n|^{N'}}\Big)dx < \infty$$

for $1 < p < (1 - \|v\|^N)^{\frac{-1}{N-1}}$.
As in the case 2, we are going to estimate $\int_B |f(x,u_n)|^q dx$.
For $\varepsilon > 0$, one has

$$\begin{aligned}
\int_B |f(x,u_n)|^q dx &= \int_{\{|u_n|\leq t_\varepsilon} |f(x,u_n)|^q dx + \int_{\{|u_n|>t_\varepsilon\}} |f(x,u_n)|^q dx \\
&\leq \omega_{N-1} \max_{B\times[-t_\varepsilon,t_\varepsilon]} |f(x,t)|^q + C\int_B \exp\Big(Ne^{\alpha_0(1+\varepsilon)|u_n|^{N'}}\Big)dx \\
&\leq C_\varepsilon + C\int_B \exp\Big(Ne^{\alpha_0(1+\varepsilon)\|u_n\|^{N'}|v_n|^{N'}}\Big)dx \leq C
\end{aligned}$$

if we have $\alpha_0(1+\varepsilon)\|u_n\|^{N'} \leq p\omega_{N-1}^{\frac{1}{N-1}}$ and $1 < p < (1 - \|v\|^N)^{\frac{-1}{N-1}}$.
Since

$$(1 - \|v\|^N)^{\frac{-1}{N-1}} = \Big(\frac{N(c + \int_B F(x,u)dx)}{N(c + \int_B F(x,u)dx) - \|u\|^N)}\Big)^{\frac{1}{N-1}} = \Big(\frac{c + \int_B F(x,u)dx}{c - J(u)}\Big)^{\frac{1}{N-1}}$$

and

$$\lim_{n\to+\infty} \|u_n\|^{N'} = (N\Big(c + \int_B F(x,u)dx\Big)\Big)^{\frac{1}{N-1}}$$

then,

$$\alpha_0(1+\varepsilon)\|u_n\|^{N'} \leq \alpha_0(1+2\varepsilon)(N\Big(c + \int_B F(x,u)dx\Big)^{\frac{1}{N-1}}.$$

But $J(u) \geq 0$ and $c < \dfrac{\omega_{N-1}}{N\alpha_0^{N-1}}$, then if we choose $\varepsilon > 0$ small enough such that

$$\frac{\alpha_0}{\omega_{N-1}^{\frac{1}{N-1}}}(1+2\varepsilon) < \Big(\frac{1}{N(c-J(u))}\Big)^{\frac{1}{N-1}}$$

which means

$$(1+2\varepsilon)\Big((c-J(u)\Big)^{\frac{1}{N-1}} < \frac{\omega_{N-1}^{\frac{1}{N-1}}}{N^{\frac{1}{N-1}}\alpha_0}$$

and so, the sequence $(f(x,u_n))$ is bounded in L^q, $q > 1$.
Since $\langle J'(u_n), u_n - u\rangle = o(1)$, we get

$$\int_B w(x)|\nabla u_n|^{N-2}\nabla u_n.(\nabla u_n - \nabla u)dx - \int_B f(x,u_n)(u_n-u)dx = o(1). \tag{37}$$

On the other hand, since $u_n \rightharpoonup u$ weakly in E then

$$\int_B w(x)|\nabla u|^{N-2}\nabla u.(\nabla u_n - \nabla u)dx = o(1). \tag{38}$$

Combining (37) and (38), we obtain

$$\int_B w(x)\Big(|\nabla u_n|^{N-2}\nabla u_n - |\nabla u|^{N-2}\nabla u\Big).(\nabla u_n - \nabla u)dx = \int_B f(x,u_n)(u_n-u)dx + o(1). \tag{39}$$

Using the well known inequality

$$(|x|^{N-2}x - |y|^{N-2}y).(x-y) \ \geq 2^{2-N}|x-y|^N, \quad \forall\ x,y \in \mathbb{R}^N \text{ and } \quad N \geq 2 \tag{40}$$

we obtain

$$0 \leq 2^{2-N}\int_B w(x)|\nabla u_n - \nabla u|^N dx \leq \int_B f(x,u_n)(u_n-u)dx + o(1). \tag{41}$$

By the Hölder inequality, we obtain

$$\begin{aligned} 2^{2-N}\int_B w(x)|\nabla u_n - \nabla u|^N dx &\leq \int_B f(x,u_n)(u_n-u)dx + o(1) \\ &\leq \Big(\int_B |f(x,u_n)|^q\Big)^{\frac{1}{q}}\Big(\int_B |u_n-u|^{q'}\Big)^{\frac{1}{q'}}dx + o(1). \end{aligned} \tag{42}$$

So,

$$\|u_n - u\| \to 0 \quad \text{as} \ \ n \to \infty.$$

By Brezis-Lieb's lemma, up to subsequence, we get

$$\lim_{n\to+\infty}\|u_n\|^N = N(c + \int_B F(x,u)dx) = \|u\|^N$$

and this contradicts (36).
And this finish the proof of the Proposition 3.2. □

Proof of Theorem 1.2
Since $f(x,t)$ satisfies the condition (7) for all $\alpha_0 > 0$, then by Proposition 3.2, the functional J satisfies the (PS) condition (at each possible level d). So, by Lemma 2.7 and Lemma 2.8, we deduce that the functional J has a nonzero critical point u in **E**. From maximum principle, the solution u of the problem (1) is positive.□

4. Proof of Theorem 1.3

In the Theorem 1.3, we suppose that the function $f(x,t)$ is critical, that is, it satisfies the condition (8) for some $\alpha_0 > 0$.
We still can use Theorem 1.1 if we prove that the mountain pass level c in this Theorem 1.1 satisfies

$$c < \frac{\omega_{N-1}}{N\alpha_0^{N-1}}.$$

For this purpose , we will prove that there exists $v \in \mathbf{E}$ such

$$\max_{t\geq 0} J(tv) < \frac{\omega_{N-1}}{N\alpha_0^{N-1}}. \tag{43}$$

As in [13], we consider a Moser-type sequence given by

$$\psi_n(t) = \begin{cases} \dfrac{\log(1+t)}{\log^{\frac{1}{N}}(1+n)} & \text{if } \ 0 \leq t \leq n \\ \log^{\frac{N-1}{N}}(1+n) & \text{if } \ t \geq n. \end{cases} \tag{44}$$

Let $v_n(x)$ the function defined by

$$\psi_n(t) = \omega_{N-1}^{\frac{1}{N}} v_n(x), \quad \text{where } \ e^{-t} = |x|. \tag{45}$$

With this choice of ψ_n, the sequence (v_n) is normalized since

$$\|v_n\|^N = \frac{1}{\omega_{N-1}} \int_B |\nabla\psi_n|^N |\log(\frac{e}{|x|})|^{N-1}\, dx = \int_0^{+\infty} |\psi'(t)|^N (1+t)^{N-1} dt = 1.$$

On the other hand, we have

$$\lim_{n\to+\infty} \int_B \exp\left(Ne^{\omega_{N-1}^{\frac{1}{N-1}}|v_n|^{N'}}\right)dx = \lim_{n\to+\infty} \omega_{N-1} \int_0^{+\infty} \exp\left(Ne^{|\psi_n|^{N'}} - Nt\right)dt. \tag{46}$$

Now, we introduce the following elementary result.

Lemma 4.1. *For the sequence ψ_n induced by (44), we have*

$$\lim_{n\to+\infty} \int_0^{+\infty} \exp\left(Ne^{|\psi_n|^{N'}} - Nt\right)dt = (\frac{N+1}{N})e^N. \tag{47}$$

Proof. If we set $s = 1 + t$ and $j = n + 1$, we get

$$\int_0^{+\infty} \exp\left(Ne^{|\psi_n|^{N'}} - Nt\right)dt = \frac{e^N}{N} + \int_0^n \exp\left(Ne^{\frac{\log^{N'}(1+t)}{\log^{\frac{1}{N-1}}(1+n)}} - Nt\right)dt,$$

where $N' = \dfrac{N}{N-1}$.
And so,

$$\int_0^{+\infty} \exp\left(Ne^{|\psi_n|^{N'}} - Nt\right)dt = \frac{e^N}{N} + \int_1^j \exp\left(Ns^{(\frac{\log s}{\log j})^{\frac{1}{N-1}}} - N(s-1)\right)ds.$$

That is,

$$\int_0^{+\infty} \exp\left(Ne^{|\psi_n|^{N'}} - Nt\right)dt = \frac{e^N}{N} + e^N \int_1^j \exp\left(Ns^{(\frac{\log s}{\log j})^{\frac{1}{N-1}}} - Ns\right)ds.$$

We claim that

$$\lim_{j\to+\infty}\int_1^j \exp\Big(Ns^{(\frac{\log s}{\log j})^{\frac{1}{N-1}}} - Ns\Big)ds = 1, \tag{48}$$

and so the formula (47) follows.
The proof of the claim can be omitted since it is technical and it is the same as the case when $N = 2$ in [13], Lemma 4.1. □

Lemma 4.2. *For the sequence v_n identified by (45), there exists $n \geq 1$ such that*

$$\max_{t\geq 0} J(tv_n) < \frac{\omega_{N-1}}{N\alpha_0^{N-1}}. \tag{49}$$

Proof. By contradiction, suppose that for all $n \geq 1$,

$$\max_{t\geq 0} J(tv_n) \geq \frac{\omega_{N-1}}{N\alpha_0^{N-1}}.$$

Therefore, for any $n \geq 1$, there exists $t_n > 0$ such that

$$\max_{t\geq 0} J(tv_n) = J(t_n v_n) \geq \frac{\omega_{N-1}}{N\alpha_0^{N-1}}$$

and so,

$$\frac{1}{N}t_n^N - \int_B F(x, t_n v_n)dx \geq \frac{\omega_{N-1}}{N\alpha_0^{N-1}}.$$

Then, by using $(H1)$

$$t_n^N \geq \frac{\omega_{N-1}}{\alpha_0^{N-1}}. \tag{50}$$

On the other hand,

$$\frac{d}{dt}J(tv_n)\Big|_{t=t_n} = t_n^{N-1} - \int_B f(x, t_n v_n)v_n dx = 0,$$

that is

$$t_n^N = \int_B f(x, t_n v_n)t_n v_n dx. \tag{51}$$

Now, we claim that the sequence (t_n) is bounded in $(0, +\infty)$.
Indeed, it follows from $(H5)$ that for all $\varepsilon > 0$, there exists $t_\varepsilon > 0$ such that

$$f(x,t)t \geq (\gamma_0 - \varepsilon)\exp\Big(Ne^{\alpha_0 t^{N'}}\Big) \ \ \forall |t| \geq t_\varepsilon, \ \text{ uniformly in } \ x \in B. \tag{52}$$

Since

$$t_n^N = \int_B f(x, t_n v_n)t_n v_k dx \geq \omega_{N-1}\int_n^{+\infty} f\Big(e^{-s}, t_n\frac{\psi_n}{\omega_{N-1}^{\frac{1}{N}}}\Big)t_n\frac{\psi_n}{\omega_{N-1}^{\frac{1}{N}}}e^{-Ns}ds, \tag{53}$$

and from (44) and (50), we have

$$t_n\frac{\psi_n}{\omega_{N-1}^{\frac{1}{N}}} = t_n\Big(\frac{\log(1+n)}{\omega_{N-1}^{\frac{1}{N-1}}}\Big)^{\frac{1}{N'}} \geq \Big(\frac{\log(1+n)}{\alpha_0}\Big)^{\frac{1}{N'}}$$

then, it follows from (52) that for all $\varepsilon > 0$, there exists n_0 such that for all $n \geq n_0$

$$t_n^N \geq \omega_{N-1}(\gamma_0 - \varepsilon) \int_n^{+\infty} \exp\Big(Ne^{\frac{\alpha_0}{\omega_{N-1}^{\frac{1}{N-1}}}(t_n\psi_n)^{N'}} - Ns\Big)ds, \tag{54}$$

that is

$$t_n^N \geq \frac{\omega_{N-1}}{N}(\gamma_0 - \varepsilon) \exp\Big(Ne^{\frac{\alpha_0}{\omega_{N-1}^{\frac{1}{N-1}}} t_n^{N'} \log(1+n)} - Nn\Big). \tag{55}$$

From (55), we obtain for n large enough

$$1 \geq \frac{\omega_{N-1}}{N}(\gamma_0 - \varepsilon) \exp\Big(Ne^{\frac{\alpha_0}{\omega_{N-1}^{\frac{1}{N-1}}} t_n^{N'} \log(1+n)} - Nn - N \log t_n\Big).$$

Therefore (t_n) is bounded. Also, we have from the formula (50) that

$$\lim_{n\to+\infty} t_n^N \geq \frac{\omega_{N-1}}{\alpha_0^{N-1}}.$$

Now, suppose that

$$\lim_{n\to+\infty} t_n^N > \frac{\omega_{N-1}}{\alpha_0^{N-1}}.$$

For n large enough, $t_n^N > \dfrac{\omega_{N-1}}{\alpha_0^{N-1}}$ and in this case, the right hand site of the inequality (54) will gives the unboundedness of the sequence (t_n). Since (t_n) is bounded, we get

$$\lim_{n\to+\infty} t_n^N = \frac{\omega_{N-1}}{\alpha_0^{N-1}}. \tag{56}$$

Now, we want to use the expression of t_n^N given by (51) and the hypothesis (52). So let

$$B_{n,+} = \{x \in B;\ \ t_n v_n(x) \geq t_\varepsilon\} \ \text{ and } \ B_{n,-} = \{x \in B;\ \ t_n v_n(x) < t_\varepsilon\}.$$

We have

$$t_n^N \geq (\gamma_0 - \varepsilon) \int_{B_{n,+}} \exp\Big(Ne^{\alpha_0 t_n^{N'} v_n^{N'}}\Big)dx + \int_{B_{n,-}} f(x, t_n v_n) t_n v_n dx,$$

and then

$$\begin{aligned} t_n^N \ &\geq \ (\gamma_0 - \varepsilon) \int_B \exp\Big(Ne^{\alpha_0 t_n^{N'} v_n^{N'}}\Big)dx - (\gamma_0 - \varepsilon) \int_{B_{n,-}} \exp\Big(Ne^{\alpha_0 t_n^{N'} v_n^{N'}}\Big)dx \\ &+ \ \int_{B_{n,-}} f(x, t_n v_n) t_n v_n dx. \end{aligned} \tag{57}$$

The sequence (v_n) converges to 0 in B and $\chi_{B_{n,-}}$ converges to 1a.e in B. By using the dominated convergence theorem, we obtain

$$\lim_{n\to+\infty} \int_{B_{n,-}} f(x, t_n v_n) t_n v_n dx = 0.$$

Also, we have

$$\int_{B_{n,-}} \exp\Big(Ne^{\alpha_0 t_n^{N'} v_n^{N'}}\Big)dx \leq \int_B \exp\Big(Ne^{\alpha_0 t_n^{N'} v_n^{N'}}\Big)dx.$$

So,

$$\begin{aligned} \lim_{n\to+\infty} \int_{B_{n,-}} \exp\Big(Ne^{\alpha_0 t_n^{N'} v_n^{N'}}\Big)dx \ &\leq \ \lim_{n\to+\infty} \int_B \exp\Big(Ne^{\alpha_0 t_n^{N'} v_n^{N'}}\Big)dx \\ &\leq \ e^N vol(B). \end{aligned}$$

From (46) and the Lemma 4.1, we have

$$\begin{aligned}\lim_{n\to+\infty}\int_B \exp\Big(Ne^{\alpha_0 t_n^{N'} v_n^{N'}}\Big)dx &\geq \lim_{n\to+\infty}\int_B \exp\Big(Ne^{\omega_{N-1}^{\frac{1}{N-1}}|v_n|^{N'}}\Big)dx \\ &\geq \omega_{N-1}(\frac{N+1}{N})e^N\end{aligned}$$

So, it follows from (57) that

$$\frac{\omega_{N-1}}{\alpha_0^{N-1}} \geq (\gamma_0 - \varepsilon)\omega_{N-1}e^N, \tag{58}$$

for all $\varepsilon > 0$. So

$$\gamma_0 \leq \frac{1}{\alpha_0^{N-1}e^N}$$

which is with contradiction of the condition on γ_0 in $(H5)$. The Theorem 1.3 is proved. □

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