



Scaled evolution of the reversed power mean inequalities

Marija Bošnjak^a, Mario Krnić^{b,*}, Tibor K. Pogány^{c,d}

^aDepartment of Mathematics, Mechanical Engineering Faculty, University of Slavonski Brod, 35000 Slavonski Brod, Croatia

^bUniversity of Zagreb, Faculty of Electrical Engineering and Computing, Unska 3, 10000 Zagreb, Croatia

^cInstitute of Applied Mathematics, John von Neumann Faculty of Informatics, Óbuda University, Bécsi út 96b, Budapest, Hungary

^dFaculty of Maritime Studies, University of Rijeka, 51000 Rijeka, Studentska 2, Croatia

Abstract. Motivated by the Hanin inequality, in this paper we study a class of reversed power mean inequalities that does not depend on a weight function. We first give the reverse of the basic power mean inequality describing the monotonic behavior of means. Then, we establish the scaled, i.e. two-parametric versions of the obtained inequalities. By scaling, we develop a new method for improving the starting reversed power mean inequalities. More precisely, we impose conditions under which the scaled inequality is sharper than the corresponding original power mean inequality. Some particular examples are also discussed. Finally, our results are compared with some previously known from the literature.

1. Introduction

A function $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$, where I is an interval, is convex if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$. The most important inequality for a convex function $f : I \rightarrow \mathbb{R}$, the famous Jensen inequality, asserts that

$$f\left(\sum_{i=1}^n w_i a_i\right) \leq \sum_{i=1}^n w_i f(a_i), \quad (1)$$

holds for all $a_1, a_2, \dots, a_n \in I$ and $w_1, w_2, \dots, w_n \in [0, 1]$, provided that $\sum_{i=1}^n w_i = 1$. Inequality (1) is extensively used in almost all fields of mathematics, especially in mathematical analysis and statistics.

2020 *Mathematics Subject Classification.* Primary 26D15.

Keywords. power mean, power mean inequality, reverse, Jensen inequality, Hanin inequality.

Received: 01 July 2024; Accepted: 21 July 2025

Communicated by Miodrag Spalević

Research of the second author was partially supported by the Institutional project RePowerFER through the Recovery and Resilience plan for Croatia (source 581). Research of the third author was partially supported by the University of Rijeka project UNIRI-IZ-25-108.

* Corresponding author: Mario Krnić

Email addresses: marija.bosnjak1@gmail.com (Marija Bošnjak), mario.krnic@fer.hr (Mario Krnić), pogany.tibor@nik.uni-obuda.hu, tibor.poganj@uniri.hr (Tibor K. Pogány)

ORCID iDs: <https://orcid.org/0009-0009-0821-1711> (Marija Bošnjak), <https://orcid.org/0000-0002-1140-7629> (Mario Krnić), <https://orcid.org/0000-0002-4635-8257> (Tibor K. Pogány)

In 2012, Krnić et al. [8] (see also [6] and [7]), established the following mutual bounds for the difference between the right-hand side and the left-hand side of the Jensen inequality:

$$n \max_{1 \leq j \leq n} w_j \mathcal{I}_n(f, \mathbf{a}) \geq \sum_{j=1}^n w_j f(a_j) - f\left(\sum_{j=1}^n w_j a_j\right) \geq n \min_{1 \leq j \leq n} w_j \mathcal{I}_n(f, \mathbf{a}). \quad (2)$$

Here, $\mathcal{I}_n$ stands for the corresponding non-weighted form of the Jensen functional, i.e.

$$\mathcal{I}_n(f, \mathbf{x}) = \frac{1}{n} \sum_{j=1}^m f(x_j) - f\left(\frac{1}{n} \sum_{j=1}^m x_j\right).$$

It should be noticed here that inequalities in (2) have been established as a consequence of the so called superadditivity of the Jensen functional, established by Dragomir et al. [3] (see also [4]). Obviously, the left inequality in (2) is the reverse, while the right inequality sign represents the refinement of the Jensen inequality. Based on (2), numerous inequalities such as the Young inequality, the Hölder inequality, power mean inequalities, etc. have been refined (see, e.g. [6, 8] and the references cited therein). In addition, for a comprehensive overview of the old and new results in connection to the Jensen inequality, the reader is referred to monographs [5, 10, 11] and the references cited therein.

One of the most interesting consequences of the Jensen inequality are power mean inequalities. Recall that a power mean is defined by

$$M_r(\mathbf{a}, \mathbf{w}) = \begin{cases} \left(\sum_{i=1}^n w_i a_i^r\right)^{\frac{1}{r}}, & r \neq 0, \\ \prod_{i=1}^n a_i^{w_i}, & r = 0, \end{cases}$$

where $\mathbf{a} = (a_1, a_2, \dots, a_n)$ stands for a non-negative n -tuple and $\mathbf{w} = (w_1, w_2, \dots, w_n)$ is a non-negative n -tuple such that $\sum_{i=1}^n w_i = 1$. Further, the case of $w_1 = w_2 = \dots = w_n = \frac{1}{n}$ yields the corresponding non-weighted mean

$$m_r(\mathbf{a}) = \begin{cases} \left(\frac{1}{n} \sum_{i=1}^n a_i^r\right)^{\frac{1}{r}}, & r \neq 0, \\ \left(\prod_{i=1}^n a_i\right)^{\frac{1}{n}}, & r = 0. \end{cases}$$

Recall that $M_{-1}(\mathbf{a}, \mathbf{w})$, $M_0(\mathbf{a}, \mathbf{w})$, $M_1(\mathbf{a}, \mathbf{w})$, $M_2(\mathbf{a}, \mathbf{w})$ provide the harmonic, geometric, arithmetic and quadratic means, respectively. Moreover, $M_{-\infty}(\mathbf{a}, \mathbf{w}) = \min_{1 \leq j \leq n} a_j$ and $M_{\infty}(\mathbf{a}, \mathbf{w}) = \max_{1 \leq j \leq n} a_j$. The basic power mean inequality, describing monotonic behavior of means, asserts that if $r < s$, then

$$M_r(\mathbf{a}, \mathbf{w}) \leq M_s(\mathbf{a}, \mathbf{w}). \quad (3)$$

This follows by putting $f(t) = t^{\frac{s}{r}}$ in (1) and by replacing n -tuple $\mathbf{a} = (a_1, a_2, \dots, a_n)$ with $\mathbf{a}^r = (a_1^r, a_2^r, \dots, a_n^r)$, provided that $s, r \neq 0$ (for more details, including the cases when one of the parameters r, s is equal to zero the reader is referred to [10, 11]). In this setting, relation (2) reduces to

$$n \max_{1 \leq j \leq n} w_j (m_s^s(\mathbf{a}) - m_r^s(\mathbf{a})) \geq M_s^s(\mathbf{a}, \mathbf{w}) - M_r^s(\mathbf{a}, \mathbf{w}) \geq n \min_{1 \leq j \leq n} w_j (m_s^s(\mathbf{a}) - m_r^s(\mathbf{a})), \quad (4)$$

providing the reverse and refinement of the basic power mean inequality (3). However, inequalities in (4) provide reverse and refinement of the weighted power means in terms of the corresponding non-weighted means. It is important to note that the above relations do not improve the non-weighted power mean inequalities.

On the other hand, Hanin [2], established the following reverse of the non-weighted arithmetic-quadratic mean inequality

$$\frac{M^2}{4} + m_1^2(\mathbf{a}) \geq m_2^2(\mathbf{a}), \quad (5)$$

where $M = \max_{1 \leq j \leq n} a_j$. The proof of the above inequality is extremely easy and reads as follows (see also [9]):

$$\left(\frac{M}{2} - m_1(\mathbf{a})\right)^2 + Mm_1(\mathbf{a}) - m_2^2(\mathbf{a}) = \left(\frac{M}{2} - m_1(\mathbf{a})\right)^2 + \frac{1}{n} \sum_{j=1}^n a_j(M - a_j) \geq 0.$$

In addition, Hanin [2], also gave the following extension of the above inequality by noting that

$$M_k^k(\mathbf{a}, \mathbf{w}) \leq M_1^k(\mathbf{a}, \mathbf{w}) + (k-1)k^{\frac{k}{1-k}} M^k, \quad (6)$$

provided that $k > 1$. Clearly, relation (6) represents the reverse of the power mean inequality $M_k(\mathbf{a}, \mathbf{w}) \geq M_1(\mathbf{a}, \mathbf{w})$, $k > 1$. It is important to note that the above reversed relation (6) does not depend on weight $\mathbf{w}$. This means that (6) also holds when $\mathbf{w} = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})$. Roughly speaking, this fact provides theoretical advantage of inequality (6) compared to the first inequality in (4).

The main focus in this article is a further study of the reversed mean inequality (6). For the reader's convenience, inequalities related to (6) will also be referred to as the Hanin-type inequalities. The outline of this article is as follows: after this introductory part, in Section 2 we extend inequality (6) to hold for all real parameters k . As an application, we obtain the reverse of the power mean inequality (3) that does not depend on weight $\mathbf{w}$. The obtained results are also compared with some related results, known from the literature. In Section 3 we establish scaled versions of the Hanin-type inequalities derived in Section 2. By scaled inequalities we mean two-parametric extensions of the corresponding Hanin-type inequalities. In such a way, in Section 4 we discuss efficiency of the established scaled inequalities. More precisely, we develop a new method for improving the reversed mean inequalities of Hanin-type by imposing conditions under which the scaled inequality is sharper than the corresponding original inequality. As an application, we study some particular examples.

Conventions

In order to summarize our further discussion, we present the following notations. Throughout this paper $\mathbf{a} = (a_1, a_2, \dots, a_n) \in \mathbb{R}_+^n$ stands for a non-negative n -tuple. In addition, if $\mathbf{a}$ is included in a mean $M_k(\mathbf{a}, \mathbf{w})$, where $k < 0$, then all coordinates in $\mathbf{a}$ are assumed to be strictly positive. By a weight $\mathbf{w}$, we mean a non-negative n -tuple $\mathbf{w} = (w_1, w_2, \dots, w_n) \in \mathbb{R}_+^n$ such that $\sum_{j=1}^n w_j = 1$. For brevity, the limit means $M_{-\infty}(\mathbf{a}, \mathbf{w}) = \min_{1 \leq j \leq n} a_j$ and $M_{\infty}(\mathbf{a}, \mathbf{w}) = \max_{1 \leq j \leq n} a_j$, will be denoted by m and M respectively, i.e.

$$m = \min_{1 \leq j \leq n} a_j \quad \text{and} \quad M = \max_{1 \leq j \leq n} a_j.$$

2. Main results

Here, our main focus is an extension of inequality (6) for all real parameters k . The first step in our study is an extension of (6) for an arbitrary non-negative parameter k . For completeness, we also give the proof when $k > 1$.

Theorem 2.1. *If $k > 1$, then holds the inequality*

$$M_k^k(\mathbf{a}, \mathbf{w}) \leq M_1^k(\mathbf{a}, \mathbf{w}) + (k-1)k^{\frac{k}{1-k}} M^k. \quad (7)$$

Further, if $0 < k < 1$, then the sign of inequality (7) is reversed, i.e.

$$M_k^k(\mathbf{a}, \mathbf{w}) \geq M_1^k(\mathbf{a}, \mathbf{w}) + (k-1)k^{\frac{k}{1-k}} M^k. \quad (8)$$

Proof. Let $k > 1$. Since $a_j^{k-1} \leq M^{k-1}$, $j = 1, 2, \dots, n$, we have that

$$M_k^k(\mathbf{a}, \mathbf{w}) = \sum_{j=1}^n w_j a_j^k \leq M^{k-1} \sum_{j=1}^n w_j a_j = M^{k-1} M_1(\mathbf{a}, \mathbf{w}), \quad (9)$$

which means that

$$M_k^k(\mathbf{a}, \mathbf{w}) - M_1^k(\mathbf{a}, \mathbf{w}) \leq M^{k-1} M_1(\mathbf{a}, \mathbf{w}) - M_1^k(\mathbf{a}, \mathbf{w}). \quad (10)$$

Now, consider the function $f : [0, \infty) \rightarrow \mathbb{R}$ defined by $f(t) = M^{k-1}t - t^k$. We have that $f'(t) = M^{k-1} - kt^{k-1}$, so $f'(t_0) = 0$ if and only if $M^{k-1} = kt_0^{k-1}$, that is, if $t_0 = Mk^{\frac{1}{1-k}}$. Moreover, $f'(t) > 0$ if and only if $t < t_0$, while $f'(t) < 0$ if and only if $t > t_0$. This means that f has a global maximum at the point $t = t_0$, i.e.

$$f(t_0) = M^{k-1}Mk^{\frac{1}{1-k}} - M^k k^{\frac{k}{1-k}} = (k-1)k^{\frac{k}{1-k}} M^k. \quad (11)$$

Now, combining (10) and (11), we arrive at the relation

$$M_k^k(\mathbf{a}, \mathbf{w}) - M_1^k(\mathbf{a}, \mathbf{w}) \leq (k-1)k^{\frac{k}{1-k}} M^k,$$

that is, we obtain (7).

It remains to consider the case of $0 < k < 1$. Then, similarly as above, it follows that $M_k^k(\mathbf{a}, \mathbf{w}) \geq M^{k-1} M_1(\mathbf{a}, \mathbf{w})$, so it holds the reversed inequality in (10). Moreover, the above function $f(t) = M^{k-1}t - t^k$ is monotonically decreasing on interval $(0, t_0)$, and monotonically increasing on (t_0, ∞) , which means that f attains global minimum at $t_0 = Mk^{\frac{1}{1-k}}$, which is equal to $(k-1)k^{\frac{k}{1-k}} M^k$. This yields the reversed inequality (8). $\square$

Remark 2.2. Both relations (7) and (8) represent the corresponding reverses of the power mean inequalities between $M_k(\mathbf{a}, \mathbf{w})$ and $M_1(\mathbf{a}, \mathbf{w})$, respectively. In particular, if $k = 2$, then relation (7) reduces to

$$M_2^2(\mathbf{a}, \mathbf{w}) \leq M_1^2(\mathbf{a}, \mathbf{w}) + \frac{M^2}{4}, \quad (12)$$

which represents the weighted form of the original Hanin inequality (5). Similarly, if $k = \frac{1}{2}$, relation (8) yields the inequality

$$M_{\frac{1}{2}}^{\frac{1}{2}}(\mathbf{a}, \mathbf{w}) \geq M_1^{\frac{1}{2}}(\mathbf{a}, \mathbf{w}) - \frac{\sqrt{M}}{4}, \quad (13)$$

which represents the reverse of the power mean inequality $M_{\frac{1}{2}}(\mathbf{a}, \mathbf{w}) \leq M_1(\mathbf{a}, \mathbf{w})$.

The next step in our study is an extension of Theorem 2.1 for the case of negative parameters k . In fact, we obtain the corresponding reverses between the power mean $M_k(\mathbf{a}, \mathbf{w})$, $k < 0$, and the harmonic mean $M_{-1}(\mathbf{a}, \mathbf{w})$. The corresponding result is an easy consequence of the previous theorem.

Corollary 2.3. If $k < -1$, then holds the inequality

$$M_k^k(\mathbf{a}, \mathbf{w}) \leq M_{-1}^k(\mathbf{a}, \mathbf{w}) - (k+1)(-k)^{-\frac{k}{k+1}} m^k. \quad (14)$$

Moreover, if $-1 < k < 0$, then it holds

$$M_k^k(\mathbf{a}, \mathbf{w}) \geq M_{-1}^k(\mathbf{a}, \mathbf{w}) - (k+1)(-k)^{-\frac{k}{k+1}} m^k. \quad (15)$$

Proof. To prove inequalities (14) and (15), we consider relations (7) and (8) with the n -tuple

$$\left(\frac{1}{a_1}, \frac{1}{a_2}, \dots, \frac{1}{a_n}\right)$$

instead of $(a_1, a_2, \dots, a_n)$. Then, by noting that

$$\max_{1 \leq j \leq n} \frac{1}{a_j} = \frac{1}{m},$$

we easily arrive at the both relations. $\square$

Remark 2.4. Relations (14) and (15) also represent reverses of the corresponding mean inequalities in the case of negative parameter k . In particular, if $k = -2$, then inequality (14) takes form

$$M_{-2}^{-2}(\mathbf{a}, \mathbf{w}) \leq M_{-1}^{-2}(\mathbf{a}, \mathbf{w}) + \frac{1}{4m^2},$$

which represents the reverse of the inequality $M_{-2}(\mathbf{a}, \mathbf{w}) \leq M_{-1}(\mathbf{a}, \mathbf{w})$. On the other hand, if $k = -\frac{1}{2}$, then (15) reads

$$M_{-\frac{1}{2}}^{-\frac{1}{2}}(\mathbf{a}, \mathbf{w}) \geq M_{-1}^{-\frac{1}{2}}(\mathbf{a}, \mathbf{w}) - \frac{1}{4\sqrt{m}},$$

which is the reverse of the relation $M_{-\frac{1}{2}}(\mathbf{a}, \mathbf{w}) \geq M_{-1}(\mathbf{a}, \mathbf{w})$.

Now, by virtue of our Theorem 2.1, we are able to derive reverse of the basic power mean inequality (3) in the case when both parameters r and s are positive or negative.

Theorem 2.5. If $0 < r < s$, then holds the inequality

$$M_s^s(\mathbf{a}, \mathbf{w}) \leq M_r^s(\mathbf{a}, \mathbf{w}) + \frac{s-r}{r} \left(\frac{s}{r}\right)^{\frac{s}{r-s}} M^s. \quad (16)$$

In addition, if $s < r < 0$, then it holds

$$M_s^s(\mathbf{a}, \mathbf{w}) \leq M_r^s(\mathbf{a}, \mathbf{w}) + \frac{s-r}{r} \left(\frac{s}{r}\right)^{\frac{s}{r-s}} m^s. \quad (17)$$

Proof. To prove (16), we consider (7) with the n -tuple $\mathbf{a}^r = (a_1^r, a_2^r, \dots, a_n^r)$, $r > 0$, instead of $\mathbf{a} = (a_1, a_2, \dots, a_n)$. Then, $M_1(\mathbf{a}^r, \mathbf{w}) = \sum_{j=1}^n w_j a_j^r$ and $M^r = \max_{1 \leq j \leq n} a_j^r$. Furthermore, by putting $k = \frac{s}{r}$, $s > r > 0$, we have that

$$M_{\frac{s}{r}}^{\frac{s}{r}}(\mathbf{a}^r, \mathbf{w}) = M_s^s(\mathbf{a}, \mathbf{w}) \quad \text{and} \quad M_1^{\frac{s}{r}}(\mathbf{a}^r, \mathbf{w}) = M_r^s(\mathbf{a}, \mathbf{w}),$$

so (16) holds.

To prove (17) we follow the same lines as above, except that we use the fact that $m^r = \max_{1 \leq j \leq n} a_j^r$, when $r < 0$. $\square$

Remark 2.6. It should be noticed here that both relations (16) and (17) can be also derived from (8), as well as, from relations (14) and (15).

Remark 2.7. If r tends to zero, then we have that $\lim_{r \rightarrow 0} \frac{s-r}{r} \left(\frac{s}{r}\right)^{\frac{s}{r-s}} = 1$, so inequalities (16) and (17) respectively reduce to trivial reverses

$$M_s^s(\mathbf{a}, \mathbf{w}) \leq M_0^s(\mathbf{a}, \mathbf{w}) + M^s, \quad s > 0,$$

and

$$M_s^s(\mathbf{a}, \mathbf{w}) \leq M_0^s(\mathbf{a}, \mathbf{w}) + m^s, \quad s < 0,$$

between the power mean $M_s(\mathbf{a}, \mathbf{w})$ and the geometric mean $M_0(\mathbf{a}, \mathbf{w})$.

Remark 2.8. If $s = 2r$, then (16) and (17) respectively reduce to

$$M_{2r}^{2r}(\mathbf{a}, \mathbf{w}) \leq M_r^{2r}(\mathbf{a}, \mathbf{w}) + \frac{M^{2r}}{4}, \quad r > 0, \quad (18)$$

and

$$M_{2r}^{2r}(\mathbf{a}, \mathbf{w}) \leq M_r^{2r}(\mathbf{a}, \mathbf{w}) + \frac{m^{2r}}{4}, \quad r < 0. \quad (19)$$

Clearly, by putting $r = 1$ in inequality (18), we obtain the basic Hanin inequality (5).

We have already discussed that the advantage of Theorem 2.1, Corollary 2.3 and Theorem 2.5, compared to relation (4), is that they provide estimates that does not depend on weights. In such a way, they also provide reverses for the non-weighted power mean inequalities. However, these two methods are generally not comparable, as it will be discussed in the next remark.

Remark 2.9. Let $k = 3$ and $n = 3$. Then, considering (7) with $\mathbf{a} = (1, 2, 3)$, we obtain the estimate

$$M_3^3(\mathbf{a}, \mathbf{w}) - M_1^3(\mathbf{a}, \mathbf{w}) \leq 6\sqrt{3}, \quad (20)$$

regardless of the weight $\mathbf{w}$. On the other hand, if $n = 3$, $s = 3$, $r = 1$ and $\mathbf{a} = (1, 2, 3)$, then the first inequality in (4) reads

$$M_3^3(\mathbf{a}, \mathbf{w}) - M_1^3(\mathbf{a}, \mathbf{w}) \leq 3 \max_{1 \leq j \leq 3} w_j (m_3^3(\mathbf{a}) - m_1^3(\mathbf{a})) = 12 \max_{1 \leq j \leq 3} w_j. \quad (21)$$

Now, if $w = (\frac{1}{6}, \frac{1}{6}, \frac{2}{3})$, then $12 \max_{1 \leq j \leq 3} w_j = 8 < 6\sqrt{3}$, which means that inequality (21) is more accurate than (20). On the other hand, if $w = (\frac{1}{24}, \frac{1}{24}, \frac{22}{24})$, then $12 \max_{1 \leq j \leq 3} w_j = 11 > 6\sqrt{3}$. So, in this case relation (20) is sharper than (21).

As we can see from the previous remark, the results we derived in this section are generally not comparable to the corresponding results developed in [6] and [8] (see also [5]). However, their advantage is that we they provide estimates for the non-weighted inequalities.

3. Scaled evolution for the reversed mean inequalities

We aim now to establish scaled versions of the results from the previous section. More precisely, we are going to extend the corresponding results by introducing two new parameters. In such a way, we will develop a new method for improving the corresponding reversed mean inequalities. It should be noticed here that some similar approaches can be found in slightly older papers [1] and [12]. In this regard, our first result is a scaled version of Theorem 2.1.

Theorem 3.1. If $k > 1$ and $0 < \beta \leq k\alpha$, then holds the inequality

$$\beta M_k^k(\mathbf{a}, \mathbf{w}) \leq \alpha^k M_1^k(\mathbf{a}, \mathbf{w}) + (k-1)M^k. \quad (22)$$

Further, if $0 < k < 1$ and $0 < k\alpha \leq \beta$, then

$$\beta M_k^k(\mathbf{a}, \mathbf{w}) \geq \alpha^k M_1^k(\mathbf{a}, \mathbf{w}) + (k-1)M^k. \quad (23)$$

Proof. We first consider the case $k > 1$. We use estimate (9) from the proof of Theorem 2.1. Then, we have that

$$\beta M_k^k(\mathbf{a}, \mathbf{w}) - \alpha^k M_1^k(\mathbf{a}, \mathbf{w}) \leq \beta M^{k-1} M_1(\mathbf{a}, \mathbf{w}) - \alpha^k M_1^k(\mathbf{a}, \mathbf{w}). \quad (24)$$

Now, similarly to the proof of Theorem 2.1, we are going to show that the function $h : [0, \infty) \rightarrow \mathbb{R}$ defined by $h(t) = \beta M^{k-1}t - \alpha^k t^k$ has global maximum on its domain. Namely, since $h'(t) = \beta M^{k-1} - k\alpha^k t^{k-1}$, it follows that h has stationary point $t_0 = k^{\frac{1}{1-k}} \alpha^{\frac{k}{1-k}} \beta^{\frac{1}{1-k}} M$. Moreover, since h is monotonically increasing on interval $(0, t_0)$, and monotonically decreasing on (t_0, ∞) , it follows that h attains global maximum at t_0 , equal to

$$h(t_0) = \left(\frac{\beta}{\alpha}\right)^{\frac{k}{k-1}} k^{\frac{k}{1-k}} (k-1) M^k. \quad (25)$$

In addition, since $\frac{\beta}{\alpha} \leq k$ by the assumptions, the previous relation yields the estimate

$$h(t_0) \leq (k-1)M^k. \quad (26)$$

Finally, combining (24) and (26), we have that

$$\beta M_k^k(\mathbf{a}, \mathbf{w}) - \alpha^k M_1^k(\mathbf{a}, \mathbf{w}) \leq (k-1)M^k,$$

as claimed.

It remains to consider the complementary case when $0 < k < 1$ and $0 < k\alpha \leq \beta$. Then, similarly to the proof of Theorem 2.1, it follows that the sign of inequality (24) is reversed, while the above defined function h possess global minimum given by (25). Then, $h(t_0) \geq (k-1)M^k$, provided that $0 < k < 1$ and $0 < k\alpha \leq \beta$. Clearly, this yields (23), as claimed. $\square$

Remark 3.2. Let $\alpha^k = \beta$, where $k > 1$. Combining this condition with the assumption $0 < \beta \leq k\alpha$, it follows that $0 < \alpha \leq k^{\frac{1}{k-1}}$. Therefore, in this case relation (22) reduces to

$$M_k^k(\mathbf{a}, \mathbf{w}) \leq M_1^k(\mathbf{a}, \mathbf{w}) + \frac{(k-1)M^k}{\alpha^k}, \quad 0 < \alpha \leq k^{\frac{1}{k-1}}.$$

Of course, the latter inequality is the best possible for $\alpha = k^{\frac{1}{k-1}}$, that is, in the case of the starting Hanin-type inequality (7). In particular, if $k = 2$, then $\alpha = 2$, so the above inequality reduces to relation (12) from Remark 2.2.

A similar conclusion can be drawn for the case of $0 < k < 1$. Then, combining $\alpha^k = \beta$ and $0 < k\alpha \leq \beta$, we also have that $0 < \alpha \leq k^{\frac{1}{k-1}}$, so relation (23) becomes

$$M_k^k(\mathbf{a}, \mathbf{w}) \geq M_1^k(\mathbf{a}, \mathbf{w}) + \frac{(k-1)M^k}{\alpha^k}, \quad 0 < \alpha \leq k^{\frac{1}{k-1}}.$$

Clearly, since $k-1 < 0$, the above inequality is the best possible in the case of inequality (8). In particular, if $k = \frac{1}{2}$, then, $\alpha = 4$, so the previous inequality becomes (13).

Similarly to the previous section, our next goal is to extend Theorem 3.1 to hold for negative parameters. In other words, we give scaled version of Corollary 2.3. More precisely, we obtain the corresponding scaled reverse between a power mean $M_k(\mathbf{a}, \mathbf{w})$, $k < 0$, and the harmonic mean $M_{-1}(\mathbf{a}, \mathbf{w})$.

Corollary 3.3. If $k < -1$ and $0 < \beta \leq -k\alpha$, then holds the inequality

$$\beta M_k^k(\mathbf{a}, \mathbf{w}) \leq \alpha^{-k} M_{-1}^k(\mathbf{a}, \mathbf{w}) - (k+1)m^k. \quad (27)$$

Further, if $-1 < k < 0$ and $0 < -k\alpha \leq \beta$, then it holds

$$\beta M_k^k(\mathbf{a}, \mathbf{w}) \geq \alpha^{-k} M_{-1}^k(\mathbf{a}, \mathbf{w}) - (k+1)m^k. \quad (28)$$

Proof. We follow the same procedure as in the proof of Corollary 2.3, except that we use relations (22) and (23), instead of (7) and (8). $\square$

Remark 3.4. Let $\alpha^{-k} = \beta$. Combining this with the assumption $0 < \beta \leq -k\alpha$, when $k < -1$, it follows that $0 < \alpha \leq (-k)^{-\frac{1}{k+1}}$. Clearly, the limit case $\alpha = (-k)^{-\frac{1}{k+1}}$ in (27) yields inequality (14). In the same way, combining $\alpha^{-k} = \beta$ and $0 < -k\alpha \leq \beta$, when $-1 < k < 0$, we also have that $0 < \alpha \leq (-k)^{-\frac{1}{k+1}}$. Consequently, the limit case in (28) yields (15), as expected.

Finally, by virtue of Theorem 3.1, we will now establish scaled reversed relations that correspond to the basic power mean inequality (3), when both parameters r, s are positive or negative.

Theorem 3.5. Let $\alpha, \beta > 0$. If $0 < r < s$ and $0 < r\beta \leq s\alpha$, then holds the inequality

$$\beta M_s^s(\mathbf{a}, \mathbf{w}) \leq \alpha^{\frac{s}{r}} M_r^s(\mathbf{a}, \mathbf{w}) + \frac{s-r}{r} M^s. \quad (29)$$

In addition, if $s < r < 0$ and $s\alpha \leq r\beta < 0$, then it holds

$$\beta M_s^s(\mathbf{a}, \mathbf{w}) \leq \alpha^{\frac{s}{r}} M_r^s(\mathbf{a}, \mathbf{w}) + \frac{s-r}{r} m^s. \quad (30)$$

Proof. We follow the same procedure as in the proof of Theorem 2.5 except that we use scaled inequality (22) instead of (7). Clearly, the condition $0 < \beta < k\alpha$ from Theorem 3.1 is equivalent to $0 < r\beta \leq s\alpha$, when both r, s are positive, i.e. to $s\alpha \leq r\beta < 0$, when both r, s are negative. The rest of the proof is obvious. $\square$

Remark 3.6. Similarly to Remarks 3.2 and 3.4, we aim now to impose conditions under which the scaled relations (29) and (30) reduce to reversed power mean inequalities (16) and (17). This happens when $\alpha^s = \beta^r$. Combining this condition with $0 < r\beta \leq s\alpha$ when both r, s are positive, or with $s\alpha \leq r\beta < 0$, when both r, s are negative, we have that $0 < \alpha \leq \left(\frac{s}{r}\right)^{\frac{r}{s-r}}$, i.e. $0 < \alpha^{\frac{s}{r}} \leq \left(\frac{s}{r}\right)^{\frac{s}{s-r}}$. Clearly, the maximum value of α yields power mean inequalities (16) and (17).

In Remarks 3.2, 3.4 and 3.6, we have obtained reversed mean inequalities from Section 2, from the corresponding scaled relations, when parameter α took the maximum value. For all smaller values of parameter α the obtained inequalities were weaker in the corresponding setting. However, the scaled relations can also be utilized in refining reversed mean inequalities from Section 2. That is the main topic of the next section.

4. Efficiency discussion for the scaled Hanin-type inequalities

Our main goal in this section is to discuss efficiency of scaled Hanin-type inequalities from the previous section. More precisely, we aim here to impose some general conditions under which the scaled Hanin-type inequalities are more accurate than the original inequalities from Section 2. The starting point in our discussion is efficiency of inequality (22). In order to summarize our further discussion, we define the quantity

$$c_k = \frac{M_1^k(\mathbf{a}, \mathbf{w})}{M_1^k(\mathbf{a}, \mathbf{w}) + (k-1)k^{\frac{k}{1-k}} M^k}, \quad k > 1. \quad (31)$$

It should be noticed here that the above parameter is always positive provided that $\mathbf{a} \neq (0, 0, \dots, 0)$, although it can be sufficiently small. For example, if $\mathbf{a} = (1, 0, 0, \dots, 0)$ and $\mathbf{w} = \left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right)$, then

$$c_k = \frac{1}{1 + (k-1)k^{\frac{k}{1-k}} n^k},$$

and consequently, $\lim_{n \rightarrow \infty} c_k = 0$. On the other hand, since $M_1(\mathbf{a}, \mathbf{w}) \leq M$, it follows that $c_k \leq (1 + (k-1)k^{\frac{k}{1-k}})^{-1}$, i.e. $c_k \in \left(0, (1 + (k-1)k^{\frac{k}{1-k}})^{-1}\right]$. In particular, if $k = 2$, then $c_2 \in \left(0, \frac{4}{5}\right]$, while for $k = 3$ we have that $c_3 \in \left(0, \frac{27-6\sqrt{3}}{23}\right]$.

Now, we are able to establish conditions under which inequality (22) is more accurate than inequality (7).

Theorem 4.1. Let $k > 1$ and let

$$D_k(c_k) = \{(u, v) \in \mathbb{R}_+^2; v \leq ku, v \geq c_k u^k + k^{\frac{k}{1-k}}(1 - c_k)\},$$

where c_k is defined by (31). If $(\alpha, \beta) \in D_k(c_k)$, then inequality (22) is sharper than (7).

Proof. If $(\alpha, \beta) \in D_k(c_k)$, then $\beta \leq k\alpha$ and

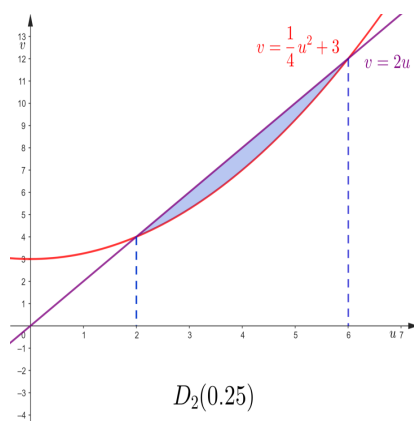
$$\beta \geq c_k \alpha^k + k^{\frac{k}{1-k}}(1 - c_k) = \frac{\alpha^k M_1^k(\mathbf{a}, \mathbf{w}) + (k-1)M^k}{M_1^k(\mathbf{a}, \mathbf{w}) + (k-1)k^{\frac{k}{1-k}}M^k},$$

which is equivalent to

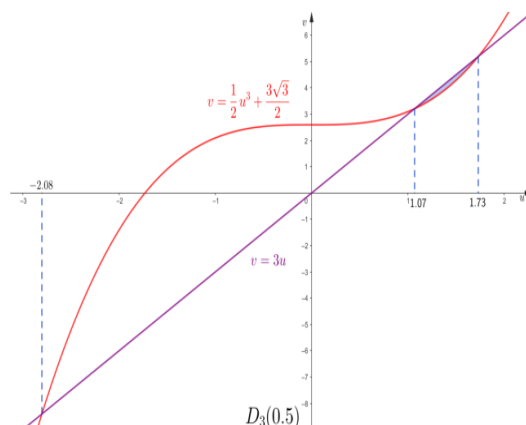
$$M_1^k(\mathbf{a}, \mathbf{w}) + (k-1)k^{\frac{k}{1-k}}M^k \geq \frac{\alpha^k M_1^k(\mathbf{a}, \mathbf{w}) + (k-1)M^k}{\beta}.$$

Obviously, the last inequality shows that inequality (22) is more accurate than (7) in this setting. $\square$

Remark 4.2. The set $D_k(c_k)$, $k > 1$, defined in the previous theorem, is a region in the first quadrant of the uv -plane bounded by the line $v = ku$ and a curve $v = c_k u^k + k^{\frac{k}{1-k}}(1 - c_k)$. In particular, if $k = 2$, then the line $v = 2u$ and parabola $v = c_2 u^2 + 4(1 - c_2)$ intersect at the points with coordinates $(2, 4)$ and $(\frac{2}{c_2} - 2, \frac{4}{c_2} - 4)$. It should be noticed here that both intersection points belong to the first quadrant. In particular, if $c_2 = \frac{1}{4}$, we have the region $D_2(\frac{1}{4})$ which represents better accuracy of inequality (22) in comparison to (7):



Further, if $k = 3$, then the line $v = 3u$ and a cubical parabola $v = c_3 u^3 + 3\sqrt{3}(1 - c_3)$ intersect at three points with u -coordinates $u_1 = \sqrt{3}$ and $u_{2,3} = \frac{-c_3 \sqrt{3} \pm \sqrt{3c_3(4-3c_3)}}{2c_3}$. It is easy to see that two intersection points belong to the first quadrant, while the remaining one is in the third quadrant. In particular, if $c_3 = \frac{1}{2}$, then the intersection points with u -coordinates $u_1 = \sqrt{3}$ and $u_2 = \frac{-\sqrt{3} + \sqrt{15}}{2}$ belong to the first quadrant, providing the region $D_3(\frac{1}{2})$ of efficiency of inequality (22):



Generally, it is easy to see that the one of intersection points of the line $v = ku$ and a curve $v = c_k u^k + k^{\frac{k}{1-k}}(1 - c_k)$ has coordinates $(k^{\frac{1}{k-1}}, k^{\frac{k}{k-1}})$.

Our next step is to discuss the case $0 < k < 1$. In other words, we discuss efficiency of the scaled relation (23), which is, in some way, complementary to (22). It should be noticed here that if $0 < k < 1$, then the value c_k , defined by (31), can take negative values. For example, if $n = 25$, $k = \frac{1}{2}$, $\mathbf{a} = (1, 0, \dots, 0)$, $\mathbf{w} = (\frac{1}{25}, \frac{1}{25}, \dots, \frac{1}{25})$, then the denominator of $c_{\frac{1}{2}}$ is equal to

$$\sqrt{M_1(\mathbf{a}, \mathbf{w})} - \frac{\sqrt{M}}{4} = -\frac{1}{20} < 0.$$

Therefore, to establish efficiency of (23), we will utilize somewhat different approach. We define

$$\bar{c}_k = \frac{M_k^k(\mathbf{a}, \mathbf{w}) + (1 - k)k^{\frac{k}{1-k}} M^k}{M_k^k(\mathbf{a}, \mathbf{w})}, \quad 0 < k < 1. \quad (32)$$

Now, $\bar{c}_k$ is always positive, moreover since $M_k(\mathbf{a}, \mathbf{w}) \leq M$, it follows that $\bar{c}_k \geq 1 + (1 - k)k^{\frac{k}{1-k}}$. On the other hand, if $\mathbf{a} = (1, 0, \dots, 0)$, $\mathbf{w} = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})$, we have that $\bar{c}_k = 1 + (1 - k)k^{\frac{k}{1-k}} n^k$, so $\bar{c}_k$ tends to infinity, as n tends to infinity. In particular, if $k = \frac{1}{2}$, then $\bar{c}_{\frac{1}{2}} \in [\frac{5}{4}, \infty)$.

The following result provides conditions under which the scaled relation (23) is more accurate than (8).

Theorem 4.3. Let $0 < k < 1$ and let

$$D_k(\bar{c}_k) = \{(u, v) \in \mathbb{R}_+^2; v \geq ku, v \leq \bar{c}_k u^k - k^{\frac{k}{k-1}}(\bar{c}_k - 1)\},$$

where $\bar{c}_k$ is defined by (32). If $(\alpha, \beta) \in D_k(\bar{c}_k)$, then inequality (23) is sharper than (8).

Proof. If $(\alpha, \beta) \in D_k(\bar{c}_k)$, then $\beta \geq k\alpha$ and

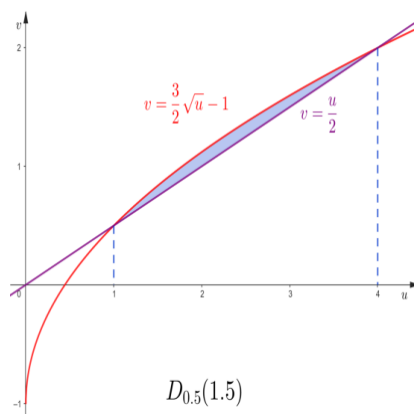
$$\beta \leq \bar{c}_k \alpha^k - k^{\frac{k}{k-1}}(\bar{c}_k - 1) = \frac{(M_k^k(\mathbf{a}, \mathbf{w}) + (1 - k)k^{\frac{k}{1-k}} M^k) \alpha^k - (1 - k) M^k}{M_k^k(\mathbf{a}, \mathbf{w})},$$

which is equivalent to

$$M_k^k(\mathbf{a}, \mathbf{w}) + (1 - k)k^{\frac{k}{1-k}} M^k \geq \frac{\beta M_k^k(\mathbf{a}, \mathbf{w}) + (1 - k) M^k}{\alpha^k}.$$

Clearly, this shows better precision of inequality (23) in comparison to (8). $\square$

Remark 4.4. Similarly to our discussion in Remark 4.2, the set $D_k(\bar{c}_k)$ is a region bounded by the line $v = ku$ and the curve $v = \bar{c}_k u^k - k^{\frac{k}{k-1}}(\bar{c}_k - 1)$, $0 < k < 1$. Let us consider the previous theorem for $k = \frac{1}{2}$. Then, $D_{1/2}(\bar{c}_{1/2})$ is a region in the first quadrant of uv -plane bounded by the line $v = \frac{u}{2}$ and the curve $v = \bar{c}_{1/2} \sqrt{u} - 2(\bar{c}_{1/2} - 1)$. By a straightforward computation, it follows that the line and curve intersect at the points with coordinates $(4, 2)$ and $(4(\bar{c}_{1/2} - 1)^2, 2(\bar{c}_{1/2} - 1)^2)$. In particular, if $\bar{c}_{1/2} = \frac{3}{2}$, we deal with the curve $v = \frac{3}{2} \sqrt{u} - 1$, which yields the region $D_{1/2}(\frac{3}{2})$ for which inequality (23) achieves better accuracy compared to inequality (8):



Now, it remains to consider the setting with negative parameters. According to our discussion from the previous two sections, it turns out that these negative cases are, in some way, symmetric to the corresponding positive cases. First, we consider the case when $k < -1$. Then, we define

$$s_k = \frac{M_{-1}^k(\mathbf{a}, \mathbf{w})}{M_{-1}^k(\mathbf{a}, \mathbf{w}) - (k+1)(-k)^{-\frac{k}{k+1}} m^k}, \quad k < -1. \quad (33)$$

Clearly, s_k is always positive. Moreover, since $M_{-1}(\mathbf{a}, \mathbf{w}) \geq m$, it follows that $s_k \leq \left(1 - (k+1)(-k)^{-\frac{k}{k+1}}\right)^{-1}$. On the other hand, if $\mathbf{a} = (1, n, \dots, n)$ and $\mathbf{w} = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})$, we have that $\lim_{n \rightarrow \infty} s_k = 0$. In particular, if $k = -2$, then $s_{-2} \in (0, \frac{4}{5}]$.

The corresponding result that provides conditions under which (27) is more accurate than (14) reads as follows:

Corollary 4.5. Let $k < -1$ and let

$$D_k(s_k) = \{(u, v) \in \mathbb{R}_+^2; v \leq -ku, v \geq s_k u^{-k} - (-k)^{\frac{k}{k+1}}(s_k - 1)\},$$

where s_k is defined by (33). If $(\alpha, \beta) \in D_k(s_k)$, then inequality (27) is sharper than (14).

Proof. The proof follows the lines of the proof of Theorem 4.1, except that we deal with inequalities (27) and (14) instead of (22) and (7). $\square$

To conclude the above discussion, we discuss the case when $-1 < k < 0$. Then, following the lines as in the corresponding positive case, we define

$$\bar{s}_k = \frac{M_k^k(\mathbf{a}, \mathbf{w}) + (k+1)(-k)^{-\frac{k}{k+1}} m^k}{M_k^k(\mathbf{a}, \mathbf{w})}, \quad -1 < k < 0. \quad (34)$$

Similarly to the case when $0 < k < 1$, it follows that $\bar{s}_k \in [1 + (k+1)(-k)^{-\frac{k}{k+1}}, \infty)$. Namely, the left endpoint of the interval is established by using the fact that $M_k(\mathbf{a}, \mathbf{w}) \geq m$, while for $\mathbf{a} = (1, n, \dots, n)$ and $\mathbf{w} = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})$, we have that $\lim_{n \rightarrow \infty} \bar{s}_k = \infty$. In particular, for $k = -\frac{1}{2}$, we have that $\bar{s}_{-\frac{1}{2}} \in [\frac{5}{4}, \infty)$.

Then, comparing inequalities (15) and (28), as in the proof of Theorem 4.3, we arrive at efficiency conditions for the scaled relation (28).

Corollary 4.6. Let $-1 < k < 0$ and let

$$D_k(\bar{s}_k) = \{(u, v) \in \mathbb{R}_+^2; v \geq -ku, v \leq \bar{s}_k u^{-k} - (-k)^{\frac{k}{k+1}}(\bar{s}_k - 1)\},$$

where $\bar{s}_k$ is defined by (34). If $(\alpha, \beta) \in D_k(\bar{s}_k)$, then inequality (28) is sharper than (15).

Remark 4.7. We have already discussed that the positive cases of parameter k , i.e. $k > 1$ and $0 < k < 1$, are, in some way, symmetric with respect to the corresponding negative cases $k < -1$ and $-1 < k < 0$. Clearly, in the negative case, the role of the arithmetic mean is taken over by the harmonic mean. One might think that the cases k and $-k$, $k > 0$ are completely identical. However, this is not true. For example, if $\mathbf{a} = (1, 2, 3)$ and $\mathbf{w} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, we have that $M_1(\mathbf{a}, \mathbf{w}) = 2$, $M_{-1}(\mathbf{a}, \mathbf{w}) = \frac{18}{11}$, and consequently, $c_2 = \frac{16}{25}$, $s_{-2} = \frac{121}{202}$. This shows that the sets

$$D_2(c_2) = \left\{ (u, v) \in \mathbb{R}_+^2; v \leq 2u, v \geq \frac{16}{25}u^2 + \frac{36}{25} \right\}$$

and

$$D_{-2}(s_{-2}) = \left\{ (u, v) \in \mathbb{R}_+^2; v \leq 2u, v \geq \frac{121}{202}u^2 + \frac{162}{101} \right\},$$

describing efficiency of relations (22) and (27), are not the same. Of course, if $\mathbf{a} = (a_1, a_2, \dots, a_n)$ and $\mathbf{w} = (w_1, w_2, \dots, w_n)$ are chosen such that $c_k = s_{-k}$, $k > 1$, then

$$D_{-k}(s_{-k}) = \{(u, v) \in \mathbb{R}_+^2; v \leq ku, v \geq s_{-k}u^k + k^{\frac{k}{k-1}}(1 - s_{-k})\} = D_k(c_k).$$

In particular, this means that the sets $D_{-2}(\frac{1}{4})$ and $D_{-3}(\frac{1}{2})$ represents the same regions as $D_2(\frac{1}{4})$ and $D_3(\frac{1}{2})$, respectively, as it has been discussed in Remark 4.2. Clearly, the similar conclusion can be drawn for the cases $0 < k < 1$ and $-1 < k < 0$.

Finally, to conclude this section, we impose conditions under which the scaled relations (29) and (30) are more precise than the reversed power mean inequalities (16) and (17). Clearly, the conclusion follows directly from Theorem 4.1.

Corollary 4.8. Let $\frac{s}{r} > 1$ and

$$D_{\frac{s}{r}}(c_{\frac{s}{r}}) = \{(u, v) \in \mathbb{R}_+^2; v \leq \frac{s}{r}u, v \geq c_{\frac{s}{r}}u^{\frac{r}{s}} + \left(\frac{s}{r}\right)^{\frac{s}{s-r}}(1 - c_{\frac{s}{r}})\},$$

where $c_{\frac{s}{r}}$ is defined by (31). If $0 < r < s$ and $(\alpha, \beta) \in D(c_{\frac{s}{r}})$, then inequality (29) is more accurate than (16). In addition, if $s < r < 0$, then (30) is sharper than (17).

Remark 4.9. As an application of Corollary 4.8, we consider the scaled power mean inequalities (29) and (30) when $s = 2r$. Then, (29) and (30) reduce respectively to

$$\beta M_{2r}^{2r}(\mathbf{a}, \mathbf{w}) \leq \alpha^2 M_r^{2r}(\mathbf{a}, \mathbf{w}) + M^{2r}, \quad r > 0, \quad (35)$$

and

$$\beta M_{2r}^{2r}(\mathbf{a}, \mathbf{w}) \leq \alpha^2 M_r^{2r}(\mathbf{a}, \mathbf{w}) + m^{2r}, \quad r < 0. \quad (36)$$

Now, according to Corollary 4.8, if

$$(\alpha, \beta) \in D_2(c_2) = \{(u, v) \in \mathbb{R}_+^2; v \leq 2u, v \geq c_2u^2 + 4(1 - c_2)\},$$

then inequality (35) is sharper than (18), while (36) is more accurate than (19). The region of efficiency of inequalities (35) and (36) is described in Remark 4.2.

Declarations

- **Ethical Approval:** Not applicable.
- **Availability of data and materials:** Not applicable.
- **Competing interests:** The authors declare that they have no competing interests.
- **Funding:** Not applicable.
- **Authors' contributions:** Authors declare that they have contributed equally to this paper. All authors have read and approved this version.

References

- [1] V. Csiszár, T.F. Móri, *The convexity method of proving moment-type inequalities*, Statistics & Probability Letters **66** (2004), 303–313.
- [2] L.G. Hanin, *Problem M 1083*, Kvant No.1 (1988), *Solution M 1083*, Kvant No. 5. (1988).
- [3] S.S. Dragomir, J.E. Pečarić, L.E. Persson, *Properties of some functionals related to Jensen's inequality*, Acta Math. Hungar. (70) **1-2** (1996), 129–143.
- [4] S.S. Dragomir, *Bounds for the normalized Jensen's functional*, Bull. Austral. Math. Soc. **74** (2006), 471–478.
- [5] M. Krnić, N. Lovričević, J. Pečarić, J. Perić, *Superadditivity and monotonicity of the Jensen-type functionals*, Element, Zagreb, 2015.
- [6] M. Krnić, N. Lovričević, J. Pečarić, *On the properties of McShane's functional and their applications*, Period. Math. Hung. **66** (2013), 159–180.
- [7] M. Krnić, N. Lovričević, J. Pečarić, *Jensen's operator and applications to mean inequalities for operators in Hilbert space*, Bull. Malays. Math. Sci. Soc. **35** (2012), 1–14.
- [8] M. Krnić, N. Lovričević, J. Pečarić, *Jessen's functional, its properties and applications*, An. Șt. Univ. Ovidius Constanța **20** (2012), 225–248.
- [9] D.S. Mitrinović, J.E. Pečarić, *Mean Values in Mathematics*, Matematički problemi i ekspozicije **14**, Naučna Knjiga, Beograd, 1989.
- [10] D.S. Mitrinović, J.E. Pečarić, A.M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht/Boston/London, 1993.
- [11] J.E. Pečarić, F. Proschan, Y.L. Tong, *Convex functions, partial orderings, and statistical applications*, Academic Press, Inc, 1992.
- [12] T.K. Pogány, *On an open problem of F. Qi*, J. Inequal. Pure and Appl. Math. **3** Art. 54 (2002), 5p.