



On Ulam stability of fractional iterative differential equations with Caputo derivative

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Abstract. This study extends stability theory by investigating the Hyers–Ulam and Hyers–Ulam–Rassias stability of a class of Caputo fractional iterative differential equations. The analysis employs a fixed point approach together with a generalized form of the Bielecki metric to derive sufficient conditions ensuring the stability of solutions. Both bounded and unbounded intervals are examined to highlight the robustness of the established results across different temporal domains. The framework accommodates the interplay between fractional–order derivatives and iterative dynamics, offering a unified treatment that generalizes several existing results in the literature. To support the theoretical analysis, several illustrative examples are presented to demonstrate the validity of the main results within the mathematical framework.

1. Introduction and Preliminaries

The concept of stability introduced by S. M. Ulam [47] in 1940 has gained increasing attention from researchers due to its vast applications. The initial results concerning this type of stability, particularly for functional equations, aimed to determine how close an approximate solution is to the exact solution and whether such a solution exists. For the additive Cauchy equation, $f(x + y) = f(x) + f(y)$, D. H. Hyers [9] provided a partial answer within Banach spaces, leading to what is now known as Hyers–Ulam stability. Later, T. M. Rassias [51] expanded upon this idea by introducing new concepts, culminating in Hyers–Ulam–Rassias stability.

Various generalizations of stability have since been developed by numerous researchers, such as Rassias [52], Gajda [59], and Aoki [53], who explored alternative norms, different classes of equations, and distinct mathematical techniques for obtaining approximate solutions. For a comprehensive overview of these advancements, refer to [23] and Brzdek et al. [15]. For more insights, refer to the papers [16, 58].

In this paper, we investigate Hyers–Ulam stability, Hyers–Ulam–Rassias stability, and a new form of stability called σ -semi-Hyers–Ulam stability for the following Caputo fractional iterative differential

2020 *Mathematics Subject Classification.* Primary 26D10; Secondary 45G10, 45M10, 47H10.

Keywords. Banach fixed–point theorem; Hyers–Ulam stability; Hyers–Ulam–Rassias stability; fractional differential equations; Caputo derivative

Received: 09 February 2025; Revised: 21 July 2025; Accepted: 24 July 2025

Communicated by Miodrag Spalević

Research supported by Kohsar University Murree.

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equation:

$$\begin{aligned} {}^C D_0^\alpha x(t) &= f(t, x(t), x(g(x(t))))), \quad 0 < \alpha < 1, \\ x(0) &= c, \quad c \geq 0. \end{aligned} \quad (1)$$

where $t \in I = [0, T]$, ${}^C D_0^\alpha$ denotes the Caputo fractional derivative of order α , $f : I \times \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ is a continuous function, and $g : \mathbb{C} \rightarrow \mathbb{C}$ is a continuous operator such that $g(x(t)) \leq t$ for all $t \in I$. Here, $\mathbb{C}$ denotes the Banach space of all real-valued continuous functions defined on the interval I .

Our interest in this problem is sparked by a recent work [1], where the authors examined the existence and uniqueness of solutions to a first-order iterative initial value problem:

$$x'(t) = f(t, x(t), x(g(x(t))))), \quad x(0) = c, \quad (2)$$

using Picard's methods under certain restrictions on f and g . Many related studies exist in fields such as electrodynamics [50], population growth [21], infectious disease modeling [10], and dynamical systems. The utility of both integer and fractional derivatives in modeling has prompted extensive research into ordinary and fractional iterative differential equations. For works on ordinary derivatives, see [8], and for fractional derivatives, refer to [28, 29]. Two widely recognized definitions of Riemann–Liouville and Caputo fractional derivatives are employed in this study, as referenced in [17, 54].

Building upon these foundational and emerging studies, several recent works have focused specifically on the stability analysis of fractional and integral equations using various generalized concepts. For instance, Irshad et al. [24] examined the stability of time-fractional nonlinear Schrödinger equations, while Shah and Irshad [30] addressed Hyers–Ulam–Mittag–Leffler stability in nonlinear fractional reaction–diffusion equations with delays. The Hyers–Ulam–Rassias stability of nonlinear convolution integral equations has been explored in the work of Irshad, Shah, and Liaquat [25]. Related contributions include studies on the Hyers–Ulam stability of Bernoulli's differential equation [31] and impulsive Fredholm integral equations on finite intervals [32]. Shah and Irshad further extended this investigation to oscillatory Volterra integral equations [33], hybrid differential equations involving Gronwall-type inequalities [34], and impulsive Hammerstein integral systems [35].

Additional advancements include the application of the Gronwall lemma to Ulam-type stability of integral equations by Shah et al. [36] and the use of fixed point methods for stability analysis of delay fractional integro-differential equations with almost sectorial operators [37]. In the context of generalized integral systems, Shah and Tanveer [38] studied (k, ψ) -fractional order quadratic integral equations, while Shah and Abbasi [39] focused on impulsive Hammerstein integral equations. Further related efforts by Shah and collaborators address the Hyers–Ulam stability of weakly singular Volterra equations [40], nonlinear Volterra–Fredholm equations [41], and impulsive Volterra integral equations using fixed point theory [42]. A foundational fixed point analysis for Volterra-type systems with delay can also be found in [43].

In a parallel line of research integrating fixed point theory with biological modeling, Turab and Sintunavarat contributed several influential works. These include the analysis of traumatic avoidance learning models via the Banach fixed point theorem [2], behavioral modeling of two-choice decision-making in fish using analytic techniques [3], and a novel study on nonlinear fractional boundary value problems defined over graph structures such as the ethane graph [4]. These cumulative efforts demonstrate the flexibility and depth of fixed point techniques and fractional analysis in addressing a wide variety of nonlinear systems across mathematical and applied domains.

Fractional differential equations provide powerful tools for modeling complex phenomena involving memory and hereditary characteristics across various scientific domains such as biology, engineering, epidemiology, and chemical graph theory. Turab et al. [5] developed solutions for nonlinear boundary value problems on the hexasilinane graph, demonstrating the relevance of fractional differential equations in structural graph modeling. In the context of epidemiology, Sintunavarat and Turab [56] formulated an extended SEIR model for COVID-19 using the Atangana–Baleanu–Caputo fractional operator to describe disease dynamics involving asymptomatic, symptomatic, and hospitalized populations. The model included existence, uniqueness, and stability analysis, supported by a suitable numerical scheme. Furthermore, [57] applied fixed point theory to analyze fractional boundary value problems emerging from

chemical graph theory. These investigations highlight the significance of fractional differential equations in real-world applications and encourage continued research in this direction.

Fractional iterative differential equations with Caputo derivatives have numerous applications across various disciplines. In physics and engineering, such equations effectively model viscoelastic materials, anomalous diffusion, and electrical circuits featuring fractional-order elements. These equations are also used to describe heat transfer, thermal analysis, and mechanical systems characterized by fractional-order derivatives.

Moreover, these equations play a significant role in biology and medicine. They help model population dynamics, tumor growth, and complex biological systems, offering insights into pharmacokinetics, pharmacodynamics, and neuronal activity.

In finance and economics, fractional iterative differential equations model financial markets, stock prices, and fractional-order stochastic processes, contributing to option pricing, risk management, and the dynamics of economic systems [7, 18, 26]. In signal processing and image analysis, fractional-order models are applied to filtering, image denoising, edge detection, and signal modeling. Ongoing research in mathematics and numerical analysis continues to develop methods for stability, convergence, and theoretical understanding of such equations.

Real-world applications extend further to modeling renewable energy systems, groundwater flow, contaminant transport, and complex networks. Notable examples include the fractional-order logistic equation, Lotka–Volterra model, diffusion equation, wave equation, and heat equation. Solutions are obtained through methods such as finite difference, finite element, Laplace transforms, Adomian decomposition, and variational iteration [27, 44, 49, 55].

The Caputo derivative, introduced by Michele Caputo in 1967, provides a robust framework for modeling memory-dependent and non-local phenomena. Its versatility has made it a key tool in fractional iterative differential equations, driving ongoing research and applications [12, 13, 22, 45].

In recent years, substantial efforts have been made to study the stability of fractional differential equations using fixed point methods and various metric frameworks. However, to the best of our knowledge, there are no existing results on the Hyers–Ulam and Hyers–Ulam–Rassias stability for Caputo fractional iterative differential equations within the setting of a generalized Bielecki metric. Existing studies primarily focus on either classical Caputo-type equations or do not account for the iterative nature of the system. Moreover, the combined effects of iterative dynamics and fractional-order derivatives have received limited attention, especially in the context of both bounded and unbounded temporal domains. These gaps motivate the present study, which aims to fill this void by establishing new stability criteria using a generalized metric approach, thus extending and unifying several known results.

The results presented in this paper offer practical benefits by providing a rigorous analytical framework for ensuring the stability of fractional iterative differential systems. Such stability guarantees are essential for validating the reliability of mathematical models that involve memory effects and iterative processes, features commonly found in real-world phenomena. By establishing Hyers–Ulam and Hyers–Ulam–Rassias stability through a generalized Bielecki metric, the study enhances the tools available for researchers dealing with complex dynamical systems. These findings help ensure that small deviations in input or initial conditions lead to predictable and bounded responses, which is crucial for accurate analysis and dependable behavior in applied mathematical modeling.

Let us consider the following standard definitions of fractional calculus.

Definition 1.1 (See [14, 46]). Let $x(t) : [0, T] \rightarrow \mathbb{C}$ be a continuous and integrable function over $[0, T]$, and let $\alpha > 0$. The fractional integral of $x(t)$ of order α is defined as

$$I_0^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \xi)^{\alpha-1} x(\xi) d\xi. \quad (3)$$

Definition 1.2 (See [14, 46]). Let $\alpha \in (n - 1, n)$ for $n \in \mathbb{N}$, and let $x(t) : [0, T] \rightarrow \mathbb{C}$ be a continuous and integrable function for $T > 0$. The Caputo fractional derivative of $x(t)$ of order α is defined as

$${}^C D_0^\alpha x(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - \xi)^{n-\alpha-1} x^{(n)}(\xi) d\xi. \quad (4)$$

In Definitions 1.1 and 1.2, the Gamma function, $\Gamma(x)$, is expressed as

$$\Gamma(x) = \int_0^{\infty} \xi^{x-1} e^{-\xi} d\xi. \quad (5)$$

It is known that $\Gamma(1) = 1$, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, $\Gamma(x+1) = x\Gamma(x)$ for $x > 0$, and $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$. Furthermore, it can be observed that $I_0^\alpha({}^C D_0^\alpha x(t)) = x(t) - x(0)$. Consequently, the integral representation of Equation (1) is given by

$$x(t) = c + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \xi)^{\alpha-1} f(\xi, x(\xi), x(g(x(\xi)))) d\xi, \quad (6)$$

where $t, \xi \in [0, T]$, $\alpha \in (0, 1)$, $c \geq 0$, and $\Gamma(\cdot)$ denotes the Gamma function.

The stabilities discussed earlier are formally defined in the context of the Caputo fractional iterative differential Equation (1).

Definition 1.3. Let x be a continuous function on $[0, T]$ such that

$$\left| {}^C D_0^\alpha x(t) - f(t, x(t), x(g(x(t)))) \right| \leq \sigma(t), \quad t \in [0, T],$$

where σ is a non-decreasing function. If there exists a solution x_0 to the Caputo fractional iterative differential equation and a constant $C > 0$, independent of x and x_0 , such that

$$|x(t) - x_0(t)| \leq C\sigma(t),$$

for all $t \in [0, T]$, then the Caputo fractional iterative differential Equation (1) is said to exhibit Hyers–Ulam–Rassias stability.

Definition 1.4. Let x be a continuous function on $[0, T]$, such that

$$\left| {}^C D_0^\alpha x(t) - f(t, x(t), x(g(x(t)))) \right| \leq \theta, \quad t \in [0, T],$$

where $\theta \geq 0$. If there exists a solution x_0 to the Caputo fractional iterative differential equation and a constant $C > 0$, independent of x and x_0 , such that

$$|x(t) - x_0(t)| \leq C\theta,$$

for all $t \in [0, T]$, then the Caputo fractional iterative differential Equation (1) is said to have Hyers–Ulam stability.

We now introduce a novel type of stability, as identified by Castro and Simões [19]. This stability lies between the two previously discussed stabilities, namely Hyers–Ulam–Rassias stability and Hyers–Ulam stability.

Definition 1.5. Let σ be a non-decreasing function defined on $[0, T]$. If for every continuous function x satisfying

$$\left| {}^C D_0^\alpha x(t) - f(t, x(t), x(g(x(t)))) \right| \leq \theta, \quad t \in [0, T], \quad (7)$$

where $\theta \geq 0$, there exists a solution x_0 of the Caputo fractional iterative differential equation and a constant $C > 0$, independent of x and x_0 , such that

$$|x(t) - x_0(t)| \leq C\sigma(t), \quad t \in [0, T], \quad (8)$$

then the Caputo fractional iterative differential Equation (1) is said to have σ -semi-Hyers–Ulam stability.

For a nonempty set X , the generalized metric on X is defined as follows:

Definition 1.6 ([48]). Let X be a nonempty set and $d : X \times X \rightarrow [0, +\infty]$ be a given mapping. We say that d is a generalized metric on X if and only if d satisfies the following conditions:

1. $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

This concept differs from the standard definition of a complete metric space, as it allows the possibility that not all pairs of points in X have a finite distance. Consequently, such a space can be described as a generalized complete metric space.

Next, we present a well-known result in fixed-point theory.

Theorem 1.7 ([11]). *Let (X, d) be a generalized complete metric space and consider a mapping $T : X \rightarrow X$ which is a strictly contractive operator, that is,*

$$d(Tx, Ty) \leq Ld(x, y), \quad x, y \in X,$$

for some Lipschitz constant $0 \leq L < 1$. If there exists a non-negative integer k such that $d(T^{k+1}x, T^kx) < \infty$ for some $x \in X$, then the following three propositions hold true:

(M1) *The sequence $(T^n x)_{n \in \mathbb{N}}$ converges to a fixed-point x^* of T ;*

(M2) *x^* is the unique fixed-point of T in $X^* = \{y \in X : d(T^k x, y) < \infty\}$;*

(M3) *If $y \in X^*$, then*

$$d(y, x^*) \leq \frac{1}{1-L} d(Ty, y). \quad (9)$$

2. Hyers–Ulam–Rassias Stability on a Bounded Interval

This section presents the necessary conditions for the Hyers–Ulam–Rassias stability of the Caputo fractional iterative differential Equation (1), where $t \in [0, T]$, for some fixed real number T .

We will explore the space of continuous functions $C([0, T])$ on $[0, T]$, which is equipped with a generalization of the Bielecki metric given by

$$d(x, y) = \sup_{t \in [0, T]} \frac{|x(t) - y(t)|}{\sigma(t)}, \quad (10)$$

where σ is a non-decreasing continuous function $\sigma : [0, T] \rightarrow (0, \infty)$. In (10), if we take $\sigma(t) = e^{p(t)}$ with $p > 0$, we obtain the well-known Bielecki metric. In this work, we have chosen to consider a more general form of the metric to make it more inclusive.

We recall that the space $C([0, T])$ with the generalized metric d is a complete metric space (see, for instance, earlier research [20], [6]).

Theorem 2.1. *Let $f : [0, T] \times \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ be a continuous function such that there exist constants $L_1, L_2 > 0$ satisfying*

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq L_1|x_1 - x_2| + L_2|y_1 - y_2| \quad (11)$$

for all $t \in [0, T]$ and $x_1, x_2, y_1, y_2 \in \mathbb{C}$. Additionally, let $g : \mathbb{C} \rightarrow \mathbb{C}$ be a continuous function such that $g(x(t)) \leq t$.

Let $\sigma : [0, T] \rightarrow (0, \infty)$ be a non-decreasing continuous function such that there exists $K > 0$ for which the following inequality holds:

$$\int_0^t (t - \xi)^{\alpha-1} \sigma(\xi) d\xi \leq K\sigma(t) \quad (12)$$

for all $t \in [0, T]$.

If $x \in C([0, T])$ satisfies

$$\left| {}^C D_0^\alpha x(t) - f(t, x(t), x(g(x(t)))) \right| \leq \sigma(t), \quad t \in [0, T], \quad (13)$$

and $\frac{K(L_1+L_2)}{\Gamma(\alpha)} < 1$, then there exists a unique function $x_0 \in C([0, T])$, a solution to Equation (1), given by

$$x_0(t) = c + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \xi)^{\alpha-1} f(\xi, x_0(\xi), x_0(g(x_0(\xi)))) d\xi, \quad (14)$$

such that

$$|x(t) - x_0(t)| \leq \frac{K}{\Gamma(\alpha) - K(L_1 + L_2)} \sigma(t) \quad (15)$$

for all $t \in [0, T]$.

Thus, under the above conditions, the Caputo fractional iterative differential Equation (1) is Hyers–Ulam–Rassias stable.

Proof. We will consider the operator $\Psi : C([0, T]) \rightarrow C([0, T])$ defined by

$$(\Psi x)(t) = c + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \xi)^{\alpha-1} f(\xi, x(\xi), x(g(x(\xi)))) d\xi \quad (16)$$

for all $t \in [0, T]$ and $x \in C([0, T])$.

It is important to note that Ψx is also continuous for every continuous function x . Indeed,

$$\begin{aligned} |(\Psi x)(t) - (\Psi x)(t_0)| &= \frac{1}{\Gamma(\alpha)} \left| \int_0^t (t - \xi)^{\alpha-1} f(\xi, x(\xi), x(g(x(\xi)))) d\xi \right. \\ &\quad \left. - \int_0^{t_0} (t_0 - \xi)^{\alpha-1} f(\xi, x(\xi), x(g(x(\xi)))) d\xi \right| \\ &= \frac{1}{\Gamma(\alpha)} \left| \int_0^t (t - \xi)^{\alpha-1} f(\xi, x(\xi), x(g(x(\xi)))) d\xi \right. \\ &\quad \left. - \int_0^t (t_0 - \xi)^{\alpha-1} f(\xi, x(\xi), x(g(x(\xi)))) d\xi \right. \\ &\quad \left. + \int_0^t (t_0 - \xi)^{\alpha-1} f(\xi, x(\xi), x(g(x(\xi)))) d\xi \right. \\ &\quad \left. - \int_0^{t_0} (t_0 - \xi)^{\alpha-1} f(\xi, x(\xi), x(g(x(\xi)))) d\xi \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \left(\int_0^t |(t - \xi)^{\alpha-1} - (t_0 - \xi)^{\alpha-1}| \right. \\ &\quad \times |f(\xi, x(\xi), x(g(x(\xi))))| d\xi \\ &\quad \left. + \left| \int_{t_0}^t (t_0 - \xi)^{\alpha-1} f(\xi, x(\xi), x(g(x(\xi)))) d\xi \right| \right) \rightarrow 0 \end{aligned}$$

when $t \rightarrow t_0$.

We shall conclude that under the present conditions, the operator Ψ is strictly contractive with respect to the metric (10)

$$\begin{aligned} d(\Psi x, \Psi y) &= \sup_{t \in [0, T]} \frac{|(\Psi x)(t) - (\Psi y)(t)|}{\sigma(t)} \\ &= \frac{1}{\Gamma(\alpha)} \sup_{t \in [0, T]} \frac{1}{\sigma(t)} \left| \int_0^t (t - \xi)^{\alpha-1} f(\xi, x(\xi), x(g(x(\xi)))) d\xi \right. \\ &\quad \left. - \int_0^t (t - \xi)^{\alpha-1} f(\xi, y(\xi), y(g(y(\xi)))) d\xi \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \sup_{t \in [0, T]} \frac{1}{\sigma(t)} \int_0^t (t - \xi)^{\alpha-1} \end{aligned}$$

$$\begin{aligned}
& \times \left| f(\xi, x(\xi), x(g(x(\xi)))) - f(\xi, y(\xi), y(g(y(\xi)))) \right| d\xi \\
& \leq \frac{1}{\Gamma(\alpha)} \sup_{t \in [0, T]} \frac{1}{\sigma(t)} \int_0^t (t - \xi)^{\alpha-1} \\
& \quad \times \left[L_1 |x(\xi) - y(\xi)| + L_2 |x(g(x(\xi))) - y(g(y(\xi)))| \right] d\xi \\
& \leq \frac{(L_1 + L_2)}{\Gamma(\alpha)} \sup_{t \in [0, T]} \frac{1}{\sigma(t)} \int_0^t (t - \xi)^{\alpha-1} |x(\xi) - y(\xi)| d\xi \\
& = \frac{(L_1 + L_2)}{\Gamma(\alpha)} \sup_{t \in [0, T]} \frac{1}{\sigma(t)} \int_0^t (t - \xi)^{\alpha-1} \frac{|x(\xi) - y(\xi)|}{\sigma(\xi)} \sigma(\xi) d\xi \\
& \leq \frac{(L_1 + L_2)}{\Gamma(\alpha)} \sup_{\xi \in [0, T]} \frac{|x(\xi) - y(\xi)|}{\sigma(\xi)} \sup_{t \in [0, T]} \frac{1}{\sigma(t)} \int_0^t (t - \xi)^{\alpha-1} \sigma(\xi) d\xi \\
& \leq \frac{K(L_1 + L_2)}{\Gamma(\alpha)} d(x, y).
\end{aligned}$$

As a consequence of $\frac{K(L_1+L_2)}{\Gamma(\alpha)} < 1$, it follows that Ψ is strictly contractive. Therefore, we can apply the previously mentioned Banach fixed-point theorem, which guarantees the Hyers–Ulam–Rassias stability of the Caputo fractional iterative differential equation.

Moreover, Equation (15) follows from (9) and (13). Indeed, from (13), we have

$$|x(t) - (\Psi x)(t)| \leq \frac{K}{\Gamma(\alpha)} \sigma(t), \quad t \in [0, T]. \quad (17)$$

The Banach fixed-point theorem can now be applied again, and from (9), we get

$$d(x, x_0) \leq \frac{1}{1 - \frac{K(L_1+L_2)}{\Gamma(\alpha)}} d(\Psi x, x). \quad (18)$$

By the definition of the metric d and from (17), it follows that

$$\sup_{t \in [0, T]} \frac{|x(t) - x_0(t)|}{\sigma(t)} \leq \frac{K}{\Gamma(\alpha) - K(L_1 + L_2)}, \quad (19)$$

and thus, (15) holds. $\square$

3. σ -semi-Hyers–Ulam and Hyers–Ulam Stabilities on a Bounded Interval

We shall now outline the necessary conditions in this section for the σ -semi-Hyers–Ulam and Hyers–Ulam stability of the Caputo fractional iterative differential Equation (1).

Theorem 3.1. Let $f : [0, T] \times \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ be a continuous given function such that there exist $L_1, L_2 > 0$ so that

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq L_1 |x_1 - x_2| + L_2 |y_1 - y_2| \quad (20)$$

for all $t \in [0, T]$ and $x_1, x_2, y_1, y_2 \in \mathbb{C}$. In addition, $g : \mathbb{C} \rightarrow \mathbb{C}$ is a continuous function with $g(x(t)) \leq t$.

Let $\sigma : [0, T] \rightarrow (0, \infty)$ be a non-decreasing continuous function such that there exists $K > 0$ so that

$$\int_0^t (t - \xi)^{\alpha-1} \sigma(\xi) d\xi \leq K \sigma(t) \quad (21)$$

for all $t \in [0, T]$.

If $x \in C([0, T])$ is such that

$$\left| {}^C D_0^\alpha x(t) - f(t, x(t), x(g(x(t)))) \right| \leq \theta, \quad t \in [0, T], \quad (22)$$

where $\theta \geq 0$ and $\frac{K(L_1+L_2)}{\Gamma(\alpha)} < 1$, then there exists a unique function $x_0 \in C([0, T])$, solution of Equation (1), that is

$$x_0(t) = c + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \xi)^{\alpha-1} f(\xi, x_0(\xi), x_0(g(x_0(\xi)))) d\xi \quad (23)$$

such that

$$|x(t) - x_0(t)| \leq \frac{T^\alpha K \theta}{\Gamma(\alpha + 1)(\Gamma(\alpha) - K(L_1 + L_2))\sigma(0)} \sigma(t), \quad (24)$$

for all $t \in [0, T]$, which means that the Caputo fractional iterative differential Equation (1) is σ -semi-Hyers-Ulam stable.

Proof. Applying the same method as before, we conclude that Ψ is strictly contractive with respect to the metric (10) since $\frac{K(L_1+L_2)}{\Gamma(\alpha)} < 1$. Consequently, we can utilize the Banach fixed-point theorem, which guarantees the σ -semi-Hyers-Ulam stability for the Caputo fractional iterative differential Equation (1).

However, keeping in mind (22) and the meaning of Ψ , we have that

$$|x(t) - (\Psi x)(t)| \leq \frac{T^\alpha}{\Gamma(\alpha + 1)} \theta, \quad t \in [0, T]. \quad (25)$$

From (9), the definition of the metric d , and from (25), it follows that

$$\sup_{t \in [0, T]} \frac{|x(t) - x_0(t)|}{\sigma(t)} \leq \frac{K}{\Gamma(\alpha) - K(L_1 + L_2)} \sup_{t \in [0, T]} \frac{\frac{T^\alpha \theta}{\Gamma(\alpha + 1)}}{\sigma(t)}, \quad (26)$$

so that (24) holds according to the definition of σ . $\square$

Corollary 3.2. Let $f : [0, T] \times \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ be a continuous given function such that there exist $L_1, L_2 > 0$ so that

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq L_1|x_1 - x_2| + L_2|y_1 - y_2| \quad (27)$$

for all $t \in [0, T]$ and $x_1, x_2, y_1, y_2 \in \mathbb{C}$. In addition, $g : \mathbb{C} \rightarrow \mathbb{C}$ is also a continuous function with $g(x(t)) \leq t$.

Let $\sigma : [0, T] \rightarrow (0, \infty)$ be a non-decreasing continuous function such that there exists $K > 0$ so that

$$\int_0^t (t - \xi)^{\alpha-1} \sigma(\xi) d\xi \leq K\sigma(t) \quad (28)$$

for all $t \in [0, T]$.

If $x \in C([0, T])$ is such that

$$|{}^C D_0^\alpha x(t) - f(t, x(t), x(g(x(t))))| \leq \theta, \quad t \in [0, T], \quad (29)$$

where $\theta \geq 0$ and $\frac{K(L_1+L_2)}{\Gamma(\alpha)} < 1$, then there exists a unique function $x_0 \in C([0, T])$, solution of Equation (1), that is

$$x_0(t) = c + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \xi)^{\alpha-1} f(\xi, x_0(\xi), x_0(g(x_0(\xi)))) d\xi \quad (30)$$

such that

$$|x(t) - x_0(t)| \leq \frac{T^\alpha K \sigma(b)}{\Gamma(\alpha + 1)(\Gamma(\alpha) - K(L_1 + L_2))\sigma(0)} \theta, \quad (31)$$

for all $t \in [0, T]$, which means that the Caputo fractional iterative differential Equation (1) is Hyers-Ulam stable.

4. Stabilities on an Unbounded Interval

Instead of analyzing the Caputo fractional iterative differential Equation (1) on a finite interval $[0, T]$, where $T \in \mathbb{R}$, we now turn our attention to studying the Hyers–Ulam–Rassias and σ –semi–Hyers–Ulam stabilities of the equation on the infinite interval $[0, \infty)$.

With the appropriate adjustments, the following results can also be extended to infinite intervals such as $(-\infty, T]$, where $T \in \mathbb{R}$, or even $(-\infty, \infty)$.

We now focus on the Caputo fractional iterative differential equation

$$\begin{aligned} {}^C D_0^\alpha x(t) &= f(t, x(t), x(g(x(t))))), \quad 0 < \alpha < 1, \\ x(0) &= c, \quad c \geq 0, \end{aligned} \quad (32)$$

for $t \in [0, \infty)$, where the function f is defined as $f : [0, \infty) \times \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ and is assumed to be continuous in all its arguments. The function $g : \mathbb{C} \rightarrow \mathbb{C}$ is also assumed to be continuous, satisfying the condition $g(x(t)) \leq t$. Here, $\mathbb{C}$ denotes the Banach space of all real-valued continuous functions defined on the interval $[0, \infty)$.

Our approach will leverage a recurrence procedure, building on the results obtained for the finite interval case.

Let us consider a non-decreasing continuous function $\sigma : [0, \infty) \rightarrow (\epsilon, \omega)$, where $\epsilon, \omega > 0$, and the space $C_b([0, \infty))$ of bounded functions, endowed with the metric

$$d_b(x, y) = \sup_{t \in [0, \infty)} \frac{|x(t) - y(t)|}{\sigma(t)}. \quad (33)$$

Theorem 4.1. Let $f : [0, \infty) \times \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ be a continuous function such that there exist constants $L_1, L_2 > 0$ satisfying

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq L_1|x_1 - x_2| + L_2|y_1 - y_2| \quad (34)$$

for all $t \in [0, \infty)$ and $x_1, x_2, y_1, y_2 \in \mathbb{C}$. Additionally, let $g : \mathbb{C} \rightarrow \mathbb{C}$ be a continuous function with the property that $g(x(t)) \leq t$.

Let $\sigma : [0, \infty) \rightarrow (0, \infty)$ be a non-decreasing continuous function such that there exists a constant $K > 0$ satisfying

$$\int_0^t (t - \xi)^{\alpha-1} \sigma(\xi) d\xi \leq K\sigma(t) \quad (35)$$

for all $t \in [0, \infty)$.

In addition, suppose that

$$\int_0^t (t - \xi)^{\alpha-1} f(\xi, x(\xi), x(g(x(\xi)))) d\xi$$

is a bounded continuous function for any bounded continuous function x .

Suppose $x \in C_b([0, \infty))$ satisfies the inequality

$$|{}^C D_0^\alpha x(t) - f(t, x(t), x(g(x(t))))| \leq \sigma(t), \quad t \in [0, \infty), \quad (36)$$

and that $\frac{K(L_1+L_2)}{\Gamma(\alpha)} < 1$. Then, there exists a unique function $x_0 \in C([0, T])$ that solves the equation

$$x_0(t) = c + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \xi)^{\alpha-1} f(\xi, x_0(\xi), x_0(g(x_0(\xi)))) d\xi \quad (37)$$

such that

$$|x(t) - x_0(t)| \leq \frac{K}{\Gamma(\alpha) - K(L_1 + L_2)} \sigma(t) \quad (38)$$

for all $t \in [0, \infty)$, which implies that the Caputo fractional iterative differential Equation (32) is Hyers–Ulam–Rassias stable.

Proof. For any $m \in \mathbb{N}$, we define $I_m = [0, m]$. By Theorem 2.1, there exists a unique continuous function $x_{0,m} \in C(I_m)$ such that

$$x_{0,m}(t) = c + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \xi)^{\alpha-1} f(\xi, x_{0,m}(\xi), x_{0,m}(g(x_{0,m}(\xi)))) d\xi \quad (39)$$

and

$$|x(t) - x_{0,m}(t)| \leq \frac{K}{\Gamma(\alpha) - K(L_1 + L_2)} \sigma(t) \quad (40)$$

for all $t \in I_m$. The uniqueness of $x_{0,m}$ implies that for any $t \in I_m$,

$$x_{0,m}(t) = x_{0,m+1}(t) = x_{0,m+2}(t) = \cdots \quad (41)$$

For any $t \in [0, \infty)$, let $m(t) \in \mathbb{N}$ be defined as

$$m(t) = \min\{m \in \mathbb{N} : t \in I_m\}.$$

We also define a function $x_0 : [0, \infty) \rightarrow \mathbb{C}$ by

$$x_0(t) = x_{0,m(t)}(t). \quad (42)$$

For any $t_1 \in [0, \infty)$, let $m_1 = m(t_1)$. Then $t_1 \in \text{Int}(I_{m_1+1})$, and there exists an $\epsilon > 0$ such that $x_0(t) = x_{0,m_1+1}(t)$ for all $t \in (t_1 - \epsilon, t_1 + \epsilon)$, where $\text{Int}(I_{m_1+1})$ denotes the interior of I_{m_1+1} . By Theorem 2.1, x_{0,m_1+1} is continuous at t_1 , and thus, so is x_0 .

Next, we show that x_0 satisfies

$$x_0(t) = c + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \xi)^{\alpha-1} f(\xi, x_0(\xi), x_0(g(x_0(\xi)))) d\xi \quad (43)$$

and

$$|x(t) - x_0(t)| \leq \frac{K}{\Gamma(\alpha) - K(L_1 + L_2)} \sigma(t). \quad (44)$$

For any $t \in [0, \infty)$, choose $m(t)$ such that $t \in I_{m(t)}$. From (39) and (42), we have

$$\begin{aligned} x_0(t) &= x_{0,m(t)}(t) \\ &= c + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \xi)^{\alpha-1} f(\xi, x_{0,m(t)}(\xi), x_{0,m(t)}(g(x_{0,m(t)}(\xi)))) d\xi \\ &= c + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \xi)^{\alpha-1} f(\xi, x_0(\xi), x_0(g(x_0(\xi)))) d\xi. \end{aligned} \quad (45)$$

Note that $m(\xi) \leq m(t)$ for any $\xi \in I_{m(t)}$, and from (41), we know $x_0(\xi) = x_{0,m(\xi)}(\xi) = x_{0,m(t)}(\xi)$, so the last equality in (45) holds.

To prove (44), from (42) and (40), we obtain for all $t \in [0, \infty)$,

$$|x(t) - x_0(t)| = |x(t) - x_{0,m(t)}(t)| \leq \frac{K}{\Gamma(\alpha) - K(L_1 + L_2)} \sigma(t). \quad (46)$$

Finally, we prove the uniqueness of x_0 . Let x_1 be another bounded continuous function that satisfies (37) and (38) for all $t \in [a, \infty)$. By the uniqueness of the solution on $I_{m(t)}$ for any $m(t) \in \mathbb{N}$, we have that $x_0|_{I_{m(t)}} = x_{0,m(t)}$ and $x_1|_{I_{m(t)}}$ satisfies (37) and (38) for all $t \in I_{m(t)}$, so

$$x_0(t) = x_0|_{I_{m(t)}}(t) = x_1|_{I_{m(t)}}(t) = x_1(t).$$

□

Now, we will provide sufficient conditions for the σ -semi-Hyers-Ulam stability of the Caputo fractional iterative differential Equation (32).

Theorem 4.2. Let $f : [0, \infty) \times \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ be a continuous function such that there exist constants $L_1, L_2 > 0$ with the property

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq L_1|x_1 - x_2| + L_2|y_1 - y_2| \quad (47)$$

for all $t \in [0, \infty)$ and $x_1, x_2, y_1, y_2 \in \mathbb{C}$. Additionally, let $g : \mathbb{C} \rightarrow \mathbb{C}$ be a continuous function such that $g(x(t)) \leq t$ for all $t \in [0, \infty)$.

Let $\sigma : [0, \infty) \rightarrow (0, \infty)$ be a non-decreasing continuous function such that there exists a constant $K > 0$ satisfying

$$\int_0^t (t - \xi)^{\alpha-1} \sigma(\xi) d\xi \leq K\sigma(t) \quad (48)$$

for all $t \in [0, \infty)$.

In addition, suppose that

$$\int_0^t (t - \xi)^{\alpha-1} f(\xi, x(\xi), x(g(x(\xi)))) d\xi$$

is a bounded continuous function for any bounded continuous function x .

Suppose $x \in C_b([0, \infty))$ is such that

$$|{}^C D_0^\alpha x(t) - f(t, x(t), x(g(x(t))))| \leq \theta, \quad t \in [0, \infty), \quad (49)$$

where $\theta \geq 0$ and $\frac{K(L_1+L_2)}{\Gamma(\alpha)} < 1$. Then there exists a unique function $x_0 \in C_b([0, T])$, which is the solution of Equation (32), given by

$$x_0(t) = c + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \xi)^{\alpha-1} f(\xi, x_0(\xi), x_0(g(x_0(\xi)))) d\xi \quad (50)$$

such that

$$|x(t) - x_0(t)| \leq \frac{K\theta}{\Gamma(\alpha) - K(L_1 + L_2)\sigma(0)} \sigma(t) \quad (51)$$

for all $t \in [0, \infty)$. This implies that the Caputo fractional iterative differential Equation (32) is σ -semi-Hyers-Ulam stable.

Proof. The proof follows the same procedure as in the previous case, combined with Theorem 3.2. Therefore, we omit the detailed proof here as it is straightforward. $\square$

5. Applications

To demonstrate that the conditions of the above results can be achieved, we will now present some examples.

Example 5.1. Consider the following Caputo fractional iterative differential equation:

$${}^C D_0^{\frac{1}{3}} x(t) = \frac{1}{100}t + \frac{1}{20}e^{-t^3} + \frac{\cos(x(t))}{30} + \frac{1}{15} \sin(x(g(x(t)))), \quad t \in [0, 1],$$

$$x(0) = 0, \quad (52)$$

along with the inequality:

$$\left| {}^C D_0^{\frac{1}{3}} x(t) - \frac{1}{100}t - \frac{1}{20}e^{-t^3} - \frac{\cos(x(t))}{30} - \frac{1}{15} \sin(x(g(x(t)))) \right| \leq e^{3t} = \sigma(t).$$

Equation (52) is structured according to the form of Equation (1) with the following parameters:

$$\alpha = \frac{1}{3}, \quad [0, T] = [0, 1], \quad c = 0,$$

$$f(t, x(t), x(g(x(t)))) = \frac{1}{100}t + \frac{1}{20}e^{-t^3} + \frac{\cos(x(t))}{30} + \frac{1}{15}\sin(x(g(x(t)))).$$

Next, we demonstrate that the conditions of Theorem 2.1 are satisfied. For this purpose, we choose $L_1 = \frac{1}{30}$, $L_2 = \frac{1}{15}$, and $K = 10$, and perform the following calculations:

$$\begin{aligned} & \left| f(t, x_1(t), y_1(g(x_1(t)))) - f(t, x_2(t), y_2(g(x_2(t)))) \right| \\ & \leq \frac{1}{30} |y_1(t) - y_2(t)| + \frac{1}{15} |y_1(t) - y_2(t)|, \\ & \left| \int_0^t (t - \xi)^{\alpha-1} \sigma(\xi) d\xi \right| = \left| \int_0^t (t - \xi)^{\frac{1}{3}-1} e^{3\xi} d\xi \right| \leq 10e^{3t}. \end{aligned}$$

Furthermore, we compute:

$$\frac{K(L_1 + L_2)}{\Gamma(\alpha)} = \frac{10\left(\frac{1}{30} + \frac{1}{15}\right)}{\Gamma\left(\frac{1}{3}\right)} \approx \frac{10\left(\frac{1}{10}\right)}{2.6789385} = 0.3732 < 1.$$

Thus, all conditions of Theorem 2.1 are satisfied. Therefore, Equation (52) is Hyers–Ulam–Rassias stable, with the bound:

$$|x(t) - x_0(t)| \leq \frac{10}{2.6789385 - 10\left[\frac{1}{30} + \frac{1}{15}\right]} e^{3t} = 5.9563 e^{3t}, \quad t \in [0, 1].$$

Thus, Example 5.1 provides a valid application of Theorem 2.1.

Example 5.2. Consider the following Caputo fractional iterative differential equation:

$$\begin{aligned} {}^C D_0^{\frac{2}{3}} x(t) &= \frac{1}{t^2 + 1} + \frac{\tan^{-1}(x(t))}{4} + \frac{\sin(x(g(x(t))))}{20(t^3 + t^2 + t + 1)}, \quad t \in [0, 2], \\ x(0) &= 0, \end{aligned} \tag{53}$$

and the inequality:

$$\left| {}^C D_0^{\frac{2}{3}} x(t) - \frac{1}{t^2 + 1} - \frac{\tan^{-1}(x(t))}{4} - \frac{\sin(x(g(x(t))))}{20(t^3 + t^2 + t + 1)} \right| \leq e^t = \sigma(t).$$

Equation (53) is written in the form of Equation (1), with the following parameters:

$$\alpha = \frac{2}{3}, \quad [0, T] = [0, 2], \quad c = 0,$$

$$f(t, x(t), x(g(x(t)))) = \frac{1}{t^2 + 1} + \frac{\tan^{-1}(x(t))}{4} + \frac{\sin(x(g(x(t))))}{20(t^3 + t^2 + t + 1)}.$$

Next, we demonstrate that the conditions of Theorem 2.1 are satisfied. For this purpose, we set $L_1 = \frac{1}{4}$, $L_2 = \frac{1}{20}$, and $K = 3$, and we perform the following calculations:

$$\begin{aligned} & \left| f(t, x_1(t), y_1(g(x_1(t)))) - f(t, x_2(t), y_2(g(x_2(t)))) \right| \\ & \leq \frac{1}{4} |x_1(t) - x_2(t)| + \frac{1}{20} |y_1(t) - y_2(t)|, \end{aligned}$$

$$\left| \int_0^t (t - \xi)^{\alpha-1} \sigma(\xi) d\xi \right| = \left| \int_0^t (t - \xi)^{\frac{2}{3}-1} e^\xi d\xi \right| \leq 3e^t.$$

Additionally, we compute:

$$\frac{K(L_1 + L_2)}{\Gamma(\alpha)} = \frac{3\left(\frac{1}{4} + \frac{1}{20}\right)}{\Gamma\left(\frac{2}{3}\right)} \approx \frac{3\left(\frac{3}{10}\right)}{1.3541179} = 0.6645 < 1.$$

Therefore, all the conditions of Theorem 2.1 are satisfied. As a result, Equation (53) is Hyers–Ulam–Rassias stable with the bound:

$$|x(t) - x_0(t)| \leq \frac{3}{1.3541179 - 3\left[\frac{1}{4} + \frac{1}{20}\right]} e^t = 6.6072 e^t, \quad t \in [0, 2].$$

Thus, Example 5.2 illustrates the application of Theorem 2.1.

Example 5.3. Consider the following Caputo fractional iterative differential equation:

$$\begin{aligned} {}^C D_0^{\frac{1}{3}} x(t) &= \sin t + 1000 + \frac{x(t)}{2} + \frac{\cot(x(g(x(t))))}{30}, \quad t \in [0, 3], \\ x(0) &= 0, \end{aligned} \tag{54}$$

and the inequality:

$$\left| {}^C D_0^{\frac{1}{3}} x(t) - \sin(t) - 1000 - \frac{x(t)}{2} - \frac{\cot(x(g(x(t))))}{30} \right| \leq e^{9t} = \sigma(t).$$

Equation (54) is written in the form of Equation (1), with the following parameters:

$$\alpha = \frac{1}{9}, \quad [0, T] = [0, 3], \quad c = 0,$$

$$f(t, x(t), x(g(x(t)))) = \sin(t) + 1000 + \frac{x(t)}{2} + \frac{\cot(x(g(x(t))))}{30}.$$

We now proceed to demonstrate that the conditions of Theorem 2.1 are satisfied. For this, we set $L_1 = \frac{1}{2}$, $L_2 = \frac{1}{30}$, and $K = 5$, and compute:

$$\begin{aligned} &|f(t, x_1(t), y_1(g(x_1(t)))) - f(t, x_2(t), y_2(g(x_2(t))))| \\ &\leq \frac{1}{2} |x_1(t) - x_2(t)| + \frac{1}{30} |y_1(t) - y_2(t)|, \\ &\left| \int_0^t (t - \xi)^{\alpha-1} \sigma(\xi) d\xi \right| = \left| \int_0^t (t - \xi)^{\frac{1}{9}-1} e^{9\xi} d\xi \right| \leq 5e^{9t}. \end{aligned}$$

Additionally, we compute:

$$\frac{K(L_1 + L_2)}{\Gamma(\alpha)} = \frac{5\left(\frac{1}{2} + \frac{1}{30}\right)}{\Gamma\left(\frac{1}{9}\right)} \approx \frac{5\left(\frac{8}{15}\right)}{4.5908} = 0.581 < 1.$$

Thus, all the conditions of Theorem 2.1 are satisfied. Consequently, Equation (54) is Hyers–Ulam–Rassias stable, with the bound:

$$|x(t) - x_0(t)| \leq \frac{5}{4.5908 - 5\left[\frac{1}{2} + \frac{1}{30}\right]} e^{9t} = 2.598 e^{9t}, \quad t \in [0, 3].$$

Thus, Example 5.3 demonstrates the application of Theorem 2.1.

Example 5.4. Consider the following Caputo fractional iterative differential equation:

$$\begin{aligned} {}^C D_0^{\frac{3}{5}} x(t) &= \ln(t+1) + \frac{e^{x(t)}}{50} + \frac{\tanh(x(g(x(t))))}{25}, \quad t \in [0, 2], \\ x(0) &= 2, \end{aligned} \quad (55)$$

and the inequality:

$$\left| {}^C D_0^{\frac{3}{5}} x(t) - \ln(t+1) - \frac{e^{x(t)}}{50} - \frac{\tanh(x(g(x(t))))}{25} \right| \leq e^t = \sigma(t).$$

Equation (55) is written in the form of Equation (1), with the following parameters:

$$\alpha = \frac{3}{5}, \quad [0, T] = [0, 2], \quad c = 2,$$

$$f(t, x(t), x(g(x(t)))) = \ln(t+1) + \frac{e^{x(t)}}{50} + \frac{\tanh(x(g(x(t))))}{25}.$$

Next, we verify that the conditions of Theorem 2.1 are satisfied. For this, we assume $L_1 = \frac{1}{50}$, $L_2 = \frac{1}{25}$, and $K = 10$. Then we compute:

$$\begin{aligned} & \left| f(t, x_1(t), y_1(g(x_1(t)))) - f(t, x_2(t), y_2(g(x_2(t)))) \right| \\ & \leq \frac{1}{50} |x_1(t) - x_2(t)| + \frac{1}{25} |y_1(t) - y_2(t)|, \\ & \left| \int_0^t (t-\xi)^{\alpha-1} \sigma(\xi) d\xi \right| = \left| \int_0^t (t-\xi)^{\frac{3}{5}-1} e^\xi d\xi \right| \leq 10e^t. \end{aligned}$$

Moreover, we calculate:

$$\frac{K(L_1 + L_2)}{\Gamma(\alpha)} = \frac{10\left(\frac{1}{50} + \frac{1}{25}\right)}{\Gamma\left(\frac{3}{5}\right)} \approx \frac{10\left(\frac{3}{50}\right)}{1.4892} = 0.403 < 1.$$

Thus, all the conditions of Theorem 2.1 are satisfied. Therefore, Equation (55) is Hyers–Ulam–Rassias stable, with the bound:

$$|x(t) - x_0(t)| \leq \frac{10}{1.4892 - 10\left[\frac{1}{50} + \frac{1}{25}\right]} e^t = 11.25 e^t, \quad t \in [0, 2].$$

Thus, Example 5.4 validates the application of Theorem 2.1 for the case $c \neq 0$.

6. Conclusion

In conclusion, this work successfully extends stability theory to Caputo fractional iterative differential equations by proving their Hyers–Ulam and Hyers–Ulam–Rassias stability. Employing the fixed–point method and a generalized Bielecki metric, the study rigorously establishes these results across both bounded and unbounded intervals. Furthermore, the inclusion of illustrative examples highlights the practical applicability and effectiveness of the proposed theoretical framework, contributing valuable insights to the study of fractional differential equations.

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