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Some new Milne-type inequalities via Katugampola fractional integrals

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Abstract. Drawing inspiration from the findings presented in references [10] and [16], this study aims to broaden the scope of Milne-type inequalities within the context of Katugampola fractional integrals, thus enriching the toolbox of fractional calculus. Through the discovery of a novel integral identity, we establish a suite of Milne-type inequalities tailored for functions whose first-order derivatives are s-convex in the second sense. To validate our theoretical advances, an example is provided, complete with graphical illustrations. The research concludes by underscoring the practical utility of these inequalities, showcasing their applicability across a wide range of fields within mathematical and applied sciences.

1. Introduction

Definition 1.1 ([28]). *A function $\Re : I \to \mathbb{R}$ is said to be convex, if*

$$\Re\left(\nu x + (1-\nu)\, y\right) \leq \nu \Re\,(x) + (1-\nu)\, \Re(y)$$

holds for all $x, y \in I$ and all $\nu \in [0,1]$.

Definition 1.2 ([7]). *A nonnegative function $\Re : I \subset [0,\infty) \to \mathbb{R}$ is said to be s-convex in the second sense for some fixed $s \in (0,1]$, if*

$$\Re(\nu x + (1-\nu) y) \leq \nu^s \Re(x) + (1-\nu)^s \Re(y)$$

holds for all $x, y \in I$ and $\nu \in [0,1]$.

The realm of fractional calculus represents a formidable extension of classical calculus, allowing for the exploration of derivatives and integrals of non-integer orders. This mathematical field has been enriched

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by the introduction of several fractional integral operators, such as the Riemann-Liouville [23], Hadamard [30], and Katugampola [20] fractional operators. These tools are instrumental in modeling phenomena with intrinsic memory and hereditary properties, finding extensive application across physics, engineering, and beyond.

Amidst numerical techniques for integral approximation, Milne's rule distinguishes itself within the Simpson's rule family, acclaimed for its precision in estimating function integrals. Its accuracy and adaptability have been the subject of extensive research, with studies exploring its application to classical [2], local fractional [4, 25, 27], and quantum integrals [5, 34], along with investigations into its effectiveness with various fractional operators [8–11, 36]. For an exhaustive review, references [6, 13–15, 29] offer comprehensive insights.

In [16], Djenaoui and Meftah established the following error bounds of Milne's rule via for differentiable convex functions:

Theorem 1.3. *[16] Let $\Re : [\mathfrak{c}_1, \mathfrak{c}_2] \to \mathbb{R}$ be a differentiable mapping on $(\mathfrak{c}_1, \mathfrak{c}_2)$ such that $\Re \in L_1([\mathfrak{c}_1, \mathfrak{c}_2])$. If $|\Re'|$ is convex on $[\mathfrak{c}_1, \mathfrak{c}_2]$, then we have*

$$\left|\frac{1}{3}\left(2\Re(\mathfrak{c}_1) - \Re\left(\frac{\mathfrak{c}_1+\mathfrak{c}_2}{2}\right) + 2\Re(\mathfrak{c}_2)\right) - \frac{1}{\mathfrak{c}_2-\mathfrak{c}_1}\int_{\mathfrak{c}_1}^{\mathfrak{c}_2}\Re(y)\,dy\right| \leq \frac{5(\mathfrak{c}_2-\mathfrak{c}_1)}{24}\left(|\Re'(\mathfrak{c}_1)| + |\Re'(\mathfrak{c}_2)|\right).$$

Recent advancements have seen the adaptation of Milne's Rule to the domain of fractional calculus, particularly through the establishment of Milne-type inequalities using Riemann-Liouville fractional integrals [8, 10].

Theorem 1.4. *[8] Let $\Re : [\mathfrak{c}_1, \mathfrak{c}_2] \to \mathbb{R}$ be a differentiable mapping on $(\mathfrak{c}_1, \mathfrak{c}_2)$ such that $\Re \in L_1([\mathfrak{c}_1, \mathfrak{c}_2])$. If $|\Re'|$ is convex on $[\mathfrak{c}_1, \mathfrak{c}_2]$, then we have*

$$\begin{aligned}&\left|\frac{1}{3}\left(2\Re(\mathfrak{c}_1) - \Re\left(\frac{\mathfrak{c}_1+\mathfrak{c}_2}{2}\right) + 2\Re(\mathfrak{c}_2)\right) - \frac{2^{\beta-1}\Gamma(\beta+1)}{(\mathfrak{c}_2-\mathfrak{c}_1)^{\beta}}\left(J^{\beta}_{\mathfrak{c}_1^+}\Re\left(\frac{\mathfrak{c}_1+\mathfrak{c}_2}{2}\right) + J^{\beta}_{\mathfrak{c}_2^-}\Re\left(\frac{\mathfrak{c}_1+\mathfrak{c}_2}{2}\right)\right)\right|\\ \leq&\frac{\mathfrak{c}_2-\mathfrak{c}_1}{12}\left(\frac{\beta+4}{\beta+1}\right)\left(|\Re'(\mathfrak{c}_1)| + |\Re'(\mathfrak{c}_2)|\right),\end{aligned}$$

where $J^{\beta}_{\mathfrak{c}_1^+}$ and $J^{\beta}_{\mathfrak{c}_2^-}$ represent the left- and right-sided Riemann-Liouville fractional integrals, respectively.

Theorem 1.5. *[10] Let $\Re : [\mathfrak{c}_1, \mathfrak{c}_2] \to \mathbb{R}$ be a differentiable mapping on $(\mathfrak{c}_1, \mathfrak{c}_2)$ such that $\Re \in L_1([\mathfrak{c}_1, \mathfrak{c}_2])$. If $|\Re'|$ is convex on $[\mathfrak{c}_1, \mathfrak{c}_2]$, then we have*

$$\begin{aligned}&\left|\frac{1}{3}\left(2\Re(\mathfrak{c}_1) - \Re\left(\frac{\mathfrak{c}_1+\mathfrak{c}_2}{2}\right) + 2\Re(\mathfrak{c}_2)\right) - \frac{2^{\beta-1}\Gamma(\beta+1)}{(\mathfrak{c}_2-\mathfrak{c}_1)^{\beta}}\left(J^{\beta}_{\left(\frac{\mathfrak{c}_1+\mathfrak{c}_2}{2}\right)^-}\Re(a) + J^{\beta}_{\left(\frac{\mathfrak{c}_1+\mathfrak{c}_2}{2}\right)^-}\Re(\mathfrak{c}_2)\right)\right|\\ \leq&\frac{\mathfrak{c}_2-\mathfrak{c}_1}{4}\left[\left(\frac{2}{3} - \frac{1}{(\beta+1)(\beta+2)}\right)\left(|\Re'(\mathfrak{c}_1)| + |\Re'(\mathfrak{c}_2)|\right) + \frac{4\beta+2}{3(\beta+2)}\left|\Re'\left(\frac{\mathfrak{c}_1+\mathfrak{c}_2}{2}\right)\right|\right],\end{aligned}$$

where $J^{\beta}_{\mathfrak{c}_1^+}$ and $J^{\beta}_{\mathfrak{c}_2^-}$ represent the Riemann-Liouville fractional integrals, respectively.

Within the spectrum of fractional integral operators, the introduction of Katugampola fractional integral operators marks a significant advancement, offering a broader theoretical base that encapsulates the features of both Riemann-Liouville and Hadamard fractional integrals. This innovative framework, proposed by Udita N. Katugampola in 2011, features versatile fractional integral operators that incorporate an adjustable parameter, facilitating customization for diverse application needs. The advent of Katugampola operators has spurred a wealth of research, particularly in the realm of integral inequalities. Notable contributions include the work by Farid et al., who formulated Ostrowski-type inequalities tailored for differentiable

h-convex functions, as documented in [18]. Additionally, Kermausuor extended these findings by deriving generalized Ostrowski-type inequalities for functions characterized by strongly (s, m)-convex second-order derivatives in [21], alongside Simpson-type inequalities for s-convex first-order derivatives in [22]. For further exploration in this domain, the interested reader is referred to [1, 12, 26, 31, 32], along with the works cited therein.

The result provided in Theorem 1.4 was extended to Katugampola fractional integrals by Lakhdari et al. in [24].

Theorem 1.6. *Let $\mathfrak{K} : \left[\mathfrak{c}_1^{\varrho}, \mathfrak{c}_2^{\varrho}\right] \to \mathbb{R}$ be a differentiable function on $\left(\mathfrak{c}_1^{\varrho}, \mathfrak{c}_2^{\varrho}\right)$ with $0 \leq \mathfrak{c}_1 < \mathfrak{c}_2$, and $\mathfrak{K}' \in L^1\left[\mathfrak{c}_1^{\varrho}, \mathfrak{c}_2^{\varrho}\right]$. If $|\mathfrak{K}'|$ is convex, then for $\beta, \varrho > 0$ we have*

$$\left|\frac{1}{3}\left(2\mathfrak{K}\left(\mathfrak{c}_1^{\varrho}\right) - \mathfrak{K}\left(\frac{\mathfrak{c}_1^{\varrho} + \mathfrak{c}_2^{\varrho}}{2}\right) + 2\mathfrak{K}\left(\mathfrak{c}_2^{\varrho}\right)\right) - \frac{\varrho^{\beta}2^{\beta-1}\Gamma(\beta+1)}{\left(\mathfrak{c}_2^{\varrho} - \mathfrak{c}_1^{\varrho}\right)^{\beta}}\left({}^{\varrho}\mathcal{K}_{\mathfrak{c}_1^+}^{\beta}\mathfrak{K}\left(\frac{\mathfrak{c}_1^{\varrho} + \mathfrak{c}_2^{\varrho}}{2}\right) + {}^{\varrho}\mathcal{K}_{\mathfrak{c}_2^-}^{\beta}\mathfrak{K}\left(\frac{\mathfrak{c}_1^{\varrho} + \mathfrak{c}_2^{\varrho}}{2}\right)\right)\right|$$
$$\leq \frac{\mathfrak{c}_2^{\varrho} - \mathfrak{c}_1^{\varrho}}{4}\left(\frac{8+\beta}{6(2+\beta)}\left|\mathfrak{K}'\left(\mathfrak{c}_1^{\varrho}\right)\right| + \frac{6+(\beta+1)(\beta+2)}{3(\beta+1)(\beta+2)}\left|\mathfrak{K}'\left(\frac{\mathfrak{c}_1^{\varrho} + \mathfrak{c}_2^{\varrho}}{2}\right)\right| + \frac{8+\beta}{6(2+\beta)}\left|\mathfrak{K}'\left(\mathfrak{c}_2^{\varrho}\right)\right|\right),$$

where ${}^{\varrho}\mathcal{K}_{\mathfrak{c}_1^+}^{\beta}\mathfrak{K}$ and ${}^{\varrho}\mathcal{K}_{\mathfrak{c}_2^-}^{\beta}\mathfrak{K}$

For related work on Katugampola fractional integral inequalities, Set and Mumcu explored Hermite–Hadamard-type inequalities for F-convex functions in [33]. In [35], Toplu et al. established similar inequalities under the framework of p-convexity. Furthermore, Gürbüz and colleagues derived Ostrowski-type inequalities for the same class of functions in [19].

In this paper, we extend the findings presented in [16] and [10] to the Katugampola fractional integrals. Our approach begins with the introduction of a new identity, from which we derive Milne-type inequalities for differentiable s-convex mappings. The outcomes of this research not only generalize previously established results but also refine some of them. The investigation concludes with an illustrative example, including graphical representations, which confirms the accuracy of our findings and discusses some applications.

2. Preliminaries

Definition 2.1 ([23]). *Let $\mathfrak{K} \in L_1[\mathfrak{c}_1, \mathfrak{c}_2]$. The fractional Riemann-Liouville integrals $J_{\mathfrak{c}_1^+}^{\beta}\mathfrak{K}$ and $J_{\mathfrak{c}_2^-}^{\beta}\mathfrak{K}$ of order $\beta > 0$ with $a \geq 0$, are defined by*

$$J_{\mathfrak{c}_1^+}^{\beta}\mathfrak{K}(x) = \frac{1}{\Gamma(\beta)}\int_{\mathfrak{c}_1}^{x}(x-z)^{\beta-1}\mathfrak{K}(z)dz, \quad x > \mathfrak{c}_1,$$
$$J_{\mathfrak{c}_2^-}^{\beta}\mathfrak{K}(x) = \frac{1}{\Gamma(\beta)}\int_{x}^{\mathfrak{c}_2}(z-x)^{\beta-1}\mathfrak{K}(z)dz, \quad \mathfrak{c}_2 > x,$$

respectively, where $\Gamma(\mu) = \int_0^{\infty} e^{-z}z^{\mu-1}\,dz$ is the Gamma function, and $J_{\mathfrak{c}_1^+}^{0}\mathfrak{K}(x) = J_{\mathfrak{c}_2^-}^{0}\mathfrak{K}(x) = \mathfrak{K}(x)$.

Definition 2.2 ([30]). *Let $\beta > 0$. The left and right Hadamard fractional integral of any order $\beta > 0$ are defined by*

$$H_{\mathfrak{c}_1^+}^{\beta}\mathfrak{K}(x) = \frac{1}{\Gamma(\beta)}\int_{\mathfrak{c}_1}^{x}(\ln x - \ln z)^{\beta-1}\frac{\mathfrak{K}(z)}{z}\,dz, \quad x > \mathfrak{c}_1,$$
$$H_{\mathfrak{c}_2^-}^{\beta}\mathfrak{K}(x) = \frac{1}{\Gamma(\beta)}\int_{x}^{\mathfrak{c}_2}(\ln z - \ln x)^{\beta-1}\frac{\mathfrak{K}(z)}{z}\,dz, \quad \mathfrak{c}_2 > x.$$

In the subsequent exposition, we define $X_{\ell}^{m}(\mathfrak{c}_1, \mathfrak{c}_2)$ (with ℓ an element of $\mathbb{R}$ and the range of m being from 1 to infinity) to represent the collection of complex-valued functions $\mathfrak{R}$ that are Lebesgue measurable and for which the norm $\|\mathfrak{R}\|_{X_{\ell}^{m}}$ remains finite. The norm is expressed as

$$\|\mathfrak{R}\|_{X_{\ell}^{m}} = \left(\int_{\mathfrak{c}_1}^{\mathfrak{c}_2} |\nu^{\ell}\mathfrak{R}(z)|^{m} dz\right)^{1/m} \text{ for } 1 \leq m < \infty,$$

and when $m = \infty$,

$$\|\mathfrak{R}\|_{X_{\ell}^{\infty}} = \operatorname*{ess\,sup}_{a \leq z \leq b} |z^{\ell}\mathfrak{R}(z)|.$$

Definition 2.3 ([20]). *Let $\mathfrak{R} \in X_{\ell}^{m}([\mathfrak{c}_1, \mathfrak{c}_2])$. The left and right side Katugampola fractional integrals of order $\beta \in \mathbb{C}$ with $Re(\beta) > 0$ and $\varrho > 0$, are defined by*

$${}^{\varrho}\mathcal{K}_{\mathfrak{c}_1^+}^{\beta}\mathfrak{R}(x) = \frac{\varrho^{1-\beta}}{\Gamma(\beta)} \int_{\mathfrak{c}_1}^{x} \frac{z^{\varrho-1}\mathfrak{R}(z)}{(x^{\varrho} - z^{\varrho})^{1-\beta}}\, dz, \quad x > \mathfrak{c}_1$$

and

$${}^{\varrho}\mathcal{K}_{\mathfrak{c}_2^-}^{\beta}\mathfrak{R}(x) = \frac{\varrho^{1-\beta}}{\Gamma(\beta)} \int_{x}^{\mathfrak{c}_2} \frac{z^{\varrho-1}\mathfrak{R}(z)}{(z^{\varrho} - x^{\varrho})^{1-\beta}}\, dz, \quad \mathfrak{c}_2 > x,$$

respectively.

Remark 2.4. *Given the conditions $\beta, \varrho > 0$ and for $x > a$, it is noted that:*

1. *When ϱ tends to 1, the Katugampola fractional integral, denoted as ${}^{\varrho}\mathcal{O}_{\mathfrak{c}_1^+}^{\beta}\mathfrak{R}(x)$, converges to the Riemann-Liouville fractional integral, represented by $\mathcal{J}_{\mathfrak{c}_1^+}^{\beta}\mathfrak{R}(x)$.*
2. *Conversely, as ϱ nears 0^+, the convergence of ${}^{\varrho}\mathcal{O}_{\mathfrak{c}_1^+}^{\beta}\mathfrak{R}(x)$ aligns with that of the Hadamard fractional integral, $\mathcal{H}_{\mathfrak{c}_1^+}^{\beta}\mathfrak{R}(x)$.*

This observation similarly applies to related results for right-sided operators.

Definition 2.5 ([23]). *For any complex numbers and nonpositive integers x, y such that $Re(x) > 0$ and $Re(y) > 0$. The beta function is defined by*

$$B(x, y) = \int_{0}^{1} z^{x-1}(1-z)^{y-1}\, dz = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

where $\Gamma(.)$ is the Euler gamma function.

Definition 2.6 ([23]). *The hypergeometric function is defined, as follows*

$${}_2F_1(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{c}_3; v) = \tfrac{1}{B(\mathfrak{c}_2, \mathfrak{c}_3-\mathfrak{c}_2)} \int_{0}^{1} z^{\mathfrak{c}_2-1}(1-z)^{\mathfrak{c}_3-\mathfrak{c}_2-1}(1-vz)^{-\mathfrak{c}_1}\, dz,$$

where $\mathfrak{c}_3 > \mathfrak{c}_2 > 0, |v| < 1$ and $B(.,.)$ is the beta function.

3. Main results

In order to prove our results, we need the following lemma.

Lemma 3.1. *Let $\Re:\left[\mathfrak{c}_1^\varrho,\mathfrak{c}_2^\varrho\right]\to\mathbb{R}$ be a differentiable function on $\left(\mathfrak{c}_1^\varrho,\mathfrak{c}_2^\varrho\right)$ with $0\le\mathfrak{c}_1<\mathfrak{c}_2$, and $\Re'\in L^1\left[\mathfrak{c}_1^\varrho,\mathfrak{c}_2^\varrho\right]$, then the following equality holds*

$$\begin{aligned}
&\frac{1}{3}\left(2\Re\left(\mathfrak{c}_1^\varrho\right)-\Re\left(\frac{\mathfrak{c}_1^\varrho+\mathfrak{c}_2^\varrho}{2}\right)+2\Re\left(\mathfrak{c}_2^\varrho\right)\right)-\frac{\varrho^\beta 2^{\beta-1}\Gamma(\beta+1)}{\left(\mathfrak{c}_2^\varrho-\mathfrak{c}_1^\varrho\right)^\beta}\,{}^\varrho O^\beta(\Re)\\
&=\frac{\varrho\left(\mathfrak{c}_2^\varrho-\mathfrak{c}_1^\varrho\right)}{4}\left(\int_0^1\left(\nu^{\varrho\beta}-\frac{4}{3}\right)\nu^{\varrho-1}\Re'\left((1-\nu^\varrho)\,\mathfrak{c}_1^\varrho+\nu^\varrho\frac{\mathfrak{c}_1^\varrho+\mathfrak{c}_2^\varrho}{2}\right)d\nu\right.\\
&\quad\left.+\int_0^1\left(\frac{4}{3}-\nu^{\varrho\beta}\right)\nu^{\varrho-1}\Re'\left(\nu^\varrho\frac{\mathfrak{c}_1^\varrho+\mathfrak{c}_2^\varrho}{2}+(1-\nu^\varrho)\,\mathfrak{c}_2^\varrho\right)d\nu\right),
\end{aligned}\tag{1}$$

where $\beta,\varrho>0$ and

$${}^\varrho O^\beta(\Re)={}^\varrho\mathcal{K}^\beta_{\left(\left(\frac{\mathfrak{c}_1^\varrho+\mathfrak{c}_2^\varrho}{2}\right)^{\frac{1}{\varrho}}\right)^-}\Re\left(\mathfrak{c}_1^\varrho\right)+{}^\varrho\mathcal{K}^\beta_{\left(\left(\frac{\mathfrak{c}_1^\varrho+\mathfrak{c}_2^\varrho}{2}\right)^{\frac{1}{\varrho}}\right)^+}\Re\left(\mathfrak{c}_2^\varrho\right).\tag{2}$$

Proof. Let

$$O_1=\int_0^1\left(\nu^{\varrho\beta}-\frac{4}{3}\right)\nu^{\varrho-1}\Re'\left((1-\nu^\varrho)\,\mathfrak{c}_1^\varrho+\nu^\varrho\frac{\mathfrak{c}_1^\varrho+\mathfrak{c}_2^\varrho}{2}\right)d\nu$$

and

$$O_2=\int_0^1\left(\frac{4}{3}-\nu^{\varrho\beta}\right)\nu^{\varrho-1}\Re'\left(\nu^\varrho\frac{\mathfrak{c}_1^\varrho+\mathfrak{c}_2^\varrho}{2}+(1-\nu^\varrho)\,\mathfrak{c}_2^\varrho\right)d\nu.$$

Integrating by parts O_1, we get

$$\begin{aligned}
O_1&=\frac{2}{\varrho\left(\mathfrak{c}_2^\varrho-\mathfrak{c}_1^\varrho\right)}\left(\nu^{\varrho\beta}-\frac{4}{3}\right)\Re\left((1-\nu^\varrho)\,\mathfrak{c}_1^\varrho+\nu^\varrho\frac{\mathfrak{c}_1^\varrho+\mathfrak{c}_2^\varrho}{2}\right)\Bigg|_{\nu=0}^{\nu=1}-\frac{2\beta}{\mathfrak{c}_2^\varrho-\mathfrak{c}_1^\varrho}\int_0^1\nu^{\varrho\beta-1}\Re\left(\nu^\varrho\mathfrak{c}_1^\varrho+(1-\nu^\varrho)\,\frac{\mathfrak{c}_1^\varrho+\mathfrak{c}_2^\varrho}{2}\right)d\nu\\
&=-\frac{2}{3\varrho\left(\mathfrak{c}_2^\varrho-\mathfrak{c}_1^\varrho\right)}\Re\left(\frac{\mathfrak{c}_1^\varrho+\mathfrak{c}_2^\varrho}{2}\right)+\frac{8}{3\varrho\left(\mathfrak{c}_2^\varrho-\mathfrak{c}_1^\varrho\right)}\Re\left(\mathfrak{c}_1^\varrho\right)-\frac{2^{\beta+1}\beta}{\left(\mathfrak{c}_2^\varrho-\mathfrak{c}_1^\varrho\right)^{\beta+1}}\int_{\mathfrak{c}_1}^{\left(\frac{\mathfrak{c}_1^\varrho+\mathfrak{c}_2^\varrho}{2}\right)^{\frac{1}{\varrho}}}\left(u^\varrho-\mathfrak{c}_1^\varrho\right)^{\beta-1}\Re\left(u^\varrho\right)u^{\varrho-1}du\\
&=\frac{8}{3\varrho\left(\mathfrak{c}_2^\varrho-\mathfrak{c}_1^\varrho\right)}\Re\left(\mathfrak{c}_1^\varrho\right)-\frac{2}{3\varrho\left(\mathfrak{c}_2^\varrho-\mathfrak{c}_1^\varrho\right)}\Re\left(\frac{\mathfrak{c}_1^\varrho+\mathfrak{c}_2^\varrho}{2}\right)-\frac{\varrho^{\beta-1}2^{\beta+1}\Gamma(\beta+1)}{\left(\mathfrak{c}_2^\varrho-\mathfrak{c}_1^\varrho\right)^{\beta+1}}\,{}^\varrho\mathcal{K}^\beta_{\left(\left(\frac{\mathfrak{c}_1^\varrho+\mathfrak{c}_2^\varrho}{2}\right)^{\frac{1}{\varrho}}\right)^-}\Re\left(\mathfrak{c}_1^\varrho\right).
\end{aligned}\tag{3}$$

Similarly, we have

$$O_2=-\frac{2}{\varrho\left(\mathfrak{c}_2^\varrho-\mathfrak{c}_1^\varrho\right)}\left(\frac{4}{3}-\nu^{\varrho\beta}\right)\Re\left(\nu^\varrho\frac{\mathfrak{c}_1^\varrho+\mathfrak{c}_2^\varrho}{2}+(1-\nu^\varrho)\,\mathfrak{c}_2^\varrho\right)\Bigg|_{\nu=0}^{\nu=1}$$

$$-\frac{2\beta}{\left(\mathfrak{c}_2^{\varrho}-\mathfrak{c}_1^{\varrho}\right)}\int_0^1 \nu^{\varrho\beta-1}\Re\left((1-\nu^{\varrho})\frac{\mathfrak{c}_1^{\varrho}+\mathfrak{c}_2^{\varrho}}{2}+\nu^{\varrho}\mathfrak{c}_2^{\varrho}\right)d\nu$$

$$=-\frac{2}{3\varrho\left(\mathfrak{c}_2^{\varrho}-\mathfrak{c}_1^{\varrho}\right)}\Re\left(\frac{\mathfrak{c}_1^{\varrho}+\mathfrak{c}_2^{\varrho}}{2}\right)+\frac{8}{3\varrho\left(\mathfrak{c}_2^{\varrho}-\mathfrak{c}_1^{\varrho}\right)}\Re\left(\mathfrak{c}_2^{\varrho}\right)-\frac{2^{\beta+1}\beta}{\left(\mathfrak{c}_2^{\varrho}-\mathfrak{c}_1^{\varrho}\right)^{\beta+1}}\int_{\left(\frac{\mathfrak{c}_1^{\varrho}+\mathfrak{c}_2^{\varrho}}{2}\right)^{\frac{1}{\varrho}}}^{\mathfrak{c}_2}\left(\mathfrak{c}_2^{\varrho}-u^{\varrho}\right)^{\beta-1}\Re\left(u^{\varrho}\right)u^{\varrho-1}du$$

$$=-\frac{2}{3\varrho\left(\mathfrak{c}_2^{\varrho}-\mathfrak{c}_1^{\varrho}\right)}\Re\left(\frac{\mathfrak{c}_1^{\varrho}+\mathfrak{c}_2^{\varrho}}{2}\right)+\frac{8}{3\varrho\left(\mathfrak{c}_2^{\varrho}-\mathfrak{c}_1^{\varrho}\right)}\Re\left(\mathfrak{c}_2^{\varrho}\right)-\frac{\varrho^{\beta-1}2^{\beta+1}\Gamma(\beta+1)}{\left(\mathfrak{c}_2^{\varrho}-\mathfrak{c}_1^{\varrho}\right)^{\beta+1}}\,{}^{\varrho}\mathcal{K}^{\beta}_{\left(\left(\frac{\mathfrak{c}_1^{\varrho}+\mathfrak{c}_2^{\varrho}}{2}\right)^{\frac{1}{\varrho}}\right)^{+}}\Re\left(\mathfrak{c}_2^{\varrho}\right). \tag{4}$$

Summing (3) and (4), then multiplying the resulting equality by $\frac{\varrho\left(\mathfrak{c}_2^{\varrho}-\mathfrak{c}_1^{\varrho}\right)}{4}$, we get the desired result. □

Remark 3.2. *By setting $\varrho=1$, Lemma 3.1 will be reduced to Lemma 2.1 from [10].*

Theorem 3.3. *Let $\Re:\left[\mathfrak{c}_1^{\varrho},\mathfrak{c}_2^{\varrho}\right]\to\mathbb{R}$ be a differentiable function on $\left(\mathfrak{c}_1^{\varrho},\mathfrak{c}_2^{\varrho}\right)$ with $0\le\mathfrak{c}_1<\mathfrak{c}_2$, and $\Re'\in L^1\left[\mathfrak{c}_1^{\varrho},\mathfrak{c}_2^{\varrho}\right]$. If $|\Re'|$ is s-convex in the second sense for some fixed $s\in(0,1]$, then we have*

$$\left|\frac{1}{3}\left(2\Re\left(\mathfrak{c}_1^{\varrho}\right)-\Re\left(\frac{\mathfrak{c}_1^{\varrho}+\mathfrak{c}_2^{\varrho}}{2}\right)+2\Re\left(\mathfrak{c}_2^{\varrho}\right)\right)-\frac{\varrho^{\beta}2^{\beta-1}\Gamma(\beta+1)}{\left(\mathfrak{c}_2^{\varrho}-\mathfrak{c}_1^{\varrho}\right)^{\beta}}\,{}^{\varrho}O^{\beta}(\Re)\right|$$
$$\le\frac{\mathfrak{c}_2^{\varrho}-\mathfrak{c}_1^{\varrho}}{4}\left(\left(\frac{4}{3(s+1)}-B(\beta+1,s+1)\right)\left(\left|\Re'\left(\mathfrak{c}_1^{\varrho}\right)\right|+\left|\Re'\left(\mathfrak{c}_2^{\varrho}\right)\right|\right)+\frac{2(4\beta+s+1)}{3(s+1)(\beta+s+1)}\left|\Re'\left(\frac{\mathfrak{c}_1^{\varrho}+\mathfrak{c}_2^{\varrho}}{2}\right)\right|\right),$$

where $\beta,\varrho>0$, ${}^{\varrho}O^{\beta}(\Re)$ is defined as in (2) and $B(.,.)$ is the beta function.

Proof. By taking the absolute value of both sides of equality 1, then applying the s-convexity of $|\Re'|$, we have

$$\left|\frac{1}{3}\left(2\Re\left(\mathfrak{c}_1^{\varrho}\right)-\Re\left(\frac{\mathfrak{c}_1^{\varrho}+\mathfrak{c}_2^{\varrho}}{2}\right)+2\Re\left(\mathfrak{c}_2^{\varrho}\right)\right)-\frac{\varrho^{\beta}2^{\beta-1}\Gamma(\beta+1)}{\left(\mathfrak{c}_2^{\varrho}-\mathfrak{c}_1^{\varrho}\right)^{\beta}}\,{}^{\varrho}O^{\beta}(\Re)\right|$$
$$\le\frac{\varrho\left(\mathfrak{c}_2^{\varrho}-\mathfrak{c}_1^{\varrho}\right)}{4}\left(\int_0^1\left(\frac{4}{3}-\nu^{\varrho\beta}\right)\nu^{\varrho-1}\left|\Re'\left((1-\nu^{\varrho})\mathfrak{c}_1^{\varrho}+\nu^{\varrho}\frac{\mathfrak{c}_1^{\varrho}+\mathfrak{c}_2^{\varrho}}{2}\right)\right|d\nu\right.$$
$$\left.+\int_0^1\left(\frac{4}{3}-\nu^{\varrho\beta}\right)\nu^{\varrho-1}\left|\Re'\left(\nu^{\varrho}\frac{\mathfrak{c}_1^{\varrho}+\mathfrak{c}_2^{\varrho}}{2}+(1-\nu^{\varrho})\mathfrak{c}_2^{\varrho}\right)\right|d\nu\right)$$
$$\le\frac{\varrho\left(\mathfrak{c}_2^{\varrho}-\mathfrak{c}_1^{\varrho}\right)}{4}\left(\int_0^1\left(\frac{4}{3}-\nu^{\varrho\beta}\right)\nu^{\varrho-1}\left((1-\nu^{\varrho})^{s}\left|\Re'\left(\mathfrak{c}_1^{\varrho}\right)\right|+\nu^{s\varrho}\left|\Re'\left(\frac{\mathfrak{c}_1^{\varrho}+\mathfrak{c}_2^{\varrho}}{2}\right)\right|\right)d\nu\right.$$
$$\left.+\int_0^1\left(\frac{4}{3}-\nu^{\varrho\beta}\right)\nu^{\varrho-1}\left(\nu^{s\varrho}\left|\Re'\left(\frac{\mathfrak{c}_1^{\varrho}+\mathfrak{c}_2^{\varrho}}{2}\right)\right|+(1-\nu^{\varrho})^{s}\left|\Re'\left(\mathfrak{c}_2^{\varrho}\right)\right|\right)d\nu\right)$$
$$=\frac{\varrho\left(\mathfrak{c}_2^{\varrho}-\mathfrak{c}_1^{\varrho}\right)}{4}\left(\left(\left|\Re'\left(\mathfrak{c}_1^{\varrho}\right)\right|+\left|\Re'\left(\mathfrak{c}_2^{\varrho}\right)\right|\right)\int_0^1\left(\tfrac{4}{3}-\nu^{\varrho\beta}\right)\nu^{\varrho-1}(1-\nu^{\varrho})^{s}d\nu+2\left|\Re'\left(\frac{\mathfrak{c}_1^{\varrho}+\mathfrak{c}_2^{\varrho}}{2}\right)\right|\int_0^1\left(\frac{4}{3}-\nu^{\varrho\beta}\right)\nu^{\varrho-1}\nu^{\varrho s}d\nu\right)$$
$$=\frac{\mathfrak{c}_2^{\varrho}-\mathfrak{c}_1^{\varrho}}{4}\left(\left(\frac{4}{3(s+1)}-B(\beta+1,s+1)\right)\left(\left|\Re'\left(\mathfrak{c}_1^{\varrho}\right)\right|+\left|\Re'\left(\mathfrak{c}_2^{\varrho}\right)\right|\right)+\frac{2(4\beta+s+1)}{3(s+1)(\beta+s+1)}\left|\Re'\left(\frac{\mathfrak{c}_1^{\varrho}+\mathfrak{c}_2^{\varrho}}{2}\right)\right|\right),$$

where we have used the facts that

$$\int_0^1 \left(\frac{4}{3} - \nu^{\varrho\beta}\right)\nu^{\varrho-1}\left(1-\nu^{\varrho}\right)^s d\nu = \frac{4}{3\varrho(s+1)} - \frac{1}{\varrho}B(\beta+1, s+1) \tag{5}$$

and

$$\int_0^1 \left(\frac{4}{3} - \nu^{\varrho\beta}\right)\nu^{\varrho-1}\nu^{\varrho s} d\nu = \frac{4\beta+s+1}{3\varrho(s+1)(\beta+s+1)}. \tag{6}$$

This completes the proof. □

Corollary 3.4. *By using the s-convexity of* $|\mathfrak{R}'|$, *i.e:* $\left|\mathfrak{R}'\left(\frac{\mathfrak{c}_1^{\varrho}+\mathfrak{c}_2^{\varrho}}{2}\right)\right| \leq \frac{\left|\mathfrak{R}'(\mathfrak{c}_1^{\varrho})\right|+\left|\mathfrak{R}'(\mathfrak{c}_2^{\varrho})\right|}{2^s}$, *we get from Theorem 3.3*

$$\begin{aligned}&\left|\frac{1}{3}\left(2\mathfrak{R}\left(\mathfrak{c}_1^{\varrho}\right) - \mathfrak{R}\left(\frac{\mathfrak{c}_1^{\varrho}+\mathfrak{c}_2^{\varrho}}{2}\right) + 2\mathfrak{R}\left(\mathfrak{c}_2^{\varrho}\right)\right) - \frac{\varrho^{\beta}2^{\beta-1}\Gamma(\beta+1)}{\left(\mathfrak{c}_2^{\varrho}-\mathfrak{c}_1^{\varrho}\right)^{\beta}}\,{}^{\varrho}O^{\beta}(\mathfrak{R})\right|\\ &\leq \frac{\mathfrak{c}_2^{\varrho}-\mathfrak{c}_1^{\varrho}}{4}\left(\left(\frac{4(\beta+s+1)+2^{1-s}(4\beta+s+1)}{3(s+1)(\beta+s+1)} - B(\beta+1,s+1)\right)\left(\left|\mathfrak{R}'\left(\mathfrak{c}_1^{\varrho}\right)\right| + \left|\mathfrak{R}'\left(\mathfrak{c}_2^{\varrho}\right)\right|\right)\right).\end{aligned}$$

Corollary 3.5. *If we attempt to take* $s = 1$, *Theorem 3.3 yields*

$$\begin{aligned}&\left|\frac{1}{3}\left(2\mathfrak{R}\left(\mathfrak{c}_1^{\varrho}\right) - \mathfrak{R}\left(\frac{\mathfrak{c}_1^{\varrho}+\mathfrak{c}_2^{\varrho}}{2}\right) + 2\mathfrak{R}\left(\mathfrak{c}_2^{\varrho}\right)\right) - \frac{\varrho^{\beta}2^{\beta-1}\Gamma(\beta+1)}{\left(\mathfrak{c}_2^{\varrho}-\mathfrak{c}_1^{\varrho}\right)^{\beta}}\,{}^{\varrho}O^{\beta}(\mathfrak{R})\right|\\ &\leq \frac{\mathfrak{c}_2^{\varrho}-\mathfrak{c}_1^{\varrho}}{12}\left(\frac{2\beta^2+6\beta+1}{(\beta+1)(\beta+2)}\left|\mathfrak{R}'\left(\mathfrak{c}_1^{\varrho}\right)\right| + \frac{4\beta+2}{\beta+2}\left|\mathfrak{R}'\left(\frac{\mathfrak{c}_1^{\varrho}+\mathfrak{c}_2^{\varrho}}{2}\right)\right| + \frac{2\beta^2+6\beta+1}{(\beta+1)(\beta+2)}\left|\mathfrak{R}'\left(\mathfrak{c}_2^{\varrho}\right)\right|\right).\end{aligned}$$

Moreover, using the fact that $\left|\mathfrak{R}'\left(\frac{\mathfrak{c}_1^{\varrho}+\mathfrak{c}_2^{\varrho}}{2}\right)\right| \leq \frac{\left|\mathfrak{R}'(\mathfrak{c}_1^{\varrho})\right|+\left|\mathfrak{R}'(\mathfrak{c}_2^{\varrho})\right|}{2}$, *we get*

$$\begin{aligned}&\left|\frac{1}{3}\left(2\mathfrak{R}\left(\mathfrak{c}_1^{\varrho}\right) - \mathfrak{R}\left(\frac{\mathfrak{c}_1^{\varrho}+\mathfrak{c}_2^{\varrho}}{2}\right) + 2\mathfrak{R}\left(\mathfrak{c}_2^{\varrho}\right)\right) - \frac{\varrho^{\beta}2^{\beta-1}\Gamma(\beta+1)}{\left(\mathfrak{c}_2^{\varrho}-\mathfrak{c}_1^{\varrho}\right)^{\beta}}\,{}^{\varrho}O^{\beta}(\mathfrak{R})\right|\\ &\leq \frac{\mathfrak{c}_2^{\varrho}-\mathfrak{c}_1^{\varrho}}{12}\left(\frac{4\beta^2+9\beta+2}{(\beta+1)(\beta+2)}\left[\left|\mathfrak{R}'\left(\mathfrak{c}_1^{\varrho}\right)\right| + \left|\mathfrak{R}'\left(\mathfrak{c}_2^{\varrho}\right)\right|\right]\right).\end{aligned}$$

Corollary 3.6. *By setting* $\varrho = 1$, *Theorem 3.3 gives*

$$\begin{aligned}&\left|\frac{1}{3}\left(2\mathfrak{R}(\mathfrak{c}_1) - \mathfrak{R}\left(\frac{\mathfrak{c}_1+\mathfrak{c}_2}{2}\right) + 2\mathfrak{R}(\mathfrak{c}_2)\right) - \frac{2^{\beta-1}\Gamma(\beta+1)}{(\mathfrak{c}_2-\mathfrak{c}_1)^{\beta}}\mathcal{J}^{\beta}(\mathfrak{R})\right|\\ &\leq \frac{\mathfrak{c}_2-\mathfrak{c}_1}{4}\left(\left(\frac{4}{3(s+1)} - B(\beta+1,s+1)\right)\left(\left|\mathfrak{R}'(\mathfrak{c}_1)\right| + \left|\mathfrak{R}'(\mathfrak{c}_2)\right|\right) + \frac{2(4\beta+s+1)}{3(s+1)(\beta+s+1)}\left|\mathfrak{R}'\left(\frac{\mathfrak{c}_1+\mathfrak{c}_2}{2}\right)\right|\right),\end{aligned}$$

where

$$\mathcal{J}^{\beta}(\mathfrak{R}) = J^{\beta}_{\left(\frac{\mathfrak{c}_1+\mathfrak{c}_2}{2}\right)^-}\mathfrak{R}(\mathfrak{c}_1) + J^{\beta}_{\left(\frac{\mathfrak{c}_1+\mathfrak{c}_2}{2}\right)^+}\mathfrak{R}(b). \tag{7}$$

Remark 3.7. *It is noteworthy to mention that under certain parameter conditions, the results presented in Theorem 3.3 align closely with established findings in the literature. Specifically,*

1. *Corollary 3.6 is effectively rendered equivalent to Theorem 2.2 as presented in [10] when one sets $s = 1$.*
2. *By setting $\beta = 1$, Corollary 3.6 directly corresponds to the result established in Theorem 2.2 from [16], indicating an equivalence in findings under these specific conditions.*

Theorem 3.8. *Let $\Re : \left[\mathfrak{c}_1^{\varrho}, \mathfrak{c}_2^{\varrho}\right] \to \mathbb{R}$ be a differentiable function on $\left(\mathfrak{c}_1^{\varrho}, \mathfrak{c}_2^{\varrho}\right)$ with $0 \leq \mathfrak{c}_1 < \mathfrak{c}_2$, and $\Re' \in L^1\left[\mathfrak{c}_1^{\varrho}, \mathfrak{c}_2^{\varrho}\right]$. If $|\Re'|^{\kappa}$ is s-convex in the second sense for some fixed $s \in (0, 1]$ where $\kappa > 1$ with $\frac{1}{\lambda} + \frac{1}{\kappa} = 1$, then we have*

$$\left|\frac{1}{3}\left(2\Re\left(\mathfrak{c}_1^{\varrho}\right) - \Re\left(\frac{\mathfrak{c}_1^{\varrho} + \mathfrak{c}_2^{\varrho}}{2}\right) + 2\Re\left(\mathfrak{c}_2^{\varrho}\right)\right) - \frac{\varrho^{\beta}2^{\beta-1}\Gamma(\beta+1)}{\left(\mathfrak{c}_2^{\varrho} - \mathfrak{c}_1^{\varrho}\right)^{\beta}}\, {}^{\varrho}O^{\beta}(\Re)\right|$$
$$\leq \frac{\mathfrak{c}_2^{\varrho} - \mathfrak{c}_1^{\varrho}}{3}\left({}_2F_1\left(-\lambda, \tfrac{1}{\beta}, \tfrac{1}{\beta}+1; \tfrac{3}{4}\right)\right)^{\frac{1}{\lambda}}\left(\left(\frac{\left|\Re'\left(\mathfrak{c}_1^{\varrho}\right)\right|^{\kappa} + \left|\Re'\left(\frac{\mathfrak{c}_1^{\varrho}+\mathfrak{c}_2^{\varrho}}{2}\right)\right|^{\kappa}}{s+1}\right)^{\frac{1}{\kappa}} + \left(\frac{\left|\Re'\left(\frac{\mathfrak{c}_1^{\varrho}+\mathfrak{c}_2^{\varrho}}{2}\right)\right|^{\kappa} + \left|\Re'\left(\mathfrak{c}_2^{\varrho}\right)\right|^{\kappa}}{s+1}\right)^{\frac{1}{\kappa}}\right),$$

where $\beta, \varrho > 0$, ${}^{\varrho}O^{\beta}(\Re)$ is defined as in (2) *and ${}_2F_1(., ., .; .)$ is the hypergeometric function.*

Proof. From Lemma 3.1, properties of modulus, Hölder's inequality, and s-convexity in the second sense of $|\Re'|^{\kappa}$, we have

$$\left|\frac{1}{3}\left(2\Re\left(\mathfrak{c}_1^{\varrho}\right) - \Re\left(\frac{\mathfrak{c}_1^{\varrho} + \mathfrak{c}_2^{\varrho}}{2}\right) + 2\Re\left(\mathfrak{c}_2^{\varrho}\right)\right) - \frac{\varrho^{\beta}2^{\beta-1}\Gamma(\beta+1)}{\left(\mathfrak{c}_2^{\varrho} - \mathfrak{c}_1^{\varrho}\right)^{\beta}}\, {}^{\varrho}O^{\beta}(\Re)\right|$$
$$\leq \frac{\varrho\left(\mathfrak{c}_2^{\varrho} - \mathfrak{c}_1^{\varrho}\right)}{4}\left(\int_0^1\left(\frac{4}{3} - \nu^{\varrho\beta}\right)\nu^{\varrho-1}\left|\Re'\left((1-\nu^{\varrho})\,\mathfrak{c}_1^{\varrho} + \nu^{\varrho}\frac{\mathfrak{c}_1^{\varrho} + \mathfrak{c}_2^{\varrho}}{2}\right)\right| d\nu\right.$$
$$\left. + \int_0^1\left(\frac{4}{3} - \nu^{\varrho\beta}\right)\nu^{\varrho-1}\left|\Re'\left(\nu^{\varrho}\frac{\mathfrak{c}_1^{\varrho} + \mathfrak{c}_2^{\varrho}}{2} + (1-\nu^{\varrho})\,\mathfrak{c}_2^{\varrho}\right)\right| d\nu\right)$$
$$= \frac{\mathfrak{c}_2^{\varrho} - \mathfrak{c}_1^{\varrho}}{4}\left(\int_0^1\left(\frac{4}{3} - \nu^{\beta}\right)\left|\Re'\left((1-v)\,\mathfrak{c}_1^{\varrho} + v\frac{\mathfrak{c}_1^{\varrho} + \mathfrak{c}_2^{\varrho}}{2}\right)\right| d\nu + \int_0^1\left(\frac{4}{3} - \nu^{\beta}\right)\left|\Re'\left(v\frac{\mathfrak{c}_1^{\varrho} + \mathfrak{c}_2^{\varrho}}{2} + (1-v)\,\mathfrak{c}_2^{\varrho}\right)\right| d\nu\right)$$
$$\leq \frac{\mathfrak{c}_2^{\varrho} - \mathfrak{c}_1^{\varrho}}{4}\left(\left(\int_0^1\left(\frac{4}{3} - \nu^{\beta}\right)^{\lambda} d\nu\right)^{\frac{1}{\lambda}}\left(\int_0^1\left|\Re'\left((1-v)\,\mathfrak{c}_1^{\varrho} + v\frac{\mathfrak{c}_1^{\varrho} + \mathfrak{c}_2^{\varrho}}{2}\right)\right|^{\kappa} d\nu\right)^{\frac{1}{\kappa}}\right.$$
$$\left. + \left(\int_0^1\left(\frac{4}{3} - \nu^{\beta}\right)^{\lambda} d\nu\right)^{\frac{1}{\lambda}}\left(\int_0^1\left|\Re'\left(v\frac{\mathfrak{c}_1^{\varrho} + \mathfrak{c}_2^{\varrho}}{2} + (1-v)\,\mathfrak{c}_2^{\varrho}\right)\right|^{\kappa} d\nu\right)^{\frac{1}{\kappa}}\right)$$
$$\leq \frac{\mathfrak{c}_2^{\varrho} - \mathfrak{c}_1^{\varrho}}{4}\left(\left(\frac{4}{3}\right)^{\lambda}\, {}_2F_1\left(-\lambda, \tfrac{1}{\beta}, \tfrac{1}{\beta}+1; \tfrac{3}{4}\right)\right)^{\frac{1}{\lambda}}$$
$$\times\left(\left(\int_0^1\left((1-v)^s\left|\Re'\left(\mathfrak{c}_1^{\varrho}\right)\right|^{\kappa} + \nu^s\left|\Re'\left(\frac{\mathfrak{c}_1^{\varrho} + \mathfrak{c}_2^{\varrho}}{2}\right)\right|^{\kappa}\right) d\nu\right)^{\frac{1}{\kappa}} + \left(\int_0^1\left(\nu^s\left|\Re'\left(\frac{\mathfrak{c}_1^{\varrho} + \mathfrak{c}_2^{\varrho}}{2}\right)\right|^{\kappa} + (1-v)^s\left|\Re'\left(\mathfrak{c}_2^{\varrho}\right)\right|^{\kappa}\right) d\nu\right)^{\frac{1}{\kappa}}\right)$$

$$=\frac{\mathfrak{c}_2^{\varrho}-\mathfrak{c}_1^{\varrho}}{3}\left({}_2F_1\left(-\lambda,\tfrac{1}{\beta},\tfrac{1}{\beta}+1;\tfrac{3}{4}\right)\right)^{\frac{1}{\lambda}}\left(\left(\frac{\left|\mathfrak{K}'\left(\mathfrak{c}_1^{\varrho}\right)\right|^{\kappa}+\left|\mathfrak{K}'\left(\frac{\mathfrak{c}_1^{\varrho}+\mathfrak{c}_2^{\varrho}}{2}\right)\right|^{\kappa}}{s+1}\right)^{\frac{1}{\kappa}}+\left(\frac{\left|\mathfrak{K}'\left(\frac{\mathfrak{c}_1^{\varrho}+\mathfrak{c}_2^{\varrho}}{2}\right)\right|^{\kappa}+\left|\mathfrak{K}'\left(\mathfrak{c}_2^{\varrho}\right)\right|^{\kappa}}{s+1}\right)^{\frac{1}{\kappa}}\right),$$

where we have used

$$\int_0^1\left(\frac{4}{3}-\nu^{\beta}\right)^{\lambda}d\nu=\frac{4^{\lambda}}{3^{\lambda}\beta}\int_0^1 y^{\frac{1}{\beta}-1}\left(1-\frac{3}{4}y\right)^{\lambda}dy=\left(\frac{4}{3}\right)^{\lambda}\,{}_2F_1\left(-\lambda,\tfrac{1}{\beta},\tfrac{1}{\beta}+1;\tfrac{3}{4}\right).$$

This completes the proof. □

Corollary 3.9. *By using the s-convexity of $|\mathfrak{K}'|^{\kappa}$, we get from Theorem 3.8*

$$\left|\frac{1}{3}\left(2\mathfrak{K}\left(\mathfrak{c}_1^{\varrho}\right)-\mathfrak{K}\left(\frac{\mathfrak{c}_1^{\varrho}+\mathfrak{c}_2^{\varrho}}{2}\right)+2\mathfrak{K}\left(\mathfrak{c}_2^{\varrho}\right)\right)-\frac{\varrho^{\beta}2^{\beta-1}\Gamma(\beta+1)}{\left(\mathfrak{c}_2^{\varrho}-\mathfrak{c}_1^{\varrho}\right)^{\beta}}\,{}^{\varrho}\mathcal{O}^{\beta}(\mathfrak{K})\right|$$
$$\leq\frac{\mathfrak{c}_2^{\varrho}-\mathfrak{c}_1^{\varrho}}{3}\left({}_2F_1\left(-\lambda,\tfrac{1}{\beta},\tfrac{1}{\beta}+1;\tfrac{3}{4}\right)\right)^{\frac{1}{\lambda}}\left(\left(\frac{(1+2^s)\left|\mathfrak{K}'\left(\mathfrak{c}_1^{\varrho}\right)\right|^{\kappa}+\left|\mathfrak{K}'\left(\mathfrak{c}_2^{\varrho}\right)\right|^{\kappa}}{2^s(s+1)}\right)^{\frac{1}{\kappa}}+\left(\frac{\left|\mathfrak{K}'\left(\mathfrak{c}_1^{\varrho}\right)\right|^{\kappa}+(1+2^s)\left|\mathfrak{K}'\left(\mathfrak{c}_2^{\varrho}\right)\right|^{\kappa}}{2^s(s+1)}\right)^{\frac{1}{\kappa}}\right).$$

Corollary 3.10. *If we attempt to take $s=1$, Theorem 3.8 becomes*

$$\left|\frac{1}{3}\left(2\mathfrak{K}\left(\mathfrak{c}_1^{\varrho}\right)-\mathfrak{K}\left(\frac{\mathfrak{c}_1^{\varrho}+\mathfrak{c}_2^{\varrho}}{2}\right)+2\mathfrak{K}\left(\mathfrak{c}_2^{\varrho}\right)\right)-\frac{\varrho^{\beta}2^{\beta-1}\Gamma(\beta+1)}{\left(\mathfrak{c}_2^{\varrho}-\mathfrak{c}_1^{\varrho}\right)^{\beta}}\,{}^{\varrho}\mathcal{O}^{\beta}(\mathfrak{K})\right|$$
$$\leq\frac{\mathfrak{c}_2^{\varrho}-\mathfrak{c}_1^{\varrho}}{3}\left({}_2F_1\left(-\lambda,\tfrac{1}{\beta},\tfrac{1}{\beta}+1;\tfrac{3}{4}\right)\right)^{\frac{1}{\lambda}}\left(\left(\frac{\left|\mathfrak{K}'\left(\mathfrak{c}_1^{\varrho}\right)\right|^{\kappa}+\left|\mathfrak{K}'\left(\frac{\mathfrak{c}_1^{\varrho}+\mathfrak{c}_2^{\varrho}}{2}\right)\right|^{\kappa}}{2}\right)^{\frac{1}{\kappa}}+\left(\frac{\left|\mathfrak{K}'\left(\frac{\mathfrak{c}_1^{\varrho}+\mathfrak{c}_2^{\varrho}}{2}\right)\right|^{\kappa}+\left|\mathfrak{K}'\left(\mathfrak{c}_2^{\varrho}\right)\right|^{\kappa}}{2}\right)^{\frac{1}{\kappa}}\right).$$

Moreover, using the convexity of $|\mathfrak{K}'|^{\kappa}$, we get

$$\left|\frac{1}{3}\left(2\mathfrak{K}\left(\mathfrak{c}_1^{\varrho}\right)-\mathfrak{K}\left(\frac{\mathfrak{c}_1^{\varrho}+\mathfrak{c}_2^{\varrho}}{2}\right)+2\mathfrak{K}\left(\mathfrak{c}_2^{\varrho}\right)\right)-\frac{\varrho^{\beta}2^{\beta-1}\Gamma(\beta+1)}{\left(\mathfrak{c}_2^{\varrho}-\mathfrak{c}_1^{\varrho}\right)^{\beta}}\,{}^{\varrho}\mathcal{O}^{\beta}(\mathfrak{K})\right|$$
$$\leq\frac{\mathfrak{c}_2^{\varrho}-\mathfrak{c}_1^{\varrho}}{3}\left({}_2F_1\left(-\lambda,\tfrac{1}{\beta},\tfrac{1}{\beta}+1;\tfrac{3}{4}\right)\right)^{\frac{1}{\lambda}}\left[\left(\frac{3\left|\mathfrak{K}'\left(\mathfrak{c}_1^{\varrho}\right)\right|^{\kappa}+\left|\mathfrak{K}'\left(\mathfrak{c}_2^{\varrho}\right)\right|^{\kappa}}{4}\right)^{\frac{1}{\kappa}}+\left(\frac{\left|\mathfrak{K}'\left(\mathfrak{c}_1^{\varrho}\right)\right|^{\kappa}+3\left|\mathfrak{K}'\left(\mathfrak{c}_2^{\varrho}\right)\right|^{\kappa}}{4}\right)^{\frac{1}{\kappa}}\right].$$

Corollary 3.11. *For $\varrho=1$, Theorem 3.8, gives*

$$\left|\frac{1}{3}\left(2\mathfrak{K}(\mathfrak{c}_1)-\mathfrak{K}\left(\frac{\mathfrak{c}_1+\mathfrak{c}_2}{2}\right)+2\mathfrak{K}(\mathfrak{c}_2)\right)-\frac{2^{\beta-1}\Gamma(\beta+1)}{(\mathfrak{c}_2-\mathfrak{c}_1)^{\beta}}\mathcal{J}^{\beta}(\mathfrak{K})\right|$$
$$\leq\frac{\mathfrak{c}_2-\mathfrak{c}_1}{3}\left({}_2F_1\left(-\lambda,\tfrac{1}{\beta},\tfrac{1}{\beta}+1;\tfrac{3}{4}\right)\right)^{\frac{1}{\lambda}}\left[\left(\frac{|\mathfrak{K}'(\mathfrak{c}_1)|^{\kappa}+\left|\mathfrak{K}'\left(\frac{\mathfrak{c}_1+\mathfrak{c}_2}{2}\right)\right|^{\kappa}}{s+1}\right)^{\frac{1}{\kappa}}+\left(\frac{\left|\mathfrak{K}'\left(\frac{\mathfrak{c}_1+\mathfrak{c}_2}{2}\right)\right|^{\kappa}+|\mathfrak{K}'(\mathfrak{c}_2)|^{\kappa}}{s+1}\right)^{\frac{1}{\kappa}}\right],$$

where $\mathcal{J}^{\beta}(\mathfrak{K})$ is defined by (7).

Corollary 3.12. *By choosing $s = 1$ in Corollary 3.11, we get*

$$\left|\frac{1}{3}\left(2\Re\left(\mathfrak{c}_1\right)-\Re\left(\frac{\mathfrak{c}_1+\mathfrak{c}_2}{2}\right)+2\Re\left(\mathfrak{c}_2\right)\right)-\frac{2^{\beta-1}\Gamma\left(\beta+1\right)}{\left(\mathfrak{c}_2-\mathfrak{c}_1\right)^{\beta}}\mathcal{J}^{\beta}\left(\Re\right)\right|$$
$$\leq\frac{\mathfrak{c}_2-\mathfrak{c}_1}{3}\left({}_2F_1\left(-\lambda,\tfrac{1}{\beta},\tfrac{1}{\beta}+1;\tfrac{3}{4}\right)\right)^{\frac{1}{\lambda}}\left[\left(\frac{\left|\Re'\left(\mathfrak{c}_1\right)\right|^{\kappa}+\left|\Re'\left(\frac{\mathfrak{c}_1+\mathfrak{c}_2}{2}\right)\right|^{\kappa}}{2}\right)^{\frac{1}{\kappa}}+\left(\frac{\left|\Re'\left(\frac{\mathfrak{c}_1+\mathfrak{c}_2}{2}\right)\right|^{\kappa}+\left|\Re'\left(\mathfrak{c}_2\right)\right|^{\kappa}}{2}\right)^{\frac{1}{\kappa}}\right],$$

where $\mathcal{J}^{\beta}(\Re)$ is defined by (7).

Remark 3.13. *Here are some critical observations regarding the alignment of Theorem 3.8 with previously established results in the literature. In particular,*

1. *Corollary 3.12 represents a refinement of Theorem 2.3 from [10].*
2. *Corollary 3.11 simplifies to Theorem 2.6 from [16] upon setting $\beta = 1$.*

Theorem 3.14. *Let $\Re : \left[\mathfrak{c}_1^{\varrho}, \mathfrak{c}_2^{\varrho}\right] \to \mathbb{R}$ be a differentiable function on $\left(\mathfrak{c}_1^{\varrho}, \mathfrak{c}_2^{\varrho}\right)$ with $0 \leq \mathfrak{c}_1 < \mathfrak{c}_2$, and $\Re' \in L^1\left[\mathfrak{c}_1^{\varrho}, \mathfrak{c}_2^{\varrho}\right]$. If $|\Re'|^{\kappa}$ is s-convex in the second sense for some fixed $s \in (0, 1]$ where $\kappa \geq 1$ with $\frac{1}{\lambda} + \frac{1}{\kappa} = 1$, then we have*

$$\left|\frac{1}{3}\left(2\Re\left(\mathfrak{c}_1^{\varrho}\right)-\Re\left(\frac{\mathfrak{c}_1^{\varrho}+\mathfrak{c}_2^{\varrho}}{2}\right)+2\Re\left(\mathfrak{c}_2^{\varrho}\right)\right)-\frac{\varrho^{\beta}2^{\beta-1}\Gamma\left(\beta+1\right)}{\left(\mathfrak{c}_2^{\varrho}-\mathfrak{c}_1^{\varrho}\right)^{\beta}}\,{}^{\varrho}\mathcal{O}^{\beta}\left(\Re\right)\right|$$
$$\leq\frac{\mathfrak{c}_2^{\varrho}-\mathfrak{c}_1^{\varrho}}{4}\left(\frac{4\beta+1}{3\left(\beta+1\right)}\right)^{1-\frac{1}{\kappa}}\left[\left(\left(\frac{4}{3\left(s+1\right)}-B\left(\beta+1,s+1\right)\right)\left|\Re'\left(\mathfrak{c}_1^{\varrho}\right)\right|^{\kappa}+\frac{4\beta+s+1}{3\left(s+1\right)\left(\beta+s+1\right)}\left|\Re'\left(\frac{\mathfrak{c}_1^{\varrho}+\mathfrak{c}_2^{\varrho}}{2}\right)\right|^{\kappa}\right)^{\frac{1}{\kappa}}\right.$$
$$\left.+\left(\frac{4\beta+s+1}{3\left(s+1\right)\left(\beta+s+1\right)}\left|\Re'\left(\frac{\mathfrak{c}_1^{\varrho}+\mathfrak{c}_2^{\varrho}}{2}\right)\right|^{\kappa}+\left(\tfrac{4}{3(s+1)}-B\left(\beta+1,s+1\right)\right)\left|\Re'\left(\mathfrak{c}_2^{\varrho}\right)\right|^{\kappa}\right)^{\frac{1}{\kappa}}\right],$$

where $\beta, \varrho > 0$, ${}^{\varrho}\mathcal{O}^{\beta}(\Re)$ is defined as in (2) and $B(.,.)$ is the beta function.

Proof. Using the modulus and power mean inequality to the equality (1), then applying the s-convexity of $|\Re'|^{\kappa}$, we get

$$\left|\frac{1}{3}\left(2\Re\left(\mathfrak{c}_1^{\varrho}\right)-\Re\left(\frac{\mathfrak{c}_1^{\varrho}+\mathfrak{c}_2^{\varrho}}{2}\right)+2\Re\left(\mathfrak{c}_2^{\varrho}\right)\right)-\frac{\varrho^{\beta}2^{\beta-1}\Gamma\left(\beta+1\right)}{\left(\mathfrak{c}_2^{\varrho}-\mathfrak{c}_1^{\varrho}\right)^{\beta}}\,{}^{\varrho}\mathcal{O}^{\beta}\left(\Re\right)\right|$$
$$\leq\frac{\varrho\left(\mathfrak{c}_2^{\varrho}-\mathfrak{c}_1^{\varrho}\right)}{4}\left[\int_0^1\left(\frac{4}{3}-\nu^{\varrho\beta}\right)\nu^{\varrho-1}\left|\Re'\left(\left(1-\nu^{\varrho}\right)\mathfrak{c}_1^{\varrho}+\nu^{\varrho}\frac{\mathfrak{c}_1^{\varrho}+\mathfrak{c}_2^{\varrho}}{2}\right)\right|d\nu\right.$$
$$\left.+\int_0^1\left(\frac{4}{3}-\nu^{\varrho\beta}\right)\nu^{\varrho-1}\left|\Re'\left(\nu^{\varrho}\frac{\mathfrak{c}_1^{\varrho}+\mathfrak{c}_2^{\varrho}}{2}+\left(1-\nu^{\varrho}\right)\mathfrak{c}_2^{\varrho}\right)\right|d\nu\right]$$
$$=\frac{\mathfrak{c}_2^{\varrho}-\mathfrak{c}_1^{\varrho}}{4}\left[\int_0^1\left(\frac{4}{3}-\nu^{\beta}\right)\left|\Re'\left(\left(1-v\right)\mathfrak{c}_1^{\varrho}+v\frac{\mathfrak{c}_1^{\varrho}+\mathfrak{c}_2^{\varrho}}{2}\right)\right|d\nu+\int_0^1\left(\frac{4}{3}-\nu^{\beta}\right)\left|\Re'\left(v\frac{\mathfrak{c}_1^{\varrho}+\mathfrak{c}_2^{\varrho}}{2}+\left(1-v\right)\mathfrak{c}_2^{\varrho}\right)\right|d\nu\right]$$
$$\leq\frac{\mathfrak{c}_2^{\varrho}-\mathfrak{c}_1^{\varrho}}{4}\left[\left(\int_0^1\left(\frac{4}{3}-\nu^{\beta}\right)d\nu\right)^{1-\frac{1}{\kappa}}\left(\int_0^1\left(\frac{4}{3}-\nu^{\beta}\right)\left|\Re'\left(\left(1-v\right)\mathfrak{c}_1^{\varrho}+v\frac{\mathfrak{c}_1^{\varrho}+\mathfrak{c}_2^{\varrho}}{2}\right)\right|^{\kappa}d\nu\right)^{\frac{1}{\kappa}}\right.$$

$$+\left(\int_0^1\left(\frac{4}{3}-\nu^\beta\right)d\nu\right)^{1-\frac{1}{\kappa}}\left(\int_0^1\left(\frac{4}{3}-\nu^\beta\right)\left|\Re'\left(v\frac{\text{c}_1^\varrho+\text{c}_2^\varrho}{2}+(1-v)\,\text{c}_2^\varrho\right)\right|^\kappa d\nu\right)^{\frac{1}{\kappa}}\Bigg]$$

$$\leq\frac{\text{c}_2^\varrho-\text{c}_1^\varrho}{4}\left(\int_0^1\left(\frac{4}{3}-\nu^\beta\right)d\nu\right)^{1-\frac{1}{\kappa}}\left[\left(\int_0^1\left(\frac{4}{3}-\nu^\beta\right)\left((1-v)^s\left|\Re'\left(\text{c}_1^\varrho\right)\right|^\kappa+\nu^s\left|\Re'\left(\frac{\text{c}_1^\varrho+\text{c}_2^\varrho}{2}\right)\right|^\kappa\right)d\nu\right)^{\frac{1}{\kappa}}\right.$$

$$\left.+\left(\int_0^1\left(\frac{4}{3}-\nu^\beta\right)\left(\nu^s\left|\Re'\left(\frac{\text{c}_1^\varrho+\text{c}_2^\varrho}{2}\right)\right|^\kappa+(1-v)^s\left|\Re'\left(\text{c}_2^\varrho\right)\right|^\kappa\right)d\nu\right)^{\frac{1}{\kappa}}\right]$$

$$=\frac{\text{c}_2^\varrho-\text{c}_1^\varrho}{4}\left(\frac{4\beta+1}{3(\beta+1)}\right)^{1-\frac{1}{\kappa}}\left[\left(\left(\frac{4}{3(s+1)}-B(\beta+1,s+1)\right)\left|\Re'\left(\text{c}_1^\varrho\right)\right|^\kappa+\frac{4\beta+s+1}{3(s+1)(\beta+s+1)}\left|\Re'\left(\frac{\text{c}_1^\varrho+\text{c}_2^\varrho}{2}\right)\right|^\kappa\right)^{\frac{1}{\kappa}}\right.$$

$$\left.+\left(\frac{4\beta+s+1}{3(s+1)(\beta+s+1)}\left|\Re'\left(\frac{\text{c}_1^\varrho+\text{c}_2^\varrho}{2}\right)\right|^\kappa+\left(\frac{4}{3(s+1)}-B(\beta+1,s+1)\right)\left|\Re'\left(\text{c}_2^\varrho\right)\right|^\kappa\right)^{\frac{1}{\kappa}}\right].$$

The proof is achieved. □

Corollary 3.15. *By using the s-convexity of $|\Re|^\kappa$, Theorem 3.14 becomes*

$$\left|\frac{1}{3}\left(2\Re\left(\text{c}_1^\varrho\right)-\Re\left(\frac{\text{c}_1^\varrho+\text{c}_2^\varrho}{2}\right)+2\Re\left(\text{c}_2^\varrho\right)\right)-\frac{\varrho^\beta 2^{\beta-1}\Gamma(\beta+1)}{\left(\text{c}_2^\varrho-\text{c}_1^\varrho\right)^\beta}\,{}^\varrho O^\beta(\Re)\right|$$

$$\leq\frac{\text{c}_2^\varrho-\text{c}_1^\varrho}{4}\left(\frac{4\beta+1}{3(\beta+1)}\right)^{1-\frac{1}{\kappa}}\left[\left(\left(\frac{4\beta+s+1+2^s\times4(\beta+s+1)}{2^s\times3(s+1)(\beta+s+1)}-B(\beta+1,s+1)\right)\left|\Re'\left(\text{c}_1^\varrho\right)\right|^\kappa+\frac{4\beta+s+1}{2^s\times3(s+1)(\beta+s+1)}\left|\Re'\left(\text{c}_2^\varrho\right)\right|^\kappa\right)^{\frac{1}{\kappa}}\right.$$

$$\left.+\left(\frac{4\beta+s+1}{2^s\times3(s+1)(\beta+s+1)}\left|\Re'\left(\text{c}_1^\varrho\right)\right|^\kappa+\left(\frac{4\beta+s+1+2^s\times4(\beta+s+1)}{2^s\times3(s+1)(\beta+s+1)}-B(\beta+1,s+1)\right)\left|\Re'\left(\text{c}_2^\varrho\right)\right|^\kappa\right)^{\frac{1}{\kappa}}\right].$$

Corollary 3.16. *If we attempt to take $s=1$, Theorem 3.14 yields*

$$\left|\frac{1}{3}\left(2\Re\left(\text{c}_1^\varrho\right)-\Re\left(\frac{\text{c}_1^\varrho+\text{c}_2^\varrho}{2}\right)+2\Re\left(\text{c}_2^\varrho\right)\right)-\frac{\varrho^\beta 2^{\beta-1}\Gamma(\beta+1)}{\left(\text{c}_2^\varrho-\text{c}_1^\varrho\right)^\beta}\,{}^\varrho O^\beta(\Re)\right|$$

$$\leq\frac{\text{c}_2^\varrho-\text{c}_1^\varrho}{4}\left(\frac{4\beta+1}{3(\beta+1)}\right)^{1-\frac{1}{\kappa}}\left[\left(\left(\frac{2}{3}-\frac{1}{(\beta+1)(\beta+2)}\right)\left|\Re'\left(\text{c}_1^\varrho\right)\right|^\kappa+\frac{2\beta+1}{3(\beta+2)}\left|\Re'\left(\frac{\text{c}_1^\varrho+\text{c}_2^\varrho}{2}\right)\right|^\kappa\right)^{\frac{1}{\kappa}}\right.$$

$$\left.+\left(\frac{2\beta+1}{3(\beta+2)}\left|\Re'\left(\frac{\text{c}_1^\varrho+\text{c}_2^\varrho}{2}\right)\right|^\kappa+\left(\frac{2}{3}-\frac{1}{(\beta+1)(\beta+2)}\right)\left|\Re'\left(\text{c}_2^\varrho\right)\right|^\kappa\right)^{\frac{1}{\kappa}}\right].$$

Moreover, by making use of the convexity of $|\Re'|^\kappa$, we get

$$\left|\frac{1}{3}\left(2\Re\left(\text{c}_1^\varrho\right)-\Re\left(\frac{\text{c}_1^\varrho+\text{c}_2^\varrho}{2}\right)+2\Re\left(\text{c}_2^\varrho\right)\right)-\frac{\varrho^\beta 2^{\beta-1}\Gamma(\beta+1)}{\left(\text{c}_2^\varrho-\text{c}_1^\varrho\right)^\beta}\,{}^\varrho O^\beta(\Re)\right|$$

$$\leq\frac{\text{c}_2^\varrho-\text{c}_1^\varrho}{4}\left(\frac{4\beta+1}{3(\beta+1)}\right)^{1-\frac{1}{\kappa}}\left[\left(\left(\frac{2}{3}-\frac{2\beta^2+3\beta-5}{6(\beta+1)(\beta+2)}\right)\left|\Re'\left(\text{c}_1^\varrho\right)\right|^\kappa+\frac{2\beta+1}{6(\beta+2)}\left|\Re'\left(\text{c}_2^\varrho\right)\right|^\kappa\right)^{\frac{1}{\kappa}}\right.$$

$$+\left(\frac{2\beta+1}{6(\beta+2)}\left|\Re'\left(\mathfrak{c}_1^{\varrho}\right)\right|^{\kappa}+\left(\frac{2}{3}-\frac{2\beta^2+3\beta-5}{6(\beta+1)(\beta+2)}\right)\left|\Re'\left(\mathfrak{c}_2^{\varrho}\right)\right|^{\kappa}\right)^{\frac{1}{\kappa}}\Bigg].$$

Corollary 3.17. *For $\varrho=1$, Theorem 3.14 becomes*

$$\left|\frac{1}{3}\left(2\Re(\mathfrak{c}_1)-\Re\left(\frac{\mathfrak{c}_1+\mathfrak{c}_2}{2}\right)+2\Re(\mathfrak{c}_2)\right)-\frac{2^{\beta-1}\Gamma(\beta+1)}{(\mathfrak{c}_2-\mathfrak{c}_1)^{\beta}}\mathcal{J}^{\beta}(\Re)\right|$$
$$\leq\frac{\mathfrak{c}_2-\mathfrak{c}_1}{4}\left(\frac{4\beta+1}{3(\beta+1)}\right)^{1-\frac{1}{\kappa}}\Bigg[\left(\left(\frac{4}{3(s+1)}-B(\beta+1,s+1)\right)|\Re'(\mathfrak{c}_1)|^{\kappa}+\frac{4\beta+s+1}{3(s+1)(\beta+s+1)}\left|\Re'\left(\frac{\mathfrak{c}_1+\mathfrak{c}_2}{2}\right)\right|^{\kappa}\right)^{\frac{1}{\kappa}}$$
$$+\left(\frac{4\beta+s+1}{3(s+1)(\beta+s+1)}\left|\Re'\left(\frac{\mathfrak{c}_1+\mathfrak{c}_2}{2}\right)\right|^{\kappa}+\left(\frac{4}{3(s+1)}-B(\beta+1,s+1)\right)|\Re'(\mathfrak{c}_2)|^{\kappa}\right)^{\frac{1}{\kappa}}\Bigg],$$

where $\mathcal{J}^{\beta}(\Re)$ is defined by (7).

Remark 3.18. *Similar to the preceding theorems, Theorem 3.14 constitutes a generalization of several previously established results. Specifically:*

1. *In Corollary 3.17, if we take $s=1$, we obtain Theorem 2.5 from [10].*
2. *In Corollary 3.17, if we take $\beta=1$, we obtain Theorem 2.9 from [16].*

4. Applications

4.1. Milne's quadrature formula

Let Υ be a partition of the interval $[\mathfrak{c}_1,\mathfrak{c}_2]$ with $a=y_0<y_1<...<y_n=b$, and consider the following quadrature rule

$$\int_{\mathfrak{c}_1}^{\mathfrak{c}_2}\Re(y)\,dy=\lambda(\Re,\Upsilon)+R(\Re,\Upsilon),$$

where

$$\lambda(\Re,\Upsilon)=\sum_{i=0}^{n-1}\tfrac{y_{i+1}-y_i}{3}\left(2\Re(y_i)-\Re\left(\tfrac{y_i+y_{i+1}}{2}\right)+2\Re(y_{i+1})\right)$$

and $R(\Re,\Upsilon)$ denotes the associated approximation error.

Proposition 4.1. *Let $n\in\mathbb{N}$ and $\Re:[\mathfrak{c}_1,\mathfrak{c}_2]\to\mathbb{R}$ be a differentiable function on $[\mathfrak{c}_1,\mathfrak{c}_2]$ with $0\leq\mathfrak{c}_1<\mathfrak{c}_2$ and $\Re'\in L^1([\mathfrak{c}_1,\mathfrak{c}_2])$. If $|\Re'|$ is a convex function, we have*

$$|R(\Re,\Upsilon)|\leq\sum_{i=0}^{n-1}\tfrac{5(y_{i+1}-y_i)^2}{24}\left(\left|\Re'(y_i)\right|+\left|\Re'(y_{i+1})\right|\right).$$

Proof. Applying Corollary 3.6 with $\beta=s=1$, on the subintervals $[y_i,y_{i+1}]$ $(i=0,1,...,n-1)$ of the partition Υ, we get

$$\left|\tfrac{1}{3}\left(2\Re(y_i)-\Re\left(\tfrac{y_i+y_{i+1}}{2}\right)+2\Re(y_{i+1})\right)-\tfrac{1}{y_{i+1}-y_i}\int_{y_i}^{y_{i+1}}\Re(y)\,dy\right|$$

$$\leq \frac{y_{i+1}-y_i}{24}\left(3\left|\Re'\left(y_i\right)\right|+4\left|\Re'\left(\frac{y_i+y_{i+1}}{2}\right)\right|+3\left|\Re'\left(y_{i+1}\right)\right|\right). \tag{8}$$

From the convexity of $|\Re'|$ we have

$$\left|\Re'\left(\frac{y_i+y_{i+1}}{2}\right)\right| \leq \frac{\left|\Re'\left(y_i\right)\right|+\left|\Re'\left(y_{i+1}\right)\right|}{2}. \tag{9}$$

Using (9) into (8), we obtain

$$\left|\frac{1}{3}\left(2\Re\left(y_i\right)-\Re\left(\frac{y_i+y_{i+1}}{2}\right)+2\Re\left(y_{i+1}\right)\right)-\frac{1}{y_{i+1}-y_i}\int_{y_i}^{y_{i+1}}\Re\left(y\right)dy\right| \leq \frac{5\left(y_{i+1}-y_i\right)}{24}\left(\left|\Re'\left(y_i\right)\right|+\left|\Re'\left(y_{i+1}\right)\right|\right). \tag{10}$$

By multiplying both sides of (10) by $y_{i+1}-y_i$, summing over $i=0$ to $n-1$, and applying the triangle inequality to the resulting sum, we obtain the required result. □

Proposition 4.2. *Let $n \in \mathbb{N}$ and $\Re : [\mathfrak{c}_1, \mathfrak{c}_2] \to \mathbb{R}$ be a differentiable function on $[\mathfrak{c}_1, \mathfrak{c}_2]$ with $0 \leq \mathfrak{c}_1 < \mathfrak{c}_2$ and $\Re' \in L^1\left([\mathfrak{c}_1, \mathfrak{c}_2]\right)$. If $|\Re'|$ is an s-convex function, we have*

$$|R\left(\Re, \Upsilon\right)| \leq \sum_{i=0}^{n-1}\frac{\left(y_{i+1}-y_i\right)^2}{4(s+1)(s+2)}\left(\frac{4s+5}{3}+\frac{2s+10}{2^s\times 3}\right)\left(\left|\Re'\left(y_i\right)\right|+\left|\Re'\left(y_{i+1}\right)\right|\right).$$

Proof. Applying Corollary 3.6 with $\beta=1$ on the subintervals $[y_i, y_{i+1}]$ $(i=0,1,...,n-1)$ of the partition Υ, we get

$$\left|\frac{1}{3}\left(2\Re\left(y_i\right)-\Re\left(\frac{y_i+y_{i+1}}{2}\right)+2\Re\left(y_{i+1}\right)\right)-\frac{1}{y_{i+1}-y_i}\int_{y_i}^{y_{i+1}}\Re\left(y\right)dy\right| \tag{11}$$
$$\leq \frac{y_{i+1}-y_i}{4(s+1)(s+2)}\left(\frac{4s+5}{3}\left|\Re'\left(y_i\right)\right|+\frac{2s+10}{3}\left|\Re'\left(\frac{y_i+y_{i+1}}{2}\right)\right|+\frac{4s+5}{3}\left|\Re'\left(y_{i+1}\right)\right|\right).$$

From the s-convexity of $|\Re'|$, we have

$$\left|\Re'\left(\frac{y_i+y_{i+1}}{2}\right)\right| \leq \frac{\left|\Re'\left(y_i\right)\right|+\left|\Re'\left(y_{i+1}\right)\right|}{2^s}. \tag{12}$$

Using (12) into (11), we obtain

$$\left|\frac{1}{3}\left(2\Re\left(y_i\right)-\Re\left(\frac{y_i+y_{i+1}}{2}\right)+2\Re\left(y_{i+1}\right)\right)-\frac{1}{y_{i+1}-y_i}\int_{y_i}^{y_{i+1}}\Re\left(y\right)dy\right|$$
$$\leq \frac{y_{i+1}-y_i}{4(s+1)(s+2)}\left(\frac{4s+5}{3}+\frac{2s+10}{2^s\times 3}\right)\left(\left|\Re'\left(y_i\right)\right|+\left|\Re'\left(y_{i+1}\right)\right|\right). \tag{13}$$

By multiplying both sides of (13) by $y_{i+1}-y_i$, summing over $i=0$ to $n-1$, and applying the triangle inequality to the resulting sum, we obtain the required result. □

4.2. *Application to special means*

For arbitrary real positive numbers $\mathfrak{c}_1, \mathfrak{c}_2$ we have:
The geometic mean: $G\left(\mathfrak{c}_1, \mathfrak{c}_2\right)=\sqrt{\mathfrak{c}_1\mathfrak{c}_2}$.
The harmonic mean: $H\left(\mathfrak{c}_1, \mathfrak{c}_2\right)=\frac{2\mathfrak{c}_1\mathfrak{c}_2}{\mathfrak{c}_1+\mathfrak{c}_2}$.
The p-logarithmic mean: $L_p\left(\mathfrak{c}_1, \mathfrak{c}_2\right)=\left(\frac{\mathfrak{c}_2^{p+1}-\mathfrak{c}_1^{p+1}}{(p+1)(\mathfrak{c}_2-\mathfrak{c}_1)}\right)^{\frac{1}{p}}$, $\mathfrak{c}_1 \neq \mathfrak{c}_2$ and $p \in \mathbb{R}\backslash\{-1,0\}$.

Proposition 4.3. *Let* $\mathfrak{c}_1, \mathfrak{c}_2 \in \mathbb{R}$ *and* $\kappa, \lambda > 1$, *with* $0 < \mathfrak{c}_1 < \mathfrak{c}_2$ *and* $\frac{1}{\kappa} + \frac{1}{\lambda} = 1$, *then we have*

$$\begin{aligned}&\left|4H^{-1}\left(\mathfrak{c}_1^3, \mathfrak{c}_2^3\right) - H^{-3}\left(\mathfrak{c}_1, \mathfrak{c}_2\right) - 3G^{-6}\left(\mathfrak{c}_1, \mathfrak{c}_2\right) L_3^3\left(\mathfrak{c}_1, \mathfrak{c}_2\right)\right| \\ \leq &\frac{3(\mathfrak{c}_2-\mathfrak{c}_1)}{4}\left(4G^{-2}\left(\mathfrak{c}_1, \mathfrak{c}_2\right) H^{-1}\left(\mathfrak{c}_1^2, \mathfrak{c}_2^2\right) + G^{-4}\left(\mathfrak{c}_1, \mathfrak{c}_2\right)\right).\end{aligned}$$

Proof. The assertion follows from Theorem 3.3 with $\varrho = \beta = s = 1$, applied to the function $\Re(y) = y^3$ on the interval $\left[\frac{1}{\mathfrak{c}_2}, \frac{1}{\mathfrak{c}_1}\right]$. $\square$

5. Conclusion

In conclusion, this study has successfully expanded existing research related to Katugampola fractional integrals by introducing a new identity that facilitates the formulation of Milne-type inequalities for differentiable convex mappings. Our findings represent a significant advancement in the field, providing a broader generalization and refinement of previously established results. The inclusion of an example with graphical demonstrations substantiates our theoretical results. Future research could further explore the implications of these findings, potentially unveiling new avenues for applying Katugampola fractional integrals in various scientific and engineering contexts. This work underscores the ongoing importance of fractional calculus in advancing mathematical research and its applications, demonstrating the dynamic interplay between theory and practice.

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