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The pseudo-Chebyshev wavelets and its applications in the error of the functions of bounded variation

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Abstract. This paper presents a novel computational approach to tackle challenges in approximation theory. The proposed method leverages pseudo-Chebyshev wavelet approximations, a concept introduced by Lal et al. in 2022, based on pseudo-Chebyshev functions. The paper provides a detailed description of the method, followed by an error analysis for a given function. Key results are illustrated through an example, highlighting the accuracy and efficiency of the pseudo-Chebyshev wavelet approximation technique. Furthermore, the paper derives error estimates for functions of bounded variation using pseudo-Chebyshev wavelets via orthogonal projection operators, demonstrating that these estimators are exceptionally precise and optimal in the context of wavelet analysis.

1. Introduction

Orthogonal functions play a vital role in addressing a wide range of problems, such as differential and integral equations, approximation theory, and dynamical systems. They facilitate the simplification of complex problems by converting them into truncated approximations, making them more manageable and easier to analyse. Among these, Chebyshev polynomials T_m, where m is non negative integers, emerge as particularly effective numerically, as emphasized in various sources [2, 3, 30–35, 38]. Ricci introduced pseudo-Chebyshev functions of fractional degree [37], which were further studied by Cesarano and Ricci, who examined their notable properties, including orthogonality, in greater detail [4].

Wavelets, a relatively recent innovation emerging in the 1980s, have undergone significant expansion, attracting the interest of numerous researchers, including Morlet et al. [27], Daubechies [7], Chui [5, 6], Strang [41], Natanson [28], Meyer [25], Daubechies and Lagarias [8], Walter [43, 44], Islam et al. [10], Mohammadi [26], Lal et al. [18, 19], Malmir[20–24], Venkatesh [42], Keshavarz et al. [11], and many others in both pure and applied mathematics. Beyond their foundations in harmonic theory and Fourier analysis,

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wavelets have also drawn inspiration from fractals and approximation theory, driving their continued development and evolution. Similarly, numerous researchers, including Rehman and Siddiqi [36], Strang [40], Lal and Kumar [15–17], Bastin [2], Biazar [3], Babolian [1], Kumar [12], Kumar et al. [13, 14] and others, have explored the applications of wavelet theory, highlighting its effectiveness as a powerful tool in science and technology. Fractals, defined by their bounded, continuous, and nowhere-differentiable nature, encompass phenomena such as Brownian trajectories, fractional Brownian motion, typical Feynman paths, complex Bernoulli spirals, and turbulent fluid dynamics.

This insight has sparked interest in the exploration of approximating functions within $f \in BV_{\Omega}(X)$ utilizing pseudo-Chebyshev wavelets and their practical implications. However, to date, there appears to be no endeavour aimed at scrutinizing the error linked with functions associated with $f \in BV_{\Omega}(X)$ using the orthogonal projection operator via pseudo-Chebyshev wavelets.

The paper is structured as follows: Section 2 presents the preliminary results and findings. Section 3 outlines the primary convergence theorems concerning pseudo-Chebyshev wavelet approximations, along with an algorithm for the pseudo-Chebyshev wavelet, which is used to solve the problems of approximation theory. Section 4 provides numerical examples that explore the approximations and applications of pseudo-Chebyshev wavelets with computational MATLAB algorithm. Section 5 offers discussions and conclusions. Finally, the references cited in this study are listed.

2. Definitions and Preliminaries

2.1. Wavelets and Multiresolution Analysis

Wavelets: A function $\psi \in L^2(\mathbb{R})$ is said to be basic wavelet. If it is satisfies the 'admissibility' condition

$$C_{\psi} = \int_{-\infty}^{\infty} \frac{\left|\hat{\psi}(\omega)\right|^2}{|\omega|} d\omega < \infty \quad [5].$$

Wavelets are a set $\{\psi_{m,n}; m, n \in \mathbb{Z}\}$ of functions constructed from translation and dilation of a single basic wavelet ψ also called the mother wavelet. If the dilation parameter a and the translation parameter b vary continuously, then the following family of continuous wavelets are

$$\psi_{a,b}(t) = |a|^{-\frac{1}{2}} \psi\left(\frac{t-b}{a}\right), a \neq 0, b \in \mathbb{R} \quad [7].$$

Multiresolution Analysis: A sequence of closed subspaces W_n of $L^2(\mathbb{R})$, $n \in \mathbb{Z}$ are said to be Multiresolution Analysis. If it is satisfying the following properties [9]:

(i) W_n is a subset of W_{n+1},
(ii) $f(t) \in W_n \Leftrightarrow f(2t) \in W_{n+1}$,
(iii) $f(t) \in W_0 \Leftrightarrow f(t+1) \in W_0$,
(iv) $\overline{\bigcup_{n=-\infty}^{\infty} W_n} = L^2(\mathbb{R})$ and $\bigcap_{n=-\infty}^{\infty} W_n = \{0\}$,
(v) there exists a function ϕ such that the set $\left\{\phi(. - k); k \in \mathbb{Z}\right\}$ is a Riesz basis of W_0. Since

$$\psi \in L^2(\mathbb{R}), \psi_{j,k} := 2^{\frac{j}{2}} \psi(2^j - k) \text{ with } V_j := clos\left\langle \psi_{j,k} : k \in \mathbb{Z} \right\rangle,$$

and this family of subspaces of $L^2(\mathbb{R})$ gives a direct sum decomposition of $L^2(\mathbb{R})$ is the same that every $g \in L^2(\mathbb{R})$ has a unique decomposition

$$g = \sum_{n\in\mathbb{Z}} g_n = \cdots + g_{-2} + g_{-1} + g_0 + g_1 + g_2 + \cdots,$$

where $g_n \in V_n$ for all $n \in \mathbb{Z}$ and we describe this by writing $L^2(\mathbb{R}) = W_n \oplus_{i=n}^{\infty} V_i$, where $W_n := \oplus_{m=-\infty}^{n-1} V_m$. $\{\psi_{n,m}; m \in \mathbb{Z}\}$, is a Riesz basis of V_n. Therefore

$$g = \sum_{n=-\infty}^{\infty} \left\langle g, \phi_{n,m} \right\rangle \phi_{n,m} + \sum_{k=m}^{\infty} \sum_{n=-\infty}^{\infty} \langle g, \psi_{n,m} \rangle \psi_{n,m} \quad [44].$$

2.2. *Pseudo Chebyshev Wavelets [18]*

In recent research articles, Ricci [37] explores the extension of classical sets of Chebyshev polynomials of the first and second kind to include fractional indices, particularly in the investigation of complex Bernoulli spirals. These extended functions are referred to as pseudo-Chebyshev functions.

The pseudo-Chebyshev functions $T_{n+1/2}(t)$ is defined as Cesarano and Ricci, [4, 37]

$$T_{(m+1/2)}(t) = \cos((m + 1/2)\,(arc\cos t)) \text{ where } m \in \mathbb{N} \cup \{0\}.$$

The recurrence relation of the pseudo-Chebyshev polynomials and its orthogonal properties are given by [4]

$$T_{m+1/2}(t) = 2tT_{m-1/2}(t) - T_{m-3/2}(t), \text{ where } T_{1/2} = T_{-1/2} = \sqrt{\frac{1+t}{2}},\ m \in \mathbb{N}.$$

The first few pseudo-Chebyshev polynomials are

$$T_{-1/2} = \sqrt{\frac{1+t}{2}},\ T_{1/2} = \sqrt{\frac{1+t}{2}} \ \&\ T_{3/2} = (2t-1)\sqrt{\frac{1+t}{2}}.$$

Since these polynomials are satisfying the following conditions

$$\int_{-1}^{1} \frac{T_{m+1/2}(t)T_{n+1/2}(t)}{\sqrt{1-t^2}} = \begin{cases} \frac{\pi}{2} & \text{for } m = n, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore the set $\{T_{m+1/2}(t); m \geq 0\}$ is an orthogonal subset of $L^2(-1, 1)$ with respect to weight functions $\omega_{k,n}(t) = \omega(2^k t - 2n + 1)$ where $\omega(t) = \frac{1}{\sqrt{1-t^2}}$.

For half-integer indices, the pseudo-Chebyshev functions exhibit similar properties to their classical counterparts, including differential equations, recurrence relations, orthogonality properties, and so on, as documented in [4, 37].

Pseudo Chebyshev Wavelets: Let $T_{m+1/2}$ be the pseudo-Chebyshev functions of indices $m + 1/2$. Then the pseudo-Chebyshev wavelets are given by Lal et al. [18] in 2022, defined by

$$\psi_{n,m}(t) := \psi_{(k,n,m)}(t) = \begin{cases} \sqrt{\frac{2^{k+1}}{\pi}} T_{m+1/2}(2^k t - 2n + 1), & \text{for } \frac{n-1}{2^{k-1}} \leq t \leq \frac{n}{2^{k-1}}, \\ 0 & \text{otherwise, where } m \geq 0,\ n = 1, 2, 3, \cdots 2^{k-1} \text{ and } k \in \mathbb{N}. \end{cases}$$

2.3. Orthogonal Projection Operators $P_n(f)$

An orthogonal projection operator $P_n(f)$ of a function $f \in L^2_\Omega(X)$ onto W_n defined as [14]

$$P_n(f) = \sum_{m=0}^{\infty} \langle f, \psi_{n,m} \rangle_{\omega_{k,n}} \psi_{n,m}, \tag{1}$$

where $\langle f, \psi_{n,m} \rangle_{\omega_{k,n}} = \int_\Omega f \psi_{n,m} \omega_{k,n} d\mu$ and fixed $n = 1, 2, 3, \cdots, 2^{k-1}$, $k \in \mathbb{N}$.

2.4. Wavelet Series

A function $f \in L^2_\Omega(X)$ is expanded by wavelet series as [24]:

$$f = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \langle f, \psi_{n,m} \rangle_{\omega_{k,n}} \psi_{n,m} = \sum_{n=1}^{\infty} \left(\sum_{m=0}^{\infty} P^m_{k,n} \right) \psi_{n,m} \text{ for each fixed } k \in \mathbb{N}, \tag{2}$$

where $P^m_{k,n} = \langle f, \psi_{n,m} \rangle_{\omega_{k,n}}$ is said to be $(n,m)^{th}$ coefficients of a function f corresponding the orthonormal wavelets $\psi_{n,m}$ of the wavelet series.

For each fixed, $n \in \mathbb{N}$, where $n = 1, 2, 3, \cdots, 2^{k-1}$, the expansions

$$\begin{aligned} \langle P_n f, \psi_{n,m} \rangle &= \left\langle \sum_{m=0}^{\infty} \langle f, \psi_{n,m} \rangle_{\omega_{k,n}} \psi_{n,m}, \psi_{n,m} \right\rangle = \sum_{m=0}^{\infty} \langle f, \psi_{n,m} \rangle_{\omega_{k,n}} \langle \psi_{n,m}, \psi_{n,m} \rangle \\ &= \sum_{m=0}^{\infty} \langle f, \psi_{n,m} \rangle_{\omega_{k,n}} = \sum_{m=0}^{\infty} P^m_{k,n}, \end{aligned}$$

are called n^{th} coefficients of an orthogonal projection operator for each fixed $k \in \mathbb{N}$.

If the wavelet series of any function f is truncated by an orthogonal projection operators

$$\left(P_{(2^{k-1},M)} f \right) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M} P^m_{k,n} \psi_{n,m}$$

then, we say that the wavelet series has sum's for each point, if the sequence of functions $\left\{ P_{(2^{k-1},M)} f(t) \right\}_{(k=0,M=0)}^{(\infty,\infty)}$ uniformly converges to s(t) i.e.,

$$s = \lim_{M \to \infty} \lim_{k \to \infty} P_{2^{k-1},M}(f).$$

It is denoted by $f \approx s$ say wavelet approximation of function f. The approximation is called best wavelet approximation, if $s(t) = f(t)$ for each points in the domain of functions.

2.5. Error of Wavelet Approximation

The error of wavelet approximation $E_{2^{k-1},M}(f)$ of a function $f \in L^2_\Omega(X)$ by the orthogonal projection operators $P_{2^{k-1},M}(f)$ is defined as [45]

$$E_{2^{k-1},M}(f) = \inf_{P_{2^{k-1},M}(f)} (f - P_{1,M}(f)) \quad \text{where } M \text{ and } k \in \mathbb{N}.$$

If $\|E_{2^{k-1},M}(f)\|_2 \to 0$ as $k \to \infty$ or $M \to \infty$ then $P_{2^{k-1},M}(f)$ is called the best wavelet approximation of a function $f \in L^2_\Omega(X)$ [45].

2.6. Function of $BV_\Omega(X)$ class

Let X and Ω be a Measurable space. Then a function f is said to be function of bounded variation, i.e.

$$f \in BV_\Omega(X) \quad \text{if} \quad \exists\, M > 0 \text{ such that } \sup_T V(f,T) < M \text{ where } T \text{ is partition of } \Omega.$$

$$\text{and} \quad V(f,T) = \sum_{k=1}^{n} \left| f(x_k) - f(x_{k-1}) \right|, [29].$$

Remark 2.1. *(i) A monotone function is a function of bounded variation.*

(ii) A Lipschitz function on the finite measurable space Ω is a function of bounded variation.

(iii) A continuous function need not be a function of bounded variation..

(iv) A bounded function on an open and connected domain Ω need not be a function of bounded variation.

(v) A differentiable function with bounded derivative is a function of bounded variation.

(vi) The space $BV_\Omega(X)$ is a real/complex linear space.

2.7. Auxiliary Lemmas

For the proof of main results, the following lemmas are required.

Lemma 2.2. ***Jordan Decomposition Theorem: [29]*** *A function $f \in BV_\Omega(X)$ iff $f = f_1 - f_2$ where f_1 and f_2 are non decreasing monotonic functions on Ω.*

Lemma 2.3. ***Generalized Mean Value Theorem: [29]*** *Let f be a finite Lebesgue integrable function on the measurable space Ω and g be a non negative non decreasing monotonic function on Ω. Then $\exists\, \alpha \in \Omega$ such that*

$$\int_\Omega fg\,d\mu \quad \leq \quad g(\alpha) \int_\Omega f\,d\mu.$$

Lemma 2.4. ***Generalized Mean Value Theorem: [29]*** *Let f be a finite Lebesgue integrable function on the measurable space Ω and g be a non negative non increasing monotonic function on Ω. Then $\exists\, \beta \in \Omega$ such that*

$$\int_\Omega fg\,d\mu \quad \geq \quad g(\beta) \int_\Omega f\,d\mu.$$

Lemma 2.5. ***Cauchy Integral Test: [39]*** *Let N be an integer and a $f : [N,\infty) \to \mathbb{R}$ be a real valued monotonic decreasing function. Then*

$$\int_N^\infty f(t)dt \leq \sum_N^\infty f(n) \leq f(N) + \int_N^\infty f(t)dt.$$

3. Main results

In this section, four new theorems have been established in the following forms:

Theorem 3.1. *If X and Ω be a Measurable space and $f \in BV_\Omega(X)$, then the Pseudo Chebyshev Wavelet coefficients of the function f corresponding the wavelet series* (2) *are*

$$\langle f, \psi_{n,m} \rangle_{\omega_{k,n}} \quad \leq \quad \sqrt{\frac{2}{\pi}} \frac{1}{2^{k/2}} \frac{(-1)^m}{m+1/2} \left\{ f_1\left(\frac{n}{2^{k-1}}\right) - f_2\left(\frac{n-1}{2^{k-1}}\right) \right\},$$

where, $f = f_1 - f_2$ given by Lemma 2.2.

Proof of Theorem 3.1. Since

$$\begin{aligned}
\left\langle f, \psi_{n,m} \right\rangle_{\omega_{k,n}} &= \left\langle f_1 - f_2, \psi_{n,m} \right\rangle_{\omega_{k,n}}, \quad \text{by Lemma 2.2,} \\
&= \left\langle f_1, \psi_{n,m} \right\rangle_{\omega_{k,n}} - \left\langle f_2, \psi_{n,m} \right\rangle_{\omega_{k,n}}.
\end{aligned}$$

Now

$$\begin{aligned}
\left\langle f_1, \psi_{n,m} \right\rangle_{\omega_{k,n}} &= \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} f_1(t)\psi_{n,m}(t)\omega_{k,n}(t)dt \\
&\leq f_1\left(\frac{n}{2^{k-1}}\right) \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \sqrt{\frac{2^{k+1}}{\pi}} T_{m+\frac{1}{2}}(2^k t - 2n + 1)\omega_{k,n}(t)dt, \quad \text{by Lemma 2.3,} \\
&= f_1\left(\frac{n}{2^{k-1}}\right) \sqrt{\frac{2^{k+1}}{\pi}} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} T_{m+\frac{1}{2}}(2^k t - 2n + 1)\omega(2^k t - 2n + 1)dt, \\
&= \sqrt{\frac{2}{\pi}} f_1\left(\frac{n}{2^{k-1}}\right) \frac{1}{2^{k/2}} \int_{-1}^{1} T_{m+\frac{1}{2}}(x)\omega(x)dx, \\
&= \sqrt{\frac{2}{\pi}} f_1\left(\frac{n}{2^{k-1}}\right) \frac{1}{2^{k/2}} \int_{0}^{\pi} cos((m+1/2)x)dx, \\
&= \sqrt{\frac{2}{\pi}} f_1\left(\frac{n}{2^{k-1}}\right) \frac{1}{2^{k/2}} \frac{(-1)^m}{m+1/2}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\left\langle f_2, \psi_{n,m} \right\rangle_{\omega_{k,n}} &= \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} f_2(t)\psi_{n,m}(t)\omega_{k,n}(t)dt \\
&\geq f_2\left(\frac{n-1}{2^{k-1}}\right) \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \sqrt{\frac{2^{k+1}}{\pi}} T_{m+\frac{1}{2}}(2^k t - 2n + 1)\omega_{k,n}(t)dt, \quad \text{by Lemma 2.4,} \\
&= \sqrt{\frac{2}{\pi}} f_2\left(\frac{n-1}{2^{k-1}}\right) \frac{1}{2^{k/2}} \frac{(-1)^m}{m+1/2}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\left\langle f, \psi_{n,m} \right\rangle_{\omega_{k,n}} &\leq \sqrt{\frac{2}{\pi}} \frac{1}{2^{k/2}} \frac{(-1)^m}{m+1/2} \left\{ f_1\left(\frac{n}{2^{k-1}}\right) - f_2\left(\frac{n-1}{2^{k-1}}\right) \right\}, \\
&\leq \sqrt{\frac{2}{\pi}} \frac{1}{2^{k/2}} \frac{(-1)^m}{m+1/2} \left\{ f_1(1) - f_2(0) \right\}, \qquad (3) \\
&\text{where} \quad f = f_1 - f_2 \ \text{given by Lemma 2.2.}
\end{aligned}$$

Thus the Theorem 3.1 is completely established.

Theorem 3.2. *If X and Ω be a Measurable space and $f \in BV_{\Omega}(X)$, then the n^{th} coefficients of an orthogonal projection operator for each fixed $k \in \mathbb{N}$ using Pseudo Chebyshev Wavelets of the function f corresponding the wavelet series* (2) *are*

$$\langle P_n f, \psi_{n,m}\rangle_{\omega_{k,n}} = \sum_{m=0}^{\infty} P_{k,n}^{m} \quad \leq \quad \sqrt{\frac{\pi}{2}}\frac{1}{2^{k/2}}\left\{f_1\left(\frac{n}{2^{k-1}}\right) - f_2\left(\frac{n-1}{2^{k-1}}\right)\right\},$$

where, $f = f_1 - f_2$ given by Lemma 2.2.

Proof of Theorem 3.2. Since

$$\begin{aligned}
\sum_{m=0}^{\infty} P_{k,n}^{m} &= \left\langle P_n f, \psi_{n,j}\right\rangle_{\omega_{k,n}} = \left\langle \sum_{m=0}^{\infty} \langle f, \psi_{n,m}\rangle_{\omega_{k,n}} \psi_{n,m}, \psi_{n,j}\right\rangle_{\omega_{k,n}}, \text{ by eq (1)} \\
&= \sum_{m=0}^{\infty} \langle f, \psi_{n,m}\rangle_{\omega_{k,n}} \left\langle \psi_{n,m}, \psi_{n,j}\right\rangle_{k,n} = \sum_{m=0}^{\infty} \langle f, \psi_{n,m}\rangle_{\omega_{k,n}}, \\
&\leq \sum_{m=0}^{\infty} \sqrt{\frac{2}{\pi}}\frac{1}{2^{k/2}}\frac{(-1)^m}{m+1/2}\left\{f_1\left(\frac{n}{2^{k-1}}\right) - f_2\left(\frac{n-1}{2^{k-1}}\right)\right\}, \text{ by eq (3)} \\
&= \sqrt{\frac{2}{\pi}}\frac{1}{2^{k/2}}\left\{f_1\left(\frac{n}{2^{k-1}}\right) - f_2\left(\frac{n-1}{2^{k-1}}\right)\right\}\sum_{m=0}^{\infty}\frac{(-1)^m}{m+1/2}, \\
&= \sqrt{\frac{2}{\pi}}\frac{1}{2^{k/2}}\left\{f_1\left(\frac{n}{2^{k-1}}\right) - f_2\left(\frac{n-1}{2^{k-1}}\right)\right\}\frac{\pi}{2}, \because \sum_{m=0}^{\infty}\frac{(-1)^m}{m+1/2} = \frac{\pi}{2}, \\
&= \sqrt{\frac{\pi}{2}}\frac{1}{2^{k/2}}\left\{f_1\left(\frac{n}{2^{k-1}}\right) - f_2\left(\frac{n-1}{2^{k-1}}\right)\right\},
\end{aligned}$$

where $f = f_1 - f_2$ given by Lemma 2.2 for each fixed $n = 1, 2, 3, \cdots, 2^{k-1}$, $k \in \mathbb{N}$.

Therefore the Theorem 3.2 is completely established.

Theorem 3.3. *If X and Ω be a Measurable space and $f \in BV_{\Omega}(X)$, then the error of wavelet approximation $E_{2^{k-1},M}(f)$ by the orthogonal projection operators $P_{2^{k-1},M}(f)$ using Pseudo Chebyshev Wavelets of the function f corresponding the wavelet series* (2) *is*

$$\| E_{2^{k-1},M}(f) \|_2 = \inf_{P_{2^{k-1},M}(f)} \|f - P_{2^{k-1},M}(f)\|_2 \quad \leq \quad \frac{2^{5/4}\left|\{f_1(1) - f_2(0)\}\right|}{\sqrt{\pi}}\frac{1}{2^{k+1/2}(M+1/2)},$$

where, $f = f_1 - f_2$ given by Lemma 2.2.

Proof of Theorem 3.3. Since

$$\begin{aligned}
f - P_{2^{k-1},M}(f) &= \left(\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} - \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}\right)\langle f, \psi_{n,m}\rangle_{\omega_{k,n}} \psi_{n,m} \\
&= \left(\left(\sum_{n=1}^{2^{k-1}} + \sum_{n=2^k}^{\infty}\right)\sum_{m=0}^{\infty} - \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}\right)\langle f, \psi_{n,m}\rangle_{\omega_{k,n}} \psi_{n,m} \\
&= \left(\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{\infty} + \sum_{n=2^k}^{\infty}\sum_{m=0}^{\infty} - \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}\right)\langle f, \psi_{n,m}\rangle_{\omega_{k,n}} \psi_{n,m} \\
&= \left(\sum_{n=1}^{2^{k-1}}\left(\sum_{m=0}^{M-1} + \sum_{m=M}^{\infty}\right) + \sum_{n=2^k}^{\infty}\sum_{m=0}^{\infty} - \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}\right)\langle f, \psi_{n,m}\rangle_{\omega_{k,n}} \psi_{n,m}
\end{aligned}$$

$$= \left(\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} + \sum_{n=1}^{2^{k-1}}\sum_{m=M}^{\infty} + \sum_{n=2^k}^{\infty}\sum_{m=0}^{\infty} - \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}\right)\langle f, \psi_{n,m}\rangle_{\omega_{k,n}}\, \psi_{n,m}$$
$$= \left(\sum_{n=1}^{2^{k-1}}\sum_{m=M}^{\infty} + \sum_{n=2^k}^{\infty}\sum_{m=0}^{\infty}\right)\langle f, \psi_{n,m}\rangle_{\omega_{k,n}}\, \psi_{n,m}$$

Now using the property of the o.n. wavelets $\{\psi_{n,m}(t)\}$ in the disjoint subsets of Ω. Then

$$\begin{aligned}
\left|f - P_{2^{k-1},M}(f)\right|^2 &= \left(\left(\sum_{n_1=1}^{2^{k-1}}\sum_{m_1=M}^{\infty} + \sum_{n_1=2^k}^{\infty}\sum_{m_1=0}^{\infty}\right)\langle f, \psi_{n_1,m_1}\rangle_{\omega_{k,n_1}}\, \psi_{n_1,m_1}\right)\\
&\times \left(\left(\sum_{n_2=1}^{2^{k-1}}\sum_{m_2=M}^{\infty} + \sum_{n_2=2^k}^{\infty}\sum_{m_2=0}^{\infty}\right)\langle f, \psi_{n_2,m_2}\rangle_{\omega_{k,n_2}}\, \psi_{n_2,m_2}(t)\right)\\
&= \sum_{n=1}^{2^{k-1}}\sum_{m=M}^{\infty}\left|\langle f, \psi_{n,m}\rangle_{\omega_{k,n}}\right|^2.
\end{aligned}$$

Hence,

$$\begin{aligned}
\left|E_{2^{k-1},M}(f)\right|^2 &= \left\|E_{n,m}(f)\right\|_2^2 = \int_{\Omega}\left|E_{n,m}(f)\right|^2 d\mu = \inf_{P_{2^{k-1},M}(f)}\int_{\Omega_{n,k}}\left|f - P_{2^{k-1},M}(f)\right|^2 d\mu\\
&= \inf_M \int_{\Omega_{n,k}}\sum_{n=1}^{2^{k-1}}\sum_{m=M}^{\infty}\left|\langle f, \psi_{n,m}\rangle_{\omega_{k,n}}\right|^2 d\mu = \inf_M \sum_{n=1}^{2^{k-1}}\sum_{m=M}^{\infty}\left|\langle f, \psi_{n,m}\rangle_{\omega_{k,n}}\right|^2 \int_{\Omega_{n,k}} d\mu\\
&\le \inf_M \sum_{n=1}^{2^{k-1}}\sum_{m=M}^{\infty}\left|\sqrt{\frac{2}{\pi}}\frac{1}{2^{k/2}}\frac{(-1)^m}{m+1/2}\{f_1(1) - f_2(0)\}\right|^2 \mu(\Omega_{n,k})\\
&= \inf_M \sum_{n=1}^{2^{k-1}}\sum_{m=M}^{\infty}\left|\sqrt{\frac{2}{\pi}}\frac{1}{2^{k/2}}\frac{(-1)^m}{m+1/2}\{f_1(1) - f_2(0)\}\right|^2 \frac{1}{2^{k-1}}\\
&= \inf_M \frac{\left|\{f_1(1) - f_2(0)\}\right|^2}{\pi}\frac{1}{2^{k-1}}\sum_{m=M}^{\infty}\frac{1}{(m+1/2)^2}\\
&= \frac{2\left|\{f_1(1) - f_2(0)\}\right|^2}{\pi}\frac{1}{2^k}\left\{\frac{1}{(M+1/2)^2} + \int_M^{\infty}\frac{dx}{(x+1/2)^2}\right\}, \text{ by Lemma 2.5}\\
&= \frac{4\left|\{f_1(1) - f_2(0)\}\right|^2}{\pi}\frac{1}{2^k}\left\{\frac{1}{(M+1/2)^2} + \frac{1}{(M+1/2)}\right\}\\
&\le \frac{2^{5/2}\left|\{f_1(1) - f_2(0)\}\right|^2}{\pi}\frac{1}{2^{k+1/2}(M+1/2)}.
\end{aligned}$$

Therefore

$$\| E_{2^{k-1},M}(f) \|_2 \le \frac{2^{5/4}\left|\{f_1(1) - f_2(0)\}\right|}{\sqrt{\pi}}\frac{1}{2^{k+1/2}(M+1/2)},$$

where $f = f_1 - f_2$ given by Lemma 2.2.

Thus the Theorem 3.3 is completely established.

Theorem 3.4. *If f is monotonic real valued function defined on the measurable set $\Omega = (0,1]$, then*

$$\langle f, \psi_{n,m}\rangle_{\omega_{k,n}} \leq \begin{cases} \sqrt{\frac{2}{\pi}}\frac{1}{2^{k/2}}\frac{(-1)^m}{m+1/2} f\left(\frac{n}{2^{k-1}}\right); \textit{ for } f \textit{ is non decreasing,} \\ \sqrt{\frac{2}{\pi}}\frac{1}{2^{k/2}}\frac{(-1)^m}{m+1/2} f\left(\frac{n-1}{2^{k-1}}\right); \textit{ for } f \textit{ is non increasing,} \end{cases}$$

$$\sum_{m=0}^{\infty} P_{k,n}^m \leq \begin{cases} \sqrt{\frac{\pi}{2}}\frac{1}{2^{k/2}} f\left(\frac{n}{2^{k-1}}\right); \textit{ for } f \textit{ is non decreasing,} \\ \sqrt{\frac{\pi}{2}}\frac{1}{2^{k/2}} f\left(\frac{n-1}{2^{k-1}}\right); \textit{ for } f \textit{ is non increasing,} \end{cases}$$

$$\| E_{2^{k-1},M}(f) \|_2 \leq \begin{cases} \sqrt{\frac{2}{\pi}}\frac{1}{2^{k/2}}\frac{1}{m+1/2} \left|f\left(\frac{n}{2^{k-1}}\right)\right|; \textit{ for } f \textit{ is non decreasing,} \\ \sqrt{\frac{2}{\pi}}\frac{1}{2^{k/2}}\frac{1}{m+1/2} \left|f\left(\frac{n-1}{2^{k-1}}\right)\right|; \textit{ for } f \textit{ is non increasing.} \end{cases}$$

Proof of Theorem 3.4. It can be developed on the lines of proofs of Theorems 3.1 , 3.2 and 3.3 considering f is monotonic function.

3.1. Corollaries

In this section, four new corollaries related to theorems 3.1, 3.2, 3.3 & 3.4 have been established in the following forms:

Corollary 3.5. *Let $f \in BV_{\Omega}(X)$ where $\Omega = (0,1]$ & $X = \mathbb{R}$ and the pseudo-Chebyshev wavelet series of the function f an order one i.e. $k = 1$ is given by*

$$\sum_{m=0}^{\infty} \langle f, \psi_{1,m}\rangle_{\omega_{1,1}} \psi_{1,m} = \langle f, \psi_{1,m}\rangle_{\omega_{1,1}} \psi_{1,0} + \langle f, \psi_{1,m}\rangle_{\omega_{1,1}} \psi_{1,1} + \langle f, \psi_{1,m}\rangle_{\omega_{1,1}} \psi_{1,2} + \cdots .$$

Then

$$\langle f, \psi_{1,m}\rangle_{\omega_{1,1}} \leq \frac{\{f_1(1) - f_2(0)\}}{\sqrt{\pi}} \frac{(-1)^m}{m+1/2},$$

where, $f = f_1 - f_2$ given by Lemma 2.2.

Corollary 3.6. *If Pf denotes the orthogonal projection operator of the function $f \in BV_{\Omega}(X)$ an order one and given by*

$$Pf = \sum_{m=0}^{\infty} \langle f, \psi_{1,m}\rangle_{\omega_{1,1}} \psi_{1,m} = \langle f, \psi_{1,0}\rangle_{\omega_{1,1}} \psi_{1,0} + \langle f, \psi_{1,1}\rangle_{\omega_{1,1}} \psi_{1,1} + \langle f, \psi_{1,2}\rangle_{\omega_{1,1}} \psi_{1,2} + \cdots ,$$

then

$$\langle Pf, \psi_{1,m}\rangle_{\omega_{1,1}} \leq \frac{(f_1(1) - f_2(0))\sqrt{\pi}}{2},$$

where, $f = f_1 - f_2$ given by Lemma 2.2.

Corollary 3.7. *Let $f \in BV_{\Omega}(X)$ where $\Omega = (0,1]$ & $X = \mathbb{R}$ and the pseudo-Chebyshev wavelet series of the function f an order one i.e. $k = 1$ is given by*

$$\sum_{m=0}^{\infty} \langle f, \psi_{1,m}\rangle_{\omega_{1,1}} \psi_{1,m} = \langle f, \psi_{1,m}\rangle_{\omega_{1,1}} \psi_{1,0} + \langle f, \psi_{1,m}\rangle_{\omega_{1,1}} \psi_{1,1} + \langle f, \psi_{1,m}\rangle_{\omega_{1,1}} \psi_{1,2} + \cdots .$$

Then

$$\| E_{1,M}(f) \|_2 \quad \leq \quad \left(\frac{\left| f_1(1) - f_2(0) \right|}{(2\pi^2)^{1/4}} \right) \frac{1}{(M+1/2)},$$

where, $f = f_1 - f_2$ *given by Lemma 2.2.*

Corollary 3.8. *Let* $f \in BV_\Omega(X)$ *where* $\Omega = (0,1]$ & $X = \mathbb{R}$ *and the pseudo-Chebyshev wavelet series of the function* f *an order one i.e.* $k = 1$ *is given by*

$$\sum_{m=0}^{\infty} \langle f, \psi_{1,m} \rangle_{\omega_{1,1}} \psi_{1,m} = \langle f, \psi_{1,m} \rangle_{\omega_{1,1}} \psi_{1,0} + \langle f, \psi_{1,m} \rangle_{\omega_{1,1}} \psi_{1,1} + \langle f, \psi_{1,m} \rangle_{\omega_{1,1}} \psi_{1,2} + \cdots .$$

Then

$$\langle f, \psi_{1,m} \rangle_{\omega_{1,1}} \quad \leq \quad \begin{cases} \frac{f(1)}{\sqrt{\pi}} \frac{(-1)^m}{m+1/2}; \text{ for } f \text{ is non decreasing,} \\ \frac{f(0)}{\sqrt{\pi}} \frac{(-1)^m}{m+1/2}; \text{ for } f \text{ is non increasing,} \end{cases}$$

$$\langle Pf, \psi_{1,m} \rangle_{\omega_{1,1}} \quad \leq \quad \begin{cases} \frac{\sqrt{\pi} f(1)}{2}; \text{ for } f \text{ is non decreasing,} \\ \frac{\sqrt{\pi} f(0)}{2}; \text{ for } f \text{ is non increasing,} \end{cases}$$

$$\| E_{1,M}(f) \|_2 \quad \leq \quad \begin{cases} \frac{|f(1)|}{(2\pi^2)^{1/4}} \frac{1}{(M+1/2)}; \text{ for } f \text{ is non decreasing,} \\ \frac{|f(0)|}{(2\pi^2)^{1/4}} \frac{1}{(M+1/2)}; \text{ for } f \text{ is non increasing.} \end{cases}$$

Proof of Corollaries 3.5, 3.6, 3.7 and 3.8. Proofs of Corollary 3.5, 3.6, 3.7 and 3.8 can be developed on the lines of proofs of Theorems 3.1, 3.2, 3.3 and 3.4 considering monotonic f respectively.

4. Illustrative Example

In this section, we calculate the approximation of a function

$$f(t) = \begin{cases} 2t^{1/2} - 3t^{3/2} + 7t^{5/2} + 10t^{7/2} - 16t^{9/2}; & \text{for} \quad 0 \leq t \leq 1, \\ 0; & \text{otherwise.} \end{cases}$$

by the pseudo-Chebyshev wavelet approximation method.

If we take in the Theorem 3.1, if $k = 1$, then $n = 1$ and

$$P_1(f)(t) = \sum_{m=0}^{\infty} \langle f, \psi_{1,m} \rangle_{\omega_{1,1}} \psi_{1,m}(t) = \sum_{m=0}^{\infty} f^{1,m} \psi_{1,m}(t) = f_0,$$

then we say that $f \approx f_0$ by the orthogonal projection operators $P_n(f)$ of an order $k = 1$.

The calculated values of the projection operators and its errors $P_{1,1}(f)$, $P_{1,2}(f)$, $P_{1,3}(f)$, $P_{1,4}(f)$, $P_{1,5}(f)$, $P_{1,6}(f)$, $P_{1,7}(f)$, $E_{1,1}(f)$, $E_{1,2}(f)$, $E_{1,3}(f)$, $E_{1,4}(f)$ $E_{1,5}(f)$ $E_{1,6}(f)$ $E_{1,7}(f)$, i.e. $P_{1,M}(f)$ & $E_{1,M}(f)$ for $1 \leq M \leq 7$ are given by Table 1 for different values of $0 \leq t \leq 1$.

$$\begin{aligned} P_1(f) &= \sum_{m=0}^{\infty} \langle f, \psi_{1,m} \rangle_{\omega_{1,1}} \psi_{1,m} \\ &\approx 1.5232\, \psi_{1,0} - 0.4708\, \psi_{1,1} - 0.6370\, \psi_{1,2} - 0.3600\, \psi_{1,3} - 0.0554\, \psi_{1,4} + 0 + \cdots + 0, \end{aligned}$$

$$\approx \quad f_0 = f, \ \ \forall\, t, \ \ 0 \le t \le 1,$$

and

$$E_{1,M}(f) \quad = \quad \inf_{P_{1,M}(f)} (f - P_{1,M}(f)) = \inf_{M} \sum_{m=M}^{\infty} \langle f, \psi_{1,m} \rangle_{\omega_{1,1}} \psi_{1,m} \approx 0, \ \text{for } M \ge 5.$$

4.1. Computaional Algorithm

Algorithm for the Solving of Computaional Problems in Symbolic Math in MATLAB

Declare the Variables: as like x, y, z, u, v
Define the Wavelets: as like here PCW
Define the exact , approximated and error valued functions:
as like symbolic f, g, h
Solve the Equations:
Use MATLAB's vpasolve: solve function to numerically
Output the Solutions:
Display the Numerical solutions in appropriate manner.

---

x	0.0000	0.1000	0.2000	0.3000	0.4000	0.5000	0.6000	0.7000	0.8000	0.9000	1.000
f_x	0.0000	0.5624	0.7756	1.0245	1.3600	1.7678	2.1738	2.4417	2.3670	1.6720	0.0000
$P_{1,1}f_x$	0.0000	0.5435	0.7686	0.9414	1.0870	1.2153	1.3313	1.4380	1.5373	1.6305	1.7188
$E_{1,1}f_x$	0.0000	0.0189	0.0070	0.0831	0.2730	0.5524	0.8425	1.0037	0.8297	0.0414	1.7187
$P_{1,2}f_x$	0.0000	0.9803	1.2913	1.4652	1.5574	1.5910	1.5782	1.5269	1.4423	1.3282	1.1875
$E_{1,2}f_x$	0.0000	0.4179	0.5157	0.4407	0.1974	0.1768	0.5956	0.9148	0.9247	0.3438	1.1875
$P_{1,3}f_x$	0.0000	0.2621	0.7642	1.2919	1.7574	2.0992	2.2686	2.2245	1.9308	1.3554	0.4688
$E_{1,3}f_x$	0.0000	0.3003	0.0115	0.2675	0.3974	0.3315	0.0948	0.2172	0.4362	0.3165	0.4687
$P_{1,4}f_x$	0.0000	0.5776	0.7220	0.9697	1.3525	1.8120	2.2359	2.4719	2.3349	1.6113	0.0625
$E_{1,4}f_x$	0.0000	0.0152	0.0536	0.0547	0.0075	0.0442	0.0620	0.0302	0.0321	0.0606	0.0625
$P_{1,5}f_x$	0.0000	0.5624	0.7756	1.0245	1.3600	1.7678	2.1738	2.4417	2.3670	1.6720	0.0000
$E_{1,5}f_x$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$P_{1,6}f_x$	0.0000	0.5624	0.7756	1.0245	1.3600	1.7678	2.1738	2.4417	2.3670	1.6720	0.0000
$E_{1,6}f_x$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$P_{1,7}f_x$	0.0000	0.5624	0.7756	1.0245	1.3600	1.7678	2.1738	2.4417	2.3670	1.6720	0.0000
$E_{1,7}f_x$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 1: $P_{1,M}f = f^{1,M}$ and f for $M = 1, 2, 3, 4$, and $k = 1$.

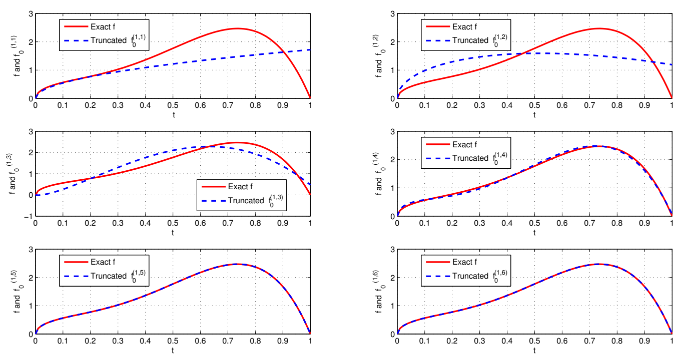

Figure 1: Graph of f & $P_{1,M}(f) = \sum_{m=o}^{M-1} f^{1,m}\psi_{1,m} = f_0^{1,M}$, $M = 1, 2, 3, 4, 5, 6$ & $k = 1$.

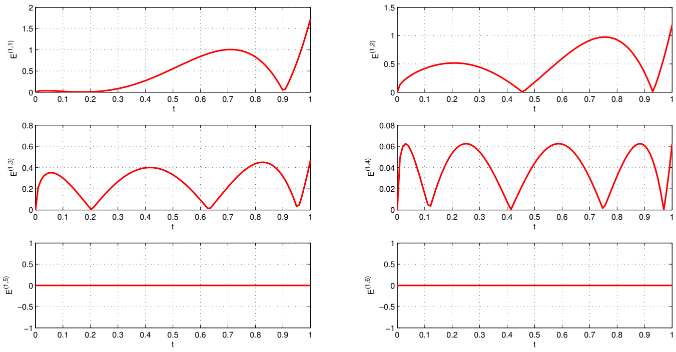

Figure 2: Graph of $E^{1,M}(f), M = 1,2,3,4,5,6$ & $k = 1$.

5. Discussion and Conclusions

Since, by Theorem 3.3 and Corollary 3.7 the absolute error $E_{2^{k-1},M}$ and $E_{1,M}$ of order k, & $k = 1$ respectively, using orthogonal projection operators, P_n, $n = 1,2,3,\cdots,2^{k-1}$ by the Pseudo-Chebyshev wavelets are

$$0 \leq \| E_{2^{k-1},M}(f) \|_2 \leq \frac{2^{5/4}\left|f_1(1) - f_2(0)\right|}{\sqrt{\pi}} \frac{1}{2^{k+1/2}(M+1/2)} \to 0 \text{ as } k \to \infty \text{ or } M \to \infty$$

$$0 \leq \| E_{1,M}(f) \|_2 \leq \frac{\left|f_1(1) - f_2(0)\right|}{(2\pi^2)^{1/4}} \frac{1}{(M+1/2)} \to 0 \text{ as } M \to \infty.$$

Therefore

$$\| E_{2^{k-1},M} \|_2 \to 0 \text{ as } k \to \infty \text{ or } M \to \infty \text{ and } \| E_{1,M} \|_2 \to 0 \text{ as } M \to \infty.$$

Thus the wavelet approximations determined in these results are best possible in the wavelet analysis [45]. More over the numerical findings in the Table 1 and Figure 1 and the absolute error in the Table 1 and Figure 2 which also shows that this approach can solve the problem effectively.

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Conflict of interest
The authors declare that there is no conflict of interest.

Data availability
This manuscript has no associated data.

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