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Durrmeyer form of the operators including Δ_ϕ-Appell polynomials

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Abstract. The present paper aims at demonstrating the Durrmeyer type modification of the operators including degenerate Appell polynomials and obtain the rate of approximation through the usual modulus of continuity and Lipschitz class functions. Moreover, we consider the convergence degree for a Peetre's $\mathcal{K}$-functional and give the Voronovskaja-type theorem.

1. Introduction and Preliminaries

In 1912, Bernstein [4] presented positive linear operators for the proof of the Weierstrass approximation theorem in [20]. These operators as follows

$$B_n(v;\gamma) = \sum_{a=0}^{n} b_{n,a}(\gamma) v\left(\frac{a}{n}\right),\ \gamma \in [0,1],\ n = 0,1,2,\dots \tag{1}$$

where the function v is continuous on $[0,1]$ and

$$b_{n,a}(\gamma) = \binom{n}{a}\gamma^a(1-\gamma)^{n-a}, a = 0,1,\dots,n.$$

By this means, Bernstein obtained a very special and simply proof. Further, Durrmeyer [8] presented a different form of the Bernstein operators as

$$D_n(v;\gamma) = (n+1)\sum_{a=0}^{n} b_{n,a}(\gamma)\left(\int_0^1 b_{n,a}(t)v(t)dt\right).$$

For the continuous functions on $[0,\infty)$, Szász [17] found the operators as

$$S_n(v;\gamma) = \sum_{a=0}^{\infty} d_{n,a}(\gamma) v\left(\frac{a}{n}\right) \tag{2}$$

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where

$$d_{n,a}(\gamma) = \frac{(n\gamma)^a}{a!}e^{-n\gamma}.$$

Then, the Durrmeyer form of the Szász operators were found by Mazhar and Totik [13] as

$$S_n^*(v;\gamma) = n\sum_{a=1}^{\infty} d_{n,a}(\gamma)\left(\int_0^{\infty} d_{n,a-1}(t)v(t)dt\right) + v(0)d_{n,0}(\gamma).$$

Many authors have widely studied about Szász operators. One of the studies, Jakimovski and Leviatan [10] introduced the operators

$$T_n(v;\gamma) = \frac{e^{-n\gamma}}{M(1)}\sum_{a=0}^{\infty}\mathcal{P}_a(n\gamma)v\left(\frac{a}{n}\right) \tag{3}$$

where $v \in C[0,\infty)$ and $C[0,\infty)$ denotes the set of continuous functions on $[0,\infty)$ such that $\left|v(\gamma)\right| \leq \alpha e^{\lambda\gamma}$ (α, λ are positive constants). In addition,

$$M(q) = \sum_{a=0}^{\infty} b_a q^a, \quad M(1) \neq 0$$

is analytic function has a formal series and $\mathcal{P}_a$ Appell polynomials [3] have the generating function as follows

$$M(q)e^{q\gamma} = \sum_{a=0}^{\infty}\mathcal{P}_a(\gamma)q^a, q \in \mathbb{R} \text{ (or } \mathbb{C}) \tag{4}$$

where $\left|q\right| < R$, $(R > 1)$. $\lim\limits_{n\to\infty} T_n(v;\gamma) = v(\gamma)$ is uniformly convergent on every compact subset of the interval $[0,\infty)$. This study is more generally version for the Szász operators. That is, if we take $M(1) = 1$ in (3), then we have the classical Szász operators (2). After the studies, Appell polynomials gained depth in many scientific fields such as physical, mathematics and engineering (see [3], [14], [15],[18],[19]). For example, İçöz et al. [9] introduced the operators with the help of generalized Appell polynomials as follows

$$\mathcal{J}_n(v;\gamma) = \frac{1}{M(k(1))E(n\gamma k(1))}\sum_{a=0}^{\infty}\mathcal{P}_a(n\gamma)v\left(\frac{a}{n}\right)$$

where $\mathcal{P}_a(\gamma)$ is the generalized Appell polynomials and they have

$$M(k(t))E(\gamma k(t)) = \sum_{a=0}^{\infty}\mathcal{P}_a(\gamma)t^a.$$

In addition, M, E and k are analytic functions as follows

$$M(t) = \sum_{a=0}^{\infty} b_a t^a,\ E(t) = \sum_{a=0}^{\infty} c_a t^a,\ k(t) = \sum_{a=1}^{\infty} d_a t^a,\ |t| < R,\ (R > 1).$$

Another example, Durrmeyer type form of the Jakimovski-Leviatan operators was presented by Karaisa [11] as

$$T_n^*(v;\gamma) = \frac{e^{-n\gamma}}{M(1)}\sum_{a=0}^{\infty}\frac{\mathcal{P}_a(n\gamma)}{B(n+1,a)}\int_0^{\infty} z_{n,a}(t)v(t)dt + \frac{e^{-n\gamma}}{M(1)}b_0 v(0),\ \gamma \geq 0$$

where $v \in C_B[0,\infty)$, $C_B[0,\infty)$ is denoted by bounded, continuous and real valued functions, $B(n+1,a)$ is the Beta function and $z_{n,a}(t) = \frac{t^{a-1}}{(1+t)^{n+a+1}}$.
Moreover, Costabile and Longo [6] have presented the degenerate Appell polynomials (or Δ_ϕ-Appell polynomials) as follows

$$M(t;\phi)(1+\phi t)^{\gamma/\phi} = \sum_{a=0}^{\infty} \mathcal{A}_a(\gamma;\phi)\frac{t^a}{a!} \tag{5}$$

where

$$M(t) := M(t;\phi) = \sum_{a=0}^{\infty} \sigma_{a,\phi}\frac{t^a}{a!}, \sigma_{0,\phi} \neq 0. \tag{6}$$

Based on the polynomials (5), Sergi et al. [16] have presented the operators as follows

$$B_n(v;\gamma) := \frac{1}{M(1;\phi)(1+\phi)^{\frac{n\gamma}{\phi}}} \sum_{a=0}^{\infty} \mathcal{A}_a(n\gamma;\phi)\frac{1}{a!}v\left(\frac{a(1+\phi)}{n}\right) \tag{7}$$

where $\gamma \in [0,\infty)$, $v \in C[0,\infty)$. The operators are positive and linear under the proper conditions and if we take limits for the function (5) and the operator (7) as $\phi \to 0^-$, we can find that (5) reduce to (4) and (7) reduce to the Jakimovski-Leviatan operators (3), respectively. In the light of these studies, we consider the Durrmeyer type modification of the operators involving degenerate Appell polynomials as follows

$$\begin{aligned} K_n(v;\gamma) \quad := \quad & \frac{1}{M(1;\phi)(1+\phi)^{\frac{n\gamma}{\phi}}} \sum_{a=1}^{\infty} \frac{\mathcal{A}_a(n\gamma;\phi)}{B(n+1,a)a!} \int_0^\infty S_{n,a}(t)v(t(1+\phi))dt \\ & + \frac{\mathcal{A}_0(n\gamma;\phi)}{M(1;\phi)(1+\phi)^{\frac{n\gamma}{\phi}}} v(0) \end{aligned} \tag{8}$$

where $S_{n,a}(t) = \frac{t^{a-1}}{(1+t)^{n+a+1}}$. Under the restrictions

$$\begin{array}{ll} (i) & M(1;\phi) \neq 0, \text{ for fixed } \forall\phi \in (-1,0), \\ (ii) & \sigma_{a,\phi}/M(1;\phi) \geq 0, \text{ for } \forall a \in \mathbb{N} \text{ and } \forall\phi \in (-1,0), \\ (iii) & (6) \text{ and } (7) \text{ are convergent in } |t| < R, (R > 1), \end{array} \tag{9}$$

the operators are linear and positive.
The aim of the paper are considering the approximation properties of the operators (8) and estimating the degree of approximation. In section 2 of the paper, we obtain some results for the operators applying test functions, computations of central moments and the Korovkin theorem. In section 3, we get the degree of approximation for the operators (8) by considering the modulus of continuity, Peetre's $\mathcal{K}$-functional, Lipschitz class and Voronovskaja-type theorem.

2. Auxiliary Results

Lemma 2.1. *For the linear operator $K_n(v;y)$, one can obtain that*

$$\begin{aligned} K_n(1;\gamma) &= 1, \\ K_n(t;\gamma) &= \gamma + \frac{1}{n}\frac{M'(1;\phi)}{M(1;\phi)}(1+\phi), \\ K_n(t^2;\gamma) &= \frac{1}{(n-1)n}\left[\frac{M''(1;\phi)+2M'(1;\phi)}{M(1;\phi)}(1+\phi)^2\right. \end{aligned}$$

$$
\begin{aligned}
&+n\gamma\left(\frac{2M'(1;\phi)}{M(1;\phi)}(1+\phi)+2+\phi\right)+(n\gamma)^2\Bigg], n>1,\\
K_n(t^3;\gamma) \;=\;& \frac{1}{(n-2)(n-1)n}\Bigg[\frac{M'''(1;\phi)+6M''(1;\phi)+6M'(1;\phi)}{M(1;\phi)}(1+k)^3\\
&+n\gamma\Bigg(\frac{3\left(M''(1;\phi)+4M'(1;\phi)\right)}{M(1;\phi)}(1+\phi)^2-\frac{3\phi M'(1;\phi)}{M(1;\phi)}(1+\phi)\\
&+2\phi^2+6\phi+6\Bigg)+(n\gamma)^2\left(\frac{3M'(1;\phi)}{M(1;\phi)}(1+\phi)+3\phi+6\right)+(n\gamma)^3\Bigg], n>2,\\
K_n(t^4;\gamma) \;=\;& \frac{1}{(n-3)(n-2)(n-1)n}\Bigg[\frac{M^{(IV)}(1;\phi)+12M'''(1;\phi)}{M(1;\phi)}(1+\phi)^4\\
&+\frac{36M''(1;\phi)+24M'(1;\phi)}{M(1;\phi)}(1+\phi)^4\\
&+n\gamma\Bigg(\frac{4\left(M'''(1;\phi)+9M''(1;\phi)+18M'(1;\phi)\right)}{M(1;\phi)}(1+\phi)^3\\
&-\frac{6\phi\left(M''(1;\phi)+6M'(1;\phi)\right)}{M(1;\phi)}(1+\phi)^2+\frac{8\phi^2M'(1;\phi)}{M(1;\phi)}(1+\phi)\\
&+6(\phi^3+4\phi^2+6\phi+4)\Bigg)\\
&+(n\gamma)^2\Bigg(\frac{6\left(M''(1;\phi)+6M'(1;\phi)\right)}{M(1;\phi)}(1+\phi)^2-\frac{12\phi M'(1;\phi)}{M(1;\phi)}(1+\phi)\\
&+11\phi^2+36\phi+36\Bigg)+(n\gamma)^3\left(\frac{4M'(1;\phi)}{M(1;\phi)}(1+\phi)+6\phi+12\right)\\
&+(n\gamma)^4\Bigg], n>3.
\end{aligned}
$$

Lemma 2.2. *The operator K_n has the central moments as*

$$K_n(t-\gamma;\gamma) = \frac{1}{n}\frac{M'(1;\phi)}{M(1;\phi)}(1+\phi), \tag{10}$$

$$
\begin{aligned}
K_n((t-\gamma)^2;\gamma) =& \frac{1}{(n-1)n}\Bigg(\frac{M''(1;\phi)+2M'(1;\phi)}{M(1;\phi)}(1+\phi)^2\\
&+\frac{2\gamma M'(1;\phi)}{M(1;\phi)}(1+\phi)+n\gamma(2+\phi+\gamma)\Bigg), n>1.
\end{aligned} \tag{11}
$$

Theorem 2.3. *For $v \in C[0,\infty)\cap G$, the operators (8) verify*

$$\lim_{n\to\infty} K_n(v;\gamma) = v(\gamma)$$

converges uniformly in every compact subset of $[0,\infty)$. *Here* $G = \left\{v \in C[0,\infty) : \lim\limits_{\gamma\to\infty}\frac{v(\gamma)}{1+\gamma^2} \text{ is finite}\right\}$.

Proof. From Lemma 2.1,

$$\lim_{n\to\infty} K_n(t^i;\gamma) = \gamma^i, i = 0,1,2$$

is uniformly on $[0,\infty)$. Considering that the universal Korovkin theorem [1], we obtain the desired result. □

3. Approximation Results

In this section, we define that $T[0,\infty)$ is the set of all function v which defined on $[0,\infty)$ and satisfied $|v(t)| \leq M_v(1+t^2)$. Here, M_v is a constant that depending only on v. $E[0,\infty)$ is the subspace of all continuous functions belong to $T[0,\infty)$. We also define that $Q[0,\infty)$ is the subspace of all functions belonging to $E[0,\infty)$ and $\lim\limits_{\gamma\to\infty} \frac{v(\gamma)}{1+\gamma^2}$ is finite. In addition, the sup norm on $Q[0,\infty)$ is

$$\|v\| = \sup_{\gamma\in[0,\infty)} \frac{|v(\gamma)|}{1+\gamma^2}.$$

For $v \in [0,\infty)$ and $\delta > 0$, the modulus of continuity [2] is defined by

$$\omega(v;\delta) = \sup_{\substack{\gamma,t\in[0,\infty),\\ |t-\gamma|\leq\delta}} |v(t)-v(\gamma)|. \tag{12}$$

The modulus of continuity has the following properties:

- $\left|v(t)-v(\gamma)\right| \leq \omega(v;\delta)\left(1+\frac{|t-\gamma|}{\delta}\right)$,
- If v is uniformly continuous on $[0,\infty)$, then $\lim\limits_{\delta\to 0}\omega(v;\delta) = 0$.

Theorem 3.1. *For $v \in C[0,\infty)\cap G$, we have the modulus of continuity of the operator K_n as*

$$\left|K_n(v;\gamma)-v(\gamma)\right| \leq 2\omega(v;\delta)$$

where $\delta(y;\phi) = \sqrt{K_n((t-\gamma)^2;\gamma)}$.

Proof. From the above properties, we obtain

$$\begin{aligned}\left|K_n(v;\gamma)-v(\gamma)\right| &\leq K_n(\left|v(t)-v(\gamma)\right|;\gamma)\\ &\leq \omega(v;\delta)\left(1+\frac{1}{\delta}K_n(\left|t-\gamma\right|;\gamma)\right).\end{aligned}$$

From the Cauchy-Schwarz inequality and definition of $\delta(\gamma;k)$, we achieve

$$\begin{aligned}\left|K_n(v;\gamma)-v(\gamma)\right| &\leq \omega(v;\delta)\left(1+\frac{1}{\delta}\sqrt{K_n((t-\gamma)^2;\gamma)}\right)\\ &= 2\omega(v;\delta).\end{aligned}$$

□

Now, we obtain the approximation degree of the operator K_n for functions belonging to Lipschitz class $Lip_Z(b)$. Lipschitz class is defined as

$$Lip_Z(b) = \left\{v \in E[0,\infty) : \left|v(t)-v(\gamma)\right| \leq M_v\frac{\left|t-\gamma\right|^b}{(t+\gamma)^{\frac{b}{2}}}\right\}$$

where $0 < b \leq 1$, $\gamma \in (0,\infty)$, $t \in [0,\infty)$ and M_v is a constant depends on v (see [17]).

Theorem 3.2. *Let $v \in Lip_Z(b)$ and $\delta(\gamma;\phi) = \sqrt{K_n((t-\gamma)^2;\gamma)}$. Then we get the following inequality*

$$\left|K_n(v;\gamma)-v(\gamma)\right| \leq M_v\frac{\delta_n^b(\gamma;\phi)}{\gamma^{\frac{b}{2}}}.$$

Proof. From the linearity and positivity of the operator K_n,

$$\begin{aligned}\left|K_n(v;\gamma)-v(\gamma)\right| &= \left|K_n(v(t)-v(\gamma);\gamma)\right| \\ &\leq K_n\left(\left|v(t)-v(\gamma)\right|;\gamma\right)\end{aligned}$$

is obtained. By considering that the operator K_n as monotonity increased and $v \in Lip_Z(b)$, the inequality

$$\left|K_n(v;\gamma)-v(\gamma)\right| \leq M_v K_n\left(\frac{\left|t-\gamma\right|^b}{(t+\gamma)^{\frac{b}{2}}};\gamma\right)$$

is found. For $p=\frac{2}{b}, q=\frac{2}{2-b}$, using the Hölder inequality, one can have the following equations

$$\begin{aligned}\left|K_n(v;\gamma)-v(\gamma)\right| &\leq M_v\left(K_n\left(\left|\frac{\left|t-\gamma\right|^2}{t+\gamma}\right|;\gamma\right)\right)^{\frac{b}{2}}\left(K_n(1;\gamma)\right)^{\frac{2-b}{2}} \\ &= M_v\left(K_n\left(\frac{(t-\gamma)^2}{t+\gamma};\gamma\right)\right)^{\frac{b}{2}} \\ &\leq M_v\left(K_n\left(\frac{(t-\gamma)^2}{\gamma};\gamma\right)\right)^{\frac{b}{2}} \\ &= M_v\frac{\delta_n^b(\gamma;\phi)}{\gamma^{\frac{b}{2}}}.\end{aligned}$$

Therefore, the proof is completed. □

Let us remember that $C_B[0,\infty)$ is the space of all bounded and continuous functions on endowed with

$$\|v\|_{C_B[0,\infty)} = \sup_{(\gamma,z)\in[0,\infty)}\left|v(\gamma,z)\right|. \tag{13}$$

From the impression, one can express the definition of Peetre's $\mathcal{K}$-functional and then demonstrate the theorem regarding this definition.

Definition 3.3. *[7] Let $C_B^2[0,\infty)$ be the space of all functions $v \in C_B[0,\infty)$ and belongs to the space $C_B[0,\infty)$ such that $\frac{\partial^i v}{\partial\gamma^i}$, $(i=1,2)$. The norm on the space $C_B^2[0,\infty)$ is defined as follows*

$$\|v\|_{C_B^2[0,\infty)} = \|v\|_{C_B[0,\infty)} + \sum_{i=1}^{2}\left\|\frac{\partial^i v}{\partial\gamma^i}\right\|_{C_B[0,\infty)}. \tag{14}$$

For $v \in C_B[0,\infty)$ and $\delta > 0$, the Peetre's $\mathcal{K}$-functional is defined by

$$\mathcal{K}(v;\delta) = \inf_{p\in C_B^2[0,\infty)}\left\{\left\|v-p\right\| + \delta\left\|p\right\|\right\}.$$

Besides,

$$\mathcal{K}(v;\delta) \leq L\left[\omega_2\left(v;\sqrt{\delta}\right) + min(1,\delta)\,\|v\|_{C_B^2[0,\infty)}\right] \tag{15}$$

(see [5]). Here L is a constant independent of v and δ. $\omega_2\left(v;\sqrt{\delta}\right)$ is defined by the second modulus of continuity [12] as follows

$$\omega_2(v;\delta) := \sup_{0<t\leq\delta}\|v(\cdot+2t)-2v(\cdot+t)+v(\cdot)\|_{C_B[0,\infty)}.$$

Now, we can give the theorem.

Theorem 3.4. *For $v \in C_B[0,\infty)$, the operator K_n satisfy the inequality as follows*

$$\begin{aligned}\left|K_n(v;\gamma)-v(\gamma)\right| &\leq 4L\left\{\omega_2\left(v;\frac{1}{2}\sqrt{\delta(\gamma;\phi)+\left(\frac{1}{n}\frac{M'(1;\phi)}{M(1;\phi)}(1+\phi)\right)^2}\right)\right.\\ &\quad\left.+\min\left(1,\delta(\gamma;\phi)+\left(\frac{1}{n}\frac{M'(1;\phi)}{M(1;\phi)}(1+\phi)\right)^2\right)\|v\|_{C_B[0,\infty)}\right\}\\ &\quad+\omega\left(v;\left(\frac{1}{n}\frac{M'(1;\phi)}{M(1;\phi)}(1+\phi)\right)\right)\end{aligned}$$

where L is a positive constant, ω_2 is the second order modulus of continuity and $\delta(\gamma;\phi)$ is defined by (11).

Proof. Let us define the operator

$$\tilde{K}_n(g;\gamma) = K_n(g;\gamma)+g(\gamma)-g\left(\gamma+\frac{1}{n}\frac{M'(1;\phi)}{M(1;\phi)}(1+\phi)\right). \tag{16}$$

We show that the equalities $\tilde{K}_n(1;\gamma)=1$ and $\tilde{K}_n(t;\gamma)=\gamma$ are satisfied. For $t\in[0,\infty)$, Taylor expansion of the function $p\in C_B^2[0,\infty)$ is as follows

$$p(t)=p(\gamma)+\frac{\partial p(\gamma)}{\partial\gamma}(t-\gamma)+\int_\gamma^t (t-v)\frac{\partial^2 p(v)}{\partial v^2}dv.$$

If we apply the operators $\tilde{K}_n$ to both side of the above equality, then we get

$$\begin{aligned}\tilde{K}_n(p;\gamma)-p(\gamma) &= \frac{\partial p(\gamma)}{\partial\gamma}\tilde{K}_n(t-\gamma;\gamma)+\tilde{K}_n\left(\int_\gamma^t (t-v)\frac{\partial^2 p(v)}{\partial v^2}dv;\gamma\right)\\ &= \tilde{K}_n\left(\int_\gamma^t (t-v)\frac{\partial^2 p(v)}{\partial v^2}dv;\gamma\right).\end{aligned}$$

From (16), we find

$$\begin{aligned}\tilde{K}_n(p;\gamma)-p(\gamma) &= K_n\left(\int_\gamma^t (t-v)\frac{\partial^2 p(v)}{\partial v^2}dv;\gamma\right)\\ &\quad-\int_\gamma^{\gamma+\frac{1}{n}\frac{M'(1;\phi)}{M(1;\phi)}(1+\phi)}\left(\gamma+\frac{1}{n}\frac{M'(1;\phi)}{M(1;\phi)}(1+\phi)-v\right)\frac{\partial^2 p(v)}{\partial v^2}dv\end{aligned}$$

and then

$$\begin{aligned}\left|\tilde{K}_n(p;\gamma)-p(\gamma)\right| &\leq K_n\left(\left|\int_\gamma^t |t-v|\left|\frac{\partial^2 p(v)}{\partial v^2}\right|dv\right|;\gamma\right)\\ &\quad+\left|\int_\gamma^{\gamma+\frac{1}{n}\frac{M'(1;\phi)}{M(1;\phi)}(1+\phi)}\left|\gamma+\frac{1}{n}\frac{M'(1;\phi)}{M(1;\phi)}(1+\phi)-v\right|\left|\frac{\partial^2 p(v)}{\partial v^2}\right|dv\right|\end{aligned}$$

$$
\begin{aligned}
&\leq K_n\left((t-\gamma)^2;\gamma\right)\left\|p\right\|_{C_B^2[0,\infty)} \\
&\quad+\left(\frac{1}{n}\frac{M'(1;\phi)}{M(1;\phi)}(1+\phi)\right)\left\|p\right\|_{C_B^2[0,\infty)}. \qquad (17)
\end{aligned}
$$

In addition, by using the equality (16), we have

$$
\begin{aligned}
\left|\tilde{K}_n(v;\gamma)\right| &\leq \left|K_n(v;\gamma)\right|+\left|v(\gamma)\right|+\left|v\left(\gamma+\frac{1}{n}\frac{M'(1;\phi)}{M(1;\phi)}(1+\phi)\right)\right| \\
&\leq 3\left\|v\right\|_{C_B[0,\infty)}.
\end{aligned}
$$

With the help of the above inequality, we can obtain

$$
\begin{aligned}
\left|K_n(v;\gamma)-v(\gamma)\right| &= \left|\tilde{K}_n(v;\gamma)-v(\gamma)+v\left(\gamma+\frac{1}{n}\frac{M'(1;\phi)}{M(1;\phi)}(1+\phi)\right)-v(\gamma)\right| \\
&\leq \left|\tilde{K}_n(v;\gamma)-\tilde{K}_n(p;\gamma)\right|+\left|p(\gamma)-v(\gamma)\right|+\left|\tilde{K}_n(p;\gamma)-p(\gamma)\right| \\
&\quad+\left|v\left(\gamma+\frac{1}{n}\frac{M'(1;\phi)}{M(1;\phi)}(1+\phi)\right)-v(\gamma)\right| \\
&\leq 4\left\|v-p\right\|_{C_B^2[0,\infty)}+\left|\tilde{K}_n(p;\gamma)-p(\gamma)\right| \\
&\quad+\left|v\left(\gamma+\frac{1}{n}\frac{M'(1;\phi)}{M(1;\phi)}(1+\phi)\right)-v(\gamma)\right|.
\end{aligned}
$$

From the equality (17), we get

$$
\begin{aligned}
\left|K_n(v;\gamma)-v(\gamma)\right| &\leq 4\left\|v-p\right\|_{C_B^2[0,\infty)}+K_n\left((t-\gamma)^2;\gamma\right)\left\|p\right\|_{C_B^2[0,\infty)} \\
&\quad+\left(\frac{1}{n}\frac{M'(1;\phi)}{M(1;\phi)}(1+\phi)\right)^2\left\|p\right\|_{C_B^2[0,\infty)} \\
&\quad+\omega\left(v;\left(\frac{1}{n}\frac{M'(1;\phi)}{M(1;\phi)}(1+\phi)\right)\right).
\end{aligned}
$$

According to the definition of Peetre's $\mathcal{K}$-functional, the desired result is obtained as follows

$$
\begin{aligned}
\left|K_n(v;\gamma)-v(\gamma)\right| &\leq 4\mathcal{K}\left(v;\frac{1}{4}\left(\delta(\gamma;\phi)+\left(\frac{1}{n}\frac{M'(1;\phi)}{M(1;\phi)}(1+\phi)\right)^2\right)\right) \\
&\quad+\omega\left(v;\left(\frac{1}{n}\frac{M'(1;\phi)}{M(1;\phi)}(1+\phi)\right)\right) \\
&\leq 4L\left\{\omega_2\left(v;\frac{1}{2}\sqrt{\delta(\gamma;\phi)+\left(\frac{1}{n}\frac{M'(1;\phi)}{M(1;\phi)}(1+\phi)\right)^2}\right)\right. \\
&\quad\left.+\min\left(1,\delta(\gamma;\phi)+\left(\frac{1}{n}\frac{M'(1;\phi)}{M(1;\phi)}(1+\phi)\right)^2\right)\left\|v\right\|_{C_B^2[0,\infty)}\right\} \\
&\quad+\omega\left(v;\left(\frac{1}{n}\frac{M'(1;\phi)}{M(1;\phi)}(1+\phi)\right)\right)
\end{aligned}
$$

where L is a positive constant and $\delta(\gamma;\phi):=K_n\left((t-\gamma)^2;\gamma\right)$. $\square$

Lastly, we evidence the following Voronovskaja-type theorem.

Theorem 3.5. *For $\gamma \in [0,\infty)$ and $v, v', v'' \in Q[0,\infty)$, the equality is satisfied*

$$\lim_{n\to\infty} n\left(K_n(v;\gamma)-v(\gamma)\right) = v'(\gamma)\frac{M'(1;\phi)}{M(1;\phi)}(1+\phi)+\frac{1}{2}v''(\gamma)\left(\gamma(2+\phi)+\gamma^2\right).$$

Proof. If we consider the Taylor's expansion formula, we obtain

$$v(t) = v(\gamma)+v'(\gamma)(t-\gamma)+\frac{1}{2}v''(\gamma)(t-\gamma)^2+\mu(t;\gamma)(t-\gamma)^2$$

where $t \in [0,\infty)$, $\mu(t;\gamma) \in Q[0,\infty)$ and $\lim_{t\to\gamma}\mu(t;\gamma) = 0$. Then, by applying $K_n(v;\gamma)$ to both side of above equality, we have

$$\begin{aligned} K_n(v;\gamma)-v(\gamma) &= v'(\gamma)K_n(t-\gamma;\gamma)+\frac{1}{2}v''(\gamma)K_n((t-\gamma)^2;\gamma) \\ &\quad +K_n\left(\mu(t;\gamma)(t-\gamma)^2;\gamma\right). \end{aligned}$$

Multiplying with n and take limit as $n\to\infty$, we find

$$\begin{aligned} \lim_{n\to\infty} n\left(K_n(v;\gamma)-v(\gamma)\right) &= v'(\gamma)\lim_{n\to\infty} nK_n((t-\gamma);\gamma) \\ &\quad +\frac{1}{2}v''(\gamma)\lim_{n\to\infty} nK_n((t-\gamma)^2;\gamma) \\ &\quad +\lim_{n\to\infty} nK_n\left(\mu(t;\gamma)(t-\gamma)^2;\gamma\right) \\ &= v'(\gamma)\frac{M'(1;\phi)}{M(1;\phi)}(1+\phi)+\frac{1}{2}v''(\gamma)\left(\gamma(2+\phi)+\gamma^2\right) \\ &\quad +\lim_{n\to\infty} nK_n\left(\mu(t;\gamma)(t-\gamma)^2;\gamma\right). \end{aligned} \tag{18}$$

In the last equality, we consider the Lemma 2.2. By means of the Cauchy-Schwarz inequality, we have

$$nK_n\left(\mu(t;\gamma)(t-\gamma)^2;\gamma\right) \le \sqrt{n^2K_n((t-\gamma)^4;\gamma)}\sqrt{K_n(\mu^2(t;\gamma);\gamma)}.$$

Since $\lim_{t\to\gamma}\mu(t;\gamma) = 0$, $\mu^2(.;\gamma) \in Q[0,\infty)$ and $K_n\left((t-\gamma)^4;\gamma\right) = O\left(n^{-2};\phi\right)$, we find that

$$\lim_{n\to\infty} K_n\left(\mu^2(t;\gamma);\gamma\right) = \mu^2(\gamma;\gamma) = 0$$

uniformly. Therefore, the equality (18) become as

$$\lim_{n\to\infty} n\left(K_n(v;\gamma)-v(\gamma)\right) = v'(\gamma)\frac{M'(1;\phi)}{M(1;\phi)}(1+\phi)+\frac{1}{2}v''(\gamma)\left(\gamma(2+\phi)+\gamma^2\right).$$

Consequently, the proof is completed. □

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