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A note on the approximate pseudospectrum of upper triangular operator matrices

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Abstract. We study the approximate pseudospectrum $\sigma_{ap,\varepsilon}(\mathcal{T})$ of a 2×2 upper triangular bounded operator matrix

$$\mathcal{T}=\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}: \mathcal{X}\times\mathcal{X}\to\mathcal{X}\times\mathcal{X},$$

on a complex Banach space, focusing on its relationship with the approximate pseudospectrum $\sigma_{ap,\varepsilon}(A)$ and $\sigma_{ap,\varepsilon}(D)$ of the diagonal entries. First, by constructing counterexamples, we show that in general there is no simple inclusion between $\sigma_{ap,\varepsilon}(\mathcal{T})$ and $\sigma_{ap,\varepsilon}(A)\cup\sigma_{ap,\varepsilon}(D)$. Next, we establish a sufficient condition: if $\lambda\in\sigma_{ap,\varepsilon}(D)$ and $\mathcal{R}(B)\subseteq\mathcal{R}(A-\lambda I)$, then $\lambda\in\sigma_{ap,\varepsilon}(\mathcal{T})$. Under this condition and an additional inequality constraint, we obtain the equality $\sigma_{ap,\varepsilon}(\mathcal{T})=\sigma_{ap,\varepsilon}(A)\cup\sigma_{ap,\varepsilon}(D)$. In addition, when the coupling operator B can be regarded as a sufficiently small perturbation, we show that $\sigma_{ap,\varepsilon}(\mathcal{T})$ may be "sandwiched" between appropriately expanded or contracted approximate pseudospectrum of the diagonal entries A and D.

1. Introduction

Let $\mathcal{X}$ be a Banach space and let $\mathcal{B}(\mathcal{X})$ denote the set of all bounded linear operators on $\mathcal{X}$. The study of the 2×2 upper triangular operator matrix

$$\mathcal{T}=\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}: \mathcal{X}\times\mathcal{X}\to\mathcal{X}\times\mathcal{X},$$

where $A,B,D\in\mathcal{B}(\mathcal{X})$, arises naturally from the following observation: Let $T\in\mathcal{B}(\mathcal{X})$ and suppose $\mathcal{M}\subset\mathcal{X}$ is a closed invariant subspace for T. If there exists a closed subspace $\mathcal{N}\subset\mathcal{X}$ such that $\mathcal{X}=\mathcal{M}\oplus\mathcal{N}$, then T admits a 2×2 upper triangular operator matrix representation.

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The spectrum of a bounded linear operator is one of the central objects in operator theory, providing key insights into its algebraic and analytic behavior. However, in many important applications, especially when dealing with non-normal operators, spectral information alone may not suffice to capture the operator's true behavior under perturbations. This has led to the development of pseudospectrum, which offer a useful tool for analyzing spectral instability and non-normal phenomena, was introduced and studied by researchers, e.g., [4, 6, 9, 10, 12, 13].

Among several variants, the approximate pseudospectrum has emerged as an intermediate concept between the spectrum and the classical pseudospectrum. For any $\varepsilon > 0$ and $T \in \mathcal{B}(\mathcal{X})$, the approximate pseudospectrum is defined by

$$\sigma_{ap,\varepsilon}(T) = \sigma_{ap}(T) \cup \left\{\lambda \in \mathbb{C} : \inf_{\|x\|=1} \|(T - \lambda I)x\| < \varepsilon\right\},$$

where

$$\sigma_{ap}(T) = \left\{\lambda \in \mathbb{C} : \inf_{\|x\|=1} \|(T - \lambda I)x\| = 0\right\}.$$

The approximate pseudospectrum was further developed by Ammar et al. [1, 2], who established

$$\sigma_{ap,\varepsilon}(T) = \bigcup_{\|E\|<\varepsilon} \sigma_{ap}(T + E),$$

which emphasizes the perturbational sensitivity of the approximate point spectrum. Moreover, they introduced the concept of the essential approximate pseudospectrum as follows

$$\sigma_{eap,\varepsilon}(T) = \bigcap_{K\in\mathcal{K}(\mathcal{X})} \sigma_{ap,\varepsilon}(T + K),$$

where $\mathcal{K}(\mathcal{X})$ is the subspace of compact operators on $\mathcal{X}$. In Banach spaces, Veeramani et al. [11] established the upper hemicontinuity of the approximate pseudospectrum and demonstrated through counterexamples that lower hemicontinuity is generally impossible. Recently, Ettayb [5] extended the theory of approximate pseudospectrum to ultrametric Banach spaces.

This paper focuses on extending some of these ideas to upper triangular operator matrices $\mathcal{T}$. For the upper triangular operator matrix, it is well-known that the spectrum satisfies $\sigma(\mathcal{T}) \subseteq \sigma(A) \cup \sigma(D)$. One might naturally expect that a similar relation holds for the approximate point spectrum, namely $\sigma_{ap}(\mathcal{T}) \subseteq \sigma_{ap}(A) \cup \sigma_{ap}(D)$. Indeed, this inclusion is valid, as was established by Hwang et al. [7]. However, when considering the approximate pseudospectrum, the situation becomes more complex. Neither the inclusion $\sigma_{ap,\varepsilon}(\mathcal{T}) \subseteq \sigma_{ap,\varepsilon}(A) \cup \sigma_{ap,\varepsilon}(D)$ nor the reverse inclusion $\sigma_{ap,\varepsilon}(A) \cup \sigma_{ap,\varepsilon}(D) \subseteq \sigma_{ap,\varepsilon}(\mathcal{T})$ holds in general. To illustrate this, we present two counterexamples:

Example 1.1 (showing $\sigma_{ap,\varepsilon}(\mathcal{T}) \nsubseteq \sigma_{ap,\varepsilon}(A) \cup \sigma_{ap,\varepsilon}(D)$). *Let $\mathcal{X} = \ell^2(\mathbb{N})$, $0.75 < \varepsilon < 1$, $A = D = \mathbf{0}$ (the zero operator), and $B = S_r$ be the right shift operator, defined by $S_r(x_n) = x_{n-1}$ for $n \geq 2$, where $\{x_n\}_{n=1}^{\infty} \in \mathcal{X}$. Consider the operator $\mathcal{T}$ defined by*

$$\mathcal{T} = \begin{pmatrix} 0 & S_r \\ 0 & 0 \end{pmatrix} : \mathcal{X} \times \mathcal{X} \longrightarrow \mathcal{X} \times \mathcal{X}.$$

Clearly, we have

$$\sigma_{ap,\varepsilon}(A) \cup \sigma_{ap,\varepsilon}(D) = \{\lambda \in \mathbb{C} : \inf_{\|x\|=1} \|\lambda x\| < \varepsilon\} = \{\lambda \in \mathbb{C} : |\lambda| < \varepsilon\}.$$

For the operator $\mathcal{T}$ and $x = (x_1, x_2)^T \in \mathcal{X} \times \mathcal{X}$, let us compute for $\lambda = 1$,

$$\begin{aligned}\inf_{\|x\|=1} \|(\mathcal{T} - I)x\| &= \inf_{\|x_1\|^2+\|x_2\|^2=1} \left\| \begin{pmatrix} -x_1 + S_r x_2 \\ -x_2 \end{pmatrix} \right\| \\ &= \inf_{\|x_1\|^2+\|x_2\|^2=1} \left(\| - x_1 + S_r x_2\|^2 + \| - x_2\|^2\right)^{\frac{1}{2}}.\end{aligned}$$

Choose $x_1 = \frac{e_2}{\sqrt{2}}$ *and* $x_2 = \frac{e_1}{\sqrt{2}}$ *where* $e_i = (0, \cdots, 0, \underset{i}{1}, 0, \cdots)$ *for* $i = 1, 2$. *Then we have*

$$\|(\mathcal{T} - I)x\| = \frac{1}{\sqrt{2}} < \varepsilon.$$

Thus, $\lambda = 1 \in \sigma_{ap,\varepsilon}(\mathcal{T})$. *However,* $1 \notin \{\lambda : |\lambda| < \varepsilon\}$ *which shows that*

$$\sigma_{ap,\varepsilon}(\mathcal{T}) \nsubseteq \sigma_{ap,\varepsilon}(A) \cup \sigma_{ap,\varepsilon}(D).$$

Example 1.2 (showing $\sigma_{ap,\varepsilon}(A) \cup \sigma_{ap,\varepsilon}(D) \nsubseteq \sigma_{ap,\varepsilon}(\mathcal{T})$**).** *Let* $\mathcal{X} = \ell^2(\mathbb{N})$, $\varepsilon < 1$, $A = S_r$, $B = P_1$ *be the projection operator defined by* $P_1(x_1, x_2, x_3, \dots) = (x_1, 0, 0, \dots)$, *where* $\{x_n\}_{n=1}^{\infty} \in \mathcal{X}$ *and* $D = S_l$ *be the left shift operator, defined by* $S_l(x_n) = x_{n+1}$ *for* $n \geq 1$, *where* $\{x_n\}_{n=1}^{\infty} \in \mathcal{X}$. *Let* $\mathcal{T}$ *be the operator defined by*

$$\mathcal{T} = \begin{pmatrix} S_r & P_1 \\ 0 & S_l \end{pmatrix} : \mathcal{X} \times \mathcal{X} \longrightarrow \mathcal{X} \times \mathcal{X}.$$

If $\lambda = 0$, *we can take* $x = e_1 = (1, 0, 0, \cdots)$ *with* $\|x\| = 1$ *such that*

$$\|Dx\| = \|S_l x\| = 0.$$

Thus,

$$0 \in \sigma_{ap,\varepsilon}(D) \subseteq \sigma_{ap,\varepsilon}(A) \cup \sigma_{ap,\varepsilon}(D).$$

For the operator $\mathcal{T}$ *and* $x = (x_1, x_2)^T \in \mathcal{X} \times \mathcal{X}$, *let us compute for* $\lambda = 0$. *Let* $P_1(x_2) = (x_{2,1}, 0, \cdots)$, *then*

$$\begin{aligned}
\inf_{\|x\|=1} \|\mathcal{T}x\| &= \inf_{\|x_1\|^2+\|x_2\|^2=1} \left(\|S_r x_1 + P_1 x_2\|^2 + \|S_l x_2\|^2\right)^{\frac{1}{2}} \\
&= \inf_{\|x_1\|^2+\|x_2\|^2=1} \left(\|x_1\|^2 + |x_{2,1}|^2 + \sum_{i=2}^{\infty} |x_{2,i}|^2\right)^{\frac{1}{2}} \\
&= \inf_{\|x_1\|^2+\|x_2\|^2=1} \left(\|x_1\|^2 + \|x_2\|^2\right)^{\frac{1}{2}} \\
&= 1 > \varepsilon,
\end{aligned}$$

i.e., $0 \notin \sigma_{ap,\varepsilon}(\mathcal{T})$. *In summary,*

$$\sigma_{ap,\varepsilon}(A) \cup \sigma_{ap,\varepsilon}(D) \nsubseteq \sigma_{ap,\varepsilon}(\mathcal{T}).$$

For this reason, the paper provides sufficient (necessary) conditions for $\sigma_{ap,\varepsilon}(\mathcal{T})$ to be contained in (to contain) the union of $\sigma_{ap,\varepsilon}(A)$ and $\sigma_{ap,\varepsilon}(D)$. In addition, when the coupling operator B can be regarded as a sufficiently small perturbation, we show that $\sigma_{ap,\varepsilon}(\mathcal{T})$ may be "sandwiched" between appropriately expanded or contracted approximate pseudospectrum of the diagonal entries A and D.

The remainder of the paper is organized as follows: Section 2 presents the preliminary results, while Section 3 includes the main results and corollaries.

2. Preliminary

In this section, we introduce the preliminary results that will be used throughout the paper.

Proposition 2.1. *(Proposition 2.3 in [1]) Let* $T \in \mathcal{B}(X)$ *and* $\varepsilon > 0$, *then*

(i) $\sigma_{ap}(T) = \bigcap_{\varepsilon > 0} \sigma_{ap,\varepsilon}(T)$.

(ii) *If* $\varepsilon_1 < \varepsilon_2$, *then* $\sigma_{ap}(T) \subset \sigma_{ap,\varepsilon_1}(T) \subset \sigma_{ap,\varepsilon_2}(T)$.
(iii) *If* $\lambda \in \sigma_{ap,\varepsilon}(T)$, *then* $|\lambda| < \varepsilon + \|T\|$.
(iv) *If* $\alpha \in \mathbb{C}$ *and* $\varepsilon > 0$, *then* $\sigma_{ap,\varepsilon}(T + \alpha) = \alpha + \sigma_{ap,\varepsilon}(T)$.
(v) *If* $\alpha \in \mathbb{C}\backslash\{0\}$ *and* $\varepsilon > 0$, *then* $\sigma_{ap,|\alpha|\varepsilon}(\alpha T) = \alpha\sigma_{ap,\varepsilon}(T)$.

Similarly to the proof of [1, Theorem 3.6], we obtain the following lemma.

Lemma 2.2. *Let* $\varepsilon > 0$ *and* $T, E \in \mathcal{B}(\mathcal{X})$ *such that* $\|E\| < \varepsilon$. *Then*

$$\sigma_{ap,\varepsilon-\|E\|}(T) \subseteq \sigma_{ap,\varepsilon}(T + E) \subseteq \sigma_{ap,\varepsilon+\|E\|}(T).$$

Similarly to the proof of [1, Theorem 3.7], we obtain the following lemma.

Lemma 2.3. *Let* $T_0 \in \mathcal{B}(\mathcal{X})$ *and* $V \in \mathcal{B}(\mathcal{X})$ *be invertible. Define* $T = V^{-1}T_0V$. *Then*

$$\sigma_{ap}(T) = \sigma_{ap}(T_0),$$

and for $\varepsilon > 0$ *and* $k = \|V^{-1}\|\|V\|$, *we have*

$$\sigma_{ap,\frac{\varepsilon}{k}}(T_0) \subseteq \sigma_{ap,\varepsilon}(T) \subseteq \sigma_{ap,k\varepsilon}(T_0).$$

Lemma 2.4. *Let* $\varepsilon > 0$ *and*

$$\mathcal{T}_0 = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} : \mathcal{X} \times \mathcal{X} \longrightarrow \mathcal{X} \times \mathcal{X},$$

where $A, D \in \mathcal{B}(\mathcal{X})$. *Then*

$$\sigma_{ap,\varepsilon}(\mathcal{T}_0) = \sigma_{ap,\varepsilon}(A) \cup \sigma_{ap,\varepsilon}(D).$$

Proof. *We begin by proving* $\sigma_{ap,\varepsilon}(\mathcal{T}_0) \subseteq \sigma_{ap,\varepsilon}(A) \cup \sigma_{ap,\varepsilon}(D)$. *Let* $\lambda \in \sigma_{ap,\varepsilon}(\mathcal{T}_0)$ *and* $x = (x_1, x_2)^T \in \mathcal{X} \times \mathcal{X}$ *with* $\|x\| = 1$. *Then*

$$\inf_{\|x\|=1} \|(\mathcal{T}_0 - \lambda I)x\| = \inf_{\|x_1\|^2+\|x_2\|^2=1} \left(\|(A - \lambda I)x_1\|^2 + \|(D - \lambda I)x_2\|^2\right)^{\frac{1}{2}} < \varepsilon.$$

Define

$$\alpha := \inf_{\|x'\|=1} \|(A - \lambda I)x'\|,$$
$$\beta := \inf_{\|x'\|=1} \|(D - \lambda I)x'\|.$$

Then, the inequalities

$$\|(A - \lambda I)x_1'\| \geq \alpha\|x_1'\|, \quad \|(D - \lambda I)x_2'\| \geq \beta\|x_2'\|$$

hold for all $x_1', x_2' \in \mathcal{X}$. *Setting* $t = \|x_1\|^2 \in [0, 1]$, *we obtain the lower bound*

$$\|(\mathcal{T}_0 - \lambda I)x\|^2 \geq \alpha^2\|x_1\|^2 + \beta^2\|x_2\|^2 = \alpha^2 t + \beta^2(1 - t).$$

Observe that the function $t \mapsto \alpha^2 t + \beta^2(1 - t)$ *achieves its minimum at the endpoints of* $[0, 1]$, *hence*

$$\min\{\alpha, \beta\} \leq \inf_{\|x\|=1} \|(\mathcal{T}_0 - \lambda I)x\| < \varepsilon,$$

which implies $\lambda \in \sigma_{ap,\varepsilon}(A) \cup \sigma_{ap,\varepsilon}(D)$.

Conversely, suppose the $\lambda \in \sigma_{ap,\varepsilon}(A) \cup \sigma_{ap,\varepsilon}(D)$. *First case, if* $\lambda \in \sigma_{ap,\varepsilon}(A)$ *and* $x_1 \in \mathcal{X}$ *with* $\|x_1\| = 1$. *Then*

$$\inf_{\|x_1\|=1} \|(A - \lambda I)x_1\| < \varepsilon.$$

Take the vectors $x' = (x_1, 0)^T \in \mathcal{X} \times \mathcal{X}$. *We have*

$$\inf_{\|x'\|=1} \|(\mathcal{T}_0 - \lambda I)x'\| = \inf_{\|x_1\|=1} \|(A - \lambda I)x_1\| < \varepsilon,$$

which implies

$$\inf_{\|x\|=1} \|(\mathcal{T}_0 - \lambda I)x\| < \varepsilon,$$

i.e., $\lambda \in \sigma_{ap,\varepsilon}(\mathcal{T}_0)$. *Second case, if* $\lambda \in \sigma_{ap,\varepsilon}(D)$ *follows analogously by considering* $x = (0, x_2)^T$. *The equality*

$$\sigma_{ap,\varepsilon}(\mathcal{T}_0) = \sigma_{ap,\varepsilon}(A) \cup \sigma_{ap,\varepsilon}(D)$$

is thus established. □

Lemma 2.5. *Let* $\varepsilon > 0$ *and*

$$\mathcal{T} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} : \mathcal{X} \times \mathcal{X} \longrightarrow \mathcal{X} \times \mathcal{X},$$

where $A, B, D \in \mathcal{B}(\mathcal{X})$. *Then*

$$\sigma_{ap,\varepsilon}(A) \subseteq \sigma_{ap,\varepsilon}(\mathcal{T}).$$

Proof. Let $\lambda \in \sigma_{ap,\varepsilon}(A)$ *and* $x_1 \in \mathcal{X}$ *with* $\|x_1\| = 1$. *Then*

$$\inf_{\|x_1\|=1} \|(A - \lambda I)x_1\| < \varepsilon.$$

Take $x = (x_1, 0)^T \in \mathcal{X} \times \mathcal{X}$. *It follows that*

$$\inf_{\|x\|=1} \|(\mathcal{T} - \lambda I)x\| = \inf_{\|x_1\|=1} \|(A - \lambda I)x_1\| < \varepsilon$$

then $\lambda \in \sigma_{ap,\varepsilon}(\mathcal{T})$. □

3. Main Results

In this section, we introduce our main results and corollaries.

Theorem 3.1. *Let* $\varepsilon > 0$ *and*

$$\mathcal{T} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} : \mathcal{X} \times \mathcal{X} \longrightarrow \mathcal{X} \times \mathcal{X},$$

where $A, B, D \in \mathcal{B}(\mathcal{X})$. *If* $\lambda \in \sigma_{ap,\varepsilon}(\mathcal{T})$, *then at least one of the following holds:*

(i) $\lambda \in \sigma_{ap,\varepsilon}(A) \cup \sigma_{ap,\varepsilon}(D)$;

(ii) *there exists a unit vector* $(x_1, x_2)^T \in \mathcal{X} \times \mathcal{X}$ *(i.e.* $\|x_1\|^2 + \|x_2\|^2 = 1$*) such that*

$$\|(A - \lambda I)x_1 + Bx_2\| < \|(A - \lambda I)x_1\|.$$

Proof. *We prove the contrapositive: if neither (i) nor (ii) holds, then*

$$\inf_{\|x\|=1} \|(\mathcal{T} - \lambda I)x\| \geq \varepsilon,$$

i.e. $\lambda \notin \sigma_{ap,\varepsilon}(\mathcal{T})$.

Assume (i) fails. That is,

$$\lambda \notin \sigma_{ap,\varepsilon}(A) \quad \text{and} \quad \lambda \notin \sigma_{ap,\varepsilon}(D).$$

By definition of the approximate pseudospectrum, we have

$$\inf_{\|x_1\|=1} \|(A - \lambda I)x_1\| \geq \varepsilon \quad \text{and} \quad \inf_{\|x_2\|=1} \|(D - \lambda I)x_2\| \geq \varepsilon.$$

Assume (ii) also fails. That is, for every unit vector $x = (x_1, x_2)^T \in \mathcal{X} \times \mathcal{X}$ *(so* $\|x_1\|^2 + \|x_2\|^2 = 1$*), we have*

$$\|(A - \lambda I)x_1 + Bx_2\| \geq \|(A - \lambda I)x_1\| \geq \varepsilon\|x_1\|.$$

We now estimate

$$\begin{aligned}\|(\mathcal{T} - \lambda I)x\| &= \left\|\Big((A - \lambda I)x_1 + Bx_2, (D - \lambda I)x_2\Big)^T\right\| \\ &= \Big(\|(A - \lambda I)x_1 + Bx_2\|^2 + \|(D - \lambda I)x_2\|^2\Big)^{\frac{1}{2}}.\end{aligned}$$

From the spectral bound on D,

$$\|(D - \lambda I)x_2\| \geq \varepsilon\|x_2\|.$$

Hence

$$\|(\mathcal{T} - \lambda I)x\| \geq \left(\varepsilon^2 \left(\|x_1\|^2 + \|x_2\|^2\right)\right)^{\frac{1}{2}} = \varepsilon,$$

so for every unit x, we conclude

$$\inf_{\|x\|=1} \|(\mathcal{T} - \lambda I)x\| \geq \varepsilon,$$

i.e. $\lambda \notin \sigma_{ap,\varepsilon}(\mathcal{T})$.

By contraposition, we have shown that if $\lambda \in \sigma_{ap,\varepsilon}(\mathcal{T})$*, then at least one of (i) or (ii) must hold. This completes the proof.* □

Theorem 3.2. *Let* $\varepsilon > 0$ *and*

$$\mathcal{T} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} : \mathcal{X} \times \mathcal{X} \longrightarrow \mathcal{X} \times \mathcal{X},$$

where $A, B, D \in \mathcal{B}(\mathcal{X})$. *If* $\lambda \in \sigma_{ap,\varepsilon}(D)$ *and* $\mathcal{R}(B) \subseteq \mathcal{R}(A - \lambda I)$, *then* $\lambda \in \sigma_{ap,\varepsilon}(\mathcal{T})$.

Proof. *Suppose* $\lambda \in \sigma_{ap,\varepsilon}(D)$. *By Lemma 2.5, the inclusion* $\sigma_{ap,\varepsilon}(A) \subseteq \sigma_{ap,\varepsilon}(\mathcal{T})$ *is immediate. Hence it suffices to consider only the case* $\lambda \in \sigma_{ap,\varepsilon}(D) n \sigma_{ap,\varepsilon}(A)$.

By $\lambda \notin \sigma_{ap,\varepsilon}(A)$, *we know* $\mathcal{R}(A - \lambda I)$ *is closed and* $A - \lambda I$ *is bounded below. Hence the inverse*

$$L := (A - \lambda I)^{-1}\Big|_{\mathcal{R}(A - \lambda I)}$$

is a bounded operator from $\mathcal{R}(A - \lambda I)$ *onto* $\mathcal{X}$. *In particular, if* $\mathcal{R}(B) \subseteq \mathcal{R}(A - \lambda I)$, *then* $L(Bx')$ *is well-defined for every* $x' \in \mathcal{X}$. *Fix any nonzero* $x_2 \in \mathcal{X}$, *there is a nonzero* $x_1 \in \mathcal{X}$ *such that*

$$x_1 = -L\Big(Bx_2\Big) \qquad \text{i.e.} \qquad (A - \lambda I)x_1 = -Bx_2.$$

Then by construction

$$(A-\lambda I)x_1 + Bx_2 = -Bx_2 + Bx_2 = 0.$$

Take $x = (x_1, x_2)^T$, *we have*

$$(\mathcal{T}-\lambda I)x = \begin{pmatrix} 0 \\ (D-\lambda I)x_2 \end{pmatrix}$$

and therefore

$$\|(\mathcal{T}-\lambda I)x\| = \|(D-\lambda I)x_2\|.$$

We only know x might not be 1, so normalize:

$$u = \frac{x_1}{\sqrt{\|x_1\|^2 + \|x_2\|^2}}, \quad v = \frac{x_2}{\sqrt{\|x_1\|^2 + \|x_2\|^2}}.$$

Suppose $\lambda \notin \sigma_{ap,\varepsilon}(\mathcal{T})$, *for every unit vector* $x' \in \mathcal{X}$, *we have*

$$\|(\mathcal{T}-\lambda I)x'\| \geq \varepsilon.$$

Set $y = (u, v)^T$, *we have*

$$\|(\mathcal{T}-\lambda I)y\| = \frac{1}{\sqrt{\|x_1\|^2 + \|x_2\|^2}}\|(D-\lambda I)x_2\| \geq \varepsilon.$$

Rearranging,

$$\|(D-\lambda I)x_2\| \geq \varepsilon\sqrt{\|x_1\|^2 + \|x_2\|^2} \geq \varepsilon\|x_2\|,$$

which implies

$$\inf_{\|x_2\|=1}\|(D-\lambda I)x_2\| \geq \varepsilon, \quad i.e. \quad \lambda \notin \sigma_{ap,\varepsilon}(D).$$

This completes the proof. □

Corollary 3.3. *Let* $\varepsilon > 0$ *and*

$$\mathcal{T} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} : \mathcal{X}\times\mathcal{X} \longrightarrow \mathcal{X}\times\mathcal{X},$$

where $A, B, D \in \mathcal{B}(\mathcal{X})$. *If the following two conditions are satisfied:*

(i) *For all* $\lambda \in \sigma_{ap,\varepsilon}(D)$, *it holds that* $\mathcal{R}(B) \subseteq \mathcal{R}(A-\lambda I)$;

(ii) *For all* $\lambda \in \sigma_{ap,\varepsilon}(\mathcal{T})$ *and unit vector* $(x_1, x_2)^T \in \mathcal{X}\times\mathcal{X}$, *it holds that*

$$\|(A-\lambda I)x_1 + Bx_2\| \geq \|(A-\lambda I)x_1\|.$$

Then

$$\sigma_{ap,\varepsilon}(\mathcal{T}) = \sigma_{ap,\varepsilon}(A) \cup \sigma_{ap,\varepsilon}(D).$$

Theorem 3.4. *Let* $\varepsilon > 0$ *and*

$$\mathcal{T} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} : \mathcal{X}\times\mathcal{X} \longrightarrow \mathcal{X}\times\mathcal{X},$$

where $A, B, D \in \mathcal{B}(\mathcal{X})$ *such that* $\|B\| < \varepsilon$, *then*

$$\sigma_{ap,\varepsilon-\|B\|}(A) \cup \sigma_{ap,\varepsilon-\|B\|}(D) \subseteq \sigma_{ap,\varepsilon}(\mathcal{T}) \subseteq \sigma_{ap,\varepsilon+\|B\|}(A) \cup \sigma_{ap,\varepsilon+\|B\|}(D).$$

Proof. *[Proof 1 (Perturbation Theory via Lemma 2.2)] We view $\mathcal{T}$ as a perturbation of the diagonal operator $\mathcal{T}_0 = \left(\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right)$, with the perturbation term $E = \left(\begin{smallmatrix} 0 & B \\ 0 & 0 \end{smallmatrix}\right)$ and $\|E\| = \|B\|$. By Lemma 2.2, we have*

$$\sigma_{ap,\varepsilon-\|B\|}(\mathcal{T}_0) = \sigma_{ap,\varepsilon-\|E\|}(\mathcal{T}_0) \subseteq \sigma_{ap,\varepsilon}(\mathcal{T}) = \sigma_{ap,\varepsilon}(\mathcal{T}_0 + E) \subseteq \sigma_{ap,\varepsilon+\|E\|}(\mathcal{T}_0) = \sigma_{ap,\varepsilon+\|B\|}(\mathcal{T}_0).$$

Using Lemma 2.4, we obtain

$$\sigma_{ap,\varepsilon-\|B\|}(A) \cup \sigma_{ap,\varepsilon-\|B\|}(D) \subseteq \sigma_{ap,\varepsilon}(\mathcal{T}) \subseteq \sigma_{ap,\varepsilon+\|B\|}(A) \cup \sigma_{ap,\varepsilon+\|B\|}(D).$$

□

We now present an alternative proof based directly on the definition of the approximate pseudospectrum.

Proof. *We begin by proving $\sigma_{ap,\varepsilon-\|B\|}(A) \cup \sigma_{ap,\varepsilon-\|B\|}(D) \subseteq \sigma_{ap,\varepsilon}(\mathcal{T})$. Let $\lambda \in \sigma_{ap,\varepsilon-\|B\|}(A) \cup \sigma_{ap,\varepsilon-\|B\|}(D)$. If $\lambda \in \sigma_{ap,\varepsilon-\|B\|}(A)$, by Lemma 2.5 and (ii) of Proposition 2.1 we conclude $\lambda \in \sigma_{ap,\varepsilon}(\mathcal{T})$. If $\lambda \in \sigma_{ap,\varepsilon-\|B\|}(D)$, then for any $x_2 \in \mathcal{X}$ with $\|x_2\| = 1$,*

$$\inf_{\|x_2\|=1} \|(D - \lambda I)x_2\| < \varepsilon - \|B\|.$$

Take $x = (0, x_2)^T \in \mathcal{X} \times \mathcal{X}$. Then

$$\|(\mathcal{T} - \lambda)x\| = (\|Bx_2\|^2 + \|(D - \lambda I)x_2\|^2)^{\frac{1}{2}} \leq \|B\| + \|(D - \lambda I)x_2\|.$$

Thus,

$$\inf_{\|x\|=1} \|(\mathcal{T} - \lambda)x\| \leq \|B\| + \inf_{\|x_2\|=1} \|(D - \lambda I)x_2\| \leq \|B\| + \varepsilon - \|B\| = \varepsilon.$$

Consequently, $\lambda \in \sigma_{ap,\varepsilon}(\mathcal{T})$. Now, we will prove that $\sigma_{ap,\varepsilon}(\mathcal{T}) \subseteq \sigma_{ap,\varepsilon+\|B\|}(A) \cup \sigma_{ap,\varepsilon+\|B\|}(D)$. Let $\lambda \in \sigma_{ap,\varepsilon}(\mathcal{T})$ and $x = (x_1, x_2)^T \in \mathcal{X} \times \mathcal{X}$ with $\|x\| = 1$,

$$\inf_{\|x\|=1} \|(\mathcal{T} - \lambda I)x\| = \inf_{\|x_1\|^2+\|x_2\|^2=1} \left(\|(A - \lambda I)x_1 + Bx_2\|^2 + \|(D - \lambda I)x_2\|^2\right)^{\frac{1}{2}} < \varepsilon.$$

Define α and β as in Lemma 2.4. An analogous argument yields

$$\min\{\alpha, \beta\} \leq \inf_{\|x\|=1} \|(\mathcal{T} - \lambda I)x\| < \varepsilon + \|B\|,$$

which immediately implies $\lambda \in \sigma_{ap,\varepsilon+\|B\|}(A) \cup \sigma_{ap,\varepsilon+\|B\|}(D)$. Consequently,

$$\sigma_{ap,\varepsilon-\|B\|}(A) \cup \sigma_{ap,\varepsilon-\|B\|}(D) \subseteq \sigma_{ap,\varepsilon}(\mathcal{T}) \subseteq \sigma_{ap,\varepsilon+\|B\|}(A) \cup \sigma_{ap,\varepsilon+\|B\|}(D).$$

□

Remark 3.5. *Under the condition $\|B\| < \varepsilon$, the off-diagonal term B is a small perturbation relative to the ε-level of the approximate pseudospectrum. Consequently, the approximate pseudospectrum of $\mathcal{T}$ is "sandwiched" between the approximate pseudospectrum of the diagonal components expanded by $\|B\|$.*

Lemma 3.6. *Let $\varepsilon > 0$, and*

$$\mathcal{T} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} : \mathcal{X} \times \mathcal{X} \longrightarrow \mathcal{X} \times \mathcal{X},$$

where $A, B, D \in \mathcal{B}(\mathcal{X})$. Let $X \in \mathcal{B}(\mathcal{X})$ such that $AX - XD = B$, then

$$\sigma_{ap}(\mathcal{T}) = \sigma_{ap}(A) \cup \sigma_{ap}(D),$$

and for $k = \|\left(\begin{smallmatrix} I & -X \\ 0 & I \end{smallmatrix}\right)\|\|\left(\begin{smallmatrix} I & X \\ 0 & I \end{smallmatrix}\right)\|$, we have

$$\sigma_{ap,\frac{\varepsilon}{k}}(A) \cup \sigma_{ap,\frac{\varepsilon}{k}}(D) \subseteq \sigma_{ap,\varepsilon}(\mathcal{T}) \subseteq \sigma_{ap,k\varepsilon}(A) \cup \sigma_{ap,k\varepsilon}(D).$$

Proof. Let $V = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}$, it is immediate that $V^{-1} = \begin{pmatrix} I & -X \\ 0 & I \end{pmatrix}$ and

$$V^{-1}\mathcal{T}_0 V = \begin{pmatrix} A & AX - XD \\ 0 & D \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \mathcal{T}.$$

By Lemma 2.3, we have

$$\sigma_{ap}(\mathcal{T}) = \sigma_{ap}(\mathcal{T}_0),$$

and for $\varepsilon > 0$ *and* $k = \|V^{-1}\|\|V\|$, *we have*

$$\sigma_{ap,\frac{\varepsilon}{k}}(\mathcal{T}_0) \subseteq \sigma_{ap,\varepsilon}(\mathcal{T}) \subseteq \sigma_{ap,k\varepsilon}(\mathcal{T}_0).$$

Using Lemma 2.4, we obtain

$$\sigma_{ap}(\mathcal{T}) = \sigma_{ap}(A) \cup \sigma_{ap}(D),$$

and

$$\sigma_{ap,\frac{\varepsilon}{k}}(A) \cup \sigma_{ap,\frac{\varepsilon}{k}}(D) \subseteq \sigma_{ap,\varepsilon}(\mathcal{T}) \subseteq \sigma_{ap,k\varepsilon}(A) \cup \sigma_{ap,k\varepsilon}(D).$$

□

The conclusion below follows directly from Lemma 3.6 and the Sylvester-Rosenblum Theorem [8] (see also [3]).

Corollary 3.7. *Let* $\varepsilon > 0$ *and*

$$\mathcal{T} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} : \mathcal{X} \times \mathcal{X} \longrightarrow \mathcal{X} \times \mathcal{X},$$

where $A, B, D \in \mathcal{B}(\mathcal{X})$. *If* $\sigma(A) \cap \sigma(D) = \emptyset$, *then*

$$\sigma_{ap}(\mathcal{T}) = \sigma_{ap}(A) \cup \sigma_{ap}(D),$$

and for $k = \| \left(\begin{smallmatrix} I & -X \\ 0 & I \end{smallmatrix}\right) \|\| \left(\begin{smallmatrix} I & X \\ 0 & I \end{smallmatrix}\right) \|$, *we have*

$$\sigma_{ap,\frac{\varepsilon}{k}}(A) \cup \sigma_{ap,\frac{\varepsilon}{k}}(D) \subseteq \sigma_{ap,\varepsilon}(\mathcal{T}) \subseteq \sigma_{ap,k\varepsilon}(A) \cup \sigma_{ap,k\varepsilon}(D).$$

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