Filomat 39:25 (2025), 8679–8688
https://doi.org/10.2298/FIL2525679W

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On the generalization of orthogonality and parallelism in C^*-algebras

Zhenquan Wang[a], Liguang Wang[a,*]

[a] *School of Mathematical Sciences, Qufu Normal University, Qufu, Shandong, 273165, China*

Abstract. In this paper, a new norm $\|\cdot\|_\lambda$ is defined on a C^*-algebra $\mathcal{A}$. Some basic results of $\|\cdot\|_\lambda$ are given. We also give characterizations of parallelism and orthogonality of operators with respect to $\|\cdot\|_\lambda$. Finally, we obtain some upper bounds of $\|\cdot\|_\lambda$.

1. Introduction

Let $\mathcal{A}$ be a C^*-algebra with unit e. An element $a \in \mathcal{A}$ is positive, denoted by $a \geq 0$, if a is self-adjoint with spectrum $\sigma(a) \subset [0, \infty)$. Let $\mathcal{A}'$ denotes the space of all continuous linear functionals on $\mathcal{A}$. A positive linear functional of $\mathcal{A}$ is a map $\varphi \in \mathcal{A}'$ such that $\varphi(a) \geq 0$ whenever $a \geq 0$. The set $S(\mathcal{A})$ of all states of $\mathcal{A}$ is the set of all positive linear functionals of $\mathcal{A}$ with norm 1. The algebraic numerical range of $a \in \mathcal{A}$ is defined as

$$V(a) = \{\varphi(a) : \varphi \in S(\mathcal{A})\}.$$

The algebraic numerical radius $v(a)$ of a is the maximum modulus of $V(a)$, i.e.,

$$v(a) := \sup\{|z| : z \in V(a)\}.$$

The algebraic numerical radius $v(\cdot) : \mathcal{A} \to \mathbb{R}$ defines an norm on $\mathcal{A}$ and satisfying $\frac{1}{2}\|a\| \leq v(a) \leq \|a\|$ for every $a \in \mathcal{A}$. We referred to [1, 5, 6, 13, 18, 19] for more results on algebraic numerical radius.

Birkhoff orthogonality is one of the most important orthogonality defined on normed linear spaces [4]. Two vectors x and y in a normed linear space X are said to be Birkhoff orthogonal, denoted by $x \perp_B y$, if $\|x + \lambda y\| \geq \|x\|$ for any $\lambda \in \mathbb{C}$.

A. Seddik introduced the parallelism relation of vectors [17] in 2006. Two vectors x and y in a normed linear space X are said to be parallel, denoted by $x\|y$, if $\|x + \lambda y\| = \|x\| + \|y\|$ holds for some $\lambda \in \mathbb{T} = \{z \in \mathbb{C}, |z| = 1\}$.

There are many extensions of Birkhoff orthogonality and parallelism such as the orthogonality and parallelism in C^*-algebras, Birkhoff orthogonality in Hilbert C^*-modules and so on (see [3, 8, 10, 12, 14, 16, 18–21]).

2020 *Mathematics Subject Classification*. Primary 47A12; Secondary 15A60, 47L05.
Keywords. C^*-algebras, orthogonality, parallelism.
Received: 22 April 2025; Revised: 16 July 2025; Accepted: 26 July 2025
Communicated by Dragan S. Djordjević
* Corresponding author: Liguang Wang
Email addresses: wangzhenquan1997@163.com (Zhenquan Wang), wangliguang0510@163.com (Liguang Wang)
ORCID iDs: https://orcid.org/0009-0009-0976-8466 (Zhenquan Wang), https://orcid.org/0000-0001-6449-7405 (Liguang Wang)

In particular, a new semi-norm which generalizes simultaneously the A-operator semi-norm and the A-numerical radius for operators on semi-Hilbertian spaces was defined in [9] and characterizations of Birkhoff orthogonality and parallelism were also given.

Inspired by results in [9], we are interested in the following generalization of Birkhoff orthogonality and parallelism in C^*-algebra.

Definition 1.1 Let $\mathcal{A}$ be a C^*-algebra and $\lambda \in [0,1]$. For $a \in \mathcal{A}$, let

$$\|a\|_\lambda = \sup\{ \sqrt{\lambda\varphi(a^*a) + (1-\lambda)|\varphi(a)|^2} : \varphi \in S(\mathcal{A})\}.$$

Now we give the definitions of Birkhoff orthogonality and parallelism about $\|\cdot\|_\lambda$.

Definition 1.2 Let $\mathcal{A}$ be a C^*-algebra and $\lambda \in [0,1]$. Given $a, b \in \mathcal{A}$.
(1) We say $a \perp_B^\lambda b$, if $\|a+\mu b\|_\lambda \geq \|a\|_\lambda$ for every $\mu \in \mathbb{C}$.
(2) We say $a \|_\lambda b$, if $\|a+\mu b\|_\lambda = \|a\|_\lambda + \|b\|_\lambda$ for some $\mu \in \mathbb{T}$.

Now, we present specific examples about Birkhoff orthogonality and parallelism with respect to the new norm $\|\cdot\|_\lambda$.

Example 1.3 Note that by Theorem 2.2 in section 2, we have $v(a) = \|a\|_\lambda = \|a\|$ when a is normal. Let $M_2(\mathbb{C})$ be the C*-algebra consisting of all 2×2 complex matrices.
(i) For every $\mu \in \mathbb{C}$, let $a = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then

$$\|a+\mu b\|_\lambda = \left\| \begin{bmatrix} 2 & 0 \\ 0 & \mu \end{bmatrix} \right\|_\lambda = \max\{2, |\mu|\} \geq 2 = \|a\|_\lambda$$

and therefore we have $a \perp_B^\lambda b$.
(ii) Let $a = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $b = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\mu = -1$. Then we have

$$\|a+\mu b\|_\lambda = \left\| \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \right\|_\lambda = 2 = \|a\|_\lambda + \|b\|_\lambda$$

and therefore $a \|_\lambda b$.

The organization of this paper is as follows. In Section 2, we prove that $\|\cdot\|_\lambda$ also defines a norm on $\mathcal{A}$ and $v(a) \leq \|a\|_\lambda \leq \|a\|$. Especially, we present the characterizations of parallelism and orthogonality about $\|\cdot\|_\lambda$. In Section 3, we obtain some upper bounds of $\|\cdot\|_\lambda$.

2. Main results

Let $\mathcal{A}$ be a C^*-algebra and $\lambda \in [0,1]$.

Theorem 2.1 The function $\|\cdot\|_\lambda : \mathcal{A} \to [0,\infty)$ defines a norm on $\mathcal{A}$.

Proof. The nonnegativity follows easily from the definition of $\|\cdot\|_\lambda$. $\|\cdot\|_\lambda$ is the supremum of a nonnegative valued function. If $\|a\|_\lambda = 0$, then

$$\lambda\varphi(a^*a) + (1-\lambda)|\varphi(a)|^2 = 0$$

for all $\varphi \in S(\mathcal{A})$. Hence $\varphi(a^*a) = 0$ and $|\varphi(a)| = 0$ for all $\varphi \in S(\mathcal{A})$. Then we have $a = 0$.

Let $a, b \in \mathcal{A}$. Then $\forall \varphi \in S(\mathcal{A})$, we have

$$\begin{aligned}
&\lambda\varphi((a+b)^*(a+b)) + (1-\lambda)|\varphi(a+b)|^2 \\
&= \lambda(\varphi(a^*a) + \varphi(b^*a) + \varphi(a^*b) + \varphi(b^*b)) + (1-\lambda)(|\varphi(a)|^2 + |\varphi(b)|^2 + \overline{\varphi(a)}\varphi(b) + \varphi(a)\overline{\varphi(b)}) \\
&\leq (\lambda\varphi(a^*a) + (1-\lambda)|\varphi(a)|^2) + (\lambda\varphi(b^*b) + (1-\lambda)|\varphi(b)|^2) + 2(\lambda|\varphi(a^*b)| + (1-\lambda)|\varphi(a)||\varphi(b)|) \\
&\leq (\lambda\varphi(a^*a) + (1-\lambda)|\varphi(a)|^2) + (\lambda\varphi(b^*b) + (1-\lambda)|\varphi(b)|^2) + 2(\lambda\sqrt{\varphi(a^*a)\varphi(b^*b)} + (1-\lambda)|\varphi(a)||\varphi(b)|) \\
&\leq (\lambda\varphi(a^*a) + (1-\lambda)|\varphi(a)|^2) + (\lambda\varphi(b^*b) + (1-\lambda)|\varphi(b)|^2) \\
&+ 2\sqrt{\lambda\varphi(a^*a) + (1-\lambda)|\varphi(a)|^2}\sqrt{\lambda\varphi(b^*b) + (1-\lambda)|\varphi(b)|^2} \\
&\leq \|a\|_\lambda^2 + \|b\|_\lambda^2 + 2\|a\|_\lambda\|b\|_\lambda \\
&= (\|a\|_\lambda + \|b\|_\lambda)^2.
\end{aligned}$$

Therefore

$$\sqrt{\lambda\varphi((a+b)^*(a+b)) + (1-\lambda)|\varphi(a+b)|^2} \leq \|a\|_\lambda + \|b\|_\lambda.$$

Taking supremum over all $\varphi \in S(\mathcal{A})$, we have

$$\|a+b\|_\lambda \leq \|a\|_\lambda + \|b\|_\lambda.$$

Finally, for every $\mu \in \mathbb{C}$,

$$\begin{aligned}
\|\mu a\|_\lambda &= \sup\{\sqrt{\lambda|\mu|^2\varphi((\mu a)^*(\mu a)) + (1-\lambda)|\varphi(\mu a)|^2} : \varphi \in S(\mathcal{A})\} \\
&= \sup\{\sqrt{\lambda|\mu|^2\varphi(a^*a) + (1-\lambda)|\mu|^2|\varphi(a)|^2} : \varphi \in S(\mathcal{A})\} \\
&= |\mu|\sup\{\sqrt{\lambda\varphi(a^*a) + (1-\lambda)|\varphi(a)|^2} : \varphi \in S(\mathcal{A})\} \\
&= |\mu|\|a\|_\lambda.
\end{aligned}$$

This completes the proof.□

Theorem 2.2 Let $\mathcal{A}$ be a C^*-algebra and $a \in \mathcal{A}$. Then the function $f : [0,1] \to [0,\infty)$ defined as $f(\lambda) = \|a\|_\lambda$ is monotonically increasing and

$$v(a) \leq \|a\|_\lambda \leq \|a\|.$$

Proof. Let $\lambda, \mu \in [0,1]$ be arbitrary such that $\lambda \leq \mu$. Then, for all $\varphi \in S(\mathcal{A})$,

$$\begin{aligned}
&\mu\varphi(a^*a) + (1-\mu)|\varphi(a)|^2 \\
&= (\lambda + \mu - \lambda)\varphi(a^*a) + (1 - \lambda - (\mu - \lambda))|\varphi(a)|^2 \\
&= \lambda\varphi(a^*a) + (1-\lambda)|\varphi(a)|^2 + (\mu - \lambda)(\varphi(a^*a) - |\varphi(a)|^2) \\
&\geq \lambda\varphi(a^*a) + (1-\lambda)|\varphi(a)|^2.
\end{aligned}$$

Taking supremum over all $\varphi \in S(\mathcal{A})$, we have $\|a\|_\lambda \leq \|a\|_\mu$. Since $\|a\|_0 = v(a)$ and $\|a\|_1 = \|a\|$, $v(a) \leq \|a\|_\lambda \leq \|a\|$.□

Corollary 2.3 Let $\mathcal{A}$ be a C^*-algebra and $a, b \in \mathcal{A}$. Suppose a is normal. Then for $\lambda, \mu \in [0,1]$, $\lambda \leq \mu$,

$$a \perp_B^\lambda b \Rightarrow a \perp_B^\mu b.$$

Proof. Let $\lambda, \mu \in [0,1]$ and $\lambda \leq \mu$. Since a is normal, $v(a) = \|a\|$. By Theorem 2.2, $v(a) \leq \|a\|_\lambda \leq \|a\|_\mu \leq \|a\| = v(a)$. Thus for every $t \in \mathbb{C}$,

$$\|a\|_\mu = \|a\|_\lambda \leq \|a + tb\|_\lambda \leq \|a + tb\|_\mu$$

and $a \perp_B^{\mu} b$.□

Now we can give characterizations of parallelism and orthogonality about $\|\cdot\|_\lambda$. We first recall the following lemma.

Lemma 2.4 ([2]) Let $\varphi \in S(\mathcal{A})$. Then there exist a Hilbert space $\mathcal{H}$, a representation $\pi : \mathcal{A} \to \mathbb{B}(\mathcal{H})$ and a unit vector $\xi \in \mathcal{H}$ such that $\varphi(a) = \langle \pi(a)\xi, \xi\rangle$ for all $a \in \mathcal{A}$.

Theorem 2.5 Let $\mathcal{A}$ be a C^*-algebra and $\lambda \in [0, 1]$. Given $a, b \in \mathcal{A}$. The following statements are equivalent.

(i) $a \parallel_\lambda b$;

(ii) There exists a $\varphi \in S(\mathcal{A})$ such that

$$|\lambda\varphi(a^*b) + (1-\lambda)\varphi(a^*)\varphi(b)| = \|a\|_\lambda\|b\|_\lambda.$$

(iii) There exist a Hilbert space $\mathcal{H}$, a representation $\pi : \mathcal{A} \to \mathbb{B}(\mathcal{H})$ and a unit vector $\xi \in \mathcal{H}$ satisfying
  (1) $\lambda\|\pi(a)\xi\|^2 + (1-\lambda)|\langle\pi(a)\xi, \xi\rangle|^2 = \|a\|_\lambda^2$;
  (2) $|\lambda\langle\pi(a^*b)\xi, \xi\rangle + (1-\lambda)\langle\pi(a^*)\xi, \xi\rangle\langle\pi(b)\xi, \xi\rangle| = \|a\|_\lambda\|b\|_\lambda$.

Proof. (i)⇒(ii) If $a \parallel_\lambda b$, then there exists $\mu \in \mathbb{T}$ such that $\|a + \mu b\|_\lambda = \|a\|_\lambda + \|b\|_\lambda$. There exists $\varphi \in S(\mathcal{A})$ such that

$$\lambda\varphi((a+\mu b)^*(a+\mu b)) + (1-\lambda)|\varphi(a+\mu b)|^2 = (\|a\|_\lambda + \|b\|_\lambda)^2.$$

Therefore

$$\begin{aligned}
(\|a\|_\lambda + \|b\|_\lambda)^2 &= \lambda\varphi((a+\mu b)^*(a+\mu b)) + (1-\lambda)|\varphi(a+\mu b)|^2\\
&= \lambda(\varphi(a^*a) + \overline{\mu}\varphi(b^*a) + \mu\varphi(a^*b) + \varphi(b^*b))\\
&+ (1-\lambda)(|\varphi(a)|^2 + |\varphi(b)|^2 + \mu\varphi(a^*)\varphi(b) + \overline{\mu}\varphi(a)\varphi(b^*))\\
&= (\lambda\varphi(a^*a) + (1-\lambda)|\varphi(a)|^2) + (\lambda\varphi(b^*b) + (1-\lambda)|\varphi(b)|^2)\\
&+ 2\Re(\lambda\mu\varphi(a^*b) + (1-\lambda)\mu\varphi(a^*)\varphi(b))\\
&\le (\lambda\varphi(a^*a) + (1-\lambda)|\varphi(a)|^2) + (\lambda\varphi(b^*b) + (1-\lambda)|\varphi(b)|^2)\\
&+ 2|\lambda\varphi(a^*b) + (1-\lambda)\varphi(a^*)\varphi(b)|\\
&\le (\lambda\varphi(a^*a) + (1-\lambda)|\varphi(a)|^2) + (\lambda\varphi(b^*b) + (1-\lambda)|\varphi(b)|^2)\\
&+ 2(\lambda|\varphi(a^*b)| + (1-\lambda)|\varphi(a)||\varphi(b)|)\\
&\le (\lambda\varphi(a^*a) + (1-\lambda)|\varphi(a)|^2) + (\lambda\varphi(b^*b) + (1-\lambda)|\varphi(b)|^2)\\
&+ 2(\lambda\sqrt{\varphi(a^*a)\varphi(b^*b)} + (1-\lambda)|\varphi(a)||\varphi(b)|)\\
&\le (\lambda\varphi(a^*a) + (1-\lambda)|\varphi(a)|^2) + (\lambda\varphi(b^*b) + (1-\lambda)|\varphi(b)|^2)\\
&+ 2\sqrt{\lambda\varphi(a^*a) + (1-\lambda)|\varphi(a)|^2}\sqrt{\lambda\varphi(b^*b) + (1-\lambda)|\varphi(b)|^2}\\
&\le \|a\|_\lambda^2 + \|b\|_\lambda^2 + 2\|a\|_\lambda\|b\|_\lambda\\
&= (\|a\|_\lambda + \|b\|_\lambda)^2.
\end{aligned}$$

Thus, we can get there exists $\varphi \in S(\mathcal{A})$ such that

$$|\lambda\varphi(a^*b) + (1-\lambda)\varphi(a^*)\varphi(b)| = \|a\|_\lambda\|b\|_\lambda.$$

(ii)⇒(i) Suppose there exists $\varphi \in S(\mathcal{A})$ such that

$$|\lambda\varphi(a^*b) + (1-\lambda)\varphi(a^*)\varphi(b)| = \|a\|_\lambda\|b\|_\lambda.$$

Then

$$\begin{aligned}
\|a\|_\lambda\|b\|_\lambda &= |\lambda\varphi(a^*b) + (1-\lambda)\varphi(a^*)\varphi(b)| \\
&\le \lambda|\varphi(a^*b)| + (1-\lambda)|\varphi(a)||\varphi(b)| \\
&\le \lambda\sqrt{\varphi(a^*a)\varphi(b^*b)} + (1-\lambda)|\varphi(a)||\varphi(b)| \\
&\le \sqrt{\lambda\varphi(a^*a) + (1-\lambda)|\varphi(a)|^2}\sqrt{\lambda\varphi(b^*b) + (1-\lambda)|\varphi(b)|^2} \\
&\le \sqrt{\lambda\varphi(a^*a) + (1-\lambda)|\varphi(a)|^2}\|b\|_\lambda \\
&\le \|a\|_\lambda\|b\|_\lambda
\end{aligned}$$

and we have

$$\sqrt{\lambda\varphi(a^*a) + (1-\lambda)|\varphi(a)|^2} = \|a\|_\lambda.$$

Similarly, we also have

$$\sqrt{\lambda\varphi(b^*b) + (1-\lambda)|\varphi(b)|^2} = \|b\|_\lambda.$$

Further, there exists $\mu \in \mathbb{T}$ such that

$$\mu(\lambda\varphi(a^*b) + (1-\lambda)\varphi(a^*)\varphi(b)) = \|a\|_\lambda\|b\|_\lambda.$$

Thus

$$\begin{aligned}
\|a+\mu b\|_\lambda^2 &\ge \lambda\varphi((a+\mu b)^*(a+\mu b)) + (1-\lambda)|\varphi(a+\mu b)|^2 \\
&= \lambda(\varphi(a^*a) + \overline{\mu}\varphi(b^*a) + \mu\varphi(a^*b) + \varphi(b^*b)) \\
&+ (1-\lambda)(|\varphi(a)|^2 + |\varphi(b)|^2 + \mu\varphi(a^*)\varphi(b) + \overline{\mu}\varphi(a)\varphi(b^*)) \\
&= (\lambda\varphi(a^*a) + (1-\lambda)|\varphi(a)|^2) + (\lambda\varphi(b^*b) + (1-\lambda)|\varphi(b)|^2) \\
&+ 2\Re(\lambda\mu\varphi(a^*b) + (1-\lambda)\mu\varphi(a^*)\varphi(b)) \\
&= \|a\|_\lambda^2 + \|b\|_\lambda^2 + 2\|a\|_\lambda\|b\|_\lambda \\
&= (\|a\|_\lambda + \|b\|_\lambda)^2.
\end{aligned}$$

Since $\|a+\mu b\|_\lambda \le \|a\|_\lambda + \|b\|_\lambda$, we have $\|a+\mu b\|_\lambda = \|a\|_\lambda + \|b\|_\lambda$, i.e., $a \parallel_\lambda b$.

(ii)⇒(iii) By (ii)⇒(i), we have

$$\lambda\varphi(a^*a) + (1-\lambda)|\varphi(a)|^2 = \|a\|_\lambda^2.$$

By lemma 2.4, we can get

$$\begin{aligned}
\|a\|_\lambda^2 &= \lambda\varphi(a^*a) + (1-\lambda)|\varphi(a)|^2 \\
&= \lambda\langle\pi(a^*a)\xi,\xi\rangle + (1-\lambda)|\langle\pi(a)\xi,\xi\rangle|^2 \\
&= \lambda\|\pi(a)\xi\|^2 + (1-\lambda)|\langle\pi(a)\xi,\xi\rangle|^2
\end{aligned}$$

and

$$\begin{aligned}
\|a\|_\lambda\|b\|_\lambda &= |\lambda\varphi(a^*b) + (1-\lambda)\varphi(a^*)\varphi(b)| \\
&= |\lambda\langle\pi(a^*b)\xi,\xi\rangle + (1-\lambda)\langle\pi(a^*)\xi,\xi\rangle\langle\pi(b)\xi,\xi\rangle| \\
&= |\lambda\langle\pi(b)\xi,\pi(a)\xi\rangle + (1-\lambda)\langle\xi,\pi(a)\xi\rangle\langle\pi(b)\xi,\xi\rangle|.
\end{aligned}$$

(iii)⇒(ii) Suppose (iii) holds. Let $\varphi : \mathcal{A} \to \mathbb{C}$ be the state associated to π by $\varphi(c) = \langle\pi(c)\xi,\xi\rangle$, $c \in \mathcal{A}$. Then

$$\begin{aligned}
\|a\|_\lambda\|b\|_\lambda &= |\lambda\langle\pi(b)\xi,\pi(a)\xi\rangle + (1-\lambda)\langle\xi,\pi(a)\xi\rangle\langle\pi(b)\xi,\xi\rangle| \\
&= |\lambda\langle\pi(a^*b)\xi,\xi\rangle + (1-\lambda)\langle\pi(a^*)\xi,\xi\rangle\langle\pi(b)\xi,\xi\rangle| \\
&= |\lambda\varphi(a^*b) + (1-\lambda)\varphi(a^*)\varphi(b)|.
\end{aligned}$$

This completes the proof.□

Remark 2.6 When $\lambda = 1$ in Theorem 2.5, we get [20, Corollary 4.4]. When $\lambda = 0$ in Theorem 2.5, we get the characterization of numerical radius parallelism in C*-algebras in A. Zamani [18].

Theorem 2.7 Let $\mathcal{A}$ be a C^*-algebra and $\lambda \in [0,1]$. Given $a, b \in \mathcal{A}$. Then the following statements are equivalent.

(i) $a \perp_B^\lambda b$;

(ii) For each $\theta \in [0, 2\pi)$, there exists $\varphi_\theta \in S(\mathcal{A})$ such that the following two conditions hold.

(1) $\lambda\varphi_\theta(a^*a) + (1-\lambda)|\varphi_\theta(a)|^2 = \|a\|_\lambda^2$;

(2) $\Re(e^{i\theta}\lambda\varphi_\theta(a^*b) + e^{i\theta}(1-\lambda)\varphi_\theta(a^*)\varphi_\theta(b)) \geq 0$.

Proof. (i)⇒(ii) If $a \perp_B^\lambda b$. Then for every $t \in \mathbb{C}$, $\|a + tb\|_\lambda \geq \|a\|_\lambda$. We may assume $\|a\|_\lambda \neq 0$, since when $\|a\|_\lambda = 0$, we have $a = 0$ and (ii) is trivial. Let $\theta \in [0, 2\pi)$. For all $n \in \mathbb{N}$, we have

$$\left\|a + \frac{e^{i\theta}}{n}b\right\|_\lambda \geq \|a\|_\lambda > \|a\|_\lambda - \frac{1}{n^2}.$$

Therefore there exists $\varphi_n^\theta \in S(\mathcal{A})$ such that

$$\lambda\varphi_n^\theta\left(\left(a + \frac{e^{i\theta}}{n}b\right)^*\left(a + \frac{e^{i\theta}}{n}b\right)\right) + (1-\lambda)\left|\varphi_n^\theta\left(a + \frac{e^{i\theta}}{n}b\right)\right|^2 = \left\|a + \frac{e^{i\theta}}{n}b\right\|_\lambda^2 > \left(\|a\|_\lambda - \frac{1}{n^2}\right)^2.$$

Hence for all $n \in \mathbb{N}$, we have

$$\begin{aligned}
\|a\|_\lambda^2 + \frac{1}{n^4} - \frac{2}{n^2}\|a\|_\lambda &< \lambda\varphi_n^\theta\left(\left(a + \frac{e^{i\theta}}{n}b\right)^*\left(a + \frac{e^{i\theta}}{n}b\right)\right) + (1-\lambda)\left|\varphi_n^\theta\left(a + \frac{e^{i\theta}}{n}b\right)\right|^2 \\
&= (\lambda\varphi_n^\theta(a^*a) + (1-\lambda)|\varphi_n^\theta(a)|^2) + \frac{1}{n^2}(\lambda\varphi_n^\theta(b^*b) + (1-\lambda)|\varphi_n^\theta(b)|^2) \\
&+ \frac{2}{n}\Re(\lambda e^{i\theta}\varphi_n^\theta(a^*b) + (1-\lambda)e^{i\theta}\varphi_n^\theta(a^*)\varphi(b)).
\end{aligned}$$

This implies

$$\begin{aligned}
&\frac{n}{2}(\|a\|_\lambda^2 - (\lambda\varphi_n^\theta(a^*a) + (1-\lambda)|\varphi_n^\theta(a)|^2)) \\
&< \frac{1}{n}\|a\|_\lambda - \frac{1}{2n^3} + \frac{1}{2n}(\lambda\varphi_n^\theta(b^*b) + (1-\lambda)|\varphi_n^\theta(b)|^2) + \Re(\lambda e^{i\theta}\varphi_n^\theta(a^*b) + (1-\lambda)e^{i\theta}\varphi_n^\theta(a^*)\varphi(b))
\end{aligned}$$

and therefore

$$0 < \frac{1}{n}\|a\|_\lambda - \frac{1}{2n^3} + \frac{1}{2n}(\lambda\varphi_n^\theta(b^*b) + (1-\lambda)|\varphi_n^\theta(b)|^2) + \Re(\lambda e^{i\theta}\varphi_n^\theta(a^*b) + (1-\lambda)e^{i\theta}\varphi_n^\theta(a^*)\varphi(b)).$$

Since the set $S(\mathcal{A})$ is weak*-compact, we may assume that $\varphi_n^\theta \xrightarrow{w^*} \varphi_\theta$ for some $\varphi_\theta \in S(\mathcal{A})$ as $n \to \infty$. Now, letting $n \to \infty$, we have

$$\Re(e^{i\theta}\lambda\varphi_\theta(a^*b) + e^{i\theta}(1-\lambda)\varphi_\theta(a^*)\varphi_\theta(b)) \geq 0.$$

We also have

$$\begin{aligned}
\|a\|_\lambda &\geq \left\|a + \frac{e^{i\theta}}{n}b\right\|_\lambda - \frac{1}{n}\|b\|_\lambda \\
&= \sqrt{\lambda\varphi_n^\theta\left(\left(a + \frac{e^{i\theta}}{n}b\right)^*\left(a + \frac{e^{i\theta}}{n}b\right)\right) + (1-\lambda)\left|\varphi_n^\theta\left(a + \frac{e^{i\theta}}{n}b\right)\right|^2} - \frac{1}{n}\|b\|_\lambda \\
&\geq \|a\|_\lambda - \frac{1}{n}\|b\|_\lambda - \frac{1}{n^2}.
\end{aligned}$$

Taking limit as $n \to \infty$ in the above inequality, we have

$$\lambda\varphi_\theta(a^*a) + (1-\lambda)|\varphi_\theta(a)|^2 = \|a\|_\lambda^2.$$

(ii)⇒(i) Given $t \in \mathbb{C}$. There exists $\theta \in [0, 2\pi)$ such that $t = |t|e^{i\theta}$. By assumption, there exists $\varphi_\theta \in S(\mathcal{A})$ satisfying conditions in (ii). Then

$$\begin{aligned}
\|a+tb\|_\lambda^2 &\geq \lambda\varphi_\theta((a+tb)^*(a+tb)) + (1-\lambda)|\varphi_\theta(a+tb)|^2 \\
&= (\lambda\varphi_\theta(a^*a) + (1-\lambda)|\varphi_\theta(a)|^2) + \frac{|t|^2}{n^2}(\lambda\varphi_\theta(b^*b) + (1-\lambda)|\varphi_\theta(b)|^2) \\
&+ \frac{2|t|}{n}\Re(\lambda e^{i\theta}\varphi_\theta(a^*b) + (1-\lambda)e^{i\theta}\varphi_\theta(a^*)\varphi_\theta(b)) \\
&\geq \lambda\varphi_\theta(a^*a) + (1-\lambda)|\varphi_\theta(a)|^2 = \|a\|_\lambda^2.
\end{aligned}$$

Thus $a \perp_B^\lambda b$. This completes the proof.□

The following result is an immediate consequence of Theorem 2.7.

Corollary 2.8 Let $\mathcal{A}$ be a C^*-algebra. For every $a, b \in \mathcal{A}$, the following statements are equivalent.

(i) $a \perp_B b$;

(ii) For each $\theta \in [0, 2\pi)$, there exists $\varphi_\theta \in S(\mathcal{A})$ such that

$$\varphi_\theta(a^*a) = \|a\|^2 \quad \text{and} \quad \Re(e^{i\theta}\varphi_\theta(a^*b)) \geq 0;$$

(iii) There exists $\varphi \in S(\mathcal{A})$ such that

$$\varphi(a^*a) = \|a\|^2 \quad \text{and} \quad \varphi(a^*b) = 0.$$

Proof. Let $\lambda = 1$ in Theorem 2.7, we get $(i) \Leftrightarrow (ii)$. $(i) \Leftrightarrow (iii)$ follows from [3, Theorem 4.1].□

Finally, let $\lambda = 0$ in Theorem 2.7, we can get the following result which was proved in [21].

Corollary 2.9 Let $\mathcal{A}$ be a C^*-algebra. For every $a, b \in \mathcal{A}$, the following statements are equivalent.

(i) $a \perp_B^v b$;

(ii) For each $\theta \in [0, 2\pi)$, there exists $\varphi_\theta \in S(\mathcal{A})$ such that

$$|\varphi_\theta(a)| = v(a) \quad \textit{and} \quad \Re(e^{i\theta}\varphi_\theta(a^*)\varphi_\theta(b)) \geq 0.$$

3. Some upper bounds of $\|\cdot\|_\lambda$

In this section, we will give some upper bounds for $\|\cdot\|_\lambda$. Firstly, we need the following lemmas.

Lemma 3.1([11]) Let f, g be nonnegative continuous functions on $[0, \infty)$ which satisfy the relation $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then

$$|\langle Ax, y\rangle| \leq \|f(|A|)x\|\|g(|A^*|)y\|$$

for all $A \in \mathcal{B}(\mathcal{H})$ and $x, y \in \mathcal{H}$. In particular,

$$|\langle Ax, y\rangle|^2 \leq \langle |A|x, x\rangle\langle |A^*|y, y\rangle$$

for all $A \in \mathcal{B}(\mathcal{H})$ and $x, y \in \mathcal{H}$.

Lemma 3.2([15]) Let $A \in \mathcal{B}(\mathcal{H})$ be positive and $x \in \mathcal{H}$ with $\|x\| = 1$. Then

$$\langle Ax, x\rangle^p \leq \langle A^p x, x\rangle, \quad p \geq 1;$$
$$\langle A^p x, x\rangle \leq \langle Ax, x\rangle^p, \quad p \in [0,1].$$

Theorem 3.3 Let $\mathcal{A}$ be a C^*-algebra and $\lambda \in [0,1]$. For every $a \in \mathcal{A}$,

$$\|a\|_\lambda^2 \leq \left\|\frac{(1+\lambda)}{2}a^*a + \frac{(1-\lambda)}{2}aa^*\right\|.$$

Proof. Let $\varphi \in S(\mathcal{A})$ and let π be the representation as in Lemma 2.4. Then

$$\begin{aligned}
\lambda\varphi(a^*a) + (1-\lambda)|\varphi(a)|^2 &= \lambda\langle\pi(a^*a)\xi,\xi\rangle + (1-\lambda)|\langle\pi(a)\xi,\xi\rangle|^2 \quad \text{(Lemma 2.4)}\\
&\leq \lambda\langle\pi(a^*a)\xi,\xi\rangle + (1-\lambda)\langle|\pi(a)|\xi,\xi\rangle\langle|\pi(a^*)|\xi,\xi\rangle \quad \text{(Lemma 3.1)}\\
&\leq \lambda\langle\pi(a^*a)\xi,\xi\rangle + \frac{1-\lambda}{2}(\langle|\pi(a)|\xi,\xi\rangle^2 + \langle|\pi(a^*)|\xi,\xi\rangle^2)\\
&\leq \lambda\langle\pi(a^*a)\xi,\xi\rangle + \frac{1-\lambda}{2}(\langle|\pi(a)|^2\xi,\xi\rangle + \langle|\pi(a^*)|^2\xi,\xi\rangle) \quad \text{(Lemma 3.2)}\\
&= \lambda\langle\pi(a^*a)\xi,\xi\rangle + \frac{1-\lambda}{2}\langle\pi(a^*a + aa^*)\xi,\xi\rangle\\
&= \left\langle\pi\left(\frac{1+\lambda}{2}a^*a + \frac{1-\lambda}{2}aa^*\right)\xi,\xi\right\rangle\\
&\leq \left\|\frac{1+\lambda}{2}a^*a + \frac{1-\lambda}{2}aa^*\right\|.
\end{aligned}$$

Therefore, taking the supremum over all $\varphi \in S(\mathcal{A})$, we have

$$\|a\|_\lambda^2 \leq \left\|\frac{1+\lambda}{2}a^*a + \frac{1-\lambda}{2}aa^*\right\|. \square$$

Remark 3.4 Taking $\lambda = 0$ in Theorem 3.3, we have for a in a C^*-algebra $\mathcal{A}$ that $v^2(a) \leq \frac{1}{2}\|a^*a + aa^*\|$.

Lemma 3.5 ([7]) Let $x, y, e \in \mathcal{H}$ with $\|e\| = 1$. Then $|\langle x, e\rangle\langle e, y\rangle| \leq \frac{1}{2}(\|x\|\|y\| + |\langle x, y\rangle|)$.

Theorem 3.6 Let $\mathcal{A}$ be a C^*-algebra and $\lambda \in [0,1]$. For every $a \in \mathcal{A}$,

$$\|a\|_\lambda^2 \leq \left\|\frac{1+3\lambda}{4}a^*a + \frac{1-\lambda}{4}aa^*\right\| + \frac{1-\lambda}{2}v(a^2).$$

Proof. Let $\varphi \in S(\mathcal{A})$. Then

$$\begin{aligned}
&\lambda\varphi(a^*a) + (1-\lambda)|\varphi(a)|^2\\
&= \lambda\langle\pi(a^*a)\xi,\xi\rangle + (1-\lambda)|\langle\pi(a)\xi,\xi\rangle|^2 \quad \text{(Lemma 2.4)}\\
&= \lambda\langle\pi(a^*a)\xi,\xi\rangle + (1-\lambda)|\langle\pi(a)\xi,\xi\rangle\langle\xi,\pi(a^*)\xi\rangle|\\
&\leq \lambda\langle\pi(a^*a)\xi,\xi\rangle + \frac{1-\lambda}{2}(\|\pi(a)\xi\|\|\pi(a^*)\xi\| + |\langle\pi(a)\xi,\pi(a^*)\xi\rangle|) \quad \text{(Lemma 3.5)}\\
&\leq \lambda\langle\pi(a^*a)\xi,\xi\rangle + \frac{1-\lambda}{4}(\|\pi(a)\xi\|^2 + \|\pi(a^*)\xi\|^2) + \frac{1-\lambda}{2}|\langle\pi(a^2)\xi,\xi\rangle|\\
&= \lambda\langle\pi(a^*a)\xi,\xi\rangle + \frac{1-\lambda}{4}\langle\pi(a^*a + aa^*)\xi,\xi\rangle + \frac{1-\lambda}{2}|\langle\pi(a^2)\xi,\xi\rangle|\\
&= \left\langle\pi\left(\frac{1+3\lambda}{4}a^*a + \frac{1-\lambda}{4}aa^*\right)\xi,\xi\right\rangle + \frac{1-\lambda}{2}|\langle\pi(a^2)\xi,\xi\rangle|\\
&= \varphi\left(\frac{1+3\lambda}{4}a^*a + \frac{1-\lambda}{4}aa^*\right) + \frac{1-\lambda}{2}|\varphi(a^2)|\\
&\leq \left\|\frac{1+3\lambda}{4}a^*a + \frac{1-\lambda}{4}aa^*\right\| + \frac{1-\lambda}{2}v(a^2).
\end{aligned}$$

Therefore, taking the supremum over all $\varphi \in S(\mathcal{A})$, we have

$$\|a\|_\lambda^2 \leq \left\| \frac{1+3\lambda}{4} a^*a + \frac{1-\lambda}{4} aa^* \right\| + \frac{1-\lambda}{2} v(a^2). \square$$

Remark 3.7 Note that let $\lambda = 0$ in Theorem 3.6, we have

$$v^2(a) \leq \frac{1}{4} \|a^*a + aa^*\| + \frac{1}{2} v(a^2)$$

for a in a C^*-algebra $\mathcal{A}$.

Theorem 3.8 Let $\mathcal{A}$ be a C^*-algebra and $\lambda \in [0,1]$. For every $a \in \mathcal{A}$,

$$\|a\|_\lambda^2 \leq \frac{1-\lambda}{4} (v^2(|a| + i|a^*|) + v(|a||a^*|)) + \left\| \frac{1+7\lambda}{8} a^*a + \frac{1-\lambda}{8} aa^* \right\|.$$

Proof. Let $\varphi \in S(\mathcal{A})$. Then we have

$$\begin{aligned}
&\lambda\varphi(a^*a) + (1-\lambda)|\varphi(a)|^2 \\
&= \lambda\langle \pi(a^*a)\xi, \xi\rangle + (1-\lambda)|\langle \pi(a)\xi, \xi\rangle|^2 \quad \text{(Lemma 2.4)} \\
&\leq \lambda\langle \pi(a^*a)\xi, \xi\rangle + (1-\lambda)\langle |\pi(a)|\xi, \xi\rangle\langle |\pi(a^*)|\xi, \xi\rangle \quad \text{(Lemma 3.1)} \\
&\leq \lambda\langle \pi(a^*a)\xi, \xi\rangle + \frac{1-\lambda}{4}(\langle |\pi(a)|\xi, \xi\rangle + \langle |\pi(a^*)|\xi, \xi\rangle)^2 \\
&= \lambda\langle \pi(a^*a)\xi, \xi\rangle + \frac{1-\lambda}{4}(\langle |\pi(a)|\xi, \xi\rangle^2 + \langle |\pi(a^*)|\xi, \xi\rangle^2 + 2\langle |\pi(a)|\xi, \xi\rangle\langle |\pi(a^*)|\xi, \xi\rangle) \\
&= \lambda\langle \pi(a^*a)\xi, \xi\rangle + \frac{1-\lambda}{4}(|\langle \pi(|a|)\xi, \xi\rangle + i\langle \pi(|a^*|)\xi, \xi\rangle|^2 + 2\langle \pi(|a|)\xi, \xi\rangle\langle \pi(|a^*|)\xi, \xi\rangle) \\
&\leq \lambda\langle \pi(a^*a)\xi, \xi\rangle + \frac{1-\lambda}{4}(|\langle \pi(|a| + i|a^*|)\xi, \xi\rangle|^2 + \|\pi(|a|)\xi\|\|\pi(|a^*|)\xi\| + |\langle \pi(|a|)\xi, \pi(|a^*|)\xi\rangle|) \\
&\leq \lambda\langle \pi(a^*a)\xi, \xi\rangle + \frac{1-\lambda}{4}(|\langle \pi(|a| + i|a^*|)\xi, \xi\rangle|^2 + \frac{1}{2}\langle \pi(a^*a + aa^*)\xi, \xi\rangle + |\langle \pi(|a||a^*|)\xi, \xi\rangle|) \\
&= \frac{1-\lambda}{4}(|\langle \pi(|a| + i|a^*|)\xi, \xi\rangle|^2 + |\langle \pi(|a||a^*|)\xi, \xi\rangle|) + \left\langle \pi\left(\frac{1+7\lambda}{8}a^*a + \frac{1-\lambda}{8}aa^*\right)\xi, \xi\right\rangle \\
&= \frac{1-\lambda}{4}(\varphi^2(|a| + i|a^*|) + |\varphi(|a||a^*|)|) + \varphi\left(\frac{1+7\lambda}{8}a^*a + \frac{1-\lambda}{8}aa^*\right) \\
&\leq \frac{1-\lambda}{4}(v^2(|a| + i|a^*|) + v(|a||a^*|)) + \left\| \frac{1+7\lambda}{8}a^*a + \frac{1-\lambda}{8}aa^* \right\|.
\end{aligned}$$

Therefore, taking the supremum over all $\varphi \in S(\mathcal{A})$, we have

$$\|a\|_\lambda^2 \leq \frac{1-\lambda}{4} (v^2(|a| + i|a^*|) + v(|a||a^*|)) + \left\| \frac{1+7\lambda}{8} a^*a + \frac{1-\lambda}{8} aa^* \right\|. \square$$

Remark 3.9 Taking $\lambda = 0$ in Theorem 3.8, we have

$$v^2(a) \leq \frac{1}{4}(v^2(|a| + i|a^*|) + v(|a||a^*|)) + \frac{1}{8} \|a^*a + aa^*\|.$$

Declarations

Competing interests The authors declare that they have no conflict of interest.

Funding This work was partially supported by NSF of China (12471124) and NSF of Shandong Province, China (ZR2025MS93).

References

[1] A. Alahmari, M. Mabrouk and A. Zamani, *Further results on the a-numerical range in C^*-algebras*, Banach J. Math. Anal. **16** (2022), no. 25.
[2] W. Arveson, *An invitation to C^*-algebras*, Springer-Verlag, New York, 1976.
[3] T. Bhattacharyya, P. Grover, *Characterization of Birkhoff-James orthogonality*, J. Math. Anal. Appl. **407(2)** (2013), 350–358.
[4] G. Birkhoff, *Orthogonality in linear metric spaces*, Duke Math. J. **1(2)** (1935), 169–172.
[5] F. F. Bonsall, J. Duncan, *Numerical ranges of operators on normed spaces and elements of normed algebras*, Vol. 2, Londonmathematical society lecture note series. London: Cambridge University Press, 1971.
[6] A. Bourhim, M. Mabrouk, *Numerical radius and product of elements in C^*-algebras*, Linear Multilinear Algebra **65(6)** (2017), 1108–1116.
[7] M. L. Buzano, *Generalizzazione della diseguaglianza di Cauchy-Schwarz*, (Italian) Rend Sem Mat Univ E Politech Torino. **31**(1971/73), 405–409.
[8] D. S. Cvetković Ilić, H. Stanković, *Numerical radius symmetry and relations of Birkhoff-James and numerical radius orthogonality for different classes of operators*, J. Math. Anal. Appl. **541(1)** (2025), 128690.
[9] S. M. Enderami, M. Abtahi and A. Zamani, *An extension of Birkhoff-James orthogonality relations in semi-Hilbertian space operators*, Mediterr. J. Math. **19** (2022), 234.
[10] S. Ghosh, A. Mal, K. Paul, D. Sain, *On symmetric points with respect to the numerical radius norm*, Banach J. Math. Anal. **17** (2023), 67.
[11] F. Kittaneh, *Notes on some inequalities for Hilbert space operators*, Publ. Res. Inst. Math. Sci. **24 (2)**(1988), 283–293.
[12] F. Kittaneh, A. Zamani, *Characterizations of w_ρ-Birkhoff-James orthogonality and w_ρ-parallelism*, Math. Inequal. Appl. **28** (2025), 173–187.
[13] M. Mabrouk, A. Zamani, *An extension of the a-numerical radius on C^*-algebras*, Banach J. Math. Anal. **17** (2023), 42.
[14] A. Mal, K. Paul, J. Sen, *Birkhoff-James orthogonality and numerical radius inequalities of operator matrices*, Monatsh. Math. **197** (2022), 717–731.
[15] C. A. McCarthy, *Cp*, Israel J. Math. **5** (1967), 249–271.
[16] M. Mehrazin, M. Amyari and A. Zamani, *Numerical radius parallelism of Hilbert space operators*, Bull. Iranian Math. Soc. **46(3)** (2020), 821–829.
[17] A. Seddik, *Rank one operators and norm of elementary operators*, Linear Algebra Appl. **424** (2007), 177–183.
[18] A. Zamani, *Characterization of numerical radius parallelism in C^*-algebras*, Positivity **23** (2019), 397–411.
[19] A. Zamani, *Numerical radius in Hilbert C^*-modules*, Math. Inequal. Appl. **24** (2021), 1017–1030.
[20] A. Zamani, M.S. Moslehian, *Exact and approximate operator parallelism*, Canad. Math. Bull. **58** (2015), 207–224.
[21] A. Zamani, P. Wójcik, *Numerical radius orthogonality in C^*-algebras*, Ann. Funct. Anal. **11**(2020), 1081–1092.