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Difference equations for a class of twice-iterated Hahn-Appell sequences of polynomials

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Abstract. In this article we consider a family of twice iterated Hahn-Appell polynomials (TI H-AP), which includes the twice iterated and the usual versions of Appell, ω-Appell and q-Appell polynomials. An equivalence theorem for the definition including the explicit representation and the generating function is given. Then determinantal representation, pure recurrence relation, lowering, rasing operators and difference equation by means of Hahn difference operator are obtained for these polynomials. As an application of the main results, we provide some results for 2-orthogonal Hahn-Appell polynomials in terms of one-orthogonal version. In the last section, we introduce the Hahn-Bernoulli, Hahn-Euler, Hahn-Genocchi and twice-iterated Hahn-Bernoulli-Euler polynomials.

1. Introduction

Hahn difference operator [8] has been considered in many recent papers [2],[7],[6] since it contains the main operators of calculus and therefore the investigated problem can be solved in a more general scale. More precisely, this operator includes the usual derivative, q-derivative, forward difference and delta operators. It is given by,

$$\mathcal{D}_{q,\omega} f(T) = \frac{f(qT+\omega) - f(T)}{(q-1)T+\omega}, \qquad q \in [0,1], \omega > 0 \tag{1}$$

where $T \neq \omega_0$ and $\omega_0 := \frac{\omega}{1-q}$.

There are 2 types of (q,ω)-exponential (Hahn exponential) functions given by [24]

$$e_q^{\omega}(t) := \sum_{n=0}^{\infty} \frac{1}{[n]_q!}(t)_{q,\omega}^n = \frac{(-\omega;q)_\infty}{((1-q)t-\omega;q)_\infty}, \qquad |(1-q)t-\omega|<1 \tag{2}$$

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and

$$E_q^{\omega}(t) := \sum_{n=0}^{\infty} \frac{[t]_{q,\omega}^n}{[n]_q!} = \frac{((q-1)t+\omega;q)_\infty}{(-\omega;q)_\infty}, \qquad |\omega|<1 \tag{3}$$

which are the solutions of

$$\mathcal{D}_{q,\omega} f(t) = f(t), \quad f(0) = 1$$

$$\mathcal{D}_{q,\omega} f(t) = f(qt+\omega), \quad f(0) = 1$$

respectively.

They satisfy the property

$$E_q^{-\omega}(-t) e_q^{\omega}(t) = 1,$$

and in the special cases we have,

$$e_q^0(t) = E_q^0(t) = e_q(t), \qquad e_1^0(t) = E_1^0(t) = e^t,$$
$$e_1^{\omega}(t) = (1+\omega)^{\frac{t}{\omega}}, \qquad E_1^{\omega}(t) = (1-\omega)^{\frac{-t}{\omega}}.$$

In (2) and (3), the symbols $[n]_q$, $(t)_{q,\omega}^n$, $[t]_{q,\omega}^n$, $(a;q)_\infty$, $(a;q)_j$ and the symbols of q-calculus which are used thoughout the paper are given as follows:

The q-analogue of the integer n;

$$[n]_q = 1 + q + q^2 + ... + q^{n-1} = \frac{(q^n - 1)}{(q-1)}.$$

The q-factorial is given by

$$(a;q)_j = \begin{cases} 1, & j = 0; \\ \prod_{m=0}^{j-1}(1-q^m a), & j \geq 1; \\ \prod_{m=0}^{\infty}(1-q^m a), & j = \infty. \end{cases}$$

The q-binomial coefficient and the q-binomial formula are respectively defined as follows:

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_{n-m}(q;q)_m}, \qquad (m \leqslant n, \quad n, m \in \mathbb{N}).$$

$$(1-a)_q^j := (a;q)_j = \sum_{m=0}^{j} \begin{bmatrix} j \\ m \end{bmatrix}_q q^{(1/2)m(m-1)} (-1)^m a^m.$$

The polynomial bases of Hahn calculus are $\{(t)_{q,\omega}^n\}_{n\geq 0}$ and $\{[t]_{q,\omega}^n\}_{n\geq 0}$ which are given by,

$$(t)_{q,\omega}^n = t(t-\omega)(t-[2]_q\omega)(t-[3]_q\omega)...(t-[n-1]_q\omega), \tag{4}$$

$$[t]_{q,\omega}^n = t(qt+\omega)(q^2t+[2]_q\omega)...(q^{n-1}t+[n-1]_q\omega). \tag{5}$$

Applying Hahn difference operator to (4) and (5), we obtain:

$$\mathcal{D}_{q,\omega}(t)_{q,\omega}^n = [n]_q (t)_{q,\omega}^{n-1},$$

$$\mathcal{D}_{q,\omega}[t]^n_{q,\omega} = [n]_q[qt+\omega]^{n-1}_{q,\omega}$$

where $\omega > 0$,$q \in (0,1)$ and $n \geq 1$.

Taking into account the fact that the Appell polynomials defined by [3]

$$d/d_z\left(A_n(z)\right) = nA_{n-1}(z)$$

has many applications in approximation theory, special function theory, analytic function theory and numerical analysis [see in [20], [19], [23], [14], [16], [13], [22], [15]], recently the authors in [24] introduced and investigated the H-AP through the definition

$$\mathcal{D}_{q,\omega}P_n(x) = [n]_qP_{n-1}(x), n \geq 1. \tag{6}$$

It has shown that these polynomials implicate the q-, usual and $\omega-$Appell polynomials. (see in [5] and [1]). They can be given by an elegant generating relation

$$A(t)e_q^{\omega t}(xt) = \sum_{n=0}^{\infty} A_n(x;\omega;q)\frac{t^n}{[n]_q!} \tag{7}$$

where $A(t)$ is an analytic function at the origin given by

$$A(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{[k]_q!}.$$

Recently, another dimension of research has been started on the TI AP which are defined by [11]

$$A_1(t)A_2(t)e^{xt} = \sum_{n=0}^{\infty} A_{n,2}(x)\frac{t^n}{n!}. \tag{8}$$

We should note that an equivalent definition of (8) has been given for the ω-Appell polynomials in [18]. For the TI A type polynomials, we further refer the following papers ([11], [21], [17], [12]).

Motivated essentially by the above mentioned applicaitions and research dimensions, in the peresent paper we introduce the TI H-AP. The main advantage of this definition is two folded. These new polynomials includes the q-Hahn Appell polynomials in their special case, they include the TI versions of usual, ω- and q-Appell polynomials. Therefore, rather then proving similar results for each special case, this new definition gives chance to establish a theory more efficiently.

We organise the paper as follows. In section 2, we consider a family of TI H-AP and obtain their explicit form and generating function. Some equivalence theorems for the definition are also obtained. Futhermore, the determinantal form of the TI HA sequences is derived in terms of the (H-AP) Hahn-Appell polynomials and a circular theorem provided by TI HA sequences is proven. Lowering operators, rasing operators are obtained, in section 3 and by using them we give the difference equation and the pure recurrence relation contented by TI H-AP. As an application of these polynomials, in the fourth section we give the explicit representation and determinantal form of the 2-orthogonal TI H-AP.

2. Twice-iterated Hahn-Appell Polynomials

Let $Q_n(x)$ be (H-A sequence) the Hahn-Appell sequence defined in [24]

$$Q_n(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q_{n-k}(x)^k_{q,\omega} \tag{9}$$

with

$$a_1(t) = \sum_{n=0}^{\infty} q_n \frac{t^n}{n!}. \tag{10}$$

Now, we give the definition of the TI HA sequences as follows.

Definition 2.1. *Let $P_n(x)$ be any polynomial given by*

$$P_n(x) = \sum_{k=0}^{n} p(n,k)(x)_{q,\omega}^{k}. \tag{11}$$

with $p(n,k)$ is the coefficients and $(x)_{q,\omega}^{k}$ is a basis of Hahn Calculus which is given in (4) . *Then $Q_k(x)$ be a H-A sequence defined in* (9) *and the polynomials ${}_HA_n^{[2]}(x)$ defined by*

$${}_HA_n^{[2]}(x) = \sum_{k=0}^{n} p(n,k)Q_k(x) \tag{12}$$

If the following relation holds true

$$D_{q,\omega\,H}A_n^{[2]}(x) = [n]_{q\,H}A_{n-1}^{[2]}(x), \tag{13}$$

${}_HA_n^{[2]}(x)$ called the TI HA sequence of polynomials.

Theorem 2.2. *[Equivalence Theorem]*
The subsequent statements are mutually satisfying and can be derived from one another.

i) *$\{\, {}_HA_n^{[2]}(x) \,\}$ is a TI HA sequence.*

ii) *$\{\, P_n(x) \,\}$ is a H-A sequence, then $p(n,k)$ represents the coefficients,*

$$p(n,k) = \begin{bmatrix} n \\ k \end{bmatrix}_q p_{n-k},$$

$a_2(t)$ is the determining function of $P_n(x)$ is defined by

$$a_2(t) = \sum_{n=0}^{\infty} p_n \frac{t^n}{[n]_q!}.$$

iii) *The explicit form of the TI HA sequence $\{\, {}_HA_n^{[2]}(x) \,\}$ is defined by*

$${}_HA_n^{[2]}(x) = \sum_{k=0}^{n} \sum_{l=0}^{k} p_{n-k} q_{k-l} (x)_{q,\omega}.$$

iv) *Generating function of TI H-AP $\{\, {}_HA_n^{[2]}(x) \,\}$ is given by*

$$a_2(t)a_1(t)e_q^{\omega t}(x) = \sum_{n=0}^{\infty} {}_HA_n^{[2]}(x) \frac{t^n}{[n]_q!}. \tag{14}$$

Proof. *Firstly we prove that* (1) $\longleftrightarrow$ (2). *Let* $\{\ _HA_n^{[2]}(x)\ \}$ *be a TI HA sequence. Applying the Hahn operator on both side of* (12) *and recalling* $\{\ _HA_n^{[2]}(x)\ \}$ *and* $\{\ Q_k(x)\ \}$ *are the H-A sequence, we have*

$$D_{q,\omega}\left[{}_HA_n^{[2]}(x)\right] = D_{q,\omega}\left[\sum_{k=0}^{n} p(n,k)Q_k(x)\right] = \left[\sum_{k=0}^{n} p(n,k)D_{q,\omega}Q_k(x)\right].$$

From (13), *we can obtain that;*

$$[n]_q{}_HA_{n-1}^{[2]}(x) = \left[\sum_{k=0}^{n} p(n,k)D_{q,\omega}Q_k\right] = \left[\sum_{k=1}^{n} p(n,k)[k]_qQ_{k-1}(x)\right] = \left[\sum_{k=0}^{n-1}[k+1]_qp(n,k+1)Q_k(x)\right].$$

Therefore,

$${}_HA_{n-1}^{[2]}(x) = \left[\frac{1}{[n]_q}\sum_{k=0}^{n-1}[k+1]_qp(n,k+1)Q_k(x)\right].$$

So, we obtain that

$${}_HA_n^{[2]}(x) = \frac{1}{[n+1]_q}\sum_{k=0}^{n}[k+1]_qp(n+1,k+1)Q_k(x) = \sum_{k=0}^{n} p(n,k)Q_k(x).$$

This gives for $k = 0,1,2,...,n$ *that*

$$\frac{[k+1]_q}{[n+1]_q} = p(n+1,k+1)p(n,k).$$

Now, through successive iterations of the preceding relation k times, we arrive at the following result:

$$p(n+1,k+1) = \frac{[n+1]_q[n]_q...[n-k+1]_q[n-k]_q!}{[k+1]_q[k]_q...[1]_q[n-k]_q!}p(n-k,0) = \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q p(n-k,0).$$

Hence

$$p(n,k) = \begin{bmatrix} n \\ k \end{bmatrix}_q p_{n-k}. \tag{15}$$

Replacing (15) *into* (11), *we have*

$$P_n(x) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q p_{n-k}(x)_{q,\omega}^n. \tag{16}$$

Now, applying the Hahn derivative which is given in (1), *on the both sides of* (16) *and by following the relation*

$$D_{q,\omega}(x)_{q,\omega}^k = [k]_q(x)_{q,\omega}^{k-1}$$

we get,

$$\begin{aligned} D_{q,\omega}\left(P_n(x)\right) &= D_{q,\omega}\left(\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q p_{n-k}(x)_{q,\omega}^k\right) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q p_{n-k}D_{q,\omega}\left((x)_{q,\omega}^k\right) \\ &= [n]_q\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q p_{n-k}(x)_{q,\omega}^{k-1} = [n]_qP_{n-1}(x). \end{aligned}$$

Therefore, $\{ P_n(x) \}$ *is a H-A sequence.*

Secondly, (2) $\longleftrightarrow$ (3)*. Take* $\{ Q_n(x) \}$ *is a H-A sequence. From assertion* (2) *and the definition of TI H-AP, we can write*

$$Q_k(x) = \sum_{l=0}^{k} \begin{bmatrix} k \\ l \end{bmatrix}_q q_{k-1}(x)^n_{q,\omega}$$

and

$${}_HA_n^{[2]}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q p_{n-k} Q_k(x).$$

Now, we substitute the $Q_k(x)$ *in the above equation, it is clearly to see that*

$${}_HA_n^{[2]}(x) = \sum_{k=0}^{n} \sum_{l=0}^{k} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} k \\ l \end{bmatrix}_q p_{n-k} q_{n-k} (x)^n_{q,\omega}.$$

Finally, we prove that (3) $\longleftrightarrow$ (4)*. Using assertion* (3)*, we get*

$$\sum_{n=0}^{\infty} {}_HA_n^{[2]}(x) \frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{l=0}^{k} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} k \\ l \end{bmatrix}_q p_{n-k} q_{n-k} (x)^n_{q,\omega} \frac{t^n}{[n]_q!}.$$

Now, by using Cauchy product, we get

$$\begin{aligned} \sum_{n=0}^{\infty} {}_HA_n^{[2]}(x) \frac{t^n}{[n]_q!} &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{l=0}^{k} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} k \\ l \end{bmatrix}_q p_{n-k} q_{n-k} (x)^n_{q,\omega} \frac{t^n}{[n]_q!} \\ &= \sum_{n=0}^{\infty} p_n \frac{t^n}{[n]_q!} \sum_{k=0}^{\infty} q_k \frac{t^k}{[k]_q!} \sum_{l=0}^{\infty} (x)^l_{q,\omega} \frac{t^l}{[l]_q!} \\ &= a_2(t) a_1(t) e_q^{\omega t}(xt) \end{aligned}$$

where

$$a_2(t) = \sum_{n=0}^{\infty} p_n \frac{t^n}{[n]_q!}, \quad a_1(t) = \sum_{k=0}^{\infty} q_k \frac{t^k}{[k]_q!}, \quad e_q^{\omega t}(xt) = \sum_{l=0}^{\infty} (x)^l_{q,\omega} \frac{t^l}{[l]_q!}.$$

where the result. □

Remark 2.3. *Note that,*

1. *Choosing* $a_1(t) = 1$ *(or* $a_2(t) = 1$*) in generating function of TI H-AP* (14)*, we get H-AP which was given by* (7) *in [24].*
2. *Letting* $q \to 1$ *and* $\omega \to 0$ *in TI H-AP, we get TI Appell polynomials which was introduced in [10].*
3. *Letting* $q \to 1$ *in TI H-AP, we get TI* ω*-Appell polynomials.*
4. *Taking* $h \to 0$ *in TI H-AP, we get TI q-appell polynomials which is obtained in [12].*

Theorem 2.4. *The determinantal representation of the TI HA sequence is given by*

$$
{}_HA_n^{[2]}(x)=\frac{(-1)^n}{(\phi_0)^{n+1}}\begin{vmatrix}
Q_0(x) & Q_1(x) & Q_2(x) & \cdots & \cdots & Q_{n-1}(x) & Q_n(x)\\
\phi_0 & \phi_1 & \phi_2 & \cdots & \cdots & \phi_{n-1} & \phi_n\\
0 & \phi_0 & \begin{bmatrix}2\\1\end{bmatrix}_q\phi_1 & \cdots & \cdots & \begin{bmatrix}n-1\\1\end{bmatrix}_q\phi_{n-2} & \begin{bmatrix}n\\1\end{bmatrix}_q\phi_{n-1}\\
0 & 0 & \phi_0 & \cdots & \cdots & \begin{bmatrix}n-1\\2\end{bmatrix}_q\phi_{n-3} & \begin{bmatrix}n\\2\end{bmatrix}_q\phi_{n-2}\\
. & . & . & . & . & . & .\\
. & . & . & . & . & . & .\\
. & . & . & . & . & . & .\\
0 & . & . & . & 0 & \phi_0 & \begin{bmatrix}n\\n-1\end{bmatrix}_q\phi_1
\end{vmatrix}
$$

where $\{\ {}_HA_n^{[2]}(x)\ \}$ *is the TI HA sequence with the generating function* (14) *and* $\phi_0,\phi_1,\ldots,\phi_n$ *are the Maclaurin coefficients of the function* $[a_2(t)]^{-1}$.

Proof. *Let the Maclaurin series of* $[a_2(t)]^{-1}$ *be given by:*

$$
[a_2(t)]^{-1}=\sum_{k=0}^{\infty}\phi_k\frac{t^k}{[k]_q!}.
$$

By multiplying each side of the equation (14) *by the reciprocal of* $[a_2(t)]$, *we obtain:*

$$
[a_2(t)]^{-1}\sum_{n=0}^{\infty}{}_HA_n^{[2]}(x)\frac{t^n}{[n]_q!}=a_1(t)e_q^{\omega t}(xt)
$$

which gives,

$$
\sum_{k=0}^{\infty}\phi_k\frac{t^k}{[k]_q!}\sum_{n=0}^{\infty}{}_HA_n^{[2]}(x)\frac{t^n}{[n]_q!}=a_1(t)e_q^{\omega t}(xt), \tag{17}
$$

H-AP generating relation is given by

$$
a_1(t)e_q^{\omega t}(xt)=\sum_{n=0}^{\infty}Q_n(x)\frac{t^n}{[n]_q!},
$$

and substitute the above equation into the (17), *we get*

$$
\sum_{k=0}^{\infty}\phi_k\frac{t^k}{[k]_q!}\sum_{n=0}^{\infty}{}_HA_n^{[2]}(x)\frac{t^n}{[n]_q!}=\sum_{n=0}^{\infty}Q_n(x)\frac{t^n}{[n]_q!}. \tag{18}
$$

Using the Cauchy product, we find that

$$
\begin{aligned}
\sum_{n=0}^{\infty}\sum_{k=0}^{n}\phi_k{}_HA_{n-k}^{[2]}(x)\frac{t^{n-k}}{[n-k]_q!}\frac{t^k}{[k]_q!}&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\phi_k{}_HA_{n-k}^{[2]}(x)\frac{t^n}{[n]_q!}\frac{[n]_q!}{[n-k]_q![k]_q!}\\
&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\phi_k{}_HA_{n-k}^{[2]}(x)\frac{t^n}{[n]_q!}\\
&=\sum_{n=0}^{\infty}Q_n(x)\frac{t^n}{[n]_q!}.
\end{aligned}
$$

We easily obtain that

$$Q_n(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \phi_{kH} A_{n-k}^{[2]}(x).$$

We obtain the system of equation with the unknowns ${}_HA_n^{[2]}(x)$ as follows

$$\begin{aligned} Q_0(x) &= \phi_{0H} A_0^{[2]}(x) \\ Q_1(x) &= \phi_{1H} A_1^{[2]}(x) + \phi_{0H} A_0^{[2]}(x) \\ Q_2(x) &= \phi_{2H} A_2^{[2]}(x) + \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \phi_{1H} A_1^{[2]}(x) + \phi_{0H} A_0^{[2]}(x) \\ &\;\;. \\ &\;\;. \\ &\;\;. \\ Q_n(x) &= \phi_{nH} A_n^{[2]}(x) + \begin{bmatrix} n \\ 1 \end{bmatrix}_q \phi_{n-1H} A_{n-1}^{[2]}(x) + \ldots + \phi_{0H} A_0^{[2]}(x). \end{aligned}$$

From Cramer's rule, we can write that

$${}_HA_n^{[2]}(x) = \begin{vmatrix} \phi_0 & 0 & 0 & \ldots & \ldots & 0 & Q_0(x) \\ \phi_1 & \phi_0 & 0 & \ldots & \ldots & 0 & Q_1(x) \\ \phi_2 & \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \phi_1 & \phi_0 & \ldots & \ldots & 0 & Q_2(x) \\ \phi_3 & \begin{bmatrix} 3 \\ 2 \end{bmatrix}_q \phi_2 & \begin{bmatrix} 3 \\ 1 \end{bmatrix}_q \phi_1 & \ldots & \ldots & 0 & Q_3(x) \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \phi_{n-1} & \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q \phi_{n-2} & \begin{bmatrix} n-1 \\ 2 \end{bmatrix}_q \phi_{n-3} & \ldots & \ldots & \phi_0 & Q_{n-1}(x) \\ \phi_n & \begin{bmatrix} n \\ 1 \end{bmatrix}_q \phi_{n-1} & \begin{bmatrix} n \\ 2 \end{bmatrix}_q \phi_{n-2} & \ldots & \ldots & \begin{bmatrix} n \\ n-1 \end{bmatrix}_q \phi_1 & Q_n(x) \end{vmatrix}$$

$$\times \begin{vmatrix} \phi_0 & 0 & 0 & \ldots & \ldots & 0 & 0 \\ \phi_1 & \phi_0 & 0 & \ldots & \ldots & 0 & 0 \\ \phi_2 & \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \phi_1 & \phi_0 & \ldots & \ldots & 0 & 0 \\ \phi_3 & \begin{bmatrix} 3 \\ 2 \end{bmatrix}_q \phi_2 & \begin{bmatrix} 3 \\ 1 \end{bmatrix}_q \phi_1 & \ldots & \ldots & 0 & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \phi_{n-1} & \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q \phi_{n-2} & \begin{bmatrix} n-1 \\ 2 \end{bmatrix}_q \phi_{n-3} & \ldots & \ldots & \phi_0 & 0 \\ \phi_n & \begin{bmatrix} n \\ 1 \end{bmatrix}_q \phi_{n-1} & \begin{bmatrix} n \\ 2 \end{bmatrix}_q \phi_{n-2} & \ldots & \ldots & \begin{bmatrix} n \\ n-1 \end{bmatrix}_q \phi_1 & \phi_0 \end{vmatrix}^{-1}$$

If we take transpose of the first determinant and observing that the second determinant is lower triangular, we *have*

$$
{}_{H}A_n^{[2]}(x) = \frac{1}{(\phi_0)^{n+1}} \begin{vmatrix}
\phi_0 & \phi_1 & \phi_2 & \cdots & \cdots & \phi_{n-1} & \phi_n \\
0 & \phi_0 & \left[{2 \atop 1}\right]_q \phi_1 & \cdots & \cdots & \left[{n-1 \atop 1}\right]_q \phi_{n-2} & \left[{n \atop 1}\right]_q \phi_{n-1} \\
0 & 0 & \phi_0 & \cdots & \cdots & \left[{n-1 \atop 2}\right]_q \phi_{n-3} & \left[{n \atop 2}\right]_q \phi_{n-2} \\
. & . & . & . & . & . & . \\
. & . & . & . & . & . & . \\
. & . & . & . & . & . & . \\
0 & . & . & . & . & \phi_0 & \left[{n \atop n-1}\right]_q \phi_1 \\
Q_0(x) & Q_1(x) & Q_2(x) & \cdots & \cdots & Q_{n-1}(x) & Q_n(x)
\end{vmatrix}
$$

$$
= \frac{(-1)^n}{(\phi_0)^{n+1}} \begin{vmatrix}
Q_0(x) & Q_1(x) & Q_2(x) & \cdots & \cdots & Q_{n-1}(x) & Q_n(x) \\
\phi_0 & \phi_1 & \phi_2 & \cdots & \cdots & \phi_{n-1} & \phi_n \\
0 & \phi_0 & \left[{2 \atop 1}\right]_q \phi_1 & \cdots & \cdots & \left[{n-1 \atop 1}\right]_q \phi_{n-2} & \left[{n \atop 1}\right]_q \phi_{n-1} \\
0 & 0 & \phi_0 & \cdots & \cdots & \left[{n-1 \atop 2}\right]_q \phi_{n-3} & \left[{n \atop 2}\right]_q \phi_{n-2} \\
. & . & . & . & . & . & . \\
. & . & . & . & . & . & . \\
. & . & . & . & . & . & . \\
0 & . & . & . & 0 & \phi_0 & \left[{n \atop n-1}\right]_q \phi_1
\end{vmatrix}
$$

As a result, we obtain the desired result which is given in the statement. □

Theorem 2.5. *Let ${}_{H}A_n^{[2]}(x)$is the sequence defined by the determinantal form in the theorem (2.4). Then*

$$D_{q,\omega}\left[{}_{H}A_n^{[2]}(x)\right] = [n]_q {}_{H}A_{n-1}^{[2]}(x).$$

Proof. Firstly, applying the Hahn Calculus to the determinantal form of ${}_{H}A_n^{[2]}(x)$ for $n \in N$ that

$$
D_{q,\omega H}A_n^{[2]}(x) = \frac{(-1)^n}{(\phi_0)^{n+1}} \begin{vmatrix}
D_{q,\omega}Q_0(x) & D_{q,\omega}Q_1(x) & D_{q,\omega}Q_2(x) & \cdots & \cdots & D_{q,\omega}Q_{n-1}(x) & D_{q,\omega}Q_n(x) \\
\phi_0 & \phi_1 & \phi_2 & \cdots & \cdots & \phi_{n-1} & \phi_n \\
0 & \phi_0 & \left[{2 \atop 1}\right]_q \phi_1 & \cdots & \cdots & \left[{n-1 \atop 1}\right]_q \phi_{n-2} & \left[{n \atop 1}\right]_q \phi_{n-1} \\
0 & 0 & \phi_0 & \cdots & \cdots & \left[{n-1 \atop 2}\right]_q \phi_{n-3} & \left[{n \atop 2}\right]_q \phi_{n-2} \\
. & . & . & . & . & . & . \\
. & . & . & . & . & . & . \\
. & . & . & . & . & . & . \\
0 & . & . & . & 0 & \phi_0 & \left[{n \atop n-1}\right]_q \phi_1
\end{vmatrix}.
$$

Now, applying the following equation due to the definition of Hahn A.P.

$$D_{q,\omega}Q_n(x) = [n]_q Q_{n-1}(x)$$

and using the some mathematical applications, as a result we obtain the following equation with the determinantal form of TI H-AP

$$D_{q,\omega}\left[{}_{H}A_n^{[2]}(x)\right] = [n]_q {}_{H}A_{n-1}^{[2]}(x)$$

□

Theorem 2.6. *Circular Theorem satisfied by the TI HA sequence, therefore the following statements are equivalent :*

1. $\{ {}_HA_n^{[2]}(x) \}$ *is a TI HA sequence.*
2. *The TI H-AP* ${}_HA_n^{[2]}(x)$ *posses a generating function given by*

$$a_2(t)a_1(t)e_q^{\omega t}(x) = \sum_{n=0}^{\infty} {}_HA_n^{[2]}(x)\frac{t^n}{[n]!}.$$

3. *The determinantal form of TI H-AP* ${}_HA_n^{[2]}(x)$ *is defined in theorem (2.5).*

Proof. The proof of (1) $\longleftrightarrow$ (2) follows from Theorem 2.1. The proof of (2) $\longleftrightarrow$ (3) follows from Theorem 2.2. The proof of (3) $\longleftrightarrow$ (1) follows from Theorem 2.3. □

3. Some properties of Twice-Iterated H-AP

Theorem 3.1. *A reccurence relation of TI H-AP is given by*

$$\begin{aligned} {}_HA_{n+1}^{[2]}(x) = {}_HA_n^{[2]}(x)\left[x + \phi_0 q^n - \omega[n]_q\right] + \sum_{k=1}^{n} \phi_{kH}A_{n-k}^{[2]}(x)q^{n-k}\frac{[n]_q!}{[n-k]_q!} \\ + \rho_{0H}A_n^{[2]}(qx) + \sum_{k=1}^{n} \rho_{kH}A_{n-k}^{[2]}(qx). \end{aligned}$$

where $\phi_k = q^k(\alpha_k + \omega t\alpha_k)$ $k = 0, 1, 2, ...$ *and*

$$\frac{D_q a_2(t)}{a_2(qt)} = \sum_{k=0}^{\infty} \alpha_k t^k, \tag{19}$$

$$\frac{D_q a_1(t)}{a_1(t)} = \sum_{k=0}^{\infty} \beta_k t^k. \tag{20}$$

$$(1 + \omega t)\sum_{k=0}^{\infty} \beta_k t^k = \sum_{k=0}^{\infty} \rho_k t^k,$$

Proof. Taking q-derivative on both sides of (14), we have

$$D_q[a_1(t)a_2(t)]e_q^{\omega qt}(xqt) + \frac{x}{1+\omega t}[a_1(t)a_2(t)e_q^{\omega t}(xt)] = \sum_{n=0}^{\infty} D_{qH}A_n^{[2]}(x)\frac{t^n}{[n]_q!}.$$

Applying the product rule of the q-derivative, we get

$$\begin{aligned} a_1(qt)D_q[a_2(t)]e_q^{\omega qt}(xqt) + \frac{x}{1+\omega t}\sum_{n=0}^{\infty} {}_HA_n^{[2]}(x)\frac{t^n}{[n]_q!} + D_q[a_1(t)]a_2(t)]e_q^{\omega qt}(xqt) \\ = \sum_{n=0}^{\infty} P_{n+1}(x)\frac{t^n}{[n]_q!} \end{aligned}$$

which gives

$$\sum_{n=0}^{\infty} {}_HA_{n+1}^{[2]}(x)\frac{t^n}{[n]_q!} = a_1(qt)a_2(qt)e_q^{\omega qt}(xqt)\frac{D_q a_2(t)}{a_2(qt)} \qquad +a_2(t)a_1(t)e_q^{\omega qt}(xqt)\frac{D_q a_1(t)}{a_1(t)}$$
$$+\frac{x}{1+\omega t}\sum_{n=0}^{\infty} {}_HA_n^{[2]}(x)\frac{t^n}{[n]_q!}.$$

Multipling both sides by $1+\omega t$ we get,

$$(1+\omega t)\sum_{n=0}^{\infty} {}_HA_{n+1}^{[2]}(x)\frac{t^n}{[n]_q!} = (1+\omega t)\frac{D_q a_2(t)}{a_2(qt)}\sum_{n=0}^{\infty} q^n {}_HA_n^{[2]}(x)\frac{t^n}{[n]_q!} + x\sum_{n=0}^{\infty} {}_HA_n^{[2]}(x)\frac{t^n}{[n]_q!}$$
$$+(1+\omega t)\frac{D_q a_1(t)}{a_1(t)}\sum_{n=0}^{\infty} {}_HA_n^{[2]}(qx)\frac{t^n}{[n]_q!}.$$

Letting,

$$\frac{D_q a_2(t)}{a_2(qt)} = \sum_{k=0}^{\infty} \alpha_k t^k,$$

$$\frac{D_q a_1(t)}{a_1(t)} = \sum_{k=0}^{\infty} \beta_k t^k$$

and

$$(1+\omega t)\sum_{k=0}^{\infty} \alpha_k t^k = \sum_{k=0}^{\infty} \phi_k t^k$$

with $\phi_k = q^k(\alpha_k + \omega t\alpha_k)$, $k = 0, 1, 2, ...$ and

$$(1+\omega t)\sum_{k=0}^{\infty} \beta_k t^k = \sum_{k=0}^{\infty} \rho_k t^k.$$

Hence we get,

$$\sum_{k=0}^{\infty} \phi_k t^k \sum_{n=0}^{\infty} q^n {}_HA_n^{[2]}(x)\frac{t^n}{[n]_q!} + \sum_{k=0}^{\infty} \rho_k t^k \sum_{n=0}^{\infty} {}_HA_n^{[2]}(qx)\frac{t^n}{[n]_q!} + x\sum_{n=0}^{\infty} {}_HA_n^{[2]}(x)\frac{t^n}{[n]_q!}$$
$$= (1+\omega t)\sum_{n=0}^{\infty} {}_HA_{n+1}^{[2]}(x)\frac{t^n}{[n]_q!}$$

For the right handside,

$$(1+\omega t)\sum_{n=0}^{\infty} {}_HA_{n+1}^{[2]}(x)\frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} {}_HA_{n+1}^{[2]}(x)\frac{t^n}{[n]_q!} + \omega t\sum_{n=1}^{\infty} {}_HA_n^{[2]}(x)\frac{t^{n-1}}{[n-1]_q!}$$
$$= \sum_{n=0}^{\infty} {}_HA_{n+1}^{[2]}(x)\frac{t^n}{[n]_q!} + \omega\sum_{n=0}^{\infty} {}_HA_n^{[2]}(x)\frac{t^n}{[n]_q![n]_q}.$$

Then,

$$\sum_{k=0}^{\infty}\phi_k t^k\sum_{n=0}^{\infty}q^n{}_HA_n^{[2]}(x)\frac{t^n}{[n]_q!}+\sum_{k=0}^{\infty}\rho_k t^k\sum_{n=0}^{\infty}{}_HA_n^{[2]}(qx)\frac{t^n}{[n]_q!}+x\sum_{n=0}^{\infty}{}_HA_n^{[2]}(x)\frac{t^n}{[n]_q!}$$
$$=\sum_{n=0}^{\infty}\left[{}_HA_{n+1}^{[2]}(x)+\omega[n]_q{}_HA_n^{[2]}(x)\right]\frac{t^n}{[n]_q!}.$$

Using the Cauchy product, we find that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\phi_k{}_HA_{n-k}^{[2]}(x)\frac{q^{n-k}t^n}{[n-k]_q!}+\sum_{n=0}^{\infty}\sum_{k=0}^{n}\rho_k{}_HA_{n-k}^{[2]}(qx)\frac{t^n}{[n-k]_q!}+x\sum_{n=0}^{\infty}{}_HA_n^{[2]}(x)\frac{t^n}{[n]_q!}$$
$$=\sum_{n=0}^{\infty}\left[{}_HA_{n+1}^{[2]}(x)+\omega[n]_q{}_HA_n^{[2]}(x)\right]\frac{t^n}{[n]_q!}.$$

Upon equating the coefficients of $\frac{t^n}{[n]_q!}$, we get for $n=0,1,2,...$ that

$$\sum_{k=0}^{n}\phi_k{}_HA_{n-k}^{[2]}(x)q^{n-k}\frac{[n]_q!}{[n-k]_q!}+\sum_{k=0}^{n}\rho_k{}_HA_{n-k}^{[2]}(qx)\frac{[n]_q!}{[n-k]_q!}+x{}_HA_n^{[2]}(x)$$
$$={}_HA_{n+1}^{[2]}(x)+\omega[n]_q{}_HA_n^{[2]}(x)$$

or equivalently

$${}_HA_{n+1}^{[2]}(x)={}_HA_n^{[2]}(x)\left[x+\phi_0q^n-\omega[n]_q\right]+\rho_0{}_HA_n^{[2]}(qx)+\sum_{k=1}^{n}\rho_k{}_HA_{n-k}^{[2]}(qx)$$
$$+\sum_{k=1}^{n}\phi_k{}_HA_{n-k}^{[2]}(x)q^{n-k}\frac{[n]_q!}{[n-k]_q!}.$$

whence the result. □

Theorem 3.2. *The shift operators of of the TI H-AP and the difference equation of the TI H-AP are given by*

$$L_n^+=x+\phi_0q^n-\omega[n]_q+\rho_0T_q+\sum_{k=1}^{n}\phi_kq^{n-k}D_{q,\omega}^k+\sum_{k=1}^{n}\rho_kD_{q,\omega}^kT_q, \tag{21}$$

$$L_n^-:=\frac{1}{[n]_q}D_{q,\omega}. \tag{22}$$

and

$$\left[D_{q,\omega}(x+\phi_0q^n-\omega[n]_q)+D_{q,\omega}\rho_0T_q+\sum_{k=1}^{n}\phi_kq^{n-k}D_{q,\omega}^{k+1}+\sum_{k=1}^{n}\rho_kD_{q,\omega}^{k+1}T_q-[n+1]_q\right]{}_HA_n^{[2]}(x)=0 \tag{23}$$

respectively,
where $T_qf(x)=f(qx)$ and $\phi_1,\phi_2,...,\phi_k$ are the same as in the statements of the Theorem 3.1. and

$$(1+\omega t)\sum_{k=0}^{\infty}\beta_kt^k=\sum_{k=0}^{\infty}\rho_kt^k. \tag{24}$$

Proof. Recalling that

$$\frac{1}{[n]_q} D_{q,\omega\,H}A_n^{[2]}(x) = {}_HA_{n-1}^{[2]}(x),$$

we easily see

$$L_n^- := \frac{1}{[n]_q} D_{q,\omega}.$$

It is clear that:

$$\begin{aligned} {}_HA_{n-k}^{[2]}(x) &= \left[L_{n-k+1}^- L_{n-k+2}^- ... L_n^-\right] {}_HA_n^{[2]}(x) \\ &= \left[\frac{1}{[n-k+1]_q} D_{q,\omega} \frac{1}{[n-k+2]_q} D_{q,\omega} \cdots \frac{1}{[n]_q} D_{q,\omega}\right] {}_HA_n^{[2]}(x) \\ &= \frac{[n-k]_q!}{[n]_q!} D_{q,\omega\,H}^k A_n^{[2]}(x). \end{aligned}$$

Considering the above equality in the recurrence relation, we get

$$\begin{aligned} {}_HA_{n+1}^{[2]}(x) ={}& {}_HA_n^{[2]}(x)\left[x + \phi_0 q^n - \omega[n]_q\right] + \rho_{0\,H}A_n^{[2]}(qx) + \sum_{k=1}^{n} \phi_k D_{q,\omega\,H}^k A_n^{[2]}(x) q^{n-k} \\ &+ \sum_{k=1}^{n} \rho_k D_{q,\omega\,H}^k A_n^{[2]}(qx). \end{aligned}$$

Let T is the q-scaling factor operator defined by $T_q f(x) = f(qx)$. Thus,

$$A_n(qx) = T_q A_n(x).$$

$${}_HA_{n+1}^{[2]}(x) = \left[x + \phi_0 q^n - \omega[n]_q + \rho_0 T_q + \sum_{k=1}^{n} \phi_k q^{n-k} D_{q,\omega}^k + \sum_{k=1}^{n} \rho_k D_{q,\omega}^k T_q\right] {}_HA_n^{[2]}(x).$$

Then, we can see that,

$$L_n^+ = x + \phi_0 q^n - \omega[n]_q + \rho_0 T_q + \sum_{k=1}^{n} \phi_k q^{n-k} D_{q,\omega}^k + \sum_{k=1}^{n} \rho_k D_{q,\omega}^k T_q.$$

Next, by applying the factorization method:

$$L_{n+1}^- L_n^+ \left[{}_HA_n^{[2]}(x)\right] = {}_HA_n^{[2]}(x),$$

we get

$$\frac{1}{[n+1]_q} D_{q,\omega} \left[x + \phi_0 q^n - \omega[n]_q + \rho_0 T_q + \sum_{k=1}^{n} \phi_k q^{n-k} D_{q,\omega}^k + \sum_{k=1}^{n} \rho_k D_{q,\omega}^k T_q\right] {}_HA_n^{[2]}(x) = {}_HA_n^{[2]}(x).$$

Using product rule of q-calculus, we get

$$\left[D_{q,\omega}(x + \phi_0 q^n - \omega[n]_q) + D_{q,\omega}\rho_0 T_q + \sum_{k=1}^{n} \phi_k q^{n-k} D_{q,\omega}^{k+1} + \sum_{k=1}^{n} \rho_k D_{q,\omega}^{k+1} T_q - [n+1]_q\right] {}_HA_n^{[2]}(x) = 0.$$

□

4. 2-orthogonal Hahn-Appell Polynomials

Recall that the d-orthogonal H-AP (see in [24], Theorem 8) are given by

$$(t(x_1(1-q)-\omega);q)_\infty ...(t(x_d(1-q)-\omega);q)_\infty e_q^{\omega t}(xt)=\sum_{n=0}^{\infty} P_{n,(x_1,...,x_d)}(x;\omega,q)\frac{t^n}{[n]_q!}. \tag{25}$$

where $x_1, x_2 \ldots, x_d \in \mathbb{C}^* = \mathbb{C}/\{0,\omega_0\}$ with $0<q<1$ and $\omega>0$. Now consider the 1-orthogonal H-AP given by

$$a_1(t)e_q^{\omega t}(xt)=(t(x_1(1-q)-\omega);q)_\infty e_q^{\omega t}(xt)=\sum_{n=0}^{\infty} P^{(1)}_{n,x_1}(x;\omega,q)\frac{t^n}{[n]_q!}, \tag{26}$$

$$a_2(t)e_q^{\omega t}(xt)=(t(x_2(1-q)-\omega);q)_\infty e_q^{\omega t}(xt)=\sum_{n=0}^{\infty} P^{(2)}_{n,x_2}(x;\omega,q)\frac{t^n}{[n]_q!}. \tag{27}$$

The TI H-AP corresponding to the above $a_1(t)$ and $a_2(t)$ will be

$$\begin{aligned}a_1(t)a_2(t)e_q^{\omega t}(xt)&=(t(x_1(1-q)-\omega);q)_\infty(t(x_2(1-q)-\omega);q)_\infty e_q^{\omega t}(xt)\\&=\sum_{n=0}^{\infty} P_{n,(x_1,x_2)}(x;\omega,q)\frac{t^n}{[n]_q!}.\end{aligned}$$

where clearly from (25), $P_{n,(x_1,x_2)}(x;\omega,q)$ is the 2-orthogonal H-AP. We will state some results for the 2-orthogonal H-AP by using some of the our main theorems of the preceding sections. Let

$$Q_n(x)=P_{n,(x_1,x_2)}(x;\omega,q)$$

and

$$a_1^{-1}(t)=\frac{1}{(t(x_1(1-q)-\omega);q)_\infty}=\sum_{n=0}^{\infty}\phi_n^{(1)}\frac{t^n}{[n]_q!},$$

$$a_2^{-1}(t)=\frac{1}{(t(x_2(1-q)-\omega);q)_\infty}=\sum_{n=0}^{\infty}\phi_n^{(2)}\frac{t^n}{[n]_q!}.$$

Recall that [24];

$$e_q(x)=\sum_{n=0}^{\infty}\frac{x^n}{[n]_q!}=\frac{1}{((1-q)x;q)_\infty}.$$

We can see that;

$$a_1^{-1}(t)=\frac{1}{(t(q-1)(\frac{\omega}{1-q}-x_1))_\infty}=\sum_{n=0}^{\infty}\phi_n^{(1)}\frac{t^n}{[n]_q!}=\sum_{n=0}^{\infty}(\frac{\omega}{1-q}-x_1))^n\frac{t^n}{[n]_q!}.$$

$$a_2^{-1}(t)=\frac{1}{(t(q-1)(\frac{\omega}{1-q}-x_2))_\infty}=\sum_{n=0}^{\infty}\phi_n^{(2)}\frac{t^n}{[n]_q!}=\sum_{n=0}^{\infty}(\frac{\omega}{1-q}-x_2))^n\frac{t^n}{[n]_q!}.$$

Then

$$\phi_n^{(1)}=(\frac{\omega}{1-q}-x_1)^n,\qquad \phi_n^{(2)}=(\frac{\omega}{1-q}-x_2)^n\qquad (n=0,1,...) \tag{28}$$

where $\frac{\omega}{1-q} = \omega_0$.

Then the determinantal form of the 2-orthogonal H-AP is given by means of 1-orthogonal H-AP as follows:

$$P_{n,(x_1,x_2)}(x;\omega,q) = \frac{(-1)^n}{(\phi_0^{(1)})^{n+1}} \begin{vmatrix} P_{0,x_2}(x;\omega,q) & P_{1,x_2}(x;\omega,q) & P_{2,x_2}(x;\omega,q) & \cdots & \cdots & P_{n-1,x_2}(x;\omega,q) & P_{n,x_2}(x;\omega,q) \\ \phi_0^{(1)} & \phi_1^{(1)} & \phi_2^{(1)} & \cdots & \cdots & \phi_{n-1}^{(1)} & \phi_n^{(1)} \\ 0 & \phi_0^{(1)} & \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \phi_1^{(1)} & \cdots & \cdots & \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q \phi_{n-2}^{(1)} & \begin{bmatrix} n \\ 1 \end{bmatrix}_q \phi_{n-1}^{(1)} \\ 0 & 0 & \phi_0^{(1)} & \cdots & \cdots & \begin{bmatrix} n-1 \\ 2 \end{bmatrix}_q \phi_{n-3}^{(1)} & \begin{bmatrix} n \\ 2 \end{bmatrix}_q \phi_{n-2}^{(1)} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & . & . & . & 0 & \phi_0^{(1)} & \begin{bmatrix} n \\ n-1 \end{bmatrix}_q \phi_1^{(1)} \end{vmatrix}$$

$$= \frac{(-1)^n}{(\phi_0^{(2)})^{n+1}} \begin{vmatrix} P_{0,x_1}(x;\omega,q) & P_{1,x_1}(x;\omega,q) & P_{2,x_1}(x;\omega,q) & \cdots & \cdots & P_{n-1,x_1}(x;\omega,q) & P_{n,x_1}(x;\omega,q) \\ \phi_0^{(2)} & \phi_1^{(2)} & \phi_2^{(2)} & \cdots & \cdots & \phi_{n-1}^{(2)} & \phi_n^{(2)} \\ 0 & \phi_0^{(2)} & \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \phi_1^{(2)} & \cdots & \cdots & \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q \phi_{n-2}^{(2)} & \begin{bmatrix} n \\ 1 \end{bmatrix}_q \phi_{n-1}^{(2)} \\ 0 & 0 & \phi_0^{(2)} & \cdots & \cdots & \begin{bmatrix} n-1 \\ 2 \end{bmatrix}_q \phi_{n-3}^{(2)} & \begin{bmatrix} n \\ 2 \end{bmatrix}_q \phi_{n-2}^{(2)} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & . & . & . & 0 & \phi_0^{(2)} & \begin{bmatrix} n \\ n-1 \end{bmatrix}_q \phi_1^{(2)} \end{vmatrix}$$

where $\phi_n^{(1)}$ and $\phi_n^{(2)}$ are given in (28).
The explicit form of the 2-orthogonal TI HA sequence is given by

$$\begin{aligned} P_{n,(x_1,x_2)}(x;\omega,q) &= \sum_{k=0}^{n} \sum_{l=0}^{k} a_{n-k} q_{k-l} (x)_l \\ &= \sum_{k=0}^{n} \sum_{l=0}^{k} q^{\frac{(n-k)(n-k-1)}{2}} \left(\frac{\omega}{1-q} - x_1 \right)^{n-k} q^{\frac{(k-l)(k-l-1)}{2}} \left(\frac{\omega}{1-q} - x_2 \right)^{k-l} (x)_l \\ &= \sum_{l=0}^{n} q^{\frac{(n-k+l)(n-k+l-1)}{2}} \left(\frac{\omega}{1-q} - x_1 \right)^{n-k+l} q^{\frac{(k-2l)(k-2l-1)}{2}} \left(\frac{\omega}{1-q} - x_2 \right)^{k-2l} (x)_l. \end{aligned}$$

Corollary 4.1. *It is asserted that, a reccurence relation satisfied by 2-orthogonal H-AP is given by*

$$\begin{aligned} P_{n+1,(x_1,x_2)}(x;\omega,q) &= P_{n,(x_1,x_2)}(x;\omega,q) \left[x + \phi_0 q^n - \omega[n]_q \right] + \rho_0 P_{n,(x_1,x_2)}(qx;\omega,q) \\ &\quad + \sum_{k=1}^{n} \phi_k P_{n-k,(x_1,x_2)}(x;\omega,q) q^{n-k} \frac{[n]_q!}{[n-k]_q!} + \sum_{k=1}^{n} \rho_k P_{n-k,(x_1,x_2)}(qx;\omega,q). \end{aligned}$$

where

$$\frac{D_q a_2(t)}{a_2(qt)} = \sum_{k=0}^{\infty} \alpha_k t^k,$$

$$\frac{D_q a_1(t)}{a_1(t)} = \sum_{k=0}^{\infty} \beta_k t^k.$$

since $a_1(t)$ and $a_2(t)$ defined in (26),(27) *and*

$$(1+\omega t)\sum_{k=0}^{\infty}\alpha_k t^k=\sum_{k=0}^{\infty}\phi_k t^k$$

and

$$(1+\omega t)\sum_{k=0}^{\infty}\beta_k t^k=\sum_{k=0}^{\infty}\rho_k t^k.$$

with $\phi_k=q^k(\alpha_k+\omega t\alpha_k)$, $k=0,1,2,\ldots$. *On the otherhand*

$$\begin{aligned}\frac{D_q a_2(t)}{a_2(qt)}=\frac{a_2(qt)-a_2(t)}{(q-1)ta_2(qt)}&=\frac{\prod_{j=0}^{\infty}(1-q^{j+1}t(x_2(1-q)-\omega)-\prod_{j=0}^{\infty}(1-q^{j}t(x_2(1-q)-\omega)}{(q-1)t.\prod_{j=0}^{\infty}(1-q^{j+1}t(x_2(1-q)-\omega)}\\&=\frac{x_2(1-q)-\omega}{1-q}=x_2-\frac{\omega}{1-q}=\sum_{k=0}^{\infty}\alpha_k t^k.\end{aligned}$$

We see that

$$\alpha_0=x_2-\frac{\omega}{1-q},\qquad \alpha_k=0\quad for\quad k=1,2,3...$$

Moreover

$$(1+\omega t)\left(x_2-\frac{\omega}{1-q}\right)=\phi_0t^0+\phi_1t^1+\phi_2t^2+\phi_3t^3+\ldots$$

$$x_2-\frac{\omega}{1-q}+x_2\omega t-\frac{\omega^2 t}{1-q}=\phi_0t^0+\phi_1t^1+\phi_2t^2+\phi_3t^3+\ldots,$$

which gives

$$\phi_0=x_2-\frac{\omega}{1-q},\qquad \phi_1=x_2\omega-\frac{\omega^2}{1-q},\qquad \phi_k=0,\quad k=2,3,\ldots$$

Also,

$$\begin{aligned}\frac{D_q a_1(t)}{a_1(t)}=\frac{a_1(qt)-a_1(t)}{(q-1)t.a_1(t)}&=\frac{\prod_{j=0}^{\infty}(1-q^{j+1}t(x_1(1-q)-\omega)-\prod_{j=0}^{\infty}(1-q^{j}t(x_1(1-q)-\omega)}{(q-1)t.\prod_{j=0}^{\infty}(1-q^{j}t(x_1(1-q)-\omega)}\\&=\frac{x_1(1-q)-\omega}{(1-q)(1-(x_1(1-q)-\omega)t)}\\&=\frac{x_1(1-q)-\omega}{(1-q)}\sum_{n=0}^{\infty}((x_1(1-q)-\omega)t)^n\\&=\sum_{n=0}^{\infty}\beta_n t^n\end{aligned}$$

where

$$\beta_n=\frac{(x_1(1-q)-\omega)^{n+1}}{(1-q)}\qquad for\quad n=0,1,2,\ldots$$

$$(1+\omega t)\sum_{k=0}^{\infty}\frac{(x_1(1-q)-\omega)^{k+1}}{(1-q)}t^k=\sum_{k=0}^{\infty}\rho_k t^k.$$

Moreover,

$$\begin{aligned}\sum_{k=0}^{\infty}\left(\beta_k t^k+\omega\beta_k t^{k+1}\right)&=\beta_0+\sum_{k=1}^{\infty}\beta_k t^k+\sum_{k=0}^{\infty}\omega\beta_k t^{k+1}\\&=\beta_0+\sum_{k=1}^{\infty}(\beta_k+\omega\beta_{k-1})t^k\\&=\rho_0+\sum_{k=1}^{\infty}\rho_k t^k\end{aligned}$$

We obtain that, $\beta_0=\rho_0$ and $\rho_0=\beta_k+\omega\beta_{k-1}$.
Therefore, we can give the reccurence relation as follows

$$\begin{aligned}P_{n+1,(x_1,x_2)}(x;\omega,q)&=P_{n,(x_1,x_2)}(x;\omega,q)\left[x+(x_2-\frac{\omega}{1-q})q^n-\omega[n]_q\right]\\&+\frac{x_1(1-q)-\omega}{(1-q)}P_{n,(x_1,x_2)}(qx;\omega,q)\\&+\left(x_2-\frac{\omega}{1-q}+x_2\omega-\frac{\omega^2}{1-q}\right)\left(P_{n-k,(x_1,x_2)}(qx;\omega,q)q^{n-k}\frac{[n]_q!}{[n-k]_q!}\right)\\&+\sum_{k=0}^{n}\left(\frac{(x_1(1-q)-\omega)^{k+1}}{(1-q)}+\omega\frac{(x_1(1-q)-\omega)^k}{(1-q)}\right)P_{n-k,(x_1,x_2)}(qx;\omega,q).\end{aligned}$$

Corollary 4.2. *From* (4.1)*, difference equation of 2-orthogonal H-AP is given by*

$$P_{n+1,(x_1,x_2)}(x;\omega,q)=P_{n,(x_1,x_2)}(x;\omega,q)\left(x+\phi_0 q^n-\omega[n]_q+\rho_0+\sum_{k=1}^{n}\phi_k D^k_{q,\omega}+\sum_{k=0}^{1}\rho_k T_q\right)$$

where

$$\phi_0=x_2-\frac{\omega}{1-q},\qquad \phi_1=x_2\omega-\frac{\omega^2}{1-q},\qquad \phi_k=0,\quad k=2,3,\ldots$$

and

$$\rho_0=\frac{x_1(1-q)-\omega}{(1-q)},\quad \rho_k=\frac{(x_1(1-q)-\omega)^{k+1}}{(1-q)}+\omega\frac{(x_1(1-q)-\omega)^k}{(1-q)}.$$

$$D^k_{q,\omega}P_{n,(x_1,x_2)}(x;\omega,q)=\frac{[n]!}{[n-k]_q!}P_{n-k,(x_1,x_2)}(x;\omega,q).$$

$$T_q f(x)=f(qx)\qquad \textit{and}\qquad P_{n,(x_1,x_2)}(x;\omega,qx)=T_qP_{n,(x_1,x_2)}(x;\omega,q).$$

5. Some Special Hahn Appell and Twice Iterated Hahn-Appell Polynomials

In this section, we introduce some of polynomials, namely the Hahn-Bernoulli, Hahn-Euler, Hahn-Genocchi and twice-iterated Hahn-Bernoulli-Euler polynomials. These definitions provides more general form of the usual ones, and we believe that they will attrach the attention of many researchers.

Definition 5.1. ***Hahn-Bernoulli Polynomials*** *The Hahn-Bernoulli Polynomials are defined by the generating relation*

$$\frac{t}{e_q^{\omega t}(t)-1}e_q^{\omega t}(xt)=\sum_{n=0}^{\infty}B_n^H(x;\omega,q)\frac{t^n}{[n]_q!}. \tag{29}$$

Defining Hahn-Bernoulli numbers as

$$\frac{t}{e_q^{\omega t}(t)-1}=\sum_{k=0}^{\infty}B_{k,\omega,q}^H\frac{t^n}{[n]_q!},$$

clearly

$$B_k^H(0;\omega,q)=B_{k,\omega,q}^H.$$

The explicit form of the Hahn-Bernoulli polynomials can be obtained from (29) and Cauchy product that

$$B_n^H(x;\omega,q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q B_{k,\omega,q}^H(x)_{k,\omega}^{n-k}.$$

Definition 5.2. ***Hahn-Euler Polynomials*** *The Hahn-Euler polynomials are defined by*

$$\frac{2}{e_q^{\omega t}(t)+1}e_q^{\omega t}(xt)=\sum_{n=0}^{\infty}E_n^H(x;\omega,q)\frac{t^n}{[n]_q!}. \tag{30}$$

Introducing the Hahn-Euler numbers $E_{k,\omega,q}^H$ as

$$\frac{2}{e_q^{\omega t}(t)+1}=\sum_{n=0}^{\infty}E_{k,\omega,q}^H\frac{t^n}{[n]_q!},$$

we clearly see that

$$E_k^H(0;\omega,q)=E_{k,\omega,q}^H.$$

The explicit form of the Hahn-Euler polynomials can be obtained from (30) *and Cauchy product that*

$$E_n^H(x;\omega,q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q E_{k,\omega,q}^H(x)_{k,\omega}^{n-k}.$$

Definition 5.3. ***Hahn-Genocchi Polynomials*** *The Hahn-Genocchi polynomials are defined by*

$$\frac{2t}{e_q^{\omega t}(t)+1}e_q^{\omega t}(xt)=\sum_{n=0}^{\infty}G_n^H(x;\omega,q)\frac{t^n}{[n]_q!}. \tag{31}$$

Defining the Hahn-Genocchi numbers $G_{k,\omega,q}^H$ by the generating relation

$$\frac{2t}{e_q^{\omega t}(t)+1}=\sum_{n=0}^{\infty}G_{k,\omega,q}^H\frac{t^n}{[n]_q!},$$

it is clearly seen that

$$G_k^H(0;\omega,q)=G_{k,\omega,q}^H.$$

The explicit form of the Hahn-Genocchi polynomials can be obtained from (31) *and Cauchy product that*

$$G_n^H(x;\omega,q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q G_{k,\omega,q}^H(x)_{k,\omega}^{n-k}.$$

Definition 5.4. ***Twice Iterated Hahn-Bernoulli-Euler Polynomials*** *Twice Iterated Hahn-Bernoulli-Euler Polynomials represent a further extension and combination of the Hahn-Bernoulli and Hahn-Euler polynomials. We introduce the Twice Iterated Hahn-Bernoulli-Euler polynomials by*

$$\left(\frac{t}{e_q^{\omega t}(t)-1}\right)\left(\frac{2}{e_q^{\omega t}(t)+1}\right)e_q^{\omega t}(xt)=\sum_{n=0}^{\infty}{}_{B,E}H_{n,q}^{[2]}(x;\omega,q)\frac{t^n}{[n]_q!}. \tag{32}$$

Corollary 5.5. *The Twice Iterated Hahn-Bernoulli-Euler polynomials ${}_{B,E}H_{n,q}^{[2]}(x;\omega,q)$ have the representation*

$${}_{B,E}H_{n,q}^{[2]}(x;\omega,q)=\sum_{k=0}^{n}\sum_{l=0}^{k}\begin{bmatrix}n\\k\end{bmatrix}_q\begin{bmatrix}k\\l\end{bmatrix}_q B_{n-k,\omega,q}^{H}E_{n,\omega,q}^{H}(x)_{q,\omega}^{n},$$

where

$$B(t)=\frac{t}{e_q^{\omega t}(t)-1},\quad E(t)=\frac{2}{e_q^{\omega t}(t)+1},\quad e_q^{\omega t}(xt)=\sum_{l=0}^{\infty}(x)_{q,\omega}^{l}\frac{t^l}{[l]_q!}.$$

Proof. The proof follows from Theorem (2.2) and definitons (29) and (30). □

Remark 5.6. *More general Twice Iterated Hahn Bernoulli and Twice Iterated Hahn Euler polynomials can be defined by*

$$A(t)\frac{t}{e_q^{\omega t}(t)-1}e_q^{\omega t}(xt)=\sum_{n=0}^{\infty}B_{n,q}^{[2]}(x;\omega,q)\frac{t^n}{[n]_q!}. \tag{33}$$

and

$$A(t)\frac{2}{e_q^{\omega t}(t)+1}e_q^{\omega t}(xt)=\sum_{n=0}^{\infty}E_{n,q}^{[2]}(x;\omega,q)\frac{t^n}{[n]_q!}, \tag{34}$$

respectively. Clearly

1. *By setting $A(t)=1$ in equation (33), we can derive equation (29).*
2. *By setting $A(t)=1$ in equation (34), we can derive equation (30).*
3. *If we take $A(t)=\frac{2}{e_q^{\omega t}(t)+1}$ in (33), we can obtain the twice iterated Hahn-Bernoulli-Euler polynomials (32).*

6. Conclusion

In this paper, we consider the TI Hahn Appell polynomials and prove an equivalence theorem for the definition. We obtain their determinantal representation, recurrence relation and difference equation. As an application of our main theorems, we state some results for the 2-orthogonal Hahn Appell polynomals including determinantal form, explicit representation, reccurence relation and difference equation. We further, introduce the Hahn-Euler, Hahn-Genocchi and twice iterated Hahn-Bernoulli-Euler polynomials. We should mention that the r-iterated version can be considered through the generating relation

$$\prod_{i=1}^{r}a_i(t)e_q^{\omega t}(x)=\sum_{n=0}^{\infty}{}_{H}A_n^{[r]}(x)\frac{t^n}{[n]_q!}.$$

and the results of this paper can be extended for this new definition.

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