Filomat 39:25 (2025), 8709–8723
https://doi.org/10.2298/FIL2525709K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Approximation properties for difference of positive linear operators

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Abstract. In the present paper, we deal with the general quantitative estimates for the differences of recently defined composition positive linear operators (p.l.o.) by Gupta with the most representative operators used in approximation theory. The differences of these composition operators with various other available composition operators and Szász operators of Durrmeyer type have been studied. The estimates for the differences of operators having:

- different basis functions but same functional
- different basis functions and different functionals
- same basis function but different functionals

are calculated. Further the theoretical results are verified using numerical examples. Moreover the quantitative estimates in terms of weighted modulus of continuity have also been discussed.

1. Introduction

Positive linear operators have been a cornerstone in approximation theory, providing valuable tools for approximating functions. Over the years, numerous p.l.o. have been introduced and extensively studied, offering a variety of methods for function approximation. Notable among these are the Baskakov, Szász–Mirakyan, and Durrmeyer operators, each with their own distinct properties and applications. Szász-Baskakov operators and genuine-Durrmeyer type operators have further enriched the landscape of approximation theory.

Several recent contributions have expanded the theory of positive linear operators by introducing novel operator classes and refining their approximation behavior. Aral et al. [4] contributed quantitative Korovkin-type theorems, establishing rigorous estimates based on new moduli of continuity and functional inequalities. Bustamante [5] focused on generalized Baskakov operators that reproduce affine functions and presented weighted approximation results, thereby refining classical approximation techniques through weight-adjusted operator constructions. Gupta and Gupta [10] studied convergence estimates for composition operators and provided error bounds that support function approximation in constructive settings.

2020 *Mathematics Subject Classification*. Primary 41A25; Secondary 41A30, 41A36.
Keywords. modulus of continuity, weighted modulus, composition operators, difference of operators, approximation operators.
Received: 21 January 2025; Revised: 30 June 2025; Accepted: 22 July 2025
Communicated by Snežana Č. Živković-Zlatanović
Research supported by NBHM-DAE, Government of India.
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Gupta et al. [14] introduced semi-exponential Gauss–Weierstrass operators, enhancing the classical framework by combining exponential-type kernels with Gaussian structures to improve approximation accuracy. Özsaraç et al. [18] proposed a modified Mellin convolution operator and investigated its approximation behavior through information potential, offering a bridge between operator theory and information measures. Lastly, Özsaraç [17] further developed Mellin-Gauss–Weierstrass operators in the Mellin–Lebesgue space, addressing convergence in weighted function spaces with emphasis on Mellin analysis. Collectively, these works offer a rich foundation for further investigations into operator differences, convergence theorems, and structure-preserving modifications. These developments also provide essential groundwork for analyzing the difference of positive linear operators, particularly in assessing how modifications in their structure affect convergence and approximation behavior.

Such studies are crucial as they provide deeper insights into the behavior and performance of these operators in various approximation scenarios. The work by [1] Acu and Raşa (2016) set the stage by offering new estimates for the differences of p.l.o. Subsequent research by [3] Aral et al. (2019) and [7] Gupta (2020) expanded on these ideas, examining the differences of operators with different basis functions and their impact on approximation quality.
In this paper, we extend this line of enquiry by focusing on the general quantitative estimates for the differences of recently defined composition p.l.o. by Gupta [9] with the most representative operators used in approximation theory. Specifically, we study the differences of these composition operators with various other available composition operators, as well as with classical operators such as the Szász–Mirakyan, and Durrmeyer operators. Additionally, we explore the Kantorovich variant of composition operators.

Our investigation is structured around three primary scenarios:

1. Different basis functions but same functional: We explore the differences of operators that utilize different basis functions while maintaining the same functional.
2. Different basis functions and different functionals: We analyze the differences between operators that differ in both their basis functions and functionals.
3. Same basis function but different functionals: We calculate the estimates for the differences of operators that share the same basis function but employ different functionals.

To validate our theoretical results, we provide numerical examples demonstrating the practical implications of our estimates. Furthermore, we discuss the quantitative estimates in terms of the weighted modulus of continuity, offering a comprehensive view of the behavior of these operators under different conditions.
This study aims to contribute to the ongoing research in approximation theory by providing a detailed analysis of the differences between various positive linear operators, thereby enhancing our understanding of their capabilities and limitations in function approximation.

Studying the differences of operators, even after individual operators have been extensively studied, offers several compelling reasons for further investigation. Here are additional solid reasons why studying operator differences is valuable:

- Sensitivity Analysis:- Analyzing the differences between operators helps in conducting sensitivity analysis. Understanding how small variations or perturbations in the operators affect their outcomes is crucial for assessing the robustness and stability of mathematical models and algorithms.

- Error Propagation:- Studying operator differences provides insights into error propagation mechanisms. By examining how errors in the input functions propagate through the operators and manifest in the output, researchers can better understand the overall error behavior of computational methods.

- Inverse Problems:- By studying how differences in operators impact the solutions of inverse problems, researchers can develop more effective regularization techniques and improve the accuracy of inverse problem solutions.

- Function Space Characterization:- Analyzing operator differences contributes to the characterization of function spaces. By studying how operators behave differently on various function spaces, researchers can gain a deeper understanding of the properties and structures of function spaces under different operators.

- Algorithm Verification:- By comparing the outcomes of different operators on test functions or data sets, researchers can verify the correctness of algorithms and ensure their reliability in practical applications.

- Optimal Operator Selection:- Understanding the differences between operators helps in selecting the most suitable operator for a given problem or application. By comparing the performance of different operators through their differences, researchers can make informed decisions on the optimal choice of operator.

- Regularization Techniques:- Analysis of operator differences is valuable for developing and refining regularization techniques. By studying how differences in operators affect regularization methods, researchers can enhance the regularization process and improve the stability and accuracy of solutions.

- Functional Analysis Insights:- By examining how operators interact and differ in their actions on functions, researchers can deepen their understanding of functional spaces, operators, and related mathematical structures.

- Interpolation and Approximation:- By analyzing how differences in operators impact interpolation schemes and approximation methods, researchers can refine these techniques and enhance their effectiveness in function reconstruction.

In conclusion, studying operator differences offers a rich source of information for sensitivity analysis, error propagation, inverse problems, function space characterization, algorithm verification, optimal operator selection, regularization techniques, functional analysis insights, and interpolation and approximation theory. It plays a crucial role in advancing mathematical analysis, computational mathematics, and various applied fields where operators are fundamental components of mathematical models and algorithms.
With the aid of Laguerre polynomials, Sucu et al. [19] established new discrete operators, constructing a new series of p.l.o. that generalizes Szász operators. The operators are defined for x in $[0,\infty)$, $\alpha > -1$ and $n \in \mathbb{N}$ as:

$$(G_n^\alpha f)(x) = e^{\frac{-nx}{2}} 2^{-\alpha-1} \sum_{k=0}^{\infty} 2^{-k} L_k^\alpha\left(\frac{-nx}{2}\right) f\left(\frac{k}{n}\right). \tag{1}$$

The modified Laguerre polynomials, denoted as $L_m^\alpha(-x)$, are specified by means of confluent hypergeometric series as:

$$L_m^\alpha(-x) := \frac{(\alpha+1)_m}{m!} \ {}_1F_1(-m;\alpha+1;-x), \ \ \alpha > -1,$$

here $(\alpha)_0 = 1$ and $(\alpha)_m = \alpha(\alpha+1)...(\alpha+m-1)$. Also, $L_m^\alpha(-y)$ can be expanded alternatively as: $\sum_{s=0}^{m} \frac{(\alpha+m)!}{(\alpha+s)!s!(m-s)!} y^s$ and ${}_1F_1(-m;\alpha+1;-x)$ is given by: $\sum_{s=0}^{m} \frac{(-m)_s\,(-x)^s}{(\alpha+1)_s\, s!}$.
As defined in [15], we take Post-Widder operators as:

$$(P_n f)(x) := \frac{n^n}{x^n} \frac{1}{\Gamma(n)} \int_0^\infty e^{-nt/x} t^{n-1} f(t) dt. \tag{2}$$

Gupta defined composition operators in [9] given by:

$$(M_n^\alpha f)(x) := (P_n \circ G_n^\alpha f)(x). \tag{3}$$

Another representation of M_n^α is given by Theorem 4 in [9] as:

$$(M_n^\alpha f)(x) = \frac{1}{(x+2)^n}\sum_{k=0}^{\infty}\frac{1}{2^{\alpha+k-n+1}} f\Big(\frac{k}{n}\Big)\binom{\alpha+k}{k}\ {}_2F_1\Big(-k, n; \alpha+1; \frac{-x}{x+2}\Big), \tag{4}$$

where ${}_2F_1\Big(-k, n; \alpha+1; \frac{-x}{x+2}\Big) = \sum_{m=0}^{k}\frac{(-k)_m\,(n)_m}{(\alpha+1)_m\ m!}\Big(\frac{-x}{x+2}\Big)^m$.
Considering the compact form of the above operators as:

$$(M_n^\alpha f)(x) = \sum_{k=0}^{\infty} B_{n,k}F(f), \tag{5}$$

where $B_{n,k} = \frac{1}{(x+2)^n}\sum_{k=0}^{\infty}\frac{1}{2^{\alpha+k-n+1}}\binom{\alpha+k}{k}\ {}_2F_1\Big(-k, n; \alpha+1; \frac{-x}{x+2}\Big)$ and $F(f) = f\Big(\frac{k}{n}\Big)$.
The integral operators due to Rathore (see [8]) are defined by:

$$(W_n f)(x) = \frac{n^{nx}}{\Gamma(nx)}\int_0^\infty e^{-nt}t^{nx-1}f(t)dt. \tag{6}$$

Operators O_n^α as mentioned in [9] are defined by:

$$(O_n^\alpha f)(x) := (W_n \circ G_n^\alpha f)(x). \tag{7}$$

Theorem 7 in [9] gives an alternative representation of the above mentioned operators as:

$$(O_n^\alpha f)(x) = \sum_{k=0}^{\infty}\frac{1}{2^{\alpha+k+nx+1}} f\Big(\frac{k}{n}\Big)\binom{\alpha+k}{k}\ {}_2F_1\Big(nx, \alpha+k+1; \alpha+1; \frac{-1}{4}\Big). \tag{8}$$

Considering the compact form of the above operators as:

$$(O_n^\alpha f)(x) = \sum_{k=0}^{\infty}\mathfrak{P}_{n,k}\, G(f), \tag{9}$$

where $\mathfrak{P}_{n,k} = \sum_{k=0}^{\infty}\frac{1}{2^{\alpha+k+nx+1}}\binom{\alpha+k}{k}\ {}_2F_1\Big(nx, \alpha+k+1; \alpha+1; \frac{-1}{4}\Big)$ and $G(f) = f\Big(\frac{k}{n}\Big)$.
One of the most popular p.l.o are Szász-Mirakyan [20] operators:

$$(U_n f)(r) = e^{-nr}\sum_{k=0}^{\infty}\frac{(nr)^k}{k!} f\left(\frac{k}{n}\right). \tag{10}$$

Mazhar and Totik [3] introduced a modification of Szász operators of Durrmeyer type; these are called Szász-Durrmeyer operators and are defined as follows:

$$(\mathcal{S}_n f)(x) = \sum_{k=0}^{\infty} S_k(nx)\, I(f), \tag{11}$$

here $S_k(t) = \frac{(t)^k}{k!}e^{-t}$ and $I(f) = n\int_0^\infty S_k(nt)f(t)dt$.

Gupta [9] recently worked in a similar direction in 2023 by introducing four new operators that are centered on modified Laguerre polynomials and further extended the work of Sucu et al. [19]. This study is primarily concerned with analyzing the approximation results of the operators listed below:

$$(Q_n^\alpha f)(x) := (\mathcal{S}_n \circ G_n^\alpha f)(x). \tag{12}$$

According to Theorem 9 in [9], Q_n^α can be written as:

$$(Q_n^\alpha f)(x) = I_0\left(\frac{2\sqrt{2nx}}{\sqrt{3}}\right)\sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\frac{e^{-nx}}{2^{\alpha+k}}\frac{L_k^\alpha\left(\frac{-1}{3}\right)}{3}. \tag{13}$$

Here $I_0(z)$ is modified Bessel's function of first kind, $\mathcal{S}_n$ and G_n^α are given by equations (11) and (1) respectively.
Considering the compact form of the above operators as:

$$(Q_n^\alpha f)(x) = \sum_{k=0}^{\infty} A_{n,k}\, H(f), \tag{14}$$

where $A_{n,k} = I_0\left(\frac{2\sqrt{2nx}}{\sqrt{3}}\right)\sum_{k=0}^{\infty}\frac{e^{-nx}}{2^{\alpha+k}}\frac{L_k^\alpha\left(\frac{-1}{3}\right)}{3}$ and $H(f) = f\left(\frac{k}{n}\right)$.

Lemma 1.1. *First few moments of the operators M_n^α are given by:*

(i) $(M_n^\alpha e_0)(x) = 1$;

(ii) $(M_n^\alpha e_1)(x) = x + \dfrac{1+\alpha}{n}$;

(iii) $(M_n^\alpha e_2)(x) = x^2 + \dfrac{3+4\alpha+\alpha^2+5nx+2\alpha nx+nx^2}{n^2}$;

(iv)
$$(M_n^\alpha e_3)(x) = x^3 + \frac{1}{n^3}\Big[13+21\alpha+9\alpha^2+\alpha^3+31nx+21\alpha nx+3\alpha^2nx+12nx^2+3\alpha nx^2 + 12n^2x^2+3\alpha n^2x^2+2nx^3+3n^2x^3\Big];$$

(v)
$$(M_n^\alpha e_4)(x) = x^4 + \frac{1}{n^4}\Big[75+138\alpha+78\alpha^2+16\alpha^3+\alpha^4+233nx+208\alpha nx+54\alpha^2nx+4\alpha^3nx + 133nx^2+60\alpha nx^2+6\alpha^2nx^2+133n^2x^2+60\alpha n^2x^2+6\alpha^2n^2x^2+44nx^3+8\alpha nx + 66n^2x^3+12\alpha n^2x^3+22n^3x^3+4\alpha n^3x^3+6nx^4+11n^2x^4+6n^3x^4\Big];$$

(vi)
$$(M_n^\alpha e_6)(x) = x^6 + \frac{1}{n^6}\Big[(4683+10208\alpha+7845\alpha^2+2750\alpha^3+465\alpha^4+36\alpha^5+\alpha^6+21305nx + 25986\alpha nx+11415\alpha^2nx+2240\alpha^3nx+195\alpha^4nx+6\alpha^5nx+19921nx^2 + 15180\alpha nx^2+3975\alpha^2nx^2+420\alpha^3nx^2+15a^4nx^2+19921n^2x^2+15180\alpha n^2x^2 + 3975\alpha^2n^2x^2+420\alpha^3n^2x^2+15\alpha^4n^2x^2+13060nx^3+6180anx^3+900\alpha^2nx^3 + 40\alpha^3nx^3+19590n^2x^3+9270\alpha n^2x^3+1350\alpha^2n^2x^3+60\alpha^3n^2x^3+6530n^3x^3 + 3090\alpha n^3x^3+450\alpha^2n^3x^3+20\alpha^3n^3x^3+5340nx^4+1440\alpha nx^4+90\alpha^2nx^4$$

$$+ 9790n^2x^4 + 2640\alpha n^2x^4 + 165\alpha^2n^2x^4 + 5340n^3x^4 + 1440\alpha n^3x^4 + 90\alpha^2n^3x^4 + 890n^4x^4 + 240\alpha n^4x^4 + 15\alpha^2n^4x^4 + 1224nx^5 + 144\alpha nx^5 + 2550n^2x^5 + 300\alpha n^2x^5 + 1785n^3x^5 + 210\alpha n^3x^5 + 510n^4x^5 + 60\alpha n^4x^5 + 51n^5x^5 + 6\alpha n^5x^5 + 120nx^6 + 274n^2x^6 + 225n^3x^6 + 85n^4x^6 + 15n^5x^6)\Big].$$

Lemma 1.2. *First few moments of the operators O_n^α are given by:*

(*i*) $(O_n^\alpha e_0)(x) = 1;$

(*ii*) $(O_n^\alpha e_1)(x) = x + \dfrac{1+\alpha}{n};$

(*iii*) $(O_n^\alpha e_2)(x) = x^2 + \dfrac{3 + 4\alpha + \alpha^2 + 6nx + 2\alpha nx}{n^2};$

(*iv*) $(O_n^\alpha e_3)(x) = x^3 + \dfrac{13 + 21\alpha + 9\alpha^2 + \alpha^3 + 45nx + 24\alpha nx + 3\alpha^2nx + 15n^2x^2 + 3\alpha n^2x^2}{n^3};$

(*v*) $$(O_n^\alpha e_4)(x) = x^4 + \frac{1}{n^4}\Big[75 + 138\alpha + 78\alpha^2 + 16\alpha^3 + \alpha^4 + 416nx + 276\alpha nx + 60\alpha^2nx + 4\alpha^3nx + 210n^2x^2 + 72\alpha n^2x^2 + 6\alpha^2n^2x^2 + 28n^3x^3 + 4\alpha n^3x^3\Big];$$

(*vi*) $$(O_n^\alpha e_6)(x) = x^6 + \frac{1}{n^6}\Big[(4683 + 10208\alpha + 7845\alpha^2 + 2750\alpha^3 + 465\alpha^4 + 36\alpha^5 + \alpha^6 + 60970nx + 48930\alpha nx + 16380\alpha^2nx + 2700\alpha^3nx + 210\alpha^4nx + 6\alpha^5nx + 52125n^2x^2 + 27390\alpha n^2x^2 + 5490\alpha^2n^2x^2 + 480\alpha^3n^2x^2 + 15\alpha^4n^2x^2 + 13880n^3x^3 + 4740\alpha n^3x^3 + 540\alpha^2n^3x^3 + 20\alpha^3n^3x^3 + 1485n^4x^4 + 300\alpha n^4x^4 + 15\alpha^2n^4x^4 + 66n^5x^5 + 6\alpha n^5x^5)\Big].$$

Lemma 1.3. *First few moments of the operators Q_n^α are given by:*

(*i*) $(Q_n^\alpha e_0)(x) = 1;$

(*ii*) $(Q_n^\alpha e_1)(x) = x + \dfrac{2+\alpha}{n};$

(*iii*) $(Q_n^\alpha e_2)(x) = x^2 + \dfrac{10 + 6\alpha + \alpha^2 + 9nx + 2\alpha nx}{n^2};$

(*iv*) $(Q_n^\alpha e_3)(x) = x^3 + \dfrac{74 + 48\alpha + 12\alpha^2 + \alpha^3 + 97nx + 33\alpha nx + 3\alpha^2nx + 21n^2x^2 + 3\alpha n^2x^2}{n^3};$

(*v*) $$(Q_n^\alpha e_4)(x) = x^4 + \frac{1}{n^4}\Big[730 + 490\alpha + 144\alpha^2 + 20\alpha^3 + \alpha^4 + 1257nx + 520\alpha nx + 78\alpha^2nx + 4\alpha^3nx + 403n^2x^2 + 96\alpha n^2x^2 + 6\alpha^2n^2x^2 + 38n^3x^3 + 4\alpha n^3x^3\Big];$$

(*vi*) $$(Q_n^\alpha e_6)(x) = x^6 + \frac{1}{n^6}\Big[(133210 + 91574\alpha + 30270\alpha^2 + 5950\alpha^3 + 690\alpha^4 + 42\alpha^5 + \alpha^6 + 338889nx + 168966\alpha nx + 36855\alpha^2nx + 4280\alpha^3nx + 255\alpha^4nx + 6\alpha^5nx + 178771n^2x^2 + 63870\alpha n^2x^2 + 9105\alpha^2n^2x^2 + 600\alpha^3n^2x^2 + 15\alpha^4n^2x^2 + 33370n^3x^3 + 8130\alpha n^3x^3 + 690\alpha^2n^3x^3 + 20\alpha^3n^3x^3 + 2615n^4x^4 + 390\alpha n^4x^4 + 15\alpha^2n^4x^4 + 87n^5x^5 + 6\alpha n^5x^5)\Big].$$

We now move onto the difference of positive linear operators that has been an active area of research since last few years. Aral-Inoan-Raşa [3], Gupta et al. [7, 11, 12], Gupta-Tachev [21] discovered some interesting results on difference of operators. The operators involved are usually on continuous functions defined on real intervals and we consider the following weighted modulus of continuity:

$$\Omega(f,\delta) = \sup_{0\le h<\delta,\ x\in[0,\infty)} \frac{|f(x+h)-f(x)|}{(1+h^2)(1+x^2)}.$$

Further $\tilde{C}_2[0,\infty)$ denotes the closed subspace of $C_2[0,\infty)$ for which $\lim\limits_{n\to\infty} \dfrac{|f(x)|}{(1+x^2)} < C$ for some constant C and $\|.\|_2 = \sup\limits_{x\in[0,\infty)} \dfrac{|f(x)|}{(1+x^2)}$.

Theorem 1.4. *[7] Let $f \in C_2[0,\infty)$ with $f'' \in \tilde{C}_2[0,\infty)$. Then, for any two p.l.o. X and Y, we have*

$$\begin{aligned}|(X-Y)(f,x)| \le{}& \frac{1}{2}\|f''\|_2(\beta_1(x)+\beta_2(x)) + 8\Omega(f'',\delta_1)(1+\beta_1(x))\\ &+ 8\Omega(f'',\delta_2)(1+\beta_2(x)) + 16\Omega(f,\delta_3)(\gamma_1(x)+1)\\ &+ 16\Omega(f,\delta_4)(\gamma_2(x)+1),\end{aligned}$$

where

$$\beta_1(x) = \sum_{k=0}^{\infty} X_{n,k}(1+F_n(e_1)^2)T_2^{F_n(f)}, \qquad \beta_2(x) = \sum_{k=0}^{\infty} Y_{n,k}(1+G_n(e_1)^2)T_2^{G_n(f)},$$

$$\delta_1^4(x) = \sum_{k=0}^{\infty} X_{n,k}(1+F_n(e_1)^2)T_6^{F_n(f)}, \qquad \delta_2^4(x) = \sum_{k=0}^{\infty} Y_{n,k}(1+G_n(e_1)^2)T_6^{G_n(f)},$$

$$\delta_3^4(x) = \sum_{k=0}^{\infty} X_{n,k}(F_n(e_1)-x)^4, \qquad \delta_4^4(x) = \sum_{k=0}^{\infty} Y_{n,k}(G_n(e_1)-x)^4,$$

$$\gamma_1(x) = \sum_{k=0}^{\infty} X_{n,k}(1+F_n(e_1)-x)^2, \qquad \gamma_2(x) = \sum_{k=0}^{\infty} Y_{n,k}(1+G_n(e_1)-x)^2.$$

and $X_{n,k}$, $Y_{n,k}$ are the basis functions of X and Y, F_n, G_n are the functionals respectively, $e_r(t) = t^r$, $r = 0,1,2,...,T_r^{F_n} = F_n[e_1 - F_n(e_1)]^r, r \in \mathbb{N}$. We consider here that $\delta_1(x) \le 1, \delta_2(x) \le 1, \delta_3(x) \le 1, \delta_4(x) \le 1$.

2. Difference of operators/Quantitative Estimates

We compute the magnitude of difference of the two operators having the different basis functions. As an application of Theorem 1 in [7] , we have the following quantitative estimates for the difference between the operators in case 1 and case 2.

Case 1: Different basis functions but same functional.
We consider the difference of M_n^α and O_n^α, that utilize different basis functions while maintaining the same functional $F(f) = G(f) = \frac{k}{n}$.

Remark 2.1. *We have $F(e_1) = \frac{k}{n}$ and for any $r \in \mathbb{N}$, we have by simple calculations*
$T_r^F := F[e_1 - F(e_1)]^r = 0.$

Remark 2.2. *We have $G(e_1) = \frac{k}{n}$ and for any $r \in \mathbb{N}$, we have by simple calculations*
$T_r^G := G[e_1 - G(e_1)]^r = 0.$

Theorem 2.3. *Let $f \in C_2[0,\infty)$ with $f'' \in \tilde{C}_2[0,\infty)$. Then for M_n^α and O_n^α*

$$|(M_n^\alpha - O_n^\alpha)(f,x)| \leq 16\Omega(f,\delta_3)(\gamma_1(x)+1) + 16\Omega(f,\delta_4)(\gamma_2(x)+1), \tag{15}$$

where

$$\delta_3^4(x) = \sum_{k=0}^{\infty} B_{n,k}(F(e_1)-x)^4, \qquad \delta_4^4(x) = \sum_{k=0}^{\infty} \mathfrak{P}_{n,k}(G(e_1)-x)^4,$$

$$\gamma_1(x) = \sum_{k=0}^{\infty} B_{n,k}(1+F(e_1)-x)^2, \qquad \gamma_2(x) = \sum_{k=0}^{\infty} \mathfrak{P}_{n,k}(1+G(e_1)-x)^2,$$

and $e_r(t) = t^r$, $r = 0,1,2,\ldots$. We consider here that $\delta_3(x) \leq 1, \delta_4(x) \leq 1$.

Proof. Following Theorem 1.4, Remark 2.1 and Remark 2.2, we get

$$\beta_1(x) = \beta_2(x) = \delta_1(x) = \delta_2(x) = 0.$$

Using Lemma 1.1 and Lemma 1.2

$$\begin{aligned}\delta_3^4(x) &= (M_n^\alpha e_6)(x) - 4x(M_n^\alpha e_5)(x) + (6x^2+1)(M_n^\alpha e_4)(x) - 4x(1+x^2)(M_n^\alpha e_3)(x) + x^2(x^2+6)(M_n^\alpha e_2)(x) - 3x^4 \\ &= \sum_{k=0}^{\infty} B_{n,k}(F(e_1)-x)^4,\end{aligned}$$

$$\begin{aligned}\delta_4^4(x) &= (O_n^\alpha e_6)(x) - 4x(O_n^\alpha e_5)(x) + (6x^2+1)(O_n^\alpha e_4)(x) - 4x(1+x^2)(O_n^\alpha e_3)(x) + x^2(x^2+6)(O_n^\alpha e_2)(x) - 3x^4 \\ &= \sum_{k=0}^{\infty} \mathfrak{P}_{n,k}(G(e_1)-x)^4,\end{aligned}$$

$$\begin{aligned}\gamma_1(x) &= 1 + x^2 + \frac{3+4\alpha+\alpha^2+5nx+2\alpha nx+nx^2}{n^2} \\ &= \sum_{k=0}^{\infty} B_{n,k}(1+F(e_1)-x)^2,\end{aligned}$$

$$\begin{aligned}\gamma_2(x) &= 1 + x^2 + \frac{3+4\alpha+\alpha^2+6nx+2\alpha nx}{n^2} \\ &= \sum_{k=0}^{\infty} \mathfrak{P}_{n,k}(1+G(e_1)-x)^2.\end{aligned}$$

□

For another example under this case, we consider the difference of O_n^α and Q_n^α, that utilize different basis functions while maintaining the same functionals $G(f) = H(f) = \frac{k}{n}$.

Remark 2.4. *We have $H(e_1) = \frac{k}{n}$ and for any $r \in \mathbb{N}$, we have by simple calculations $T_r^H := H[e_1 - H(e_1)]^r = 0$.*

Theorem 2.5. *Let $f \in C_2[0,\infty)$ with $f'' \in \tilde{C}_2[0,\infty)$. Then for O_n^α and Q_n^α*

$$|(O_n^\alpha - Q_n^\alpha)(f,x)| \leq 16\Omega(f,\delta_3)(\gamma_1(x)+1) + 16\Omega(f,\delta_4)(\gamma_2(x)+1), \tag{16}$$

where

$$\delta_3^4(x) = \sum_{k=0}^{\infty} \mathfrak{P}_{n,k}(G(e_1)-x)^4, \qquad \delta_4^4(x) = \sum_{k=0}^{\infty} A_{n,k}(H(e_1)-x)^4,$$

$$\gamma_1(x) = \sum_{k=0}^{\infty} \mathfrak{P}_{n,k}(1+G(e_1)-x)^2, \qquad \gamma_2(x) = \sum_{k=0}^{\infty} A_{n,k}(1+H(e_1)-x)^2,$$

and $e_r(t) = t^r$, $r = 0,1,2,\ldots.$ We consider here that $\delta_3(x) \leq 1, \delta_4(x) \leq 1$.

Proof. Here $F(e_1)^2 = \left(\frac{1+\alpha}{n} + x\right)^2$.
Following Theorem 1.4, Remark 2.2 and Remark 2.4 we get $\beta_1(x) = \beta_2(x) = \delta_1(x) = \delta_2(x) = 0$.
Following Lemma 1.2 and Lemma 1.3

$$\begin{aligned}\delta_3^4(x) &= (O_n^\alpha e_6)(x) - 4x(O_n^\alpha e_5)(x) + (6x^2+1)(O_n^\alpha e_4)(x) - 4x(1+x^2)(O_n^\alpha e_3)(x) + x^2(x^2+6)(O_n^\alpha e_2)(x) - 3x^4\\ &= \sum_{k=0}^{\infty} \mathfrak{P}_{n,k}(G(e_1)-x)^4,\end{aligned}$$

$$\begin{aligned}\delta_4^4(x) &= (Q_n^\alpha e_6)(x) - 4x(Q_n^\alpha e_5)(x) + (6x^2+1)(Q_n^\alpha e_4)(x) - 4x(1+x^2)(Q_n^\alpha e_3)(x) + x^2(x^2+6)(Q_n^\alpha e_2)(x) - 3x^4\\ &= \sum_{k=0}^{\infty} A_{n,k}(H(e_1)-x)^4,\end{aligned}$$

$$\begin{aligned}\gamma_1(x) &= 1 + x^2 + \frac{3+4\alpha+\alpha^2+6nx+2\alpha nx}{n^2}\\ &= \sum_{k=0}^{\infty} \mathfrak{P}_{n,k}(1+G(e_1)-x)^2,\end{aligned}$$

$$\begin{aligned}\gamma_2(x) &= 1 + x^2 + \frac{10+6\alpha+\alpha^2+9nx+2\alpha nx}{n^2}\\ &= \sum_{k=0}^{\infty} A_{n,k}(1+H(e_1)-x)^2.\end{aligned}$$

□

Case 2: Different basis functions and different functionals.
Considering the difference of O_n^α and $\mathcal{S}_n$, that utilize different basis functions $\mathfrak{P}_{n,k} = \sum_{k=0}^{\infty} \frac{1}{2^{\alpha+k+n+1}}\binom{\alpha+k}{k}\ {}_2F_1\left(nx, \alpha+k+1; \alpha+1; \frac{-1}{4}\right)$, $S_k(t) = \frac{(t)^k}{k!}e^{-t}$ and different functionals $G(f) = \frac{k}{n}$, $I(f) = n\int_0^\infty S_k(nt)f(t)dt$ respectively.

Remark 2.6. *We have $I(e_1) = \frac{k+1}{n}$ and for any $r \in \mathbb{N}$, we have by simple calculations*

$$T_2^I := I[e_1 - I(e_1)]^2 = \frac{k+1}{n^2},$$

$$T_6^I := I[e_1 - I(e_1)]^6 = \frac{\mu_6}{n^6},$$

where $\mu_6 = (k+6)(k+5)(k+4)(k+3)(k+2)(k+1)$.

Theorem 2.7. *Let* $f \in C_2[0,\infty)$ *with* $f'' \in \tilde{C}_2[0,\infty)$. *Then for* O_n^α *and* $\mathcal{S}_n$

$$\begin{aligned}|(O_n^\alpha - \mathcal{S}_n)(f,x)| \le{}& \frac{1}{2}\|f''\|_2\beta_2(x) + 8\Omega(f'',\delta_2)(1+\beta_2(x))\\ &+ 16\Omega(f,\delta_3)(\gamma_1(x)+1) + 16\Omega(f,\delta_4)(\gamma_2(x)+1),\end{aligned}$$

where

$$\beta_2(x) = \sum_{k=0}^{\infty} S_k(1+I(e_1)^2)T_2^{I(f)}, \qquad \delta_2^4(x) = \sum_{k=0}^{\infty} S_k(1+I(e_1)^2)T_6^{I(f)},$$

$$\delta_3^4(x) = \sum_{k=0}^{\infty} \mathfrak{P}_{n,k}(G(e_1)-x)^4, \qquad \delta_4^4(x) = \sum_{k=0}^{\infty} S_k(I(e_1)-x)^4,$$

$$\gamma_1(x) = \sum_{k=0}^{\infty} \mathfrak{P}_{n,k}(1+G(e_1)-x)^2, \qquad \gamma_2(x) = \sum_{k=0}^{\infty} S_k(1+I(e_1)-x)^2.$$

and $e_r(t) = t^r,\ r = 0,1,2,...,T_r^I = I[e_1 - I(e_1)]^r, r \in \mathbb{N}$. *We consider here that* $\delta_2(x) \le 1, \delta_3(x) \le 1, \delta_4(x) \le 1$.

Proof. Following Theorem 1.4 and Remark 2.2, we get $\beta_1(x) = \delta_1(x) = 0$.
From Remark 2.6, we have

$$\begin{aligned}\beta_2(x) &= \frac{k+1}{n^2}\Big(1 + \frac{n^2x^2+3nx+1}{n^2}\Big)\\ &= \sum_{k=0}^{\infty} S_k(1+I(e_1)^2)T_2^{I(f)},\end{aligned}$$

$$\begin{aligned}\delta_2^4(x) &= \frac{(k+6)(k+5)(k+4)(k+3)(k+2)(k+1)}{n^6}\Big(1 + \frac{n^2x^2+3nx+1}{n^2}\Big)\\ &= \sum_{k=0}^{\infty} S_k(1+I(e_1)^2)T_6^{I(f)},\end{aligned}$$

$$\begin{aligned}\delta_3^4(x) &= (O_n^\alpha e_6)(x) - 4x(O_n^\alpha e_5)(x) + (6x^2+1)(O_n^\alpha e_4)(x) - 4x(1+x^2)(O_n^\alpha e_3)(x) + x^2(x^2+6)(O_n^\alpha e_2)(x) - 3x^4\\ &\sum_{k=0}^{\infty} \mathfrak{P}_{n,k}(G(e_1)-x)^4,\end{aligned}$$

$$\begin{aligned}\delta_4^4(x) &= (S_n e_6)(x) - 4x(S_n e_5)(x) + (6x^2+1)(S_n e_4)(x) - 4x(1+x^2)(S_n e_3)(x) + x^2(x^2+6)(S_n e_2)(x) - 3x^4\\ &= \sum_{k=0}^{\infty} S_k(I(e_1)-x)^4,\end{aligned}$$

$$\begin{aligned}\gamma_1(x) &= 1 + x^2 + \frac{3+4\alpha+\alpha^2+6nx+2\alpha nx}{n^2}\\ &= \sum_{k=0}^{\infty} \mathfrak{P}_{n,k}(1+G(e_1)-x)^2,\end{aligned}$$

$$\gamma_2(x) = 1 + x^2 + \frac{4x}{n} + \frac{2}{n^2}$$
$$= \sum_{k=0}^{\infty} S_k(1 + I(e_1) - x)^2.$$

Here $(\mathcal{S}_n e_i)(x)$ are the moments of Szász-Durrmeyer operators as mentioned in [13]. □

Defining New Operator O_n^{α}-Kantorovich: Defining the new operator $\mathcal{K}_n^{\alpha}$:

$$(\mathcal{K}_n^{\alpha} f)(x) = \sum_{k=0}^{\infty} \mathfrak{P}_{n,k}\, J(f), \tag{17}$$

where $\mathfrak{P}_{n,k}$ is the basis function of the operator O_n^{α} and $J(f) = n \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)dt$.

Considering the difference of O_n^{α} and K_n^{α}, that utilize same basis functions

$\mathfrak{P}_{n,k} = \sum_{k=0}^{\infty} \frac{1}{2^{\alpha+k+n+1}} \binom{\alpha+k}{k} \, {}_2F_1\left(nx, \alpha+k+1; \alpha+1; \frac{-1}{4}\right)$, and different functionals

$G(f) = \frac{k}{n}$, $J(f) = n \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)dt$. respectively.

Case 3: Same basis function but different functionals.

Remark 2.8. *We have* $J(e_1) = \dfrac{2k+1}{2n}$ *and for any* $r \in \mathbb{N}$, *we have by simple calculations*

$T_2^J := J[e_1 - J(e_1)]^2 = \dfrac{1}{12n^2}$,

$T_6^J := J[e_1 - J(e_1)]^6 = \dfrac{1}{448n^6}$.

Let $\mathfrak{P}_{n,k} \in C[0,\infty)$, $\mathfrak{P}_{n,k} \geq 0$, $k \in K$, where K be a set of non-negative integers such that $\sum_{k=0}^{\infty} \mathfrak{P}_{n,k} = e_0$. Then for each $k \in K$, let $G(f) : E[0,\infty) \to \mathbb{R}$ and $J(f) : E[0,\infty) \to \mathbb{R}$ where $E[0,\infty)$ is a space of real-valued continuous functions on $[0,\infty)$ containing the polynomials. $E_b[0,\infty)$ will be the space of all $f \in E[0,\infty)$ with

$$\|f\| := \sup |f(x) : x \in [0,\infty)| < \infty.$$

Let $D[0,\infty)$ be the set of all $f \in E[0,\infty)$ for which $\sum_{k=0}^{\infty} \mathfrak{P}_{n,k}\, G(f) \in C[0,\infty)$ and

$\sum_{k=0}^{\infty} \mathfrak{P}_{n,k}\, J(f) \in C[0,\infty)$.

Theorem 2.9. *[1] Consider O_n^{α} and $\mathcal{K}_n^{\alpha}$, defined for $f \in D[0,\infty)$ with $f'' \in E_b[0,\infty)$. Then*

$$|(O_n^{\alpha} - \mathcal{K}_n^{\alpha})(f,x)| \leq \frac{1}{2}\|f''\|\beta(x) + \omega_1(f,\delta). \tag{18}$$

where

$$\beta(x) = \sum_{k=0}^{\infty} \mathfrak{P}_{n,k}\left(T_2^{G(f)} + T_2^{J(f)}\right) \text{and } \omega_1 \text{ is first order modulus of smoothness with}$$
$$\delta := \sup |G(e_1) - J(e_1)|.$$

Proof.

$$
\begin{aligned}
|(O_n^\alpha - \mathcal{K}_n^\alpha)(f,x)| &\leq \sum_{k=0}^{\infty} \mathfrak{P}_{n,k}|G(f) - J(f)| \\
&\leq \sum_{k=0}^{\infty} \mathfrak{P}_{n,k}\Big(|G(f) - f(G(e_1))| + |J(f) - f(J(e_1))| + |f(G(e_1)) - f(J(e_1))|\Big) \\
&\leq \frac{1}{2}\sum_{k=0}^{\infty} \mathfrak{P}_{n,k}\Big[\Big(T_2^{G(f)} + T_2^{J(f)}\Big)\|f''\| + \omega_1(f, |G(e_1) - J(e_1)|)\Big] \\
&\leq \frac{1}{2}\|f''\|\beta(x) + \omega_1(f,\delta).
\end{aligned}
$$

□

3. Voronovskaya type inequalities for the difference of two positive linear operators

In this section, we investigate the quantitative Voronovskaya theorem for the difference of operators.

Remark 3.1. *If $\mu_m^{O_n^\alpha}$ represents the m^{th} central moments of the operator O_n^α where, $\mu_m^{O_n^\alpha} = (O_n^\alpha(t-x)^m; x)$ then,*

$$
\begin{aligned}
&\mu_0^{O_n^\alpha} = 1, && \mu_1^{O_n^\alpha} = 0,\\
&\mu_2^{O_n^\alpha} = \frac{x(1-x) + (1+\alpha)}{n}, && \mu_3^{O_n^\alpha} = \frac{(1-2x)(1+\alpha)}{n^2},\\
&\mu_4^{O_n^\alpha} = \frac{x^2(1-x^2) + 6x(1-x)(1+\alpha) + 3(1+\alpha)(2+\alpha)}{n^2}, &&\\
&\mu_5^{O_n^\alpha} = \frac{(1-2x)(1-x)(1+\alpha)}{n^3}, && \mu_6^{O_n^\alpha} = \frac{(1+6x(1-x)+3x^2)(1+\alpha)}{n^4}.
\end{aligned}
$$

Remark 3.2. *[6] If $\mu_m^{U_n}$ represents the m^{th} central moments of the operator U_n where, $\mu_m^{U_n} = (U_n(t-x)^m; x)$ then,*

$$
\begin{aligned}
&\mu_0^{U_n} = 1, && \mu_1^{U_n} = 0,\\
&\mu_2^{U_n} = \frac{x}{n}, && \mu_3^{U_n} = \frac{x}{n^2},\\
&\mu_4^{U_n} = \frac{3x^2}{n^2} + \frac{x}{n^3}, && \mu_5^{U_n} = \frac{10x^2}{n^3} + \frac{x}{n^4},\\
&\mu_6^{U_n} = \frac{15x^3}{n^3} + \frac{25x^2}{n^4} + \frac{x}{n^5}. &&
\end{aligned}
$$

Theorem 3.3. *Let $f \in C_2[0,\infty)$ with $f'' \in \tilde{C}_2[0,\infty)$. For O_n^α and U_n such that $\Big(O_n^\alpha - U_n\Big)\Big((t-x)^i; x\Big) = 0$ for $i = 0, 1$, the following inequality holds true:*

$$
\begin{aligned}
\Big|(O_n^\alpha - U_n)(f(t);x)\Big| \leq \frac{|f''(x)|}{2}\frac{x(2-x)+1+\alpha}{n} &+ 16(1+x^2)\frac{x(2-x)+1+\alpha}{n} \\
&\times \Omega\bigg(f''; \Big(\frac{x}{n^4(1+\alpha)} + \frac{(1+6x+22x^2)}{n^3} + \frac{15x^3}{n^2(1+\alpha)}\Big)^{1/4}\bigg).
\end{aligned} \tag{19}
$$

Proof. By Taylor's series, we have

$$
(O_n^\alpha - U_n)(f(t);x) = \frac{f''(x)}{2!}(O_n^\alpha - U_n)\Big((t-x)^2; x\Big) + (O_n^\alpha - U_n)\Big(R_2(f;t,x);x\Big),
$$

where $R_2(f;t,x)$ is given by

$$R_2(f;t,x) = \frac{(t-x)^2}{2}\Big(f''(\xi) - f''(x)\Big), \quad \xi \in (t,x). \tag{20}$$

Using $\Big(O_n^\alpha - U_n\Big)\Big((t-x)^i;x\Big) = 0$ for $i = 0,1$, and the equation (2.5) at page 28 in [2] we have:

$$\begin{aligned}
&\Big|\Big(O_n^\alpha - U_n\Big)(f(t);x)\Big| \\
&\leq \frac{|f''(x)|}{2!}(O_n^\alpha + U_n)\Big((t-x)^2;x\Big) + 8(1+x^2)\Omega(f'';\delta) \\
&\times \left\{\Big(O_n^\alpha + U_n\Big)\Big(|t-x|^2 + \frac{|t-x|^6}{\delta^4};x\Big)\right\} \\
&\leq \frac{|f''(x)|}{2!}(O_n^\alpha + U_n)\Big((t-x)^2;x\Big) \\
&+ 8(1+x^2)\Omega(f'';\delta)(O_n^\alpha + U_n)\Big((t-x)^2;x\Big)\left(1 + \frac{(O_n^\alpha + U_n)\Big((t-x)^6;x\Big)}{\delta^4(O_n^\alpha + U_n)\Big((t-x)^2;x\Big)}\right).
\end{aligned}$$

Choosing $\delta = \left(\frac{(O_n^\alpha + U_n)((t-x)^6;x)}{(O_n^\alpha + U_n)((t-x)^2;x)}\right)^{1/4}$ and using $\mu_m^{(O_n^\alpha+U_n)}(x) = (O_n^\alpha + U_n)((t-x)^m;x)$:

$$|(O_n^\alpha - U_n)(f(t);x)| \leq \frac{f''(x)}{2!}\mu_2^{O_n^\alpha+U_n}(x) + 16(1+x^2)\mu_2^{O_n^\alpha+U_n}(x)\Omega\left(f'';\left(\frac{\mu_6^{O_n^\alpha+U_n}(x)}{\mu_2^{O_n^\alpha+U_n}(x)}\right)^{1/4}\right). \tag{21}$$

From Remark 3.1 and Remark 3.2, we calculate

$$\mu_2^{O_n^\alpha+U_n}(x) = \frac{x(2-x)+1+\alpha}{n},$$

$$\mu_6^{O_n^\alpha+U_n}(x) = \frac{x}{n^5} + \frac{1+6x+22x^2}{n^4} + \frac{(1+6x-3x^2)\alpha}{n^4} + \frac{15x^3}{n^3}.$$

Here $x(2-x)+1+\alpha \geq (1+\alpha)$ for $x \in [0,1]$. Hence

$$\frac{\mu_6^{O_n^\alpha+U_n}(x)}{\mu_2^{O_n^\alpha+U_n}(x)} \leq \frac{x}{n^4(x(2-x)+1+\alpha)} + \frac{(1+6x+22x^2)}{n^3(x(2-x)+1+\alpha)} + \frac{(1+6x-3x^2)\alpha}{n^3(x(2-x)+1+\alpha)} + \frac{15x^3}{n^2(x(2-x)+1+\alpha)}$$

$$\frac{\mu_6^{O_n^\alpha+U_n}(x)}{\mu_2^{O_n^\alpha+U_n}(x)} \leq \frac{x}{n^4(1+\alpha)} + \frac{(1+6x+22x^2)}{n^3} + \frac{15x^3}{n^2(1+\alpha)}.$$

Therefore

$$\begin{aligned}
\Big|(O_n^\alpha - U_n)(f(t);x)\Big| \leq{}& \frac{|f''(x)|}{2}\frac{x(2-x)+1+\alpha}{n} + 16(1+x^2)\frac{x(2-x)+1+\alpha}{n} \\
&\times \Omega\left(f'';\left(\frac{x}{n^4(1+\alpha)} + \frac{(1+6x+22x^2)}{n^3} + \frac{15x^3}{n^2(1+\alpha)}\right)^{1/4}\right).
\end{aligned}$$

□

4. Numerical results

This section deals with some numerical examples to validate the theoretical results.

Example 1 Let $f(x) = x^4 - 5.5x^3 + 10.5x^2 - 5x$ for which the convergence of the difference of operators $E_n^\alpha(f,x) = |(Q_n^\alpha - O_n^\alpha)(f,x)|$ has to be shown. The impact of varying α is demonstrated in Figure 1 for the function f, Q_n^α and O_n^α for $n = 100$. For $n \in \{100, 120, 150\}$, the absolute values of the differences are illustrated in Figure 2.

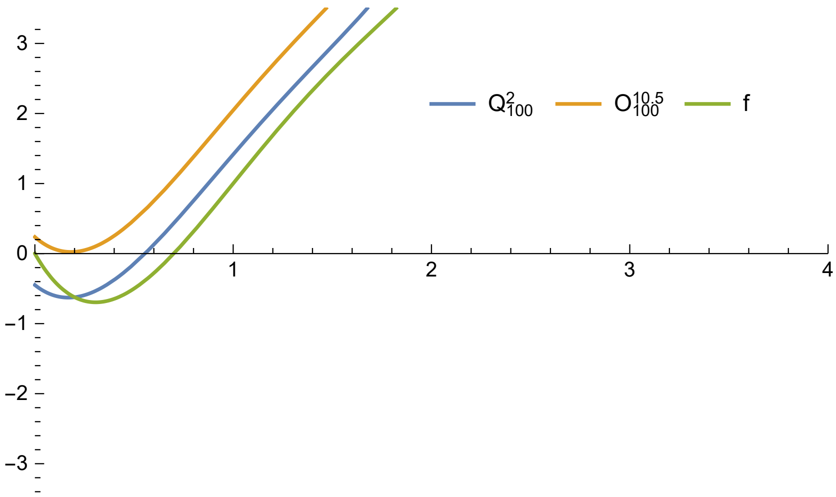

Figure 1: Approximation process by Q_n^α and O_n^α for $n = 100$.

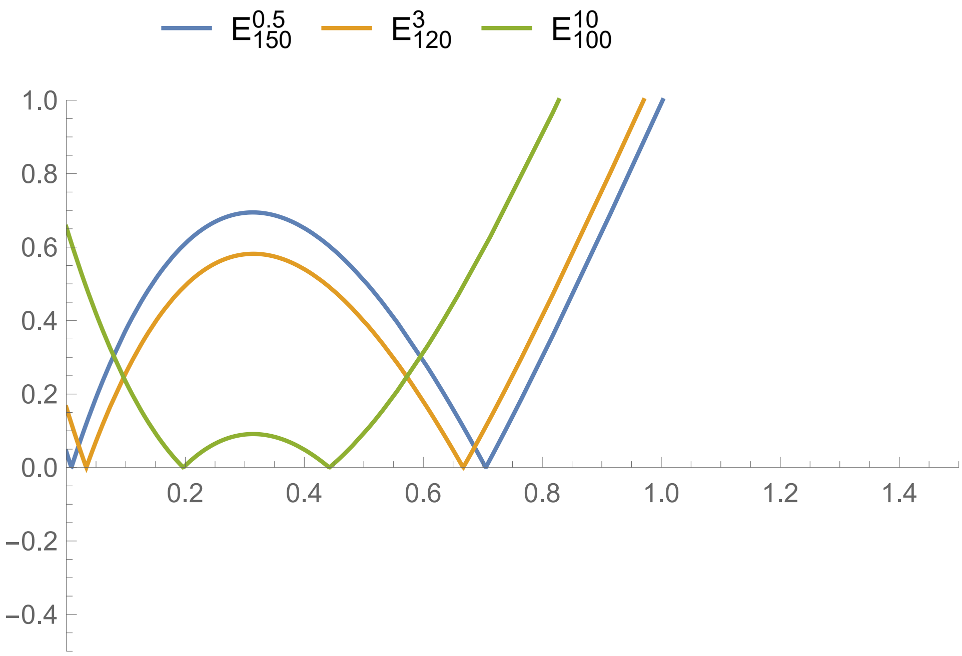

Figure 2: Difference of operators E_n^α for $n \in \{100, 120, 150\}$ and $\alpha \in \{0.5, 3, 10\}$.

Declaration of Competing Interest

The authors affirm that they have no known financial or personal affiliations that would have appeared to conflict with the work described in this study.

5. Acknowledgments

This work is supported by National Board of Higher Mathematics - Department of Atomic Energy (NBHM-DAE), Government of India (Sanction No. 02011/25/2021/NBHM(RP)/R&D II/7997) and Department of Mathematics, Thapar Institute of Engineering and Technology, Patiala, India. We are also thankful to Ministry of Science & Technology, Department of Science & Technology, Government of India, for providing the optimal infrastructure facilities under FIST project.

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