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Convex fuzzy cones of hypervector spaces

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Abstract. In this paper, we introduce and study convex fuzzy cones of hypervector spaces (or simply hyperspaces) as a generalization of both vector spaces and fuzzy vector spaces. We present a new characterization of fuzzy subhyperspaces as decompositions of special subhyperspaces. Moreover, we explore convex cones spanned by non-empty subsets of hypervector spaces over the real numbers field. We use fuzzy convex cones to generate fuzzy subhyperspaces from fuzzy subsets. That is, for a given convex fuzzy cone μ on a real hypervector space V, we determine the smallest fuzzy subhyperspace of V containing μ and the largest fuzzy subhyperspace of V contained in μ.

1. Introduction

Uncertainty is an inherent aspect of real-life phenomena. In nearly every problem, some degree of uncertainty naturally arises. Classical set theory is designed to handle only deterministic situations, and therefore, it is inadequate for dealing with vague or imprecise information.

To address this limitation, the concept of fuzzy sets was introduced by Zadeh [28] in 1965 as a powerful mathematical tool for modeling uncertainty. Fuzzy sets deal exclusively with the grade of membership, representing partial belonging of elements to sets. They serve as a natural generalization of classical (crisp) sets. In fact, classical sets can be seen as special cases of fuzzy sets under specific membership assignments. This generalization explains the widespread applicability and popularity of fuzzy set theory compared to classical set theory.

Following Zadeh's foundational work, researchers extended fuzzy set theory to algebraic structures. Rosenfeld [22] was among the first to apply fuzzy set theory to groups, which opened the door to the development of fuzzy algebraic systems. Since then, numerous studies have been conducted on fuzzy algebraic structures ([19], [20], [23]). In particular, fuzzy vector spaces have been actively studied by scholars such as Katsaras and Liu [15], Nanda [21], Malik and Mordeson [17], and Kumar [16].

Parallel to these developments, algebraic hyperstructures, originally introduced by Marty [18] in 1934, have emerged as a generalization of classical algebraic systems. In a hyperstructure, the hyperoperation between two elements yields a set instead of a single element. This approach enables the modeling of

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more flexible and non-deterministic mathematical systems, where operations are multi-valued rather than strictly functional. Algebraic hyperstructure is an important field of study not only in mathematics, but also in applied sciences such as physics, chemistry, biology, engineering, artificial intelligence, computer science, social science and etc. (for more details see [7], [8], [10] and [27]).

A class of hypervector spaces, which is a specific type of algebraic hyperstructures, was introduced and studied by Tallini [25] in 1990. These spaces extend classical vector spaces by allowing scalar multiplication to result in sets of vectors rather than a single vector. This theory has been investigated by the authors [2], [14] and Sedghi [24]. Especially, the authors in [3] introduced and studied dimensions of hypervector spaces.

In recent years, researchers have sought to unify the concepts of fuzziness and algebraic hyperstructures (for more details see [9]), leading to the development of fuzzy hypervector spaces. These structures integrate the flexibility of fuzzy sets with the multivalued operations inherent in hyperstructures. In particular, the notion of fuzzy hypervector spaces have been studied extensively in the works of Ameri ([1], [4], [5], [6]) and Dehghan ([11], [12]).

The aim of this paper is to study convex fuzzy cones in hypervector spaces. To this end, we first show that the fuzzy subhyperspaces introduced by Ameri [1] admit an important decomposition into special subhyperspaces, which are distinct from level subhyperspaces. We also investigate the relationship between affine fuzzy subsets of hypervector spaces and fuzzy superhyperplanes. Subsequently, we explore several properties of cones as special subsets of real hypervector spaces, including the characterization of convex cones generated by non-empty subsets. Furthermore, we generalize the concept of convex cones to the fuzzy subhyperspaces. In particular, Proposition 5.5, provides a characterization of convex fuzzy cones in terms of their level subsets. Finally, for a given convex fuzzy cone μ in a finite dimensional real hypervector space V, we determine the smallest fuzzy subhyperspace containing μ, and the largest fuzzy subhyperspace contained in μ, as established in Theorems 5.8, and 5.9.

2. Preliminaries

In this section, we present several fundamental definitions and previously established results that will be used throughout the paper.

Definition 2.1. *[25] Let K be a field, $(V,+)$ an Abelian group and let $P_*(V)$ denote the collection of all non-empty subsets of V. A hypervector space over K is defined as a quadruple $(V,+,\circ,K)$, where "$\circ$" is an external hyperoperation*

$$\circ : K \times V \longrightarrow P_*(V),$$

satisfying the following axioms for all $a, b \in K$ and $x, y \in V$:

(H_1) $a \circ (x+y) \subseteq a \circ x + a \circ y$, right distributive law,
(H_2) $(a+b) \circ x \subseteq a \circ x + b \circ x$, left distributive law,
(H_3) $a \circ (b \circ x) = (ab) \circ x$,
(H_4) $a \circ (-x) = (-a) \circ x = -(a \circ x)$,
(H_5) $x \in 1 \circ x$,

where in (H_1), $a \circ x + a \circ y = \{p+q : p \in a \circ x, q \in a \circ y\}$. Similarly it is in (H_2). Also in (H_3), $a \circ (b \circ x) = \bigcup_{t \in b \circ x} a \circ t$.

V is called strongly right distributive, if we have equality in (H_1). In a similar way we define the strongly left distributive hypervector spaces. V is called strongly distributive, if it is strongly right and left distributive.

A non-empty subset W of V is called a subhyperspace of V, denoted by $W \leqslant V$, if W is itself a hypervector space with the external hyperoperation on V, i.e. for all $a \in K$ and $x, y \in W$, $x - y \in W$ and $a \circ x \subseteq W$.

Example 2.2. *[25] In $(\mathbb{R}^2,+)$ define the external hyperoperation $\circ : \mathbb{R} \times \mathbb{R}^2 \longrightarrow P_*(\mathbb{R}^2)$ by setting:*

$$a \circ x = \begin{cases} \text{line } ox & \text{if } x \neq (0,0), \\ \{(0,0)\} & \text{if } x = (0,0). \end{cases}$$

Then $(\mathbb{R}^2,+,\circ,\mathbb{R})$ is a hypervector space.

Example 2.3. *[4] In classical vector space $(\mathbb{R}^3, +, ., \mathbb{R})$ we define the external hyperoperation $\circ : \mathbb{R} \times \mathbb{R}^3 \longrightarrow P_*(\mathbb{R}^3)$ by $a \circ (x_0, y_0, z_0) = l$, where l is a line with the parametric equations:*

$$l : \begin{cases} x = ax_0, \\ y = ay_0, \\ z = t. \end{cases}$$

Then $V = (\mathbb{R}^3, +, \circ, \mathbb{R})$ is a strongly left distributive hypervector space and the z-axis is a subhyperspace of V.

In the sequel of this paper, V denotes a hypervector space over the field K, unless otherwise is specified. Also, the zero of V is denoted by $\underline{0}$.

Definition 2.4. *[1] A fuzzy subset μ of V, $\mu \in FS(V)$, is called a fuzzy subhyperspace of V, if for all $a \in K$ and $x, y \in V$, the following conditions are satisfied:*

1) $\mu(x + y) \geq \mu(x) \wedge \mu(y)$,
2) $\mu(-x) \geq \mu(x)$,
3) $\bigwedge_{t \in a \circ x} \mu(t) \geq \mu(x)$.

Example 2.5. *[12] Consider the hypervector space $V = (\mathbb{R}^3, +, \circ, \mathbb{R})$ in Example 2.3. Define a fuzzy subset μ of V by:*

$$\mu(x, y, z) = \begin{cases} t_1 & (x, y, z) \in \{0\} \times \{0\} \times \mathbb{R}, \\ t_2 & (x, y, z) \in (\mathbb{R} \times \{0\} \times \mathbb{R}) \backslash (\{0\} \times \{0\} \times \mathbb{R}), \\ t_3 & \text{otherwise}, \end{cases}$$

where $0 \leq t_3 < t_2 < t_1 \leq 1$. Then μ is a fuzzy subhyperspace of V.

Theorem 2.6. *[1] $\mu \in FS(V)$ is a fuzzy subhyperspace of V if and only if $\mu_\alpha = \{x \in V : \mu(x) \geq \alpha\}$ is a subhyperspace of V, for all $\alpha \in Im\ \mu$. μ_α is called an α-level subhyperspace of V.*

Definition 2.7. *[5] Let μ, μ_1, μ_2 be fuzzy subsets of V and $a \in K$. We define $\mu_1 + \mu_2$ and $a \circ \mu$ to be the fuzzy subsets of V, whose membership functions are given by:*

$$(\mu_1 + \mu_2)(x) = \bigvee_{x = x_1 + x_2} (\mu_1(x_1) \wedge \mu_2(x_2)),$$

and

$$(a \circ \mu)(x) = \begin{cases} \bigvee_{x \in a \circ t} \mu(t) & \exists t \in V; x \in a \circ t, \\ 0 & \text{otherwise}. \end{cases}$$

3. A new characterization of fuzzy subhyperspaces

In this section, we present a novel structural characterization of fuzzy subhyperspaces as introduced by Ameri [1]. Specifically, we demonstrate that such fuzzy subhyperspaces can be decomposed into a sequence of nested special subhyperspaces, distinct from their level subhyperspaces. We also explore the relationship between affine fuzzy subsets in hypervector spaces and fuzzy superhyperplanes.

A subset S of V is called linearly independent ([2]) if for every vectors $v_1, \ldots, v_n$ in S, and $c_1, \ldots, c_n \in K$, $\underline{0} \in c_1 \circ v_1 + \cdots + c_n \circ v_n$, implies that $c_1 = \cdots = c_n = 0$. S is called linearly dependent if it is not linearly independent. A basis for V is a linearly independent subset of V such that spans V, i.e. $V = \langle S \rangle$, where

$$\begin{aligned} \langle S \rangle &= \left\{ t \in V : t \in \sum_{i=1}^{n} a_i \circ s_i, a_i \in K, s_i \in S, n \in \mathbb{N} \right\} \\ &= \{t_1 + \cdots + t_n : t_i \in a_i \circ s_i, a_i \in K, s_i \in S, n \in \mathbb{N}\}. \end{aligned}$$

We say that V is finite-dimensional if it has a finite basis. If V is strongly left distributive, invertible (V is said to be invertible if and only if $u \in a \circ v$ implies that $v \in a^{-1} \circ u$) and finite-dimensional, then every two basis of V have the same cardinality. In this case the cardinal of any basis of V is called the dimension of V and denoted by $\dim V$. Note that some hypervector spaces V (some set W of vectors) may not have any collection of linearly independent vectors. Such hypervector space (set) is called independentless. Clearly if V is independentless, then V has not any basis and for such hypervector spaces dimension is not defined.

Lemma 3.1. *Let V be finite-dimensional. If U and W are subhyperspaces of V such that $U \subsetneq W$ and $\dim W - \dim U \geqslant 2$, then there exist subhyperspaces $E_1, \ldots, E_k$ of V, $k = \dim W - 1$, such that $U \subsetneq E_1 \subsetneq \cdots \subsetneq E_k \subsetneq W$ and $\dim E_d = \dim U + d$, $1 \le d \le k$.*

Proof. Let β be a basis for U and $x_1 \in W \setminus U$. Put $E_1 = \langle \beta \cup \{x_1\} \rangle$. Then E_1 is a subhyperspace of V such that $U \subsetneq E_1 \subsetneq W$ and $\dim E_1 = \dim U + 1$. If $\dim W = \dim E_1 + 1$, then there is nothing to prove. If $\dim W - \dim E_1 \ge 2$, then similarly by considering $E_2 = \langle \beta \cup \{x_1, x_2\} \rangle$, where $x_2 \in W \setminus E_1$, E_2 is a subhyperspace of V such that $U \subsetneq E_1 \subsetneq E_2 \subsetneq W$ and $\dim E_2 = \dim E_1 + 1 = \dim U + 2$. By repeating the above, the proof is completed. □

The properties of affine subsets and affine fuzzy subsets of hypervector spaces have been studied ([12]). A non-empty subset E of V is called affine if $a \circ x + (1 - a) \circ y \subseteq E$, for all $x, y \in E$, $a \in K$. It is clear that every subhyperspace of V is an affine subset of V. A fuzzy subset μ of V is said to be affine if $a \circ \mu + (1 - a) \circ \mu \subseteq \mu$, for all $a \in K$.

Theorem 3.2. *[12] Let V be a hypervector space over the field K. Then*

1) *every fuzzy subhyperspace of V is an affine fuzzy subset of V;*
2) *$\mu \in FS(V)$ is affine if and only if $\mu_\alpha = \{x \in V : \mu(x) \ge \alpha\}$ is affine, for all $\alpha \in [0, 1]$;*
3) *if V is strongly right distributive and μ is an affine fuzzy subset of V, such that $\mu(\underline{0}) = \bigvee\limits_{t \in V} \mu(t)$, then μ is a fuzzy subhyperspace of V.*
4) *if V is strongly right distributive and $\mu \in FS(V)$, then μ is affine if and only if it is the translation of a fuzzy subhyperspace ν of V, i.e. $\mu = \nu + x_0$ for some $x_0 \in V$, where $(\nu + x_0)(x) = \nu(x - x_0)$.*

Theorem 3.3. *[2] Let V be invertible. Then for every $v_1, \ldots, v_n \in V$, either $v_1, \ldots, v_n$ are linearly independent or for some $1 \le j \le n$, v_j is in a linear combination of the others.*

Theorem 3.4. *Let V be invertible n-dimensional strongly right distributive and $\mu \in FS(V)$. Then μ is a fuzzy subhyperspace of V if and only if there exist a sequence of nested subhyperspaces $\{\underline{0}\} = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = V$ of V and $\alpha_0, \alpha_1, \ldots, \alpha_n \in [0, 1]$ such that $\dim V_i = d_i$ $(0 \le i \le n)$, $0 \le d_0 \le d_1 \le \cdots \le d_n = n$, $1 \ge \alpha_0 \ge \alpha_1 \ge \cdots \ge \alpha_n \ge 0$ and $\mu = \alpha_0 1_{V_0} \vee \cdots \vee \alpha_n 1_{V_n}$.*

Proof. Let $\mu = \alpha_0 1_{V_0} \vee \cdots \vee \alpha_n 1_{V_n}$ as stated above. Then every level subset of μ is a subhyperspace and so is an affine subset of V. Thus by Theorem 3.2(2), μ is an affine fuzzy subset of V. Also,

$$\begin{aligned}
\mu(\underline{0}) &= (\alpha_0 1_{V_0} \vee \ldots \vee \alpha_n 1_{V_n})(\underline{0}) \\
&= (\alpha_0 1_{V_0})(\underline{0}) \vee \ldots \vee (\alpha_n 1_{V_n})(\underline{0}) \\
&= \alpha_0 \vee \ldots \vee \alpha_n \\
&= \alpha_0 \\
&= \bigvee_{x \in V} \mu(x).
\end{aligned}$$

Hence, by Theorem 3.2(3), μ is a fuzzy subhyperspace of V.
Conversely, let μ be a fuzzy subhyperspace of V. At first, we prove that μ can achieve at most n different

values on $V \setminus \{\underline{0}\}$. Suppose $\{x_0, \ldots, x_n\} \subseteq V \setminus \{\underline{0}\}$ is not independentless and $\mu(x_0) < \mu(x_1) < \cdots < \mu(x_n)$. If $x_0 \in \langle x_1, \ldots, x_n \rangle$, then $x_0 \in a'_1 \circ x_1 + \cdots + a'_n \circ x_n$, for some $a'_1, \ldots, a'_n \in K$. By Definition 2.4, it follows that:

$$\mu(x_0) \geq \bigwedge_{t \in a'_1 \circ x_1 + \cdots + a'_n \circ x_n} \mu(t) \geq \mu(x_1) \wedge \cdots \wedge \mu(x_n),$$

which is impossible. Thus $x_0 \notin \langle x_1, \ldots, x_n \rangle$. Similarly, one can show that

$$x_1 \notin \langle x_2, \ldots, x_n \rangle, \ldots, x_{n-1} \notin \langle x_n \rangle.$$

Since all $x_i \neq \underline{0}$, by Theorem 3.3, it follows that:

$$\dim\langle x_0, \ldots, x_n \rangle = 1 + \dim\langle x_1, \ldots, x_n \rangle = \cdots = n + \dim\langle x_n \rangle = n + 1,$$

which is impossible, since $\dim V = n$. Consequently, $|Im\ \mu| \leq n + 1$, μ achieves n values at vectors of $V \setminus \{\underline{0}\}$ and the maximum is attained at $\underline{0}$.
Set $\mu(V) = \{\beta_0, \ldots, \beta_k\}$, where $k \leq n$, $\beta_i > \beta_{i+1}$, $0 \leq i \leq k-1$. Put $W_i = \mu_{\beta_i}$, for any $0 \leq i \leq k$. It is clear that $W_k = V$. Then by Theorem 2.6, for all $0 \leq i \leq k$, μ_{β_i} is a subhyperspace of V, such that $W_i \subsetneq W_{i+1}$ and so $\dim W_i < \dim W_{i+1}$, $0 \leq i \leq k-1$. Now put $\dim W_i = d_i$, $0 \leq i \leq k$. Then $0 \leq d_0 \leq d_1 \leq \cdots \leq d_k = n$. For each $0 \leq i \leq k$, suppose $V_{d_i} = W_i$. Then by Lemma 3.1, for any $d \in \{0, \ldots, n\} \setminus \{d_0, \ldots, d_k\}$, there exists a subhyperspace V_d of dimension d such that the family $\{V_d\}_{d \in \{0,\ldots,n\}}$ is nested, i.e. for all $0 \leq d \leq n-1$, $V_d \subsetneq V_{d+1}$. Now put

$$\alpha_d = \begin{cases} \mu(\underline{0}) = \beta_0 & d = d_0 \\ \beta_1 & d_0 < d \leq d_1 \\ \vdots & \\ \beta_k & d_{k-1} < d \leq d_n = n \end{cases}$$

Let $x \in V$. If d is the smallest integer such that $x \in V_d$, then $x \in W_{l+1} \setminus W_l$, for some $0 \leq l \leq k-1$, such that $d_l < d \leq d_{l+1}$. Thus $\beta_{l+1} \leq \mu(x) < \beta_l$ and so $\mu(x) = \beta_{l+1}$. On the other hand,

$$\alpha_0 1_{V_0}(x) \vee \cdots \vee \alpha_n 1_{V_n}(x) = \alpha_d 1_{V_d}(x) \vee \cdots \vee \alpha_n 1_{V_n}(x) = \alpha_d \vee \cdots \vee \alpha_n = \alpha_d = \beta_{l+1}.$$

Thus the theorem is proved. □

In the above decomposition of μ, $n+1$ subhyperspace V_i are nested and some of these levels coincide, so we can leave out those with the smaller dimension and keep the largest subhyperspace with the same level. This gives the following formulation of representation theorem.

Corollary 3.5. *The decomposition in Theorem 3.4, is minimal:*
Let V be invertible n-dimensional strongly right distributive and $\mu \in FS(V)$. Then μ is a fuzzy subhyperspace of V if and only if there exist nested subhyperspaces $V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_k = V$ of V, $k \leq n$ and $\alpha_0, \alpha_1, \ldots, \alpha_k \in [0,1]$ such that $\dim V_d < \dim V_{d+1}$, $0 \leq d \leq k-1$, $\alpha_0 > \alpha_1 > \cdots > \alpha_k$ *and* $\mu = \alpha_0 1_{V_0} \vee \cdots \vee \alpha_k 1_{V_k}$. *In this decomposition μ can not obtain with less components.*

Definition 3.6. *Let $\mu = \bigvee_{i=0}^{n} \alpha_i 1_{V_i}$ be a fuzzy subhyperspace of V. Then μ is called a fuzzy superhyperspace if and only if* $\dim V_{n-1} = (\dim V) - 1$, *i.e. there is a non-trivial $(n-1)$-dimensional component in the decomposition of μ. $\mu \in FS(V)$ is called a fuzzy superhyperplane if and only if it is the translation of some fuzzy superhyperspace.*

Theorem 3.7. *[13] Let V be invertible n-dimensional strongly left distributive and $W \leqslant V$ such that* $\dim W = d$. *Then W is the intersection of $(n-d)$ superhyperspaces in V (any subhyperspace of dimension $n-1$ is called a superhyperspace).*

Theorem 3.8. *Let V be invertible n-dimensional strongly right distributive. Then every affine fuzzy subset μ of V is a finite intersection of fuzzy superhyperplanes.*

Proof. By Theorem 3.2(4), $\mu = \mu' + x_0$, where μ' is a fuzzy subhyperspace of V and $x_0 \in V$. Thus by Corollary 3.5, there exist subhyperspaces $V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_k = V$ of V, $k \leq n$ and $1 \geq \alpha_0 > \alpha_1 > \cdots > \alpha_k \geq 0$ such that $\mu' = \bigvee\limits_{i=0}^{k} \alpha_i 1_{V_i}$. If $\dim V_{k-1} = n-1$, then μ' is a fuzzy superhyperspace and the proof is completed. If $0 \leq \dim V_{k-1} < n-1$, then by Theorem 3.7, $V_{k-1} = W_1 \cap \cdots \cap W_p$, where $W_1, \ldots, W_p$ are superhyperspaces of V. For all $1 \leq i \leq p$ put

$$\mu'_i = \alpha_0 1_{V_0} \vee \cdots \vee \alpha_{k-1} 1_{W_i} \vee \alpha_k 1_{V_k}.$$

Then for all $1 \leq i \leq p$, μ'_i is a fuzzy superhyperspace.
Now let $x \in E$ and l be the smallest number that $x \in V_l$. If $l \leq k-1$, then for all $1 \leq i \leq p$, $\mu'_i(x) = \alpha_l$ and so $\bigwedge\limits_{i=1}^{p} \mu'_i(x) = \alpha_l = \mu'(x)$. Thus in this case $\mu' = \bigcap\limits_{i=1}^{p} \mu'_i$. If $l = k$, then $x \notin V_{k-1}$ and $x \notin W_{i_0}$, for some $1 \leq i_0 \leq p$ and so $\bigwedge\limits_{i=1}^{p} \mu'_i(x) \leq \mu'_{i_0}(x) = \alpha_k$. Thus $\bigwedge\limits_{i=1}^{p} \mu'_i(x) = \alpha_k = \mu'_i(x)$, since α_k is the smallest value in $Im\ (\bigwedge\limits_{i=1}^{p} \mu'_i)$. Hence $\mu' = \bigwedge\limits_{i=1}^{p} \mu'_i$ and so

$$\mu = (\bigwedge_{i=1}^{p} \mu'_i) + x_0 = \bigwedge_{i=1}^{p} (\mu'_i + x_0),$$

and the proof is completed. □

4. Convex cones

The concepts of convex and balanced subsets of hypervector spaces were previously studied by the first author in [11] and [12]. In this section, we examine the properties of cones as a special class of subsets in real hypervector spaces. We show that every subhyperspace in a real hypervector space forms a cone, and every cone is necessarily balanced. Furthermore, we construct the convex cone spanned by non-empty subsets of real hypervector spaces and identify the smallest and largest subhyperspaces associated with a given convex cone.
Let K be a field. Then the mapping $|\,| : K \to \mathbb{R}$ is called a valuation on K if and only if for all $a, b \in K$, the followings hold:

1) $|a| \geq 0$ and we have equality iff $a = 0$;
2) $|ab| = |a|\,|b|$;
3) $|a + b| \leq |a| + |b|$.

A field K together with a valuation is called a valued field.
Let V be a hypervector space over valued field K. Then a non-empty subset E of V is called

1. convex if $a \circ x + (1-a) \circ y \subseteq E$, for all $x, y \in E$, $a \in K$, $|a| \leq 1$,
2. balanced if $a \circ E \subseteq E$, for all $a \in K$, $|a| \leq 1$.

A particular type of convex sets in real hypervector spaces is the convex cones.

Definition 4.1. *Let V be a real hypervector space. Then a non-empty subset $C \subseteq V$ is called a cone if*

$$\forall a \in \mathbb{R}^+,\ \forall x \in C,\ a \circ x \subseteq C.$$

It is clear that in a real hypervector space V, every subhyperspace is a cone, and every cone is balanced, when equipped with the standard absolute value on $\mathbb{R}$. The next examples show, in general, there is not any other relation between the concepts cone, subhyperspace and balanced subset of real hypervector spaces.

Example 4.2. *Consider the hypervector space $(\mathbb{R}^2, +, \circ, \mathbb{R})$ in Example 2.2, with any valuation on $\mathbb{R}$. Then for all $r_1, r_2 \in \mathbb{R}^+$, such that $r_1 < r_2$, the sub $B = \{(x,y) \in \mathbb{R}^2; r_1 x \leq y \leq r_2 x\} \subseteq \mathbb{R}^2$ is a cone and balanced, but not a subhyperspace.*

Example 4.3. *Consider the real hypervector space $V = (\mathbb{R}^2, +, \circ, \mathbb{R})$, where $\circ : \mathbb{R} \times \mathbb{R}^2 \longrightarrow P_*(\mathbb{R}^2)$ defined by $a \circ (x,y) = ax \times \mathbb{R}$ with the usual absolute value $|\cdot|$ on $\mathbb{R}$. Then the sub $B = \{(x,y) \in \mathbb{R}^2; 0 \leq x \leq 1\} \subseteq \mathbb{R}^2$ is balanced but is neither a subhyperspace nor a cone.*

Proposition 4.4. *Let V be a real hypervector space. Then*

1. *For any $x \in V$, the set $C = \mathbb{R}^+ \circ x = \bigcup_{a \in \mathbb{R}^+} a \circ x$ is a cone of V.*
2. *If C is a cone, then $C = \bigcup_{a \in \mathbb{R}^+, x \in C} a \circ x$.*

Proof. 1) If $b \in \mathbb{R}^+$ and $t \in \mathbb{R}^+ \circ x$, then $t \in a_0 \circ x$, for some $a_0 \in \mathbb{R}^+$ and so $b \circ t \subseteq b \circ (a_0 \circ x) = (ba_0) \circ x \subseteq \mathbb{R}^+ \circ x = C$. Thus $\mathbb{R}^+ \circ x$ is a cone.
2) If $x_0 \in C$, then $x_0 \in 1 \circ x_0 \subseteq \bigcup_{a \in \mathbb{R}^+, x \in C} a \circ x$. Conversely, $a \circ x \subseteq C$, for all $a \in \mathbb{R}^+$ and $x \in C$. Thus $\bigcup_{a \in \mathbb{R}^+, x \in C} a \circ x \subseteq C$. □

Definition 4.5. *A convex cone in a real hypervector space V is a cone which is a convex set.*

Example 4.6. *The set $C = \{(x,y) \in \mathbb{R}^2 : \ x \geq 0\}$ is a convex cone of the hypervector space in Example 4.3, but it is not a subhyperspace.*

Theorem 4.7. *A subset C of real hypervector space $V = (V, +, \circ, \mathbb{R})$ is a convex cone if and only if it is closed under addition and positive external hyperoperation, i.e.*

$$\forall x, y \in C, a \in \mathbb{R}^+; \ x + y \in C \text{ and } a \circ x \subseteq C.$$

Proof. Let C be a convex cone and $x, y \in C$. Then $\frac{1}{2} \circ (x+y) \subseteq \frac{1}{2} \circ x + \frac{1}{2} \circ y \subseteq C$. Thus $x + y \in 1 \circ (x+y) = 2 \circ (\frac{1}{2} \circ (x+y)) \subseteq C$. Conversely, if $x, y \in C$ and $a \in \mathbb{R}^+$ such that $|a| \leq 1$, then $(1-a) \circ x \subseteq C$ and $a \circ y \subseteq C$. Hence $(1-a) \circ x + a \circ y \subseteq C$, which means that C is convex. □

Corollary 4.8. *A subset C of real hypervector space $V = (V, +, \circ, \mathbb{R})$ is a convex cone if and only if $a_1 \circ x_1 + \cdots + a_n \circ x_n \subseteq C$, for all $a_1, \ldots, a_n \in \mathbb{R}^+$ and $x_1, \ldots, x_n \in C$.*

Corollary 4.9. *Let S be a non-empty subset of real hypervector space $V = (V, +, \circ, \mathbb{R})$. Then*

$$C = \left\{ t \in \sum_{i=1}^{n} a_i \circ x_i : a_i \in \mathbb{R}^+, x_i \in S, n \in \mathbb{N} \right\}$$

is the smallest convex cone containing S.

Proof. Let $a \in \mathbb{R}^+$ and $t \in C$. Then

$$a \circ t \subseteq a \circ \left(\sum_{i=1}^{n} a_i \circ x_i \right) \subseteq \sum_{i=1}^{n} (aa_i) \circ x_i \subseteq C,$$

and so C is a cone. Also, if $t, t' \in C$ and $a \in \mathbb{R}$ such that $|a| \leq 1$, then $t \in \sum_{i=1}^{n} a_i \circ x_i$ and $t' \in \sum_{j=1}^{m} a'_j \circ x'_j$, where $a_i, a'_j \in \mathbb{R}^+$ and $x_i, x'_j \in S$. Thus

$$\begin{aligned} a \circ t + (1-a) \circ t' &\subseteq a \circ \sum_{i=1}^{n} a_i \circ x_i + (1-a) \circ \sum_{j=1}^{m} a'_j \circ x'_j \\ &\subseteq \sum_{i=1}^{n} (aa_i) \circ x_i + \sum_{j=1}^{m} ((1-a)a'_j) \circ x'_j \\ &\subseteq C. \end{aligned}$$

Hence, C is convex. Also for all $x \in S$, $x \in 1 \circ x$ and so $x \in C$, which implies that $S \subseteq C$. Now suppose C' is a convex cone containing S and $t \in C$. Then $t \in \sum_{i=1}^{n} a_i \circ x_i$, for some $a_i \in \mathbb{R}^+$, $x_i \in S$ and $n \in \mathbb{N}$. Thus $t \in \sum_{i=1}^{n} a_i \circ x_i \subseteq C'$, by Theorem 4.7, and so $C \subseteq C'$. □

If S is a non-empty subset of real hypervector space $V = (V, +, \circ, \mathbb{R})$, then the convex cone

$$Cone\ S = \left\{ t \in \sum_{i=1}^{n} a_i \circ x_i : a_i \in \mathbb{R}^+, x_i \in S, n \in \mathbb{N} \right\}$$

is called the convex cone spanned by S.

Theorem 4.10. *Let C be a convex cone of real hypervector space $V = (V, +, \circ, \mathbb{R})$ containing $\underline{0}$. Then $C - C = \{x - y : x, y \in C\}$ is the smallest subhyperspace of V containing C and $(-C) \cap C$ is the largest subhyperspace of V contained in C.*

Proof. By Theorem 4.7, C is closed under addition and positive external hyperoperation. If $x-y, x'-y' \in C-C$, then $(x-y)-(x'-y') = (x+y')-(y+x') \in C-C$. Now let $x-y \in C-C$ and $a \in \mathbb{R}$. If $a \geq 0$, then $a \circ (x-y) \subseteq a \circ x - a \circ y \subseteq C - C$. If $a < 0$, then

$$\begin{aligned} a \circ (x-y) &= (-a) \circ ((-1) \circ (x-y)) \\ &\subseteq (-a) \circ ((-1) \circ x + (-1) \circ (-y)) \\ &= (-a) \circ (1 \circ y - 1 \circ x) \\ &\subseteq (-a) \circ (1 \circ y) - (-a) \circ (1 \circ x) \\ &= (-a) \circ y - (-a) \circ x \\ &\subseteq C - C. \end{aligned}$$

Thus $C - C \leqslant V$. Clearly, $C \subseteq C - C$. Now if C' is a subhyperspace of V containing C, then for all $x - y \in C - C$, $x - y \in C' - C' \subseteq C'$, and so $C - C \subseteq C'$, i.e. $C - C$ is the smallest subhyperspace of V containing C. Similarly, the second part of the theorem is proved. □

5. Convex fuzzy cones

The concept of convex fuzzy subsets in hypervector spaces was studied by the first author in [12]. In this section, we introduce the notion of fuzzy cones in real hypervector spaces as a specific class of convex fuzzy subsets, and study several of their properties. We also present characterizations for the smallest fuzzy subhyperspace containing a convex fuzzy cone, as well as the largest fuzzy subhyperspace contained within it.

Let V be a hypervector space over valued field K. Then a fuzzy subset μ of V is called convex if $a \circ \mu + (1-a) \circ \mu \subseteq \mu$, for all $a \in K$ with $|a| \leq 1$.

Theorem 5.1. *[12] Let V be a hypervector space over valued field K and $\mu \in FS(V)$. Then the following conditions are equivalent:*

1. *μ is convex;*
2. *$\forall x, y \in V$, $\forall a \in K$, $|a| \leq 1$, $\bigwedge_{t \in a \circ x + (1-a) \circ y} \mu(t) \geq \mu(x) \wedge \mu(y)$;*
3. *for all $\alpha \in [0, 1]$, μ_α is a convex subset of V.*

Proposition 5.2. *[12] Let V be a hypervector space over valued field K and $\{\mu_i\}_{i \in I}$ be a family of convex fuzzy subsets of V. Then $\bigcap_{i \in I} \mu_i$ is a convex fuzzy subset of V.*

Definition 5.3. *A fuzzy subset μ of real hypervector space V is called a fuzzy cone if*

$$\forall x \in V,\ a \in \mathbb{R}^+,\ \bigwedge_{t \in a \circ x} \mu(t) \geq \mu(x).$$

$\mu \in FS(V)$ is called convex fuzzy cone if it is convex and fuzzy cone.

Proposition 5.4. *If μ is a fuzzy cone of real hypervector space V, then $\bigwedge_{t \in a \circ x} \mu(t) = \mu(x)$, for all $x \in V$, $a \in \mathbb{R}^+$.*

Proof. If $x \in V$ and $a \in \mathbb{R}^+$, then

$$\begin{aligned}
\bigwedge_{t \in a \circ x} \mu(t) &\geq \mu(x) \\
&\geq \bigwedge_{y \in 1 \circ x} \mu(y) \\
&= \bigwedge_{y \in a^{-1} \circ (a \circ x)} \mu(y) \\
&= \bigwedge_{y \in a^{-1} \circ z, z \in a \circ x} \mu(y) \\
&\geq \bigwedge_{z \in a \circ x} \mu(z).
\end{aligned}$$

□

Proposition 5.5. *If μ is a fuzzy subset of real hypervector space V, then the following conditions are equivalent:*

1. *μ is a convex fuzzy cone;*
2. *$\forall x, y \in V$, $\forall a, a' \in \mathbb{R}^+$, $|a| \leq 1$, $\bigwedge_{t \in a \circ x + (1-a) \circ y} \mu(t) \geq \mu(x) \wedge \mu(y)$ and $\bigwedge_{t \in a' \circ x} \mu(t) \geq \mu(x)$;*
3. *for all $\alpha \in [0, 1]$, μ_α is a convex cone of V.*

Proof. 1 ⇔ 2) It follows from Theorem 5.1, and Definition 5.4.
2 ⇔ 3) By Theorem 5.1, μ is convex if and only if μ_α is convex. Now if μ is a fuzzy cone, $\alpha \in [0, 1]$, $y \in \mu_\alpha$ and $z \in a \circ y$, then $\mu(z) \geq \bigwedge_{t \in a \circ y} \mu(t) \geq \mu(y) \geq \alpha$ and so $z \in \mu_\alpha$, i.e. $a \circ \mu_\alpha \subseteq \mu_\alpha$, which means that μ_α is a cone. Conversely, let $a \in \mathbb{R}^+$ such that $0 < a \leq 1$, $x \in V$ and $t \in a \circ x$. Put $\mu(x) = \alpha$. Then $x \in \mu_\alpha$ and so $a \circ x \subseteq \mu_\alpha$. Thus $t \in \mu_\alpha$ and $\mu(t) \geq \alpha$. Hence $\bigwedge_{t \in a \circ x} \mu(t) \geq \alpha$. □

Proposition 5.6. *The intersection of an arbitrary family $\{\mu_i\}_{i \in I}$ of (convex) fuzzy cones is a (convex) fuzzy cone.*

Proof. By Proposition 5.2, $\bigcap_{i\in I}\mu_i$ is convex. Also

$$\begin{aligned}\bigwedge_{t\in a\circ x}\left(\bigcap_{i\in I}\mu_i\right)(t) &= \bigwedge_{t\in a\circ x}\left(\bigwedge_{i\in I}\mu_i(t)\right)\\ &= \bigwedge_{i\in I}\left(\bigwedge_{t\in a\circ x}\mu_i(t)\right)\\ &\geq \bigwedge_{i\in I}\mu_i(x)\\ &= \left(\bigcap_{i\in I}\mu_i\right)(x),\end{aligned}$$

for all $a \in \mathbb{R}^+$ and $x \in V$. □

Similar to other structures, the previous proposition allows us to define the cone of a fuzzy subset.

Definition 5.7. *Let μ be a fuzzy subset of real hypervector space V. Then the convex fuzzy cone of μ, denoted by Cone μ, is the infimum of all convex fuzzy cones of V containing μ, i.e. the smallest convex fuzzy cone of V containing μ.*

Theorem 5.8. *Let V be an invertible n-dimensional strongly right distributive real hypervector space and μ be a convex fuzzy cone of V such that $\mu(\underline{0}) = \bigvee_{x\in V}\mu(x)$. Then $\mu - \mu \in FS(V)$ defined by*

$$(\mu-\mu)(x) = \bigvee_{x=x_1-x_2}(\mu(x_1)\wedge\mu(x_2)),$$

is the smallest fuzzy subhyperspace of V containing μ.

Proof. At first we shall construct the smallest fuzzy subhyperspace of V from μ (the intersection of all fuzzy subhyperspaces of V containing μ) and then show that it is equal to $\mu - \mu$.
By Proposition 5.5, for all $\alpha \in [0,1]$, μ_α is a convex cone containing $\underline{0}$, since $\mu(\underline{0}) = \bigvee_{x\in V}\mu(x)$. Suppose $V_\alpha = \langle\mu_\alpha\rangle$. Then $\{V_\alpha\}_{\alpha\in I}$ is a chain of at most $n+1$ different subhyperspaces of V with the largest element $V = V_0 = \langle\mu_0\rangle$. Denote this chain by $W_0 \subseteq W_1 \subseteq \cdots \subseteq W_k$, where $k \leq n$. Now we partition I in $k+1$ subsets

$$I_i = \{\alpha \in I : V_\alpha = W_i\}, \ \ 0 \leq i \leq k.$$

Put $\alpha_i = \bigvee I_i$, for all $0 \leq i \leq k$. Then by Representation Theorem (Corollary 3.5), $\nu = \bigvee_{i=0}^{k}\alpha_i 1_{W_i}$ is a fuzzy subhyperspace of V. Also, if $x \in V$ and $\mu(x) = \alpha$, then $\alpha \in I_i$, for some $0 \leq i \leq k$ and so

$$\nu(x) \geq \alpha_i = \bigvee I_i \geq \alpha = \mu(x).$$

Now by construction of α_i's and W_i's it follows that ν is the smallest fuzzy subhyperspace of V containing μ.
Finally, we show that $\nu = \mu - \mu$. Let $x \in W_i \setminus W_{i-1}$, for some $0 \leq i \leq k$. Then $\nu(x) = \alpha_i$ and by Theorem 4.10, $x = y_0 - z_0$ for some $y_0, z_0 \in \mu_\alpha$. Thus

$$\begin{aligned}(\mu-\mu)(x) &= \bigvee_{x=y-z}(\mu(y)-\mu(z))\\ &\geq \mu(y_0)-\mu(z_0)\\ &\geq \alpha\wedge\alpha\\ &= \alpha.\end{aligned}$$

Hence $(\mu - \mu)(x) \geq \bigvee I_i = \alpha_i = \nu(x)$. Note that for all $\alpha \in I_{i-1}$, there exist no $y, z \in \mu_\alpha$ such that $x = y - z$, since $x \notin W_{i-1}$. By repeating the above it follows that for all $\alpha > \alpha_i$, there are no $y, z \in \mu_\alpha$ such that $x = y - z$. Then $(\mu - \mu)(x) = \alpha_i = \nu(x)$. □

Theorem 5.9. *Let V be an invertible n-dimensional strongly right distributive real hypervector space and μ be a convex fuzzy cone of V such that $\mu(\underline{0}) = \bigvee\limits_{x \in V} \mu(x)$. Then $\mu \wedge (-\mu) \in FS(V)$ defined by $(\mu \wedge (-\mu))(x) = \mu(x) \wedge \mu(-x)$ is the largest fuzzy subhyperspace of V contained in μ.*

Proof. The proof is similar to the proof of Theorem 5.8.
By Proposition 5.5, for all $\alpha \in [0, 1]$, μ_α is a convex cone containing $\underline{0}$, since $\mu(\underline{0}) = \bigvee\limits_{x \in V} \mu(x)$. From Theorem 4.10, there exists a largest subhyperspace $V_\alpha \subseteq \mu_\alpha$. Let $\{V_\alpha\}_{\alpha \in I}$ be the chain of subhyperspaces of V and $W_0 \subseteq W_1 \subseteq \cdots \subseteq W_k$, $k \leq n$, be the at most $n + 1$ different ones. Here we have again a partition of I in $I_i = \{\alpha \in I : V_\alpha = W_i\}$, $0 \leq i \leq k$. Suppose $\alpha_i = \bigvee I_i$, $0 \leq i \leq k$ and $\nu = \bigvee\limits_{i=0}^{k} \alpha_i 1_{W_i}$. Then by Representation Theorem (Corollary 3.5) and by construction of α_i's and W_i's, ν is the largest fuzzy subhyperspace of V contained in μ. Now we show that $\nu = \mu \cap (-\mu)$. Let $x \in W_i \setminus W_{i-1}$, for some $0 \leq i \leq k$. Then $\nu(x) = \alpha_i \in I_i$ and by Theorem 4.10, $W_i = \mu_{\alpha_i} \cap (-\mu_{\alpha_i})$. Thus

$$(\mu \cap (-\mu))(x) = \mu(x) \wedge \mu(-x)) \geq \alpha_i \wedge \alpha_i = \alpha_i = \nu(x).$$

It is clear that for all $\alpha \in I_{i-1}$, $\mu(x) < \alpha$ or $\mu(-x) < \alpha$. Thus in this case $\bigwedge I_{i-1} = \alpha_i$ and so $(\mu \wedge (-\mu))(x) = \alpha_i = \nu(x)$. □

6. Conclusion

In real hypervector spaces, every subhyperspace is a cone, and every cone is balanced. In this paper, we introduced the concept of convex cones spanned by non-empty subsets of real hypervector spaces. Through some examples, we demonstrated that cones, subhyperspaces, and balanced subsets are generally distinct concepts. Moreover, the smallest subhyperspace of V containing C and the largest subhyperspace of V contained in C is achieved, for a convex cone C of a real hypervector space V.

In the context of hypervector spaces, a convex fuzzy cone combines elements of convexity with fuzzy representations. A fuzzy cone is a generalization of a cone in vector spaces, where instead of precise points, we deal with fuzzy sets, allowing for degrees of membership. In the paper, a characterization of fuzzy convex cones of real subhyperspaces based on its level subsets was given. Also, for a convex fuzzy cone μ of a real finite dimensional hypervector space V, the smallest (resp. largest) fuzzy subhyperspace of V containing (resp. contained in) μ was obtained.

Convex fuzzy cone can have the following applications:

- It can be used to model constraints in optimization problems, allowing for more flexible and realistic solutions that account for uncertainty or imprecision.
- In fields like economics, inventory management, and resource allocation, convex fuzzy cones help in making decisions based on incomplete or uncertain information.
- Fuzzy logic concepts, including fuzzy cones, are utilized in algorithms for processing images and recognizing patterns where traditional binary classifications may fail.

For further research, the following topics are suggested:

- Fuzzy Set Theory: Investigate foundational texts that cover the principles of fuzzy sets, fuzzy relations, and fuzzy logic.
- Topology and Functional Analysis: Explore the topological properties of fuzzy cones and their connections to functional spaces. This includes studying compactness, connectedness, and continuity within hypervector spaces.
- Fuzzy Geometry: Investigate how geometric concepts are extended in a fuzzy context. This includes studying fuzzy distances, fuzzy metrics, and their implications in the discussion of convexity.

- Algebraic Structures: Delve into the algebraic properties of fuzzy cones in the framework of lattice theory and ordered sets. This gives insights into the categorical aspects and morphisms between fuzzy structures.

- Convex Analysis: Examine how traditional convex analysis can be extended to the fuzzy context. This involves studying fuzzy convex functions and their generalizations and the implications for optimization.

- Optimization: In operations research, fuzzy optimization techniques can be employed to deal with uncertainty in parameters. Study models such as fuzzy linear programming, fuzzy multi-objective optimization, and decision-making under fuzzy constraints.

- Artificial Intelligence: Explore applications of fuzzy logic in AI, especially in expert systems, decision support systems, and machine learning. Fuzzy rules and inference systems often use concepts akin to fuzzy cones for reasoning under uncertainty.

- Economics and Game Theory: Investigate how fuzzy set theory and convex fuzzy cones can be applied to economic models, particularly in areas such as fuzzy game theory, where multiple players make decisions based on uncertain information.

- Control Systems: In engineering, study how fuzzy logic can be applied to control systems, especially in systems where precise models are difficult to obtain. Fuzzy control mechanisms can utilize concepts from convex fuzzy cones to create flexible systems.

- Image Processing and Pattern Recognition: Research applications in computer vision where fuzzy techniques improve the handling of noise and uncertainty in image data. Techniques like fuzzy clustering and fuzzy neural networks often employ these principles.

References

[1] R. Ameri, *Fuzzy hypervector spaces over valued fields*, Iran. J. Fuzzy Syst. **2** (2005), 37–47.
[2] R. Ameri, O. R. Dehghan, *On dimension of hypervector spaces*, Eur. J. Pure Appl. Math. **1(2)** (2008), 32–50.
[3] R. Ameri, O. R. Dehghan, *Fuzzy hypervector spaces*, Adv. Fuzzy Syst. **2008** (2008), 295649.
[4] R. Ameri, O. R. Dehghan, *Fuzzy basis of fuzzy hypervector spaces*, Iran. J. Fuzzy Syst. **7(3)** (2010), 97–113.
[5] R. Ameri, O. R. Dehghan, *Fuzzy hypervector spaces based on fuzzy singletons*, Comput. Math. Appl. **61** (2011), 2933–2943.
[6] R. Ameri, O. R. Dehghan, *Dimension of fuzzy hypervector spaces*, Iran. J. Fuzzy Syst. **8(5)** (2011), 149–166.
[7] P. Corsini, *Prolegomena of hypergroup theory*, (second edition), Aviani editor, 1993.
[8] P. Corsini, V. Leoreanu, *Applications of hyperstructure theory*, Kluwer Academic Publications, 2003.
[9] B. Davvaz, I. Cristea, *Fuzzy algebraic hyperstructures*, Springer, 2015.
[10] B. Davvaz, V. Leareanu-Fotea, *Hyperring theory and applications*, International Academic Press, 2007.
[11] O. R. Dehghan, *Balanced and absorbing fuzzy subsets of hypervector spaces*, Comput. Appl. Math. **39(2)** (2020), 1–12 (Article 53).
[12] O. R. Dehghan, *Affine and convex fuzzy subsets of hypervector spaces*, Comput. Appl. Math. **40(8)** (2021), 1–14 (Article 303).
[13] O. R. Dehghan, *Linear functionals on hypervector spaces*, Filomat **34(9)** (2020), 3031–3043.
[14] O. R. Dehghan, R. Ameri, H. A. Ebrahimi Aliabadi, *Some results on hypervector spaces*, Ital. J. Pure Appl. Math. **41** (2019), 23–41.
[15] A. K. Katsaras, D. B. Liu, *Fuzzy vector spaces and fuzzy topological vector spaces*, J. Math. Anal. Appl. **58** (1977), 135–146.
[16] R. Kumar, *Fuzzy vector space and fuzzy cosets*, Fuzzy Sets and Systems **45** (1992), 109–116.
[17] D. S. Malik, J. N. Mordeson, *Fuzzy vector space*, Inform. Sci. **55** (1991), 271–281.
[18] F. Marty, *Sur une generalization de la notion de groupe*, 8th congress des Mathematiciens Scandinaves, Stockholm (1934), 45–49.
[19] J. N. Mordeson, D. S. MAlik, *Fuzzy commutative algebra*, World Scientific, 1998.
[20] J. N. Mordeson, D. S. MAlik, N. Kuroki, *Fuzzy semigroups*, Springer, 2003.
[21] S. Nanda, *Fuzzy linear spaces over valued fields*, Fuzzy Sets and Systems **42** (1991), 351–354.
[22] A. Rosenfeld, *Fuzzy groups*, J. Math. Anal. Appl., 35 (1971) 512-517.
[23] A. Rosenfeld, John N. Mordeson, K. R. Bhutani, *Fuzzy group theory*, Springer, 2005.
[24] M. Sedghi, O. R. Dehghan, M. Norouzi, *n-Normed hypervector spaces*, J. Math. Sci: Adv. Appl. **45** (2017), 41–59.
[25] M. S. Tallini, *Hypervector spaces*, Fourth Int. Congress on AHA (1990), 167–174.
[26] R. Tyrrell Rockafellar, *Convex analysis*, Princeton University Press, 1970.
[27] T. Vougiouklis, *Hyperstructures and their representations*, Hardonic Press Inc., 1994.
[28] L. A. Zadeh, *Fuzzy sets*, Inf. Control **8** (1965), 338–353.