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Characterizations of EP elements via w-core inverses

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Abstract. Let R be a ring with an involution. It has been proven that $a \in R^{\mathrm{EP}}$ if and only if $(aa^{\#})^* = aa^{\#}$, where $a^{\#}$ is the group inverse of a, and R^{EP} is the set of all EP elements in R. In this paper, we use w-core inverses and core inverses to describe EP elements in a ring with an involution, by making full use of the result mentioned before.

1. Introduction

Let R be an associative ring with identity. We call an element $a \in R$ group invertible if there exists $a^{\#} \in R$ such that

$$a = aa^{\#}a, \ \ a^{\#} = a^{\#}aa^{\#}, \ \ aa^{\#} = a^{\#}a.$$

The element $a^{\#}$ is called the group inverse of a [3], which is unique if it exists. The symbol $R^{\#}$ stands for the set of all group invertible elements in R. In particular, if $a = a^2b = ca^2$ for some $b, c \in R$, then $a^{\#} = cab = c^2a = ab^2$.

A map $* : R \to R, a \mapsto a^*$ is called an involution of R if for any $a, b \in R$,

$$(a^*)^* = a, (ab)^* = b^*a^*, (a+b)^* = a^* + b^*.$$

An element $a \in R$ is called Hermitian, if $a = a^*$. The set of of all Hermitian elements in R is denoted by R^{Her}.

An element $a^{\dagger} \in R$ is said to be Moore-Penrose invertible (or MP-invertible) if there exists the Moore-Penrose inverse (or MP-inverse) $a^{\dagger}$ of a, such that

$$aa^{\dagger}a = a, \ \ a^{\dagger}aa^{\dagger} = a^{\dagger}, \ \ (aa^{\dagger})^* = aa^{\dagger}, \ \ (a^{\dagger}a)^* = a^{\dagger}a.$$

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If $a^{\dagger}$ exists, then it is unique [8]. We use $R^{\dagger}$ to denote the set of all MP-invertible elements in R. Furthermore, an element $a \in R^{\#} \cap R^{\dagger}$ is called EP if $a^{\#} = a^{\dagger}$. The set of all EP elements of R is denoted by R^{EP}. There are numerous works concerning the EP elements in rings with involution. Such as in [4–7], Mosić et al. investigated EP elements and gave many new properties of them. In [2], Koliha and Patrício proved that $a \in R^{\#} \cap R^{\dagger}$ is EP if and only if $aa^{\#} \in R^{\mathrm{Her}}$.

Let $a, w \in R$. Then a is called w-core invertible if there exists $a_w^{\textcircled{\#}} \in R$ such that

$$a_w^{\textcircled{\#}} = aw(a_w^{\textcircled{\#}})^2, \ a = a_w^{\textcircled{\#}}awa, \ (awa_w^{\textcircled{\#}})^* = awa_w^{\textcircled{\#}},$$

where $a_w^{\textcircled{\#}}$ is called the w-core inverse of a [9]. If $a_w^{\textcircled{\#}}$ exists, then it is unique. The set of all w-core invertible elements in R is written by $R_w^{\textcircled{\#}}$. In particular, a is called core invertible if there exists $a^{\textcircled{\#}} \in R$, such that

$$a^{\textcircled{\#}} = a(a^{\textcircled{\#}})^2, \ a = a^{\textcircled{\#}}a^2, \ (aa^{\textcircled{\#}})^* = aa^{\textcircled{\#}},$$

where $a^{\textcircled{\#}}$ is said to be the core inverse of a. We denote the set of all core invertible elements of R by $R^{\textcircled{\#}}$. Dually, we call $a \in R$ dual core invertible if there exists $a_{\textcircled{\#}} \in R$ such that

$$a_{\textcircled{\#}} = (a_{\textcircled{\#}})^2a, \ a = a^2a_{\textcircled{\#}}, \ (a_{\textcircled{\#}}a)^* = a_{\textcircled{\#}}a,$$

where $a_{\textcircled{\#}}$ is called the dual core inverse of a. The set of all dual core invertible elements of R is denoted by $R_{\textcircled{\#}}$.

The following result follows from [9, Theorem 2.10].

Lemma 1.1. *Let $a, w \in R$ and $a_w^{\textcircled{\#}} = x$. Then $(aw)^{\textcircled{\#}}$ exists and $(aw)^{\textcircled{\#}} = x$.*

In [1], Cao et al. used w-core inverses to characterize some generalized inverses in rings with involution. Inspired by the works in [1] and [2], we will firstly use w-core inverses and core inverses to describe EP elements in a ring with an involution.

2. New characterizations of EP elements

In this section, we shall give some new characterizations of EP elements. Denote by $\psi_a := \{a^{\#}, a^{\dagger}, a^{\textcircled{\#}}, a_{\textcircled{\#}}\}$. It has been shown that $a^{\textcircled{\#}} = a^{\#}aa^{+}$, $a_{\textcircled{\#}} = a^{+}aa^{\#}$. Moreover if $a \in R^{\mathrm{EP}}$, then $a^{\#} = a^{\dagger} = a^{\textcircled{\#}} = a_{\textcircled{\#}}$. In this case, $\psi_a = \{a^{\#}\}$.

The following result follows from [2, Theorem 7.3] or [3, Theorem 1.1.3], which is a very useful tool to investigate EP elements in $*$-rings.

Lemma 2.1. *Let R be a $*$-ring and $a \in R^{\#}$. Then $a \in R^{\mathrm{EP}}$ if and only if one of the following equivalent conditions holds:*
(1) $aa^{\#} \in R^{\mathrm{Her}}$;
(2) $(a^{\#})^* = aa^{\#}(a^{\#})^*$;
(3) $(a^{\#})^* = (a^{\#})^*a^{\#}a$;
(4) $a^{\#}(1 - aa^{\#})^* = (1 - aa^{\#})(a^{\#})^*$.

From Lemma 2.1, one gets the following conclusion at once.

Lemma 2.2. *Let R be a $*$-ring and $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{\mathrm{EP}}$ if and only if $aa^{\#} = aa^{\dagger}$.*

Proposition 2.3. *Let R be a $*$-ring and $a \in R^{\#} \cap R^{\dagger}$. Then $a_x^{\textcircled{\#}} = (ax)^{\textcircled{\#}} = aa^{\dagger}$ for all $x \in \psi_a$.*

Proof. (1) If $x = a^\#$, then $a^{\textcircled{\#}}_{a^\#} = aa^\dagger$. In fact,

$$aa^\#(aa^\dagger)(aa^\dagger) = aa^\dagger, \ \ (aa^\dagger)aa^\#a = a,$$

and

$$((aa^\#(aa^\dagger))^* = (aa^\dagger)^* = aa^\dagger = (aa^\#(aa^\dagger).$$

Hence, $a^{\textcircled{\#}}_{a^\#} = aa^\dagger$.

(2) If $x = a^\dagger$, then $a^{\textcircled{\#}}_{a^\dagger} = aa^\dagger$. Actually,

$$aa^\dagger(aa^\dagger)(aa^\dagger) = aa^\dagger, \ \ (aa^\dagger)aa^\dagger a = a,$$

and

$$((aa^\dagger(aa^\dagger))^* = (aa^\dagger)^* = aa^\dagger = (aa^\dagger(aa^\dagger).$$

It follows that $a^{\textcircled{\#}}_{a^\dagger} = aa^\dagger$.

(3) If $x = a^{\textcircled{\#}} = a^\# aa^\dagger$, then $a^{\textcircled{\#}}_{a^\# aa^\dagger} = aa^\dagger$. It is easy to compute

$$a(a^\# aa^\dagger)(aa^\dagger)(aa^\dagger) = aa^\dagger, \ \ (aa^\dagger)a(a^\# aa^\dagger)a = aa^\# aa^\dagger a = aa^\dagger a = a,$$

and

$$(a(a^\# aa^\dagger)(aa^\dagger))^* = (aa^\dagger)^* = aa^\dagger = a(a^\# aa^\dagger)(aa^\dagger).$$

So, $a^{\textcircled{\#}}_{a^\# aa^\dagger} = aa^\dagger$.

(4) If $x = a_{\textcircled{\#}} = a^\dagger aa^\#$, then $a^{\textcircled{\#}}_{a^\dagger aa^\#} = aa^\dagger$. A straightforward computation shows

$$a(a^\dagger aa^\#)(aa^\dagger)(aa^\dagger) = aa^\dagger, \ \ (aa^\dagger)a(a^\dagger aa^\#)a = a,$$

and

$$(a(a^\dagger aa^\#)(aa^\dagger))^* = (aa^\dagger)^* = aa^\dagger = a(a^\dagger aa^\#)(aa^\dagger).$$

Hence, $a^{\textcircled{\#}}_{a^\dagger aa^\#} = aa^\dagger$.

The second equality follows from Lemma 1.1. □

Theorem 2.4. *Let R be a $*$-ring and $a \in R^\# \cap R^\dagger$. Then $a \in R^{\mathrm{EP}}$ if and only if $a^{\textcircled{\#}}_x = aa^\#$ for some $x \in \psi_a$.*

Proof. ⇒ It follows from $a \in R^{\mathrm{EP}}$ and Proposition 2.3.

⇐ (1) If $x = a^\#$, then

$$(aa^\#)^* = (aa^\#(aa^\#))^* = aa^\#(aa^\#) = aa^\#.$$

By Lemma 2.1 (1), $a \in R^{\mathrm{EP}}$.

(2) If $x = a^\dagger$, one has

$$(aa^\#)^* = (aa^\dagger(aa^\#))^* = aa^\dagger(aa^\#) = aa^\#.$$

It gives $a \in R^{\mathrm{EP}}$ by Lemma 2.1 (1).

(3) If $x = a^{\textcircled{\#}} = a^\# aa^\dagger$, one gets

$$(aa^\#)^* = (a(a^\# aa^\dagger)aa^\#)^* = a(a^\# aa^\dagger)aa^\# = aa^\#.$$

So, by Lemma 2.1 (1), $a \in R^{\mathrm{EP}}$.

(4) If $x = a_{\textcircled{\#}} = a^{\dagger}aa^{\#}$, then

$$(aa^{\#})^{*} = (a(a^{\dagger}aa^{\#})(aa^{\#}))^{*} = a(a^{\dagger}aa^{\#})(aa^{\#}) = aa^{\#}.$$

Hence, $a \in R^{\mathrm{EP}}$ by Lemma 2.1 (1). □

Remark 2.5. *In fact, the proof of Theorem 2.4 can be obtained by Lemma 2.2 and Proposition 2.3 directly. However, we choose to retain the proof in this style.*

Note that in the proof of "only if" part in Theorem 2.4, we only use $(awx)^{*} = awx$, where $(aw)^{\textcircled{\#}} = x$. So, from Theorem 2.4, one gets the following corollary immediately.

Corollary 2.6. *Let R be a $*$-ring and $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{\mathrm{EP}}$ if and only if one the following statements holds:*

(1) $(aa^{\#})^{\textcircled{\#}} = aa^{\#}$;

(2) $(aa^{\dagger})^{\textcircled{\#}} = aa^{\#}$;

(3) $a^{\textcircled{\#}}_{a^{\#}} = (aa^{\#})^{*}$;

(4) ${a^{\#}}^{\textcircled{\#}}_{a} = (aa^{\#})^{*}$;

(5) $(aa^{\#})^{\textcircled{\#}} = (aa^{\#})^{*}$.

Note that for any $x \in \psi_a$, it is easy to know $axa = a$.

Proposition 2.7. *Let R be a $*$-ring and $a \in R^{\#} \cap R^{\dagger}$. Then $a^{\textcircled{\#}}_{xa} = (axa)^{\textcircled{\#}} = a^{\textcircled{\#}} = a^{\#}aa^{\dagger}$ for all $x \in \psi_a$.*

Proof. (1) If $x = a^{\#}$, then $a^{\textcircled{\#}}_{a^{\#}a} = a^{\#}aa^{\dagger}$. In this condition,

$$a(a^{\#}a)(a^{\#}aa^{\dagger})(a^{\#}aa^{\dagger}) = a^{\#}aa^{\dagger}, \quad (a^{\#}aa^{\dagger})a(a^{\#}a)a = a,$$

and

$$(a(a^{\#}a)(a^{\#}aa^{\dagger}))^{*} = (aa^{\dagger})^{*} = aa^{\dagger} = a(a^{\#}a)(a^{\#}aa^{\dagger}).$$

It follows that $a^{\textcircled{\#}}_{a^{\#}a} = a^{\#}aa^{\dagger}$.

(2) If $x = a^{\dagger}$, then $a^{\textcircled{\#}}_{a^{\dagger}a} = a^{\#}aa^{\dagger}$. In this case,

$$a(a^{\dagger}a)(a^{\#}aa^{\dagger})(a^{\#}aa^{\dagger}) = a^{\#}aa^{\dagger}, \quad (a^{\#}aa^{\dagger})a(a^{\dagger}a)a = a,$$

and

$$(a(a^{\dagger}a)(a^{\#}aa^{\dagger}))^{*} = (aa^{\dagger})^{*} = aa^{\dagger} = a(a^{\dagger}a)(a^{\#}aa^{\dagger}).$$

Hence, $a^{\textcircled{\#}}_{a^{\dagger}a} = a^{\#}aa^{\dagger}$.

(3) If $x = a^{\textcircled{\#}} = a^{\#}aa^{\dagger}$, then $a^{\textcircled{\#}}_{a^{\#}a} = a^{\#}aa^{\dagger}$. Hence, it follows by (1).

(4) If $x = a_{\textcircled{\#}} = a^{\dagger}aa^{\#}$, then $a^{\textcircled{\#}}_{a^{\dagger}a} = a^{\#}aa^{\dagger}$. Therefore, it follows by (2). □

Theorem 2.8. *Let R be a $*$-ring and $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{\mathrm{EP}}$ if and only if $a^{\textcircled{\#}}_{xa} = a^{\#}$ for some $x \in \psi_a$.*

Proof. $\Rightarrow$ Since $a \in R^{\mathrm{EP}}$, $\chi_a = \{a^{\#}\}$, and $(aa^{\#})^{*} = aa^{\#}$ by Lemma 2.1 (1). Now, it is easy to compute

$$a(a^{\#}a)a^{\#}a^{\#} = a^{\#}, \quad a^{\#}a(a^{\#}a)a = a,$$

and

$$(a(a^{\#}a)a^{\#})^{*} = (aa^{\#})^{*} = aa^{\#} = a(a^{\#}a)a^{\#}.$$

This gives $a^{\textcircled{\#}}_{a^{\#}a} = a^{\#}$.

$\Leftarrow$ It is similar to the "$\Leftarrow$" part in the proof of Theorem 2.4. $\square$

Similarly to Corollary 2.6, Proposition 2.7 and Theorem 2.8 implies the following conclusion.

Corollary 2.9. *Let R be a $*$-ring and $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{\mathrm{EP}}$ if and only if one of the following statements holds:*

(1) $(ax)^{\textcircled{\#}}_{a} = a^{\#}aa^{\dagger}$ for all $x \in \psi_a$;

(2) $(ax)^{\textcircled{\#}}_{a} = a^{\#}$ for some $x \in \psi_a$;

(3) $a^{\textcircled{\#}} = a^{\#}$ for some $x \in \psi_a$.

The following result is easy but useful in the later study, so we give a proof of it.

Lemma 2.10. *Let R be a $*$-ring and $a \in R^{\#} \cap R^{\dagger}$. Then $aa^{\#}(a^{\dagger})^{*} = (a^{\dagger})^{*}aa^{\#} = (a^{\dagger})^{*}$. Dually, $(aa^{\#})^{*}a^{\dagger} = a^{\dagger}(aa^{\#})^{*} = a^{\dagger}$.*

Proof. We only show $aa^{\#}(a^{\dagger})^{*} = (a^{\dagger})^{*}$, and the rest proofs are similar. In fact,

$$\begin{aligned}
aa^{\#}(a^{\dagger})^{*} &= aa^{\#}(a^{\dagger}aa^{\dagger})^{*} \\
&= aa^{\#}(aa^{\dagger})^{*}(a^{\dagger})^{*} \\
&= aa^{\#}aa^{\dagger}(a^{\dagger})^{*} \\
&= aa^{\dagger}(a^{\dagger})^{*} \\
&= (aa^{\dagger})^{*}(a^{\dagger})^{*} \\
&= (a^{\dagger}aa^{\dagger})^{*} \\
&= (a^{\dagger})^{*}.
\end{aligned}$$

This completes the proof. $\square$

Proposition 2.11. *Let R be a $*$-ring and $a \in R^{\#} \cap R^{\dagger}$. Then $((a^{\#})^{*})^{\textcircled{\#}}_{a^{*}} = ((a^{\#})^{*}a^{*})^{\textcircled{\#}} = a^{\dagger}a$.*

Proof. A straightforward computation shows

$$(a^{\#})^{*}a^{*}(a^{\dagger}a)(a^{\dagger}a) = a^{\dagger}a, \quad a^{\dagger}a(a^{\#})^{*}a^{*}(a^{\#})^{*} = (a^{\dagger}a)^{*}(a^{\#})^{*} = (a^{\#}a^{\dagger}a)^{*} = (a^{\#})^{*},$$

and

$$((a^{\#})^{*}a^{*}(a^{\dagger}a))^{*} = (a^{\dagger}a)^{*} = a^{\dagger}a = (a^{\#})^{*}a^{*}(a^{\dagger}a).$$

Hence, $((a^{\#})^{*})^{\textcircled{\#}}_{a^{*}} = a^{\dagger}a$.

The second equality follows from Lemma 1.1. $\square$

Theorem 2.12. *Let R be a $*$-ring and $a \in R^{\#}$. Then $a \in R^{\mathrm{EP}}$ if and only if $((a^{\#})^{*})^{\textcircled{\#}}_{a^{*}} = aa^{\#}$.*

Proof. By $a \in R^{\mathrm{EP}}$ and Proposition 2.11, $((a^{\#})^{*})^{\textcircled{\#}}_{a^{*}} = aa^{\#}$.

$\Leftarrow$ By hypothesis,

$$(a^{\#})^{*} = (aa^{\#})(a^{\#})^{*}a^{*}(a^{\#})^{*} = aa^{\#}(a^{\#})^{*}.$$

By Lemma 2.1 (2), $a \in R^{\mathrm{EP}}$. $\square$

Note that in the proof of "only if" part in Theorem 2.12, we only use $a = xawa$, where $a^{\textcircled{\#}}_{w} = x$. The following result shows Theorem 2.12 can be generalized to core inverses.

Corollary 2.13. *Let R be a $*$-ring and $a \in R^{\#}$. Then $a \in R^{\mathrm{EP}}$ if and only if $((a^{\#})^{*}a^{*})^{\textcircled{\#}} = aa^{\#}$.*

Proof. $\Rightarrow$ It follows from the "if" part of the proof in Theorem 2.12.

$\Leftarrow$ Since $((a^{\#})^{*}a^{*})^{\textcircled{\#}} = aa^{\#}$,

$$(a^{\#})^{*}a^{*} = (aa^{\#})((a^{\#})^{*}a^{*})((a^{\#})^{*}a^{*}) = aa^{\#}(a^{\#})^{*}a^{*}. \tag{1}$$

Multiplying the equality (1) on the right by $(a^{\#})^{*}$, then

$$(a^{\#})^{*} = (a^{\#})^{*}a^{*}(a^{\#})^{*} = aa^{\#}(a^{\#})^{*}a^{*}(a^{\#})^{*} = aa^{\#}(a^{\#})^{*}.$$

Hence, by Lemma 2.1 (2), $a \in R^{\mathrm{EP}}$. $\square$

Proposition 2.14. *Let R be a $*$-ring and $a \in R^{\#} \cap R^{\dagger}$. Then $(aa^{\#})_{a^{*}}^{\textcircled{\#}} = (aa^{\#}a^{*})^{\textcircled{\#}} = (a^{\dagger})^{*}aa^{\dagger}$.*

Proof. It is easy to compute

$$\begin{aligned}
((aa^{\#})a^{*}((a^{\dagger})^{*}aa^{\dagger})))^{*} &= (aa^{\#}(a^{\dagger}a)^{*}aa^{\dagger})^{*} \\
&= (a(a^{\#}a^{\dagger}a)aa^{\dagger})^{*} \\
&= (aa^{\#}aa^{\dagger})^{*} \\
&= (aa^{\dagger})^{*} \\
&= aa^{\dagger} \\
&= (aa^{\#})a^{*}((a^{\dagger})^{*}aa^{\dagger})),
\end{aligned}$$

$$\begin{aligned}
(aa^{\#})a^{*}((a^{\dagger})^{*}aa^{\dagger})((a^{\dagger})^{*}aa^{\dagger}) &= aa^{\dagger}(a^{\dagger})^{*}aa^{\dagger} \\
&= (aa^{\dagger})^{*}(a^{\dagger})^{*}aa^{\dagger} \\
&= (a^{\dagger}aa^{\dagger})^{*}aa^{\dagger} \\
&= (a^{\dagger})^{*}aa^{\dagger},
\end{aligned}$$

and

$$\begin{aligned}
((a^{\dagger})^{*}aa^{\dagger})(aa^{\#})a^{*}(aa^{\#}) &= ((a^{\dagger})^{*}aa^{\#})a^{*}(aa^{\#}) \\
&= (a^{\dagger})^{*}a^{*}aa^{\#} \\
&= (aa^{\dagger})^{*}aa^{\#} \\
&= aa^{\dagger}aa^{\#} \\
&= aa^{\#}.
\end{aligned}$$

Hence, $(aa^{\#})_{a^{*}}^{\textcircled{\#}} = (a^{\dagger})^{*}aa^{\dagger}$.

The second equality follow from Lemma 1.1. $\square$

Theorem 2.15. *Let R be a $*$-ring and $a \in R^{\#}$. Then $a \in R^{\mathrm{EP}}$ if and only if $(aa^{\#})_{a^{*}}^{\textcircled{\#}} = (a^{\#})^{*}$.*

Proof. $\Rightarrow$ It follows from $a \in R^{\mathrm{EP}}$ and Proposition 2.14.

$\Leftarrow$ By hypothesis,

$$(a^{\#})^{*} = (aa^{\#})a^{*}(a^{\#})^{*}(a^{\#})^{*} = aa^{\#}(a^{\#})^{*}. \tag{2}$$

Multiplying the equality (2) on the right by a^{*}, then

$$(a^{\#})^{*}a^{*} = aa^{\#}(a^{\#})^{*}a^{*}.$$

Since $aa^{\#}(a^{\#})^{*}a^{*} \in R^{\mathrm{Her}}$, $aa^{\#} \in R^{\mathrm{Her}}$. Hence, by Lemma 2.1, $a \in R^{\mathrm{EP}}$. □

In Theorem 2.15, we only use $x = awx^{2}$, where $a_{w}^{\textcircled{\#}} = x$. Therefore, Theorem 2.15 implies the following conclusion.

Corollary 2.16. *Let R be a $*$-ring and $a \in R^{\#}$. Then $a \in R^{\mathrm{EP}}$ if and only if $(aa^{\#}a^{*})^{\textcircled{\#}} = (a^{\#})^{*}$.*

Proposition 2.17. *Let R be a $*$-ring and $a \in R^{\#} \cap R^{\dagger}$. Then $(a^{*})_{(a^{\#})^{*}}^{\textcircled{\#}} = (a^{*}(a^{\#})^{*})^{\textcircled{\#}} = a^{\dagger}a$.*

Proof. By a direct computation,

$$a^{*}(a^{\#})^{*}(a^{\dagger}a)(a^{\dagger}a) = a^{\dagger}a, \quad (a^{\dagger}a)a^{*}(a^{\#})^{*}a^{*} = (a^{\dagger}a)^{*}a^{*} = (aa^{\dagger}a)^{*} = a^{*},$$

and

$$(a^{*}(a^{\#})^{*}a^{\dagger}a)^{*} = (a^{\dagger}a)^{*} = a^{\dagger}a = a^{*}(a^{\#})^{*}a^{\dagger}a.$$

Hence, $(a^{*})_{(a^{\#})^{*}}^{\textcircled{\#}} = a^{\dagger}a$.

The latter equality follows from Lemma 1.1. □

Theorem 2.18. *Let R be a $*$-ring and $a \in R^{\#}$. Then $a \in R^{\mathrm{EP}}$ if and only if $(a^{*})_{(a^{\#})^{*}}^{\textcircled{\#}} = aa^{\#}$.*

Proof. $\Rightarrow$ It follows from $a \in R^{\mathrm{EP}}$ and Proposition 2.17.

$\Leftarrow$ By the assumption,

$$aa^{\#} = a^{*}(a^{\#})^{*}(aa^{\#})(aa^{\#}) = a^{*}(a^{\#})^{*}aa^{\#}.$$

Since $a^{*}(a^{\#})^{*}aa^{\#} \in R^{\mathrm{Her}}$, $aa^{\#} \in R^{\mathrm{Her}}$. So, by Lemma 2.1 (1), $a \in R^{\mathrm{EP}}$. □

Corollary 2.19. *Let R be a $*$-ring and $a \in R^{\#}$. Then $a \in R^{\mathrm{EP}}$ if and only if $((aa^{\#})^{*})^{\textcircled{\#}} = aa^{\#}$.*

Proposition 2.20. *Let R be a $*$-ring and $a \in R^{\#} \cap R^{\dagger}$. Then $(aa^{\dagger})_{ax}^{\textcircled{\#}} = aa^{\dagger}$ for all $x \in \psi_{a}$.*

Proof. (1) If $x = a$ or $x = a_{\textcircled{\#}} = a^{\dagger}aa^{\#}$, then $(aa^{\dagger})_{aa^{\#}}^{\textcircled{\#}} = aa^{\dagger}$. In this case,

$$(aa^{\dagger})(aa^{\#})(aa^{\dagger})(aa^{\dagger}) = aa^{\dagger}, \quad (aa^{\dagger})(aa^{\dagger})(aa^{\#})(aa^{\dagger}) = aa^{\dagger},$$

and

$$((aa^{\dagger})(aa^{\#})(aa^{\dagger}))^{*} = (aa^{\dagger})^{*} = aa^{\dagger} = (aa^{\dagger})(aa^{\#})(aa^{\dagger}).$$

Hence, $(aa^{\dagger})_{aa^{\#}}^{\textcircled{\#}} = aa^{\dagger}$.

(2) If $x = a^{\dagger}$ or $x = a^{\textcircled{\#}} = a^{\#}aa^{\dagger}$, then $(aa^{\dagger})_{aa^{\dagger}}^{\textcircled{\#}} = aa^{\dagger}$. It follows from a straightforward verification. □

Theorem 2.21. *Let R be a $*$-ring and $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{\mathrm{EP}}$ if and only if $(aa^{\dagger})_{ax}^{\textcircled{\#}} = aa^{\#}$ for some $x \in \psi_{a}$.*

Proof. $\Rightarrow$ It follows from $a \in R^{\mathrm{EP}}$ and Proposition 2.20.

$\Leftarrow$ (1) If $x = a^{\#}$, then

$$(aa^{\#})^{*} = ((aa^{\dagger})(aa^{\#})(aa^{\#}))^{*} = (aa^{\dagger})(aa^{\#})(aa^{\#}) = aa^{\#}.$$

Hence, $a \in R^{\mathrm{EP}}$ by Lemma 2.1 (1).

(2) If $x = a^{\dagger}$, one gets

$$(aa^{\#})^* = ((aa^{\dagger})(aa^{\dagger})(aa^{\#}))^* = (aa^{\dagger})(aa^{\dagger})(aa^{\#}) = aa^{\#}.$$

So, $a \in R^{\mathrm{EP}}$ by Lemma 2.1 (1).

(3) If $x = a^{\textcircled{\#}} = a^{\#}aa^{\dagger}$, one has

$$(aa^{\#})^* = ((aa^{\dagger})(aa^{\#}aa^{\dagger})(aa^{\#}))^* = (aa^{\dagger})(aa^{\#}aa^{\dagger})(aa^{\#}) = aa^{\#}.$$

By Lemma 2.1 (1), $a \in R^{\mathrm{EP}}$.

(4) If $x = a_{\textcircled{\#}} = a^{\dagger}aa^{\#}$, then

$$(aa^{\#})^* = ((aa^{\dagger})(aa^{\dagger}aa^{\#})(aa^{\#}))^* = (aa^{\dagger})(aa^{\dagger}aa^{\#})(aa^{\#}) = aa^{\#}.$$

Therefore, by Lemma 2.1 (1), $a \in R^{\mathrm{EP}}$. $\square$

Theorem 2.22. *Let R be a $*$-ring and $a \in R^{\#}$. Then $a \in R^{\mathrm{EP}}$ if and only if $(aa^{\#})^{\textcircled{\#}}_{a^*}$ exists, and $(aa^{\#})^{\textcircled{\#}}_{a^*} = (a^{\#})^*$.*

Proof. $\Rightarrow$ By $a \in R^{\mathrm{EP}}$, $a^{\#} = a^{\dagger}$. Then

$$(aa^{\#})a^*(a^{\#})^*(a^{\#})^* = aa^{\#}(a^{\#})^* = aa^{\#}(a^{\dagger})^* = (a^{\dagger})^* = (a^{\#})^*,$$

$$(a^{\#})^*(aa^{\#})a^*(aa^{\#}) = (a^{\dagger})^*(a^{\dagger}a)^*a^*(aa^{\dagger})^* = (aa^{\dagger}aa^{\dagger}aa^{\dagger})^* = (aa^{\dagger})^* = (aa^{\#})^*,$$

and

$$((aa^{\#})a^*(a^{\#})^*)^* = aa^{\#}a^*(a^{\#})^*.$$

Hence, $(aa^{\#})^{\textcircled{\#}}_{a^*} = (a^{\#})^*$.

$\Leftarrow$ By hypothesis,

$$(a^{\#})^* = (aa^{\#})a^*(a^{\#})^*(a^{\#})^* = aa^{\#}(a^{\#})^*. \tag{3}$$

Multiplying the equality (3) on the right by a^*, then

$$(a^{\#})^*a^* = aa^{\#}(a^{\#})^*a^*.$$

Since, $aa^{\#}(a^{\#})^*a^* \in R^{\mathrm{Her}}$, $(a^{\#})^*a^* \in R^{\mathrm{Her}}$, i.e., $aa^{\#} \in R^{\mathrm{Her}}$. So, $a \in R^{\mathrm{EP}}$.

Corollary 2.23. *Let R be a $*$-ring and $a \in R^{\#}$. Then $a \in R^{\mathrm{EP}}$ if and only if $(aa^{\#}a^*)^{\textcircled{\#}}$ exists, and $(aa^{\#}a^*)^{\textcircled{\#}} = (a^{\#})^*$.*

Theorem 2.24. *Let R be a $*$-ring and $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{\mathrm{EP}}$ if and only if $(aa^{\dagger})^{\textcircled{\#}}_{(a^{\#})^*}$ exists, and $(aa^{\dagger})^{\textcircled{\#}}_{(a^{\#})^*} = a^*$.*

Proof. By $a \in R^{\mathrm{EP}}$, $a^{\#} = a^{\dagger}$ and $(aa^{\#})^* = aa^{\#}$. Hence,

$$(aa^{\dagger})(a^{\#})^*a^*a^* = (aa^{\dagger})a^* = (aa^{\#})^*a^* = a^*,$$

$$a^*(aa^{\dagger})(a^{\#})^*(aa^{\dagger}) = a^*(aa^{\dagger})^*(a^{\#})^*(aa^{\dagger})^* = (aa^{\#}a^{\#}aa^{\#}a)^* = (aa^{\#})^* = (aa^{\dagger})^* = aa^{\dagger},$$

and

$$aa^{\#} = ((aa^{\dagger})(a^{\#})^*a^*)^* = ((aa^{\dagger})^*(a^{\#})^*a^*)^* = ((aa^{\#})^*(a^{\#})^*a^*)^* = (aa^{\dagger})(a^{\#})^*a^* = (aa^{\#})^*.$$

It follows that $(aa^{\dagger})^{\textcircled{\#}}_{(a^{\#})^*} = a^*$.

$\Leftarrow$ Since $(aa^{\dagger})^{\textcircled{\#}}_{(a^{\#})^*} = a^*$,

$$a^* = (aa^{\dagger})(a^{\#})^* a^* a^* = aa^{\dagger}a^*. \tag{4}$$

Multiplying the equality (4) on the right by $(a^{\#})^*$, then

$$a^*(a^{\#})^* = aa^{\dagger}a^*(a^{\#})^* = (aa^{\dagger})^* a^*(a^{\#})^* = (a^{\#}aaa^{\dagger})^* = (aa^{\dagger})^* = aa^{\dagger}.$$

Since $aa^{\dagger} \in R^{\mathrm{Her}}$, $a^*(a^{\#})^* \in R^{\mathrm{Her}}$, and so $a \in R^{\mathrm{EP}}$. $\square$

Corollary 2.25. *Let R be a $*$-ring and $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{\mathrm{EP}}$ if and only if $(aa^{\dagger}(a^{\#})^*)^{\textcircled{\#}}$ exists, and $(aa^{\dagger}(a^{\#})^*)^{\textcircled{\#}} = a^*$.*

Theorem 2.26. *Let R be a $*$-ring and $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{\mathrm{EP}}$ if and only if $(a^{\dagger}a)^{\textcircled{\#}}_{(a^{\#})^*}$ exists, and $(a^{\dagger}a)^{\textcircled{\#}}_{(a^{\#})^*} = a^*$.*

Proof. $\Rightarrow$ It is similar to the "if" part of the proof of Theorem 2.24.

$\Leftarrow$ By $(a^{\dagger}a)^{\textcircled{\#}}_{(a^{\#})^*} = a^*$, we have

$$a^{\#}a = (a^{\dagger}a(a^{\#})^*a^*)^* = (a^{\dagger}a)^*(a^{\#})^*a^* = (aa^{\#}a^{\dagger}a)^* = ((a^{\#}a)^*)^* = a^{\dagger}a(a^{\#})^*a^* = (a^{\#}a)^*.$$

It follows that $a \in R^{\mathrm{EP}}$. $\square$

Corollary 2.27. *Let R be a $*$-ring and $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{\mathrm{EP}}$ if and only if $(a^{\dagger}a(a^{\#})^*)^{\textcircled{\#}}$ exists, and $(a^{\dagger}a(a^{\#})^*)^{\textcircled{\#}} = a^*$.*

Remark 2.28. *In fact, we think there are much more results can be considered about EP elements and w-core inverses than appeared in this paper, we only list part of them. Furthermore, the conclusions given in this paper can yield some other questions, such as describe EP elements by a certain set of related equations. We will study these questions in the future.*

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Conflict of Interest

The authors declared that they have no conflict of interest.

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