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New representations of the g-Drazin inverse for anti-triangular matrices

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Abstract. We provide representations for the generalized Drazin inverse of an anti-triangular matrix of the form $\begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}$ in a Banach algebra $\mathcal{A}$ with commuting elements $a, b \in \mathcal{A}$. In particular, a new formula for the Drazin inverse of an anti-triangular matrix is established by employing Catalan numbers.

1. Introduction

Let $\mathcal{A}$ be a Banach algebra with identity 1 and $M_2(\mathcal{A})$ be the Banach algebra with identity I_2 of all 2×2 matrices over $\mathcal{A}$. An element $a \in \mathcal{A}$ has generalized Drazin inverse (g-Drazin inverse) if there exists $x \in \mathcal{A}$ such that

$$ax^2 = x, ax = xa, a - xa^2 \in \mathcal{A}^{qnil}.$$

If such an x exists, it is unique and is denoted by a^d. Here, $\mathcal{A}^{qnil} = \{x \in \mathcal{A} \mid 1 + \lambda x \in \mathcal{A}$ is invertible for all $\lambda \in \mathbb{C}\}$. It is well known that $x \in \mathcal{A}^{qnil}$ if and only if $\lim\limits_{n\to\infty} \|x^n\|^{\frac{1}{n}} = 0$. If we replace the quasinilpotent set $\mathcal{A}^{qnil}$ with the set of all nilpotent elements in $\mathcal{A}$, we refer to the unique x as the Drazin inverse of a, and denote it by a^D. Both the Drazin and g-Drazin inverses play significant roles in ring and matrix theory (see [6]) and graph theory (see [15]).

It is intriguing to investigate the Drazin and g-Drazin inverses of the anti-triangular matrix $M = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} \in M_2(\mathcal{A})$. One motivation for exploring this problem is the quest for a closed-form solution to systems of second-order linear differential equations, which can be expressed in the following vector-valued form: $Ax''(t) + Bx'(t) + Cx(t) = 0$ where $A, B, C \in \mathbb{C}^{n\times n}$ (with A being potentially singular) and x is an $\mathbb{C}^n$-valued function. Clearly, the solutions to singular systems of differential equations are determined by the Drazin inverse of the aforementioned anti-triangular matrix M (see [3, 4]). Recently, the generalized Drazin inverse of block operator matrices is used to find general solutions for Cauchy problems for Riccati and Lyapunov operator differential equations (see [1]). Although the Drazin and g-Drazin inverses of anti-triangular matrices are valuable tools in the study of differential equations, finding representations for such generalized inverses remains a challenging task.

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In 2005, Castro-González and Dopazo gave the representations of the Drazin inverse for a class of complex matrices $\begin{pmatrix} I & F \\ I & 0 \end{pmatrix}$ (see [10, Theorem 3.3]).

In 2011, Bu et al. investigated the Drazin inverse of the complex matrix $\begin{pmatrix} E & F \\ I & 0 \end{pmatrix}$ under the condition $EF = FE$ (see [3, Theorem 3.3]).

In 2013, Xu, Song and Zhang studied an expression of the Drazin inverse of the operator matrix $\begin{pmatrix} E & F \\ I & 0 \end{pmatrix} \in M_2\big(\mathcal{B}(X)\big)$ under the condition $EF = FE$, where $\mathcal{B}(X)$ is the Banach algebra of bounded linear operators on a complex Banach space X (see [18, Theorem 3.8]).

In 2016, Yu, Wang and Deng characterized the Drazin invertibility of the anti-triangular operator matrix $\begin{pmatrix} E & F \\ I & 0 \end{pmatrix} \in M_2\big(\mathcal{B}(\mathcal{H})\big)$ under the conditions $F^{\pi}EF^{D} = 0, F^{\pi}EF = F^{\pi}FE$, where $\mathcal{B}(\mathcal{H})$ is the Banach algebra of bounded linear operators on a complex Hilbert space $\mathcal{H}$ (see [20, Theorem 4.1]).

Recently, many authors have explored various conditions under which representations of the Drazin (or g-Drazin) inverse of such anti-triangular matrices can be established. For additional references, we refer the reader to [11, 12, 22, 24, 25, 27].

The motivation of this paper is to investigate the representation of the g-Drazin inverse of the anti-triangular matrix $M = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}$ in a Banach algebra $\mathcal{A}$. We begin by examining the solvability of a quadratic equation in the Banach algebra $\mathcal{A}$ using Catalan numbers C_n. Next, we study the representation of M under the conditions $ab = ba, a \in \mathcal{A}$ is invertible and $b \in \mathcal{A}^{qnil}$. We then employ the Morita context ring and the Pierce decomposition of a Banach algebra element as tools to extend the previous special case to the more general conditions $ab = ba, a, b \in \mathcal{A}^{d}$. Consequently, the known results are extended to a broader context within a Banach algebra. In particular, we establish a new formula for the Drazin inverse of an anti-triangular matrix, employing Catalan numbers as a key tool. This formula offers a new approach to addressing related difference equation problems in matrix structure.

Throughout this paper, all Banach algebras are considered to be complex and possess an identity element. We use $\mathcal{A}^{-1}, \mathcal{A}^{D}$ and $\mathcal{A}^{d}$ to stand for the sets of all invertible, Drazin invertible and g-Drazin invertible elements in $\mathcal{A}$, respectively. For $a \in \mathcal{A}^{d}$, we define $a^{\pi} = 1 - aa^{d}$. Let $a, p^2 = p \in \mathcal{A}$. Then a has the Pierce decomposition given by $pap + pap^{\pi} + p^{\pi}ap + p^{\pi}ap^{\pi}$, which we denote in matrix form as $\begin{pmatrix} pap & pap^{\pi} \\ p^{\pi}ap & p^{\pi}ap^{\pi} \end{pmatrix}_p$. The inverse of x in the corner ring $p\mathcal{A}p$ is denoted by x_p^{-1}.

2. Key lemmas

In this section, we present some necessary lemmas which will be used in the sequel. We start by

Lemma 2.1. *Let $a, b \in \mathcal{A}^{d}$. If $ab = 0$, then $a + b \in \mathcal{A}^{d}$ and*

$$(a + b)^{d} = \sum_{i=0}^{\infty} (a^{d})^{i+1} b^{i} b^{\pi} + \sum_{i=0}^{\infty} a^{i} a^{\pi} (b^{d})^{i+1}.$$

Proof. See [6, Lemma 15.2.2]. □

Lemma 2.2. *Let $a, b \in \mathcal{A}^{d}$. If $ab^2 = 0$ and $aba = 0$, then $a + b \in \mathcal{A}^{d}$ and*

$$\begin{aligned}(a + b)^{d} &= \sum_{i=0}^{\infty} (b^{d})^{i+1} a^{i} a^{\pi} + \sum_{i=0}^{\infty} b^{i} b^{\pi} (a^{d})^{i+1} + \sum_{i=0}^{\infty} b^{i} b^{\pi} (a^{d})^{i+2} b \\ &+ \sum_{i=0}^{\infty} (b^{d})^{i+3} a^{i+1} a^{\pi} b - b^{d} a^{d} b - (b^{d})^{2} a a^{d} b.\end{aligned}$$

Proof. See [6, Corollary 15.2.4] and [19, Theorem 2.1]. □

Lemma 2.3. *Let*

$$x = \begin{pmatrix} a & 0 \\ c & b \end{pmatrix} \text{ or } \begin{pmatrix} b & c \\ 0 & a \end{pmatrix}.$$

Then

$$x^d = \begin{pmatrix} a^d & 0 \\ z & b^d \end{pmatrix} \text{ or } \begin{pmatrix} b^d & z \\ 0 & a^d \end{pmatrix},$$

where $z = \sum\limits_{i=0}^{\infty}(b^d)^{i+2}ca^ia^{\pi} + \sum\limits_{i=0}^{\infty} b^ib^{\pi}c(a^d)^{i+2} - b^dca^d$.

Proof. See [25, Lemma 2.3]. □

Lemma 2.4. *Let $\mathcal{A}$ be a Banach algebra and $a \in \mathcal{A}^{-1}, b \in \mathcal{A}^{qnil}$. If $ab = ba$, then the equation $ax + x^2 = b$ has a solution x such that $a + x \in \mathcal{A}^{-1}, x \in \mathcal{A}^{qnil}$.*

Proof. Let $x = \sum\limits_{i=0}^{\infty} c_ia^{\alpha_i}b^{i+1}$, where $c_i \in \mathbb{C}, \alpha_i \in \mathbb{Z}$. Choose $\alpha_i = -(2i+1)$, Since $ab = ba$, we have

$$\begin{array}{rcl} ax + x^2 & = & \sum\limits_{i=0}^{\infty} c_ia^{\alpha_i+1}b^{i+1} + [\sum\limits_{i=0}^{\infty} c_ia^{\alpha_i}b^{i+1}][\sum\limits_{i=0}^{\infty} c_ia^{\alpha_i}b^{i+1}] \\ & = & c_0a^{\alpha_0+1}b + [c_1a^{\alpha_1+1} + c_0^2a^{2\alpha_0}]b^2 \\ & + & [c_2a^{\alpha_2+1} + c_0c_1a^{\alpha_0+\alpha_1} + c_1c_0a^{\alpha_1+\alpha_0}]b^3 \\ & + & [c_3a^{\alpha_3+1} + c_0c_2a^{\alpha_0+\alpha_2} + c_1c_1a^{\alpha_1+\alpha_1} + c_2c_0a^{\alpha_2+\alpha_0}]b^4 \\ & + & \cdots \\ & = & c_0b + [c_1 + c_0^2]a^{-2}b^2 + [c_2 + c_0c_1 + c_1c_0]a^{-4}b^3 \\ & + & [c_3 + c_0c_2 + c_1c_1 + c_2c_0]a^{-6}b^4 + \cdots \\ & = & b, \end{array}$$

hence, we choose

$$\begin{array}{c} c_0 = 1, c_1 = -1, c_2 = 2, c_3 = -5, c_4 = 14, c_5 = -42, \cdots \\ c_i = -(c_0c_{i-1} + c_1c_{i-2} + \cdots + c_{i-1}c_0)(i \in \mathbb{N}). \end{array}$$

Let $\{C_n\}$ be the series of Catalan numbers, i.e.,

$$\begin{array}{c} C_0 = 1, C_1 = 1, C_2 = 2, C_3 = 5, C_4 = 14, C_5 = 42, \cdots, \\ C_n = C_0C_{n-1} + \cdots + C_{n-1}C_0(n \in \mathbb{N}). \end{array}$$

Then $c_0 = C_0, c_1 = -C_1$. By induction, we claim that $c_{2n} = C_{2n}, c_{2n+1} = -C_{2n+1}(n \geq 0)$. Hence, $|c_n| = C_n(n \geq 1)$. By using the asymptotic expression of the Catalan numbers C_n, we have

$$\lim_{n\to\infty} C_n/\Big(\frac{4^n}{\sqrt{\pi}(n)^{\frac{3}{2}}}\Big) = 1.$$

Therefore

$$\lim_{n\to\infty} \sqrt[n]{|c_n|} = \lim_{n\to\infty} \frac{4}{\pi^{\frac{1}{2n}}(\sqrt[n]{n})^{\frac{3}{2}}} = 4.$$

Since $b \in \mathcal{A}^{qnil}$, we have $\lim\limits_{n\to\infty} \| b^n \|^{\frac{1}{n}} = 0$. Since

$$\sqrt[n]{\| c_na^{-(2n+1)}b^{n+1} \|} \leq \sqrt[n]{|c_n|} \| a^{-1} \|^{2+\frac{1}{n}} \sqrt[n]{\| b \|} \| b^n \|^{\frac{1}{n}},$$

we deduce that

$$\lim_{n\to\infty} \sqrt[n]{\| c_na^{-(2n+1)}b^{n+1} \|} = 0.$$

This implies that $\sum\limits_{i=0}^{\infty} c_i a^{-(2i+1)} b^{i+1}$ absolutely converges.

Accordingly, the equation $ax+x^2=b$ has a solution $x=\sum\limits_{i=0}^{\infty} c_i a^{-(2i+1)} b^{i+1}$, where $c_0=1, c_{k+1}=-\sum\limits_{i=0}^{k} c_i c_{k-i} (k\geq 0)$. Clearly, $c_n=(-1)^n C_n=(-1)^n\frac{(2n)!}{n!(n+1)!}$. Since $[\sum\limits_{i=0}^{\infty} c_i a^{-(2i+1)} b^i]b=b[\sum\limits_{i=0}^{\infty} c_i a^{-(2i+1)} b^i]$ and $b\in\mathcal{A}^{qnil}$, it follows by [6, Lemma 15.1.1] that $x=[\sum\limits_{i=0}^{\infty} c_i a^{-(2i+1)} b^i]b\in\mathcal{A}^{qnil}$. As $a^{-1}x=xa^{-1}$, we see that $1+a^{-1}x\in\mathcal{A}^{-1}$; hence, $a+x=a[1+a^{-1}x]\in\mathcal{A}^{-1}$. This completes the proof. □

Lemma 2.5. *Let $\mathcal{A}$ be a Banach algebra and $M=\begin{pmatrix} a & b\\ 1 & 0\end{pmatrix}$ with $a\in\mathcal{A}^{-1}, b\in\mathcal{A}^{qnil}$. If $ab=ba$, then $M\in M_2(\mathcal{A})^d$ and*

$$M^d=\begin{pmatrix} (a+x)^{-1}-xy & (a+x)^{-1}x-xyx\\ y & yx\end{pmatrix},$$

where $x=\sum\limits_{i=0}^{\infty}(-1)^i\frac{(2i)!}{i!(i+1)!}a^{-(2i+1)}b^{i+1}, y=\sum\limits_{i=0}^{\infty}(-1)^i(a+x)^{-(i+2)}x^i$.

Proof. In view of Lemma 2.4, the equation $ax+x^2=b$ has a solution x such that $a+x\in\mathcal{A}^{-1}$ and $x\in\mathcal{A}^{qnil}$. Here,

$$x=\sum_{i=0}^{\infty}(-1)^i\frac{(2i)!}{i!(i+1)!}a^{-(2i+1)}b^{i+1}.$$

It is easy to verify that

$$M=\begin{pmatrix} 1 & -x\\ 0 & 1\end{pmatrix}\begin{pmatrix} a+x & 0\\ 1 & -x\end{pmatrix}\begin{pmatrix} 1 & x\\ 0 & 1\end{pmatrix}.$$

Since $x\in\mathcal{A}^{qnil}$, we see that $a+x\in\mathcal{A}^{-1}$. Then $\begin{pmatrix} a+x & 0\\ 1 & x\end{pmatrix}$ has g-Drazin inverse. Therefore M has g-Drazin inverse. Exactly, we have

$$\begin{aligned} M^d &= \begin{pmatrix} 1 & -x\\ 0 & 1\end{pmatrix}\begin{pmatrix} a+x & 0\\ 1 & -x\end{pmatrix}^d\begin{pmatrix} 1 & x\\ 0 & 1\end{pmatrix}\\ &= \begin{pmatrix} 1 & -x\\ 0 & 1\end{pmatrix}\begin{pmatrix} (a+x)^{-1} & 0\\ y & 0\end{pmatrix}\begin{pmatrix} 1 & x\\ 0 & 1\end{pmatrix}\\ &= \begin{pmatrix} (a+x)^{-1}-xy & (a+x)^{-1}x-xyx\\ y & yx\end{pmatrix},\end{aligned}$$

where $y=\sum\limits_{i=0}^{\infty}(-1)^i(a+x)^{-(i+2)}x^i$. □

Lemma 2.6. *Let $\mathcal{A}$ be a Banach algebra and $M=\begin{pmatrix} a & b\\ 1 & 0\end{pmatrix}$ with $a\in\mathcal{A}, b\in\mathcal{A}^{-1}$. Then $M\in M_2(\mathcal{A})^{-1}$ and*

$$M^{-1}=\begin{pmatrix} 0 & 1\\ b^{-1} & -b^{-1}a\end{pmatrix}.$$

Proof. Straightforward. □

Let $p^2=p\in\mathcal{A}$ and let $\mathcal{A}_1=p\mathcal{A}p, \mathcal{A}_2=p^{\pi}\mathcal{A}p^{\pi}$. Let T be the ring of Morita context $(M_2(\mathcal{A}_1), M_2(\mathcal{A}_2),\varphi,\psi)$, i.e.,

$$T=\begin{pmatrix} M_2(\mathcal{A}_1) & M_2\big(p\mathcal{A}p^{\pi}\big)\\ M_2\big(p^{\pi}\mathcal{A}p\big) & M_2(\mathcal{A}_2)\end{pmatrix}_{(\varphi,\psi)}$$

with the bimodule homomorphisms of the form

$$\varphi : M_2\Big(p\mathcal{A}p^\pi\Big) \times M_2\Big(p^\pi\mathcal{A}p\Big) \to M_2(\mathcal{A}_1),$$
$$\psi : M_2\Big(p^\pi\mathcal{A}p\Big) \times M_2\Big(p\mathcal{A}p^\pi\Big) \to M_2(\mathcal{A}_2).$$

Then we have a natural isomorphism of rings given by

$$\rho : M_2(\mathcal{A}) \cong T,$$
$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto \begin{pmatrix} \begin{pmatrix} pa_{11}p & pa_{12}p \\ pa_{21}p & pa_{22}p \end{pmatrix} & \begin{pmatrix} pa_{11}p^\pi & pa_{12}p^\pi \\ pa_{21}p^\pi & pa_{22}p^\pi \end{pmatrix} \\ \begin{pmatrix} p^\pi a_{11}p & p^\pi a_{12}p \\ p^\pi a_{21}p & p^\pi a_{22}p \end{pmatrix} & \begin{pmatrix} p^\pi a_{11}p^\pi & p^\pi a_{12}p^\pi \\ p^\pi a_{21}p^\pi & p^\pi a_{22}p^\pi \end{pmatrix} \end{pmatrix}_{(\varphi,\psi)}.$$

Lemma 2.7. *Let $\mathcal{A}$ be a Banach algebra and $M = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}$ with $a \in \mathcal{A}^{-1}, b \in \mathcal{A}^d$. If $ab = ba$, then $M \in M_2(\mathcal{A})^d$ and*

$$M^d = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}$$

with z_{ij} formulated by

$$\begin{aligned} z_{11} &= (ab^\pi + x)^{-1}b^\pi - xy, \\ z_{12} &= bb^d + (ab^\pi + x)^{-1}b^\pi x - xyx, \\ z_{21} &= b^d + y, \\ z_{22} &= -ab^d + yx, \end{aligned}$$

where

$$\begin{aligned} x &= \sum_{i=0}^{\infty}(-1)^i \tfrac{(2i)!}{i!(i+1)!}a^{-(2i+1)}b^{i+1}b^\pi, \\ y &= \sum_{i=0}^{\infty}(-1)^i(ab^\pi + x)^{-(i+2)}b^\pi x^i. \end{aligned}$$

Proof. Let $p = bb^d$. Since $ab = ba$, we have

$$a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}_p, b = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}_p \in \mathcal{A}.$$

We note that every Pierce matrix relative to p is an element in $\mathcal{A}$. Then

$$M = \begin{pmatrix} \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} & \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} \\ \begin{pmatrix} p & 0 \\ 0 & p^\pi \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \in M_2(\mathcal{A}).$$

Set $\mathcal{A}_1 = p\mathcal{A}p$ and $\mathcal{A}_2 = p^\pi\mathcal{A}p^\pi$. Hence, we have

$$\rho(M) = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}_{(\varphi,\psi)},$$

where

$$M_1 = \begin{pmatrix} a_1 & b_1 \\ p & 0 \end{pmatrix} \in M_2(\mathcal{A}_1), M_2 = \begin{pmatrix} a_2 & b_2 \\ p^\pi & 0 \end{pmatrix} \in M_2(\mathcal{A}_2).$$

Claim 1. $M_1 \in M_2(\mathcal{A}_1)^d$. Clearly, $a_1 = abb^d, b_1 = b^2b^d \in \mathcal{A}_1^{-1}$. By Lemma 2.6, we have

$$M_1^d = M_1^{-1} = \begin{pmatrix} 0 & bb^d \\ b_1^{-1} & -b_1^{-1}a_1 \end{pmatrix}.$$

Claim 2. $M_2 \in M_2(\mathcal{A}_2)^d$. Clearly, $a_2 = ab^\pi \in \mathcal{A}_2^{-1}, b_2 = bb^\pi \in \mathcal{A}_2^{qnil}$. By virtue of Lemma 2.5, we have

$$M_2^d = \begin{pmatrix} (a_2+x)^{-1}b^\pi - xy & (a_2+x)^{-1}b^\pi x - xyx \\ y & yx \end{pmatrix},$$

where $x = \sum\limits_{i=0}^{\infty}(-1)^i \frac{(2i)!}{i!(i+1)!} a_2^{-(2i+1)} b_2^{i+1}, y = \sum\limits_{i=0}^{\infty}(-1)^i (a_2+x)^{-(i+2)} b^\pi x^i$. Therefore $\rho(M) \in T^d$ and

$$[\rho(M)]^d = \begin{pmatrix} \begin{pmatrix} 0 & bb^d \\ b_1^{-1} & -b_1^{-1}a_1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} (a_2+x)^{-1}b^\pi - xy & (a_2+x)^{-1}b^\pi x - xyx \\ y & yx \end{pmatrix} \end{pmatrix}_{(\varphi,\psi)}.$$

Thus, $M \in M_2(\mathcal{A})^d$. Obviously, we have $b_1^{-1} = (b^2b^d)^{-1} = b^d \in \mathcal{A}_1$ and $b_1a_1 = b^2b^d abb^d = ab^d$. By virtue of [23, Lemma 3.3], we have

$$\begin{aligned} M^d &= \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & (a_2+x)^{-1}b^\pi - xy \end{pmatrix} & \begin{pmatrix} bb^d & 0 \\ 0 & (a_2+x)^{-1}b^\pi x - xyx \end{pmatrix} \\ \begin{pmatrix} b_1^{-1} & 0 \\ 0 & y \end{pmatrix} & \begin{pmatrix} -b_1^{-1}a_1 & 0 \\ 0 & yx \end{pmatrix} \end{pmatrix} \\ &= \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \end{aligned}$$

with z_{ij} are formulated by

$$\begin{aligned} z_{11} &= (ab^\pi + x)^{-1}b^\pi - xy, \\ z_{12} &= bb^d + (ab^\pi + x)^{-1}b^\pi x - xyx, \\ z_{21} &= b^d + y, \\ z_{22} &= -ab^d + yx, \end{aligned}$$

where

$$\begin{aligned} x &= \sum_{i=0}^{\infty}(-1)^i \frac{(2i)!}{i!(i+1)!} a^{-(2i+1)} b^{i+1} b^\pi, \\ y &= \sum_{i=0}^{\infty}(-1)^i (ab^\pi + x)^{-(i+2)} b^\pi x^i. \end{aligned}$$

This completes the proof. □

Lemma 2.8. *Let $\mathcal{A}$ be a Banach algebra and $M = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}$ with $a \in \mathcal{A}^{qnil}, b \in \mathcal{A}^d$. If $ab = ba$, then $M \in M_2(\mathcal{A})^d$ and*

$$M^d = \begin{pmatrix} 0 & bb^d \\ b^d & -ab^d \end{pmatrix}.$$

Proof. Let $X = \begin{pmatrix} 0 & bb^d \\ b^d & -ab^d \end{pmatrix}$. One directly verifies that

$$\begin{aligned} MX &= \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & bb^d \\ b^d & -ab^d \end{pmatrix} = \begin{pmatrix} bb^d & 0 \\ 0 & bb^d \end{pmatrix} \\ &= \begin{pmatrix} 0 & bb^d \\ b^d & -ab^d \end{pmatrix}\begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} = XM, \\ MX^2 &= (MX)X = X, \\ M - (MX)M &= \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} bb^d & 0 \\ 0 & bb^d \end{pmatrix}\begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} a(1-bb^d) & b - b^2b^d \\ 1 - bb^d & 0 \end{pmatrix} \in M_2(\mathcal{A})^{qnil}. \end{aligned}$$

Therefore M has g-Drazin inverse and $M^d = X$, as desired. □

3. Main Results

We now present the main results of this paper, which extend [18, Theorem 3.8] and [20, Theorem 4.1] to anti-triangular matrices in Banach algebras.

Theorem 3.1. *Let $\mathcal{A}$ be a Banach algebra and $M = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}$ with $a, b \in \mathcal{A}^d$. If $ab = ba$, then $M \in M_2(\mathcal{A})^d$ and*

$$M^d = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

with $\alpha, \beta, \gamma, \delta$ formulated by

$$\begin{array}{rcl} \alpha & = & (a^2a^db^\pi + x)^{-1}_{aa^d} - xy, \\ \beta & = & aa^dbb^d + (a^2a^db^\pi + x)^{-1}_{aa^d}x - xyx + a^\pi bb^d, \\ \gamma & = & aa^db^d + y + a^\pi b^d, \\ \delta & = & -a^2a^db^d + yx - aa^\pi b^d. \end{array}$$

where

$$\begin{array}{rcl} x & = & \sum\limits_{i=0}^{\infty}(-1)^i\frac{(2i)!}{i!(i+1)!}(a^d)^{2i+1}b^{i+1}b^\pi, \\ y & = & \sum\limits_{i=0}^{\infty}(-1)^i[(a^2a^db^\pi + x)^{-1}_{aa^d}]^{i+2}x^i. \end{array}$$

Proof. Let $q = aa^d$. Since $ab = ba$, we have

$$a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}_q, b = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}_q.$$

Then

$$M = \begin{pmatrix} \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} & \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} \\ \begin{pmatrix} q & 0 \\ 0 & q^\pi \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \in M_2(\mathcal{A}).$$

Set $\mathcal{A}_1 = q\mathcal{A}q$ and $\mathcal{A}_2 = q^\pi\mathcal{A}q^\pi$. By using the isomorphism ρ between the matrix ring $M_2(\mathcal{A})$ and the the ring of Morita context $(\mathcal{A}_1, \mathcal{A}_2, \varphi, \psi)$ mentioned above, we have

$$\rho(M) = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}_{(\varphi,\psi)},$$

where

$$M_1 = \begin{pmatrix} a_1 & b_1 \\ q & 0 \end{pmatrix} \in M_2(\mathcal{A}_1), M_2 = \begin{pmatrix} a_2 & b_2 \\ q^\pi & 0 \end{pmatrix} \in M_2(\mathcal{A}_2).$$

Claim 1. $M_1 \in M_2(\mathcal{A}_1)^d$. Obviously, $a_1 \in \mathcal{A}_1^{-1}, b_1 \in \mathcal{A}_1^d$. In view of Lemma 2.7, we have

$$M_1^d = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}$$

with z_{ij} are formulated by

$$\begin{array}{rcl} z_{11} & = & (a^2a^db^\pi + x)^{-1}_{aa^d} - xy, \\ z_{12} & = & aa^dbb^d + (a^2a^db^\pi + x)^{-1}_{aa^d}x - xyx, \\ z_{21} & = & aa^db^d + y, \\ z_{22} & = & -a^2a^db^d + yx, \end{array}$$

where

$$\begin{array}{rcl} x & = & \sum\limits_{i=0}^{\infty}(-1)^i\frac{(2i)!}{i!(i+1)!}(a^d)^{2i+1}b^{i+1}b^{\pi}, \\ y & = & \sum\limits_{i=0}^{\infty}(-1)^i[(a^2a^db^{\pi}+x)^{-1}_{aa^d}]^{i+2}x^i. \end{array}$$

Claim 2. $M_2 \in M_2(\mathcal{A}_2)^d$. Obviously, $a_2 \in \mathcal{A}_2^{qnil}, b_2 \in \mathcal{A}_2^d$. By virtue of Lemma 2.8, we derive that

$$M_2^d = \begin{pmatrix} 0 & b_2b_2^d \\ b_2^d & -a_2b_2^d \end{pmatrix}.$$

Therefore $\rho(M) \in T^d$ and

$$[\rho(M)]^d = \begin{pmatrix} M_1^d & 0 \\ 0 & M_2^d \end{pmatrix}_{(\varphi,\psi)}.$$

Therefore

$$M^d = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M_2(\mathcal{A}),$$

where

$$\begin{array}{rcl} \alpha & = & \begin{pmatrix} z_{11} & 0 \\ 0 & 0 \end{pmatrix}_p = z_{11}, \\ \beta & = & \begin{pmatrix} z_{12} & 0 \\ 0 & b_2b_2^d \end{pmatrix}_p = z_{12} + a^{\pi}bb^d, \\ \gamma & = & \begin{pmatrix} z_{21} & 0 \\ 0 & b_2^d \end{pmatrix}_p = z_{21} + a^{\pi}b^d, \\ \delta & = & \begin{pmatrix} z_{22} & 0 \\ 0 & -a_2b_2^d \end{pmatrix}_p = z_{22} - aa^{\pi}b^d. \end{array}$$

This completes the proof. □

Corollary 3.2. *Let $\mathcal{A}$ be a Banach algebra and $M = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}$ with $a, b \in \mathcal{A}^D$. If $ab = ba$, then $M \in M_2(\mathcal{A})^D$ and*

$$M^D = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

with $\alpha, \beta, \gamma, \delta$ are formulated by

$$\begin{array}{rcl} \alpha & = & (a^2a^Db^{\pi}+x)^{-1}_{aa^D} - xy, \\ \beta & = & aa^Dbb^D + (a^2a^Db^{\pi}+x)^{-1}_{aa^D}x - xyx + a^{\pi}bb^D, \\ \gamma & = & aa^Db^D + y + a^{\pi}b^D, \\ \delta & = & -a^2a^Db^D + yx - aa^{\pi}b^D. \end{array}$$

where

$$\begin{array}{rcl} x & = & \sum\limits_{i=0}^{ind(b)-1}(-1)^i\frac{(2i)!}{i!(i+1)!}(a^D)^{2i+1}b^{i+1}b^{\pi}, \\ y & = & \sum\limits_{i=0}^{ind(b)-1}(-1)^i[(a^2a^Db^{\pi}+x)^{-1}_{aa^D}]^{i+2}x^i. \end{array}$$

Proof. Evidently, $z \in \mathcal{A}^D$ if and only if $z \in \mathcal{A}^d$ and $a - a^2a^d \in \mathcal{A}$ is nilpotent. In this case, $z^D = z^d$. Therefore we complete the proof by Theorem 3.1. □

Since every complex matrix can be viewed as a matrix within the Banach algebra $\mathbb{C}^{n\times n}$ comprising all $n \times n$ matrices, Corollary 3.2 introduces a new formula for the Drazin inverse of an anti-triangular complex matrix, utilizing Catalan numbers as a pivotal tool. This formula presents a fresh method for tackling related difference equation problems concerning matrix structures. We give a numerical example to illustrate Corollary 3.2.

Example 3.3.

Let $M = \begin{pmatrix} A & B \\ I_3 & 0 \end{pmatrix} \in \mathbb{C}^{6\times 6}$, where

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then we have

$$\begin{array}{rcl} A^D & = & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, B^D = 0, \\ A^\pi & = & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, B^\pi = I_3. \end{array}$$

Since $AB = BA = 0$, we have

$$\begin{array}{rcl} X & = & \sum\limits_{i=0}^{ind(B)-1} (-1)^i \frac{(2i)!}{i!(i+1)!} (A^D)^{2i+1} B^{i+1} B^\pi = 0, \\ Y & = & \sum\limits_{i=0}^{ind(B)-1} (-1)^i [(A^2 A^D B^\pi + X)^{-1}_{AA^D}]^{i+2} X^i = A^D. \end{array}$$

Hence,

$$\begin{array}{rcl} \Lambda & = & (A^2 A^D B^\pi + X)^{-1}_{AA^D} - XY = A^D, \\ \Xi & = & AA^D BB^D + (A^2 A^D B^\pi + X)^{-1}_{AA^D} X - XYX + A^\pi BB^D = 0, \\ \Gamma & = & AA^D B^D + Y + A^\pi B^D = A^D, \\ \Delta & = & -A^2 A^D B^D + YX - AA^\pi B^D = 0. \end{array}$$

By virtue of Corollary 3.2, we have

$$M^D = \begin{pmatrix} \Lambda & \Xi \\ \Gamma & \Delta \end{pmatrix} = \begin{pmatrix} A^D & 0 \\ A^D & 0 \end{pmatrix}.$$

We are now ready to prove:

Theorem 3.4. *Let $\mathcal{A}$ be a Banach algebra and $M = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}$ with $a, b, b^\pi a \in \mathcal{A}^d$. If $b^\pi ab^d = 0$ and $b^\pi(ab) = b^\pi(ba)$, then $M \in M_2(\mathcal{A})^d$ and*

$$M^d = \sum_{i=0}^{\infty} P^i [I - PP^d](Q^d)^{i+1},$$

where

$$\begin{array}{rcl} P & = & \begin{pmatrix} b^\pi a & b^\pi b \\ b^\pi & 0 \end{pmatrix}, P^d = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \\ Q & = & \begin{pmatrix} bb^d a & b^2 b^d \\ bb^d & 0 \end{pmatrix}, Q^d = \begin{pmatrix} 0 & bb^d \\ b^d & -b^d a \end{pmatrix} \end{array}$$

with $\alpha, \beta, \gamma, \delta$ formulated by

$$\begin{array}{rcl} \alpha & = & (b^\pi a^2 a^d + x)^{-1}_{b^\pi a a^d} - xy, \\ \beta & = & (b^\pi a^2 a^d + x)^{-1}_{b^\pi a a^d} x - xyx, \\ \gamma & = & y, \\ \delta & = & yx. \end{array}$$

where

$$\begin{array}{rcl} x & = & \sum\limits_{i=0}^{\infty} (-1)^i \frac{(2i)!}{i!(i+1)!} (b^\pi a^d)^{2i+1} b^{i+1} b^\pi, \\ y & = & \sum\limits_{i=0}^{\infty} (-1)^i [(b^\pi a^2 a^d + x)^{-1}_{b^\pi a a^d}]^{i+2} x^i. \end{array}$$

Proof. Let $M = P + Q$, where

$$P = \begin{pmatrix} b^\pi a & b^\pi b \\ b^\pi & 0 \end{pmatrix}, Q = \begin{pmatrix} bb^d a & b^2 b^d \\ bb^d & 0 \end{pmatrix}.$$

Step 1. P has g-Drazin inverse. By hypothesis, we verify that

$$(b^\pi a)(b^\pi b) = b^\pi (ab) = b^\pi (ba) = (b^\pi b)(b^\pi a).$$

In light of Theorem 3.1, we have

$$P^d = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

with $\alpha, \beta, \gamma, \delta$ are formulated by

$$\begin{array}{rcl} \alpha & = & (b^\pi a^2 a^d + x)^{-1}_{b^\pi a a^d} - xy, \\ \beta & = & (b^\pi a^2 a^d + x)^{-1}_{b^\pi a a^d} x - xyx, \\ \gamma & = & y, \\ \delta & = & yx. \end{array}$$

where

$$\begin{array}{rcl} x & = & \sum\limits_{i=0}^{\infty} (-1)^i \frac{(2i)!}{i!(i+1)!} (b^\pi a^d)^{2i+1} b^{i+1} b^\pi, \\ y & = & \sum\limits_{i=0}^{\infty} (-1)^i [(b^\pi a^2 a^d + x)^{-1}_{b^\pi a a^d}]^{i+2} x^i. \end{array}$$

Step 2. Q has g-Drazin inverse. By virtue of Lemma 2.6,

$$Q^d = \begin{pmatrix} 0 & bb^d \\ b^d & -b^d a \end{pmatrix}.$$

Step 3. Since $PQ = 0$, it follows by Lemma 2.1 that

$$\begin{array}{rcl} M^d & = & (P+Q)^d \\ & = & \sum\limits_{i=0}^{\infty} (P^d)^{i+1} Q^i Q^\pi + \sum\limits_{i=0}^{\infty} P^i P^\pi (Q^d)^{i+1} \\ & = & \sum\limits_{i=0}^{\infty} P^i P^\pi (Q^d)^{i+1}. \end{array}$$

This completes the proof. □

Corollary 3.5. *Let $\mathcal{A}$ be a Banach algebra and $M = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}$ with $a, b, b^\pi a \in \mathcal{A}^D$. If $b^\pi ab^D = 0$ and $b^\pi(ab) = b^\pi(ba)$, then $M \in M_2(\mathcal{A})^D$ and*

$$M^D = \sum_{i=0}^{ind(P)} P^i [I - PP^D](Q^D)^{i+1},$$

where

$$\begin{array}{rcl} P & = & \begin{pmatrix} b^{\pi}a & b^{\pi}b \\ b^{\pi} & 0 \end{pmatrix}, P^D = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \\ Q & = & \begin{pmatrix} bb^Da & b^2b^D \\ bb^D & 0 \end{pmatrix}, Q^D = \begin{pmatrix} 0 & bb^D \\ b^D & -b^Da \end{pmatrix} \end{array}$$

with $\alpha, \beta, \gamma, \delta$ *formulated by*

$$\begin{array}{rcl} \alpha & = & (b^{\pi}a^2a^D + x)^{-1}_{b^{\pi}aa^D} - xy, \\ \beta & = & (b^{\pi}a^2a^D + x)^{-1}_{b^{\pi}aa^D}x - xyx, \\ \gamma & = & y, \\ \delta & = & yx. \end{array}$$

where

$$\begin{array}{rcl} x & = & \sum\limits_{i=0}^{\infty}(-1)^i\frac{(2i)!}{i!(i+1)!}(b^{\pi}a^D)^{2i+1}b^{i+1}b^{\pi}, \\ y & = & \sum\limits_{i=0}^{\infty}(-1)^i[(b^{\pi}a^2a^D + x)^{-1}_{b^{\pi}aa^D}]^{i+2}x^i. \end{array}$$

Proof. This is the specific information obtained from Theorem 3.4. □

It is convenient at this stage to derive the following:

Theorem 3.6. *Let $\mathcal{A}$ be a Banach algebra and* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *with* $a, d, bc \in \mathcal{A}^d$. *If* $abc = bca, bdc = 0$ *and* $bd^2 = 0$, *then* $M \in M_2(\mathcal{A})^d$ *and*

$$\begin{array}{rcl} M^d & = & \sum\limits_{i=0}^{\infty}(Q^d)^{i+1}P^i(I - PP^d) + \sum\limits_{i=0}^{\infty}Q^iQ^{\pi}(P^d)^{i+1} + \sum\limits_{i=0}^{\infty}Q^i(I - QQ^d)(P^d)^{i+2}Q \\ & + & \sum\limits_{i=0}^{\infty}(Q^d)^{i+3}P^{i+1}(I - PP^d)Q - Q^dP^dQ - (Q^d)^2PP^dQ, \end{array}$$

where

$$\begin{array}{rcl} P & = & \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}, P^d = \begin{pmatrix} \alpha^2a + \alpha\beta + \beta\gamma a + \beta\delta & \alpha^2b + \beta\gamma b \\ c\gamma\alpha a + c\gamma\beta + c\delta\gamma a + c\delta^2 & c\gamma\alpha b + c\delta\gamma b \end{pmatrix}; \\ Q & = & \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}, Q^d = \begin{pmatrix} 0 & 0 \\ 0 & d^d \end{pmatrix}. \end{array}$$

with $\alpha, \beta, \gamma, \delta$ *formulated by*

$$\begin{array}{rcl} \alpha & = & (a^2a^d(bc)^{\pi} + x)^{-1}_{aa^d} - xy, \\ \beta & = & aa^d(bc)(bc)^d + (a^2a^d(bc)^{\pi} + x)^{-1}_{aa^d}x - xyx + a^{\pi}(bc)(bc)^d, \\ \gamma & = & aa^d(bc)^d + y + a^{\pi}(bc)^d, \\ \delta & = & -a^2a^d(bc)^d + yx - aa^{\pi}(bc)^d. \end{array}$$

where

$$\begin{array}{rcl} x & = & \sum\limits_{i=0}^{\infty}(-1)^i\frac{(2i)!}{i!(i+1)!}(a^d)^{2i+1}(bc)^{i+1}(bc)^{\pi}, \\ y & = & \sum\limits_{i=0}^{\infty}(-1)^i[(a^2a^d(bc)^{\pi} + x)^{-1}_{aa^d}]^{i+2}x^i. \end{array}$$

Proof. Let $P = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}$. In view of Theorem 3.1, we have

$$\begin{pmatrix} a & bc \\ 1 & 0 \end{pmatrix}^d = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

Here, $\alpha, \beta, \gamma, \delta$ are formulated by

$$\begin{array}{rcl}\alpha & = & (a^2a^d(bc)^{\pi}+x)^{-1}_{aa^d}-xy,\\ \beta & = & aa^d(bc)(bc)^d+(a^2a^d(bc)^{\pi}+x)^{-1}_{aa^d}x-xyx+a^{\pi}(bc)(bc)^d,\\ \gamma & = & aa^d(bc)^d+y+a^{\pi}(bc)^d,\\ \delta & = & -a^2a^d(bc)^d+yx-aa^{\pi}(bc)^d.\end{array}$$

where

$$\begin{array}{rcl}x & = & \sum\limits_{i=0}^{\infty}(-1)^i\frac{(2i)!}{i!(i+1)!}(a^d)^{2i+1}(bc)^{i+1}(bc)^{\pi},\\ y & = & \sum\limits_{i=0}^{\infty}(-1)^i[(a^2a^d(bc)^{\pi}+x)^{-1}_{aa^d}]^{i+2}x^i.\end{array}$$

One easily verifies that

$$\begin{array}{rcl}\begin{pmatrix} a & bc\\ 1 & 0\end{pmatrix} & = & \begin{pmatrix} a & b\\ 1 & 0\end{pmatrix}\begin{pmatrix} 1 & 0\\ 0 & c\end{pmatrix},\\ \begin{pmatrix} a & b\\ c & 0\end{pmatrix} & = & \begin{pmatrix} 1 & 0\\ 0 & c\end{pmatrix}\begin{pmatrix} a & b\\ 1 & 0\end{pmatrix}.\end{array}$$

By using Cline's formula (see [16, Theorem 2.2]), P has g-Drazin inverse and

$$\begin{array}{rcl}P^d & = & \begin{pmatrix} 1 & 0\\ 0 & c\end{pmatrix}\Big[\begin{pmatrix} a & bc\\ 1 & 0\end{pmatrix}^d\Big]^2\begin{pmatrix} a & b\\ 1 & 0\end{pmatrix}\\ & = & \begin{pmatrix} 1 & 0\\ 0 & c\end{pmatrix}\begin{pmatrix} \alpha & \beta\\ \gamma & \delta\end{pmatrix}^2\begin{pmatrix} a & b\\ 1 & 0\end{pmatrix}\\ & = & \begin{pmatrix} \alpha & \beta\\ c\gamma & c\delta\end{pmatrix}\begin{pmatrix} \alpha a+\beta & \alpha b\\ \gamma a+\delta & \gamma b\end{pmatrix}\\ & = & \begin{pmatrix} \alpha^2a+\alpha\beta+\beta\gamma a+\beta\delta & \alpha^2b+\beta\gamma b\\ c\gamma\alpha a+c\gamma\beta+c\delta\gamma a+c\delta^2 & c\gamma\alpha b+c\delta\gamma b\end{pmatrix}.\end{array}$$

Obviously, we have

$$Q^d=\begin{pmatrix} 0 & 0\\ 0 & d^d\end{pmatrix}, Q^{\pi}=\begin{pmatrix} 1 & 0\\ 0 & d^{\pi}\end{pmatrix}.$$

One easily checks that

$$\begin{array}{rcl}PQ^2 & = & \begin{pmatrix} a & b\\ c & 0\end{pmatrix}\begin{pmatrix} 0 & 0\\ 0 & d^2\end{pmatrix}=\begin{pmatrix} 0 & bd^2\\ 0 & 0\end{pmatrix}=0,\\ PQP & = & \begin{pmatrix} a & b\\ c & 0\end{pmatrix}\begin{pmatrix} 0 & 0\\ 0 & d\end{pmatrix}\begin{pmatrix} a & b\\ c & 0\end{pmatrix}\\ & = & \begin{pmatrix} bdc & 0\\ 0 & 0\end{pmatrix}=0.\end{array}$$

According to Lemma 2.2, we derive that

$$\begin{array}{rcl}M^d & = & (P+Q)^d\\ & = & \sum\limits_{i=0}^{\infty}(Q^d)^{i+1}P^iP^{\pi}+\sum\limits_{i=0}^{\infty}Q^iQ^{\pi}(P^d)^{i+1}+\sum\limits_{i=0}^{\infty}Q^iQ^{\pi}(P^d)^{i+2}Q\\ & + & \sum\limits_{i=0}^{\infty}(Q^d)^{i+3}P^{i+1}P^{\pi}Q-Q^dP^dQ-(Q^d)^2PP^dQ,\end{array}$$

as asserted. □

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