Filomat 39:26 (2025), 9113–9121
https://doi.org/10.2298/FIL2526113M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Multiple solutions for a class of Robin problem involving the $p(\cdot)$-Laplacian-like

Noureddine Moujane[a,*], Mohamed El Ouaarabi[b], Said Melliani[a]

[a] *Laboratory LMACS, Faculty of Science and Technics, Sultan Moulay Slimane University, Beni Mellal, BP 523, 23000, Morocco*
[b] *Mathematical Analysis, Algebra and Applications Laboratory, Faculty of Sciences Aïn Chock, Hassan II University, BP 5366, 20100, Casablanca, Morocco*

Abstract. In this paper, we present some results concerning the existence and multiplicity of solutions for a class of $p(x)$-Laplacian-like problems with Robin boundary conditions. Specifically, we establish the existence of three solutions in the generalized Sobolev space.

1. Introduction

The purpose of this paper is to establish the existence of at least three weak solutions to the following quasilinear elliptic problem with Robin boundary condition

$$\begin{cases} -\operatorname{div}\left(|\nabla\varphi|^{p(x)-2}\nabla\varphi + \dfrac{|\nabla\varphi|^{2p(x)-2}\nabla\varphi}{\sqrt{1+|\nabla\varphi|^{2p(x)}}}\right) = \theta(x)|\varphi|^{p(x)-2}\varphi & \text{in } Q, \\ \left(|\nabla\varphi|^{p(x)-2} + \dfrac{|\nabla\varphi|^{2p(x)-2}}{\sqrt{1+|\nabla\varphi|^{2p(x)}}}\right)\nu(x) - \lambda h(x,\varphi) = \mu f(x,\varphi) & \text{in } \partial Q, \end{cases} \tag{1.1}$$

where Q is a bounded smooth domain in $\mathbb{R}^N (N \geq 2)$, $\nu(x)$ is the outer unit normal derivative on ∂Q, λ, $\mu > 0$, $p \in C_+(\overline{Q})$ satisfy log-Hölder continuity condition, and θ is a potential function in $L^\infty(Q)$ with $\theta^- = \inf_{x\in Q}\theta(x) > 0$. On the boundary of the problem (1.1), we have the competing effects of two parametric terms $\lambda h(x,\varphi)$ and $\mu f(x,\varphi)$, with $h(\cdot,\cdot)$ and $f(\cdot,\cdot)$ being Carathéodory functions.

This research was motivated by the application of analogous problems in the field of physics, such as the modeling of continuum mechanics, elastic mechanics [37], image restoration [10] and electrorheological fluids [34, 5] and polycrystal plasticity[6]. These challenges are also of great interest from a purely mathematical point of view. For in-depth knowledge and the latest findings, we recommend that readers refer to [3, 4, 1, 31, 8, 11, 23, 6].

2020 *Mathematics Subject Classification*. Primary 46E35; Secondary 35J92, 35J30.
Keywords. Robin condition, $p(\cdot)$-Laplacian-Like, variable exponent, Ricceri's variational methods.
Received: 24 March 2024; Revised: 05 August 2025; Accepted: 06 August 2025
Communicated by Maria Alessandra Ragusa
* Corresponding author: Noureddine Moujane
Email addresses: moujanenoureddine95@gmail.com (Noureddine Moujane), mohamedelouaarabi93@gmail.com (Mohamed El Ouaarabi), s.melliani@usms.ma (Said Melliani)
ORCID iDs: https://orcid.org/0009-0004-9623-1256 (Noureddine Moujane), https://orcid.org/0000-0001-5184-9889 (Mohamed El Ouaarabi), https://orcid.org/0000-0002-5150-1185 (Said Melliani)

However, in comprehending the significance of the variable exponent, although most materials can be precisely represented using classical Lebesgue and Sobolev spaces where p is a constant, there are certain nonhomogeneous materials, e.g. the electrorheological fluids mentioned before, where this approach proves insufficient. These materials are distinguished by their capacity to undergo significant alterations in mechanical properties when influenced by an external electromagnetic field. Thereby necessitating the utilization of spaces with variable exponents, see [34, 22, 21, 25, 26, 27].

In the realm of investigating nonlinear elliptic problems, Ricceri [32] introduced a novel variational principle that has found widespread application in addressing various nonlinear eigenvalue problems [33, 19, 12].

The investigation of Problem (1.1) has been a subject of research for numerous authors over the past few decades.[14, 24, 30, 20, 9, 35]. For example, Omari and Obersnel in [30], we explore the multiplicity of positive solutions of the parametric equations

$$\begin{cases} -\mathrm{div}\Big(\dfrac{\nabla\varphi}{\sqrt{1+|\nabla\varphi|^{2p(x)}}}\Big) = \lambda f(x,\varphi) & \text{in } Q, \\ \varphi = 0 & \text{in } \partial Q, \end{cases} \tag{1.2}$$

where $\lambda > 0$ and $f : Q \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function with a potential exhibiting an appropriate oscillatory behavior at zero. Moreover, Shao-Gao Deng in [13] studied the $p(x)$-Laplacian Robin problem related to (1.1) when $\theta(x) = 1$, $\mu = 0$ and $h(x,\varphi) = |\varphi|^{p(x)-2}\varphi$, obtained the existence of an infinite number of sequences of eigenvalues.

This paper aims to build upon prior studies by demonstrating the existence of a minimum of three weak solutions for the problem (1.1), by employing the three critical points theorem from B. Ricceri [32, Theorem A] and the theory of Sobolev spaces with variable exponent.

The structure of the rest of the article is as follows. In the section (2), we review some fundamental preliminaries on variable exponent Sobolev spaces $W^{1,p(x)}(Q)$. Next, in the section (3), we give our basic assumptions and prove the main result of this work.

2. Preliminaries

In this section, we explore essential characteristics of Sobolev spaces with variable exponents. For further foundational notations on this topic, readers are directed to [18, 22, 17, 15, 16, 28, 29].
In the sequel, let Q be a bounded open subset of $\mathbb{R}^N$ with a smooth boundary ∂Q and $p \in C_+(\overline{Q})$ where

$$C_+(\overline{Q}) = \Big\{p : p \in C(\overline{Q}) \, , \, p(x) > 1 \text{ for every } x \in \overline{Q}\Big\}.$$

We establish by $p^- = \inf_{x\in Q} p(x)$ and $p^+ = \sup_{x\in Q} p(x)$.
The variable exponent Lebesgue space $L^{p(x)}(Q)$ is defined by

$$L^{p(x)}(Q) = \left\{\varphi \middle| \varphi : Q \to \mathbb{R} \text{ is measurable and } \int_Q |\varphi(x)|^{p(x)} dx < \infty\right\}.$$

Endowed with the Luxemburg norm

$$\|\varphi\|_{p(x)} = \inf\left\{\kappa > 0 : \; \rho_{p(x)}\left(\frac{\varphi}{\kappa}\right) \le 1\right\},$$

where

$$\rho_{p(x)}(\varphi) = \int_Q |\varphi(x)|^{p(x)} dx.$$

Proposition 2.1. *[22, 2] The space $(L^{p(x)}(Q), \|\cdot\|_{p(x)})$ is a separable and reflexive Banach space.*

Additionally, the space $L^{p(x)}(\partial Q)$ can be defined by

$$L^{p(x)}(\partial Q) = \left\{\varphi \middle| \varphi : \partial Q \to \mathbb{R} \text{ measurable and } \int_{\partial Q} |\varphi(x)|^{p(x)} d\sigma < +\infty \right\},$$

equipped with

$$\|\varphi\|_{p(x),\partial Q} = \inf\left\{\kappa > 0 : \int_{\partial Q} \left|\frac{\varphi(x)}{\kappa}\right|^{p(x)} d\sigma \leq 1\right\},$$

then $(L^{p(x)}(\partial Q)\ , \|.\|_{p(x),\partial Q})$ is a Banach space.

Proposition 2.2. *[22, Theorem 2.1] The conjugate space of $L^{p(x)}(Q)$ is $L^{p'(x)}(Q)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ for all $x \in Q$. For every $\varphi_0 \in L^{p(x)}(Q)$ and $\varphi_1 \in L^{p'(x)}(Q)$, we present the following Hölder-type inequality*

$$\left|\int_Q \varphi_0\varphi_1\, dx\right| \leq \Big(\frac{1}{p-} + \frac{1}{p'^-}\Big)\|\varphi_0\|_{p(x)}\|\varphi_1\|_{p'(x)} \leq 2\|\varphi_0\|_{p(x)}\|\varphi_1\|_{p'(x)}. \tag{2.1}$$

Now, the generalized Sobolev space $W^{1,p(x)}(Q)$ is defined by

$$W^{1,p(x)}(Q) = \left\{\varphi \in L^{p(x)}(Q) \text{ such that } |\nabla\varphi| \in L^{p(x)}(Q)\right\}.$$

Endowed with the norm

$$\|\varphi\|_{1,p(x)} = \|\varphi\|_{p(x)} + \|\nabla\varphi\|_{p(x)}.$$

The topological dual space of $W^{1,p(x)}(Q)$ denoted $W^{-1,p'(x)}(Q)$.

Proposition 2.3. *[22] The space $\left(W^{1,p(x)}(Q), \|\cdot\|_{1,p(x)}\right)$ is a separable and reflexive Banach space .*

Proposition 2.4. *for $p, \alpha \in C_+(Q)$ such that $\alpha(x) \leq p^*(x)$ for every $x \in \overline{Q}$, there is a continuous and compact embedding $W^{1,p(x)}(Q)$ into $L^{\alpha(x)}(Q)$ where*

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & if\ p(x) < N, \\ +\infty, & if\ p(x) \geq N. \end{cases}$$

Obviously, $p(x) \leq p*(x)$ for all $x \in Q$.

Let $\theta \in L^\infty(Q)$ with $\theta^- = \inf\limits_{x\in Q} \theta(x) > 0$. Therefore, the norm $\|\varphi\|_\theta$ established by

$$\|\varphi\|_\theta = \inf\left\{\kappa > 0 : \int_Q \left(\left|\frac{\nabla\varphi(x)}{\kappa}\right|^{p(x)} + \theta(x)\left|\frac{\varphi(x)}{\kappa}\right|^{p(x)}\right) dx \leq 1\right\},$$

for every $\varphi \in W^{1,p(x)}(Q)$. Moreover, it is readily apparent that $\|.\|_\theta$ and $\|.\|_{1,p(x)}$ are equivalent on $W^{1,p(x)}(Q)$.

Remark 2.5. *As indicated in the document [18], $W^{1,p(x)}(Q)$ is continuously embedded in $W^{1,p^-}(Q)$ and, since $p^- > N, W^{1,p^-}(Q)$ is compactly embedded in $C(\bar{Q})$. Consequently, $W^{1,p(x)}(Q)$ is compactly embedded in $C(\bar{Q})$. Then, there exists a positive constant $\varrho > 0$ such that*

$$\|\varphi\|_\infty \leq \varrho\|\varphi\|_\theta,$$

for every $\varphi \in W^{1,p(x)}(Q)$.

Proposition 2.6. *[7] Let*

$$\mathcal{I}_{\theta}(\varphi) = \int_{Q} \left(|\nabla \varphi(x)|^{p(x)} + \theta(x)|\varphi(x)|^{p(x)} \right) dx.$$

For every $\varphi \in W^{1,p(x)}(Q)$, with $\theta^{-} > 0$, we get

- $\|\varphi\|_{\theta} < 1 \,(= 1, > 1) \Longleftrightarrow \mathcal{I}_{\theta}(\varphi) < 1 \,(= 1, > 1)$;
- $\|\varphi\|_{\theta} \geq 1 \Rightarrow \|\varphi\|_{\theta}^{p^{-}} \leq \mathcal{I}_{\theta}(\varphi) \leq \|\varphi\|_{\theta}^{p^{+}}$;
- $\|\varphi\|_{\theta} \leq 1 \Rightarrow \|\varphi\|_{\theta}^{p^{+}} \leq \mathcal{I}_{\theta}(\varphi) \leq \|\varphi\|_{\theta}^{p^{-}}$;
- $\min\left\{\|\varphi\|_{\theta}^{p^{-}}, \|\varphi\|_{\theta}^{p^{+}}\right\} \leq \mathcal{I}_{\theta}(\varphi) \leq \max\left\{\|\varphi\|_{\theta}^{p^{-}}, \|\varphi\|_{\theta}^{p^{+}}\right\}$.

The proposition that follows can be demonstrated using [20, Proposition 2.2].

Lemma 2.7. *[35] Let us consider the following mapping $\Upsilon_{\theta} : W^{1,p(x)}(Q) \to \mathbb{R}$ defined by*

$$\Upsilon_{\theta}(\varphi) = \int_{Q} \frac{1}{p(x)} \left(|\nabla \varphi|^{p(x)} + \sqrt{1 + |\nabla \varphi|^{2p(x)}} + \theta(x)|\varphi(x)|^{p(x)} \right) dx,$$

for every $\varphi \in W^{1,p(x)}(Q)$.
Then, we obtain $\Upsilon_{\theta} \in C^{1}(W^{1,p(x)}(Q), \mathbb{R})$ and $\Upsilon'_{\theta} : W^{1,p(x)}(Q) \to W^{-1,p'(x)}(Q)$ defined by

$$\begin{aligned} \langle \Upsilon'_{\theta}(\varphi), \zeta \rangle = & \int_{Q} \left(|\nabla \varphi|^{p(x)-2} \nabla \varphi + \frac{|\nabla \varphi|^{2p(x)-2} \nabla \varphi}{\sqrt{1 + |\nabla \varphi|^{2p(x)}}} \right) \nabla \zeta dx \\ & + \int_{Q} \theta(x)|\varphi(x)|^{p(x)-2} \varphi \zeta dx, \end{aligned}$$

for every $\zeta \in W^{1,p(x)}(Q)$. In additionis, $\Upsilon'_{\theta}(\varphi)$ is bounded, continuous, homeomorphism, strictly monotone and is of type (S_{+}) .

Theorem 2.8. *(see [32]) Let A be a separable and reflexive real Banach space; $\Upsilon : A \to \mathbb{R}$ a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous Functional with a Gâteaux derivative having a continuous inverse on A^{*}; $\mathcal{H} : A \to \mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Additionally, consider the assumption:*

(i) $\lim\limits_{\|\varphi\| \to \infty} (\Upsilon(\varphi) + \lambda \mathcal{H}(\varphi)) = \infty$ *for all* $\lambda > 0$,
(ii) there are $r \in \mathbb{R}$ and $\varphi_{0},\ \varphi_{1} \in A$ such that $\Upsilon(\varphi_{0}) < r < \Upsilon(\varphi_{1})$,
(iii)

$$\inf_{\varphi \in \Upsilon^{-1}((-\infty, r])} \mathcal{H}(\varphi) > \frac{(\Upsilon(\varphi_{1}) - r)\mathcal{H}(\varphi_{0}) + (r - \Upsilon(\varphi_{0}))\mathcal{H}(\varphi_{1})}{\Upsilon(\varphi_{1}) - \Upsilon(\varphi_{0})}.$$

Then there exist an open interval $\Lambda \subset (0,\ \infty)$ and a positive constant $\rho > 0$ with the following property: for every $\lambda \in \Lambda$ and every C^{1} functional $\mathcal{F}$ with compact derivative, there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$ the equation

$$\Upsilon'(\varphi) + \lambda \mathcal{H}'(\varphi) + \mu \mathcal{F}'(\varphi) = 0,$$

has at least three solutions in A whose norms are less than ρ.

3. Hypotheses and Main results

In this paper, we assume that $h : \partial Q \times \mathbb{R} \to \mathbb{R}$ and $f : \partial Q \times \mathbb{R} \to \mathbb{R}$ are two Carathéodory functions such that:

(A_1) $|h(x,z)| \leq \mu_1(x) + c_1|z|^{s_1(x)-1}$ for any $(x,z) \in \partial Q \times \mathbb{R}, c_1 > 0$ where $\mu_1(x) \in L^{\frac{s_1(x)}{s_1(x)-1}}(\partial Q)$, $\mu_1(x) \geq 0$ and $s_1(x) \in C_+(\partial Q)$, with $s_2^+ < p^-$ for every $x \in \partial Q$.

(A_2) $h(x,z) < 0$ for all $(x,z) \in \partial Q \times \mathbb{R}$, and $|z| \in (0,1)$.
$h(x,z) \geq R_1 > 0$, when $|z| \in (z_0,\ \infty)$, $z_0 > 1$.

(A_3) $|f(x,z)| \leq \mu_2(x) + c_2|z|^{s_2(x)-1}$ for any $(x,z) \in \partial Q \times \mathbb{R}$, $c_2 > 0$ where $\mu_2(x) \in L^{\frac{s_2(x)}{s_2(x)-1}}(\partial Q)$, $\mu_2(x) \geq 0$ and $s_2(x) \in C_+(\partial Q)$ with $s_2^+ < p^-$ for every $x \in \partial Q$.

Subsequently, we will employ the definition of a weak solution for problem (1.1) in the following way:

Definition 3.1. *We say that $\varphi \in W^{1,p(x)}(Q)$ is a weak solution of problem* (1.1) *if*

$$\begin{aligned}&\int_Q \Big(|\nabla\varphi|^{p(x)-2}\nabla\varphi + \frac{|\nabla\varphi|^{2p(x)-2}\nabla\varphi}{\sqrt{1+|\nabla\varphi|^{2p(x)}}}\Big)\nabla\zeta dx + \int_Q \theta(x)|\varphi|^{p(x)-2}\varphi\zeta dx\\ &= \mu\int_{\partial Q} f(x,\varphi)\zeta d\sigma + \lambda\int_{\partial Q} h(x,\varphi)\zeta d\sigma,\end{aligned}$$

for all $\zeta \in W^{1,p(x)}(Q)$.

Let $\varphi \in W^{1,p(x)}(Q)$. The energy functional related to the problem (1.1) is defined by

$$\mathcal{J}_{\lambda,\mu}(\varphi) = \Upsilon_\theta(\varphi) + \lambda\mathcal{H}(\varphi) + \mu\mathcal{F}(\varphi),$$

where

$$\Upsilon_\theta(\varphi) = \int_Q \frac{1}{p(x)}\Big(|\nabla\varphi|^{p(x)} + \sqrt{1+|\nabla\varphi|^{2p(x)}} + \theta(x)|\varphi(x)|^{p(x)}\Big)dx,$$

and

$$\mathcal{F}(\varphi) = -\int_{\partial Q} F(x,\varphi)d\sigma,$$

$$\mathcal{H}(\varphi) = -\int_{\partial Q} H(x,\varphi)d\sigma,$$

$$F(x,\varphi) = \int_0^\varphi f(x,z)dz,\ H(x,\varphi) = \int_0^\varphi h(x,z)dz.$$

It is clear that $(\Upsilon'_\theta)^{-1} : W^{-1,p'(x)}(Q) \to W^{1,p(x)}(Q)$ exists and continuous, we apply a classical result due to Minty-Browder (see [36, Theorem 26 A]), because $\Upsilon'_\theta : W^{1,p(x)}(Q) \to W^{-1,p'(x)}(Q)$ is a homeomorphism by Lemma (2.7). Furthermore, in view of (A_1) and (A_3), it is established knowledge that $\mathcal{H},\mathcal{F} \in C^1(W^{1,p(x)}(Q),\mathbb{R})$ with the derivative given by

$$\langle \mathcal{H}'(\varphi)\,,\zeta\rangle = -\int_{\partial Q} h(x,\varphi)\zeta d\sigma,$$

for any $\varphi,\ \zeta \in W^{1,p(x)}(\partial Q)$, and $\mathcal{H}' : W^{1,p(x)}(Q) \to W^{-1,p'(x)}(Q)$ are completely continuous by [4, Theorem 2.9]. Therefore, $\mathcal{H}' : W^{1,p(x)}(Q) \to W^{-1,p'(x)}(Q)$ is compact.
We acknowledge that the operator $\mathcal{J}_{\lambda,\mu}$ is a $C^1(W^{1,p(x)}(Q),\mathbb{R})$ functional, and the critical points of $\mathcal{J}_{\lambda,\mu}$ are identified as weak solutions to the problem 1.1.

Presently, we articulate our primary outcome as follows.

Theorem 3.2. *Assume the provided assumptions* $(A_1)-(A_3)$, *then There exist an open interval* $\Lambda \subset (0, \infty)$ *and a positive constant* $\rho > 0$ *such that for any* $\lambda \in \Lambda$, *there exists* $\delta > 0$ *such that, for every* $\mu \in [0, \delta]$ *problem* (1.1) *has at least three solutions in* $W^{1,p(x)}(Q)$ *whose norms are less than* ρ.

Proof. In substantiating our result, we use Theorem (2.8), the precondition of this theorem is fulfilled. In what follows, we need to check that conditions (*i*), (*ii*) and (*iii*) are satisfied. Using Proposition (2.7), we find

$$\begin{aligned}\Upsilon_\theta(\varphi) &= \int_Q \frac{1}{p(x)}\left(|\nabla\varphi|^{p(x)} + \theta(x)|\varphi(x)|^{p(x)}\right)dx + \int_Q \frac{\sqrt{1+|\nabla\varphi|^{2p(x)}}}{p(x)}dx \\ &\geq \frac{1}{p^+}\mathcal{I}_\theta(\varphi) + \int_Q \frac{\sqrt{1+|\nabla\varphi|^{2p(x)}}}{p(x)}dx \\ &\geq \frac{1}{p^+}\|\varphi\|_\theta^{p^-},\end{aligned} \tag{3.1}$$

for every $\varphi \in W^{1,p(x)}(Q)$ with $\|\varphi\|_\theta > 1$.
What's more, from A_1, A_3 and Hölder inequality, we get

$$\begin{aligned}-\mathcal{H}(\varphi) &= \int_{\partial Q} H(x,\varphi)d\sigma \\ &= \int_{\partial Q}\left(\int_0^{\varphi(x)} h(x,z)dz\right)d\sigma \\ &\leq \int_{\partial Q}\left(\mu_2(x)|\varphi(x)| + \frac{c_2}{s_1(x)}|\varphi(x)|^{s_1(x)}\right)d\sigma \\ &\leq 2\|\mu_2\|_{\frac{s_1(x)}{s_1(x)-1},\partial Q}\|\varphi\|_{s_1(x),\partial Q} + \frac{c_2}{s_1^-}\int_{\partial Q}|\varphi(x)|^{s_1(x)}d\sigma.\end{aligned} \tag{3.2}$$

Since $W^{1,p(x)}(Q)$ is continuously embedded in $L^{s_1(x)}(\partial Q)$ and the inequality

$$\begin{aligned}\int_{\partial Q}|\varphi(x)|^{s_1(x)}dx &\leq \max\left\{\|\varphi\|_{s_1(x),\partial Q}^{s_1^-}, \|\varphi\|_{s_1(x),\partial Q}^{s_1^+}\right\} \\ &\leq C\left(\|\varphi\|_\theta^{s_1^+} + \|\varphi\|_\theta^{s_1^+}\right).\end{aligned} \tag{3.3}$$

If we use (3.2) and (3.3), then we get

$$-\mathcal{H}(\varphi) \leq C'\|\mu_1\|_{\frac{s_1(x)}{s_1(x)-1},Q}\|\varphi\|_\theta + C\frac{1}{s_1^-}\|\varphi\|_\theta^{s_1^+}. \tag{3.4}$$

This implies that for any $\lambda > 0$

$$\Upsilon_\theta(\varphi) + \lambda\mathcal{H}(\varphi) \geq \frac{1}{p^+}\|\varphi\|_\theta^{p^-} - C'\lambda\|\mu_1\|_{\frac{s_1(x)}{s_1(x)-1},Q}\|\varphi\|_\theta - C\frac{1}{s_1^-}\|\varphi\|_\theta^{s_1^+}.$$

By (3.1) and (3.4). Since $1 < s_1^+ < p^-$, then

$$\lim_{\|\varphi\|_\theta\to\infty}(\Upsilon_\theta(\varphi) + \lambda\mathcal{H}(\varphi)) = \infty,$$

then (*i*) of Theorem (2.8) is verified.
Now remains to show (*ii*) and (*iii*). As a result of A_2 and $\frac{\partial H(x,z)}{\partial z} = h(x,z)$, It is readily apparent that $H(x,z)$ is decreasing and increasing for $(0,1)$ and $z \in (z_0, \infty)$ uniformly for $x \in Q$. In addition, Since $H(x,z) \geq R_1 z$

uniformly for x, we have $H(x,z) \to \infty$ as $z \to \infty$.
By using this fact, we obtain that there exists $\delta > z_0$ where

$$H(x,z) \geq 0 = H(x,0) \geq H(x,s),\ \ \forall x \in Q,\ z > \delta \text{ and } s \in (0,1). \tag{3.5}$$

Let η, ε be two real numbers such that $0 < \eta < \min\{1, \varrho\}$ where ϱ is given in remark (2.5), and $\varepsilon > \max(\delta, \delta')$ satisfies $\varepsilon^{p^-}\|\theta\|_{1,Q} > 1$.
When employing the relation (3.5) we have

$$\int_{\partial Q} \sup_{0\leq z\leq\eta} H(x,z)d\sigma \leq \int_{\partial Q} H(x,0)d\sigma = 0. \tag{3.6}$$

Since $\varepsilon > \delta$ and (3.5), we know that

$$\int_{\partial Q} H(x,\varepsilon)d\sigma > 0.$$

Moreover,

$$\frac{1}{\varrho^{p^+}}\frac{\eta^+}{\varepsilon^{p-}}\Big(\int_{\partial Q} H(x,\varepsilon)d\sigma\Big) > 0. \tag{3.7}$$

If we apply the given inequalities in (3.6) and (3.7), we get

$$\int_{\partial Q} \sup_{0\leq z\leq\eta} H(x,z)d\sigma \leq 0 < \frac{1}{\varrho^{p^+}}\frac{\eta^+}{\varepsilon^{p}-}\int_{\partial Q} H(x,\varepsilon)d\sigma.$$

Next, Define $\varphi_0,\ \varphi_1 \in W^{1,p(x)}(Q)$ with $\varphi_0(x) = 0$ and $\varphi_1(x) = \varepsilon$ for any $x \in \overline{Q}$.
It is clear that, $\Upsilon_\theta(\varphi_0) = \mathcal{H}(\varphi_0) = 0$ and

$$\begin{aligned}\Upsilon_\theta(\varphi_1) = \int_Q \frac{1}{p(x)}dx + \int_Q \frac{\theta(x)}{p(x)}\varepsilon^{p(x)}dx &\geq \frac{\varepsilon^{p^-}}{p^+}\int_Q \theta(x)dx + \frac{1}{p^+}mes(Q)\\ &= \frac{1}{p^+}\left(\varepsilon^{p^-}\|\theta\|_{1,Q} + mes(Q)\right)\\ &\geq \frac{1}{p^+}.\end{aligned}$$

So, if we set $r = \frac{1}{p^+}\left(\frac{\eta}{\varrho}\right)^{p+}$, we have

$$\Upsilon_\theta(\varphi_0) < r < \Upsilon_\theta(\varphi_1),$$

and

$$\mathcal{H}(\varphi_1) = -\int_{\partial Q} H(x,\varphi_1)d\sigma = -\int_{\partial Q} H(x,\varepsilon)d\sigma < 0.$$

Thus, the demonstration of (*ii*) is achieved.

Finally, we will demonstrate the fulfillment of condition (*iii*).

$$\begin{aligned}-\frac{(\Upsilon_\theta(\varphi_1) - r)\mathcal{H}(\varphi_0) + (r - \Upsilon_\theta(\varphi_0))\mathcal{H}(\varphi_1)}{\Upsilon_\theta(\varphi_1) - \Upsilon_\theta(\varphi_0)} &= -r\frac{\mathcal{H}(\varphi_1)}{\Upsilon_\theta(\varphi_1)}\\ &= r\frac{\int_{\partial Q} H(x,\varepsilon)d\sigma}{\int_Q \frac{\theta(x)}{p(x)}\varepsilon^{p(x)}dx + \int_Q \frac{1}{p(x)}dx} > 0.\end{aligned}$$

Now, let $\varphi \in W^{1,p(x)}(Q)$ with $\varphi \in \Upsilon_\theta^{-1}([r,\infty))$.
Since

$$\frac{1}{p^+}\mathcal{I}_{\theta(x)}(\varphi) \leq \Upsilon_\theta(\varphi) \leq r,$$

we obtain

$$\mathcal{I}_{\theta(x)}(\varphi) \leq p^+ r = \left(\frac{\eta}{\varrho}\right)^{p^+} < 1.$$

As indicated by Proposition (2.6), it follows that $\|\varphi\|_{\theta(x)} < 1$,
and

$$\frac{1}{p^+}\|\varphi\|_{\theta(x)}^{p^+} \leq \frac{1}{p^+}\mathcal{I}_{\theta(x)}(\varphi) \leq \Upsilon_\theta(\varphi) \leq r.$$

Then using remark (2.5), we get

$$|\varphi(x)| \leq \varrho\|\varphi\|_{\theta(x)} \leq \varrho(p^+ r)^{\frac{1}{p^+}} = \eta,$$

for all $\varphi \in W^{1,p(x)}(Q)$ and $x \in Q$ with $\Upsilon_\theta(\varphi) \leq r$. The last inequality implies that

$$\begin{aligned} -\inf_{\varphi\in\Upsilon_\theta^{-1}((-\infty,r])}\mathcal{H}(\varphi) &= \sup_{\varphi\in\Upsilon_\theta^{-1}((-\infty,r])} -\mathcal{H}(\varphi) \\ &\leq \int_{\partial Q}\sup_{0\leq z\leq\eta} H(x,z)d\sigma \\ &\leq 0. \end{aligned}$$

Therefore, we get

$$-\inf_{\varphi\in\Upsilon_\theta^{-1}((-\infty,r])}\mathcal{H}(\varphi) < r\frac{\int_{\partial Q} H(x,\varepsilon)d\sigma}{\int_Q \frac{\theta(x)}{p(x)}\varepsilon^{p(x)}dx + \int_Q \frac{1}{p(x)}dx},$$

and $\inf_{\varphi\in\Upsilon_\theta^{-1}((-\infty,r])}\mathcal{H}(\varphi) > \dfrac{(\Upsilon_\theta(\varphi_1) - r)\mathcal{H}(\varphi_0) + (r - \Upsilon_\theta(\varphi_0))\mathcal{H}(\varphi_1)}{\Upsilon_\theta(\varphi_1) - \Upsilon_\theta(\varphi_0)}$. This completes the proof. □

Acknowledgements

The authors would like to thank the referees for the useful comments and suggestions that substantially helped improving the quality of the paper.

References

[1] Abdalmonem, A., Scapellato, A.: Fractional operators with homogeneous kernels in weighted Herz spaces with variable exponen. Applicable Analysis. **101**(6), 1953—1962 (2022).
[2] Allalou, M., El Ouaarabi, M., Raji, A.: On a class of nonhomogeneous anisotropic elliptic problem with variable exponents. Rend. Circ. Mat. Palermo, II. Ser **73**, 3195–3209 (2024).
[3] Alves, C.O., Souto, M.A.S.: Existence of solutions for a class of problems in $\mathbb{R}^N$ involving the $p(x)$-Laplacian, Prog. Nonlinear Differential Equations Appl. **66** 17–32 (2005).
[4] Allaoui, M., El Amrouss, A. R., Ourraoui, A.: Existence and multiplicity of solutions for a Steklov problem involving the $p(x)$-Laplace operator. Electron. J. Differential Equations. **132**(2012), 1—12 (2012).
[5] Antontsev, S. N., Rodrigues, J. F.: On stationary thermo-rheological viscous flows.Ann. Univ. Ferrara Sez. VII Sci. Mat. **52**, 19–36 (2006).
[6] Bocea M., Mihăilescu M.: Γ-convergence of power-law functionals with variable exponents. Nonlinear Anal.: Theory, Methods, Appl. **73**(1), 110–121 (2010).
[7] Bouaam, H., El Ouaarabi, M., Melliani, S.: Kirchhoff-type double-phase problems with variable exponents and logarithmic nonlinearity in Musielak-Orlicz Sobolev spaces. Rend. Circ. Mat. Palermo, II. Ser **74**, 164 (2025).
[8] Butakin, G., Piskin, E., Celik, E.: Blowup and global solutions of a fourth-order parabolic equation with variable exponent logarithmic nonlinearity. Journal of Function Spaces. **2024**(1), (2024).
[9] Cammaroto, F., Chinnì, A., and Di Bella, B.: Multiple solutions for a Neumann problem involving the $p(x)$-Laplacian. Nonlinear Analysis. Theory, Methods and Applications. **71**(10), 4486–4492 (2009).

[10] Chen, Y., Levine, S., Rao, R.: Variable exponent, linear growth functionals in image restoration, SIAM J. Appl. Math. **66**, 1383–1406 (2006).

[11] Chems Eddine, N., Ragusa, M.A., Repovs, D.D.: On the concentration-compactness principle for anisotropic variable exponent Sobolev spaces and its applications. Fractional Calculus and Applied Analysis. **27**(2), 725–756, (2024).

[12] Deng, S. G.: Eigenvalues of the $p(x)$-Laplacian Steklov problem. J. Math. Anal. Appl. **339**, 925—937 (2008).

[13] Deng, S. G.: Positive solutions for Robin problem involving the $p(x)$-Laplacian. Journal of mathematical analysis and applications. **360**(2), 548–560 (2009).

[14] Deng, S. G.: A local mountain pass theorem and applications to a double perturbed $p(x)$-Laplacian equations. Applied Mathematics and Computation. **211**(1), 234–241 (2009).

[15] El Ouaarabi, M., Moujane, N., Allalou, C., Melliani, S.: On a Neumann problem driven by $p(x)$-Laplacian-like operators in variable-exponent Sobolev spaces. Palestine Journal of Mathematics, **13**(1) (2024).

[16] El Ouaarabi, M., Moujane, N., Melliani, S.: Existence of three solutions to a $p(z)$-Laplacian-Like Robin problem. ANNALI DELL'UNIVERSITA'DI FERRARA, **70**(4), 1375–1388 (2024).

[17] Fan, X.L.: On nonlocal $p(x)$-Laplacian Dirichlet problems. Nonlinear Anal. **72**, 3314–3323 (2010).

[18] Fan, X.L., Zhao, D.: On the Spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$. J Math Anal Appl. **263**, 424–446 (2001). Theory Methods Appl. **52**(8), 1843–1852 (2003).

[19] Faraci, F.: Multiple solutions for two nonlinear problems involving the p-Laplacian. Nonlinear Anal. **63**, 1017–1029 (2005).

[20] Ge, B., Zhou, Q. M.: Multiple solutions for a Robin-type differential inclusion problem involving the $p(x)$-Laplacian. Mathematical Methods in the Applied Sciences, **40**(18), 6229–6238 (2017).

[21] Ji, C.: Remarks on the existence of three solutions for the $p(x)$-Laplacian equations. Nonlinear Analysis: Theory; Methods and Applications. **74**(9), 2908–2915 (2011).

[22] Kováčik, O., Rákosník, J.: On spaces $L^{p(x)}$ and $W^{k,p(x)}$, Czechoslovak Math. J. **41** (1991), 592–618.

[23] Li, C., Tang, C. L.: Three solutions for a class of quasilinear elliptic systems involving the (p, q)-Laplacian. Nonlinear Analysis: Theory, Methods and Applications. **69**(10), 3322–3329 (2008).

[24] Mihăilescu, M.: Existence and multiplicity of solutions for a Neumann problem involving the $p(x)$-Laplace operator, Nonlinear Anal. **67**, 1419–1425 (2007).

[25] Moujane, N., El Ouaarabi, M., Allalou, C.: Study of some elliptic system of $(p(x), q(x))$-Kirchhoff type with convection. Journal of Elliptic and Parabolic Equations, **9**(2), 687–704 (2023).

[26] Moujane, N., El Ouaarabi, M., Allalou, C.: Elliptic Kirchhoff-type system with two convections terms and under Dirichlet boundary conditions. Filomat **37**(28), 9693–9707 (2023).

[27] Moujane, N., El Ouaarabi, M., Melliani, S.: On a class of Leray-Lions type problem in Musielak-Orlicz-Sobolev spaces. Dynamics of Continuous Discrete and Impulsive Systems Series A Mathematical Analysis, **31**(6), 431–445 (2024).

[28] Moujane, N., El Ouaarabi, M., Allalou, C., Melliani, S.: Kirchhoff-type system involving the $(\alpha_1(m); \alpha_2(m))$-kirchhoff-Laplacian operator on the variable exponent sobolev spaces. Memoirs on Differential Equations and Mathematical Physics, **94**, 83–99 (2025).

[29] Moujane, N., El Ouaarabi, M.: On a class of Schrödinger–Kirchhoff-double phase problems with convection term and variable exponents. Communications in Nonlinear Science and Numerical Simulation, 141, 108453 (2025).

[30] Obersnel F., Omari P.: Positive solutions of the Dirichlet problem for the prescribed mean curvature equation. J Differ Equ. **249**, 1674–1725 2010.

[31] Ouidirne, F., Allalou, C., Oukessou, M.: Global existence theorem for the 3-D generalized micropolar fluid system in critical Fourier-Besov-Morrey spaces with variable exponent, Filomat, **38** (20), 7161—7171, (2024).

[32] Ricceri, B.: A three critical points theorem revisited. Nonlinear Anal. **70**, 3084–3089 (2009).

[33] Ricceri, B.: Sublevel sets and global minima of coercive functionals and local minima of their perturbations. J. Nonlinear Convex Anal. **5**, 157–168 (2004).

[34] Rủzicka, M.: Electrorheological fuids: modeling and mathematical theory. Springer Science & Business Media, (2000).

[35] Rodrigues, M. M.: Multiplicity of solutions on a nonlinear eigenvalue problem for $p(x)$-Laplacian-like operators. Mediterranean journal of mathematics. **9**, 211–223 (2012).

[36] Zeidler E.: Nonlinear Functional Analysis and its Applications II/B. Springer-Verlag. New York, (1990).

[37] Zhikov, V.V.: Averaging of functionals in the calculus of variations and elasticity. Math. USSR Izv. **29**, 33–66 (1987).