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Existence and uniqueness of mild solutions for conformable fractional differential equations using new generalized conformable fractional derivative and semigroup

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Abstract. In this work, we generalize the notion of the conformable derivative. Then, we present the new definition of conformable strongly continuous semigroup and its infinitesimal generator. Further, we study the existence and uniqueness of the mild solution for a conformable differential equation with our generalized conformable C^0-semigroup. Finally, we investigate the continuous dependence between initial data and mild solutions.

1. Introduction

One of the natural ways for modeling dynamical systems is to apply fractional calculus, which is in fact a generalization of the classic differentiation and integration to non integer order (see e.g. [1, 4, 6] and references therein). It has several applications in various fields and is considered as a powerful tool to model physical phenomena such as processes with memory. The majority of attempts to define fractional derivative used an integral form. So, they all have several failures, namely the Riemann-Liouville derivative does not vanish for constants. All fractional derivatives do not satisfy the known formulas giving the derivative of the product, the quotient and the composition of two functions. Further, the Caputo definition requires that the function f be differentiable [7]. To overcome all these setbacks, Khalil et al. in [7] and then Abdeljawad in [2] presented the conformable derivative and integral. Moreover, AL Horani et al. introduced in [3] the conformable semigroup and its generator to solve a conformable abstract Cauchy problem under their novel derivative.

In this paper, we extend the definition in [7] of the conformable derivative by using a parametric function. Then, we present an extension of the definition of the conformable semigroup and its infinitesimal generator studied in [3], in which there are still some classic results that have not been treated and that we will try to investigate. First of all, the definition due to Alhorani et al. of the α-infinitesimal generator of a conformable semigroup remains implicit. We will propose here an explicit expression of this generator, which will help us to prove its closure and the density of its domain.

In addition, for one of the main results concerning the α-conformable derivative of the mapping $g : t \mapsto T(t)x$, the authors of [3] assumed that the α-conformable semigroup $\{T(t)|t \geq 0\}$ is continuously α-differentiable

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to prove that $g^{(\alpha)}(t) = AT(t)x = T(t)xA$. But, as we will show later, this condition is useless even in the conformable case.
Finally, we study the existence and uniqueness of the mild solution for the following conformable differential equation on a Banach space X:

$$\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + f(t, x(t)),\ t \in [0, a] \quad \text{and } x(0) = x_0,$$

where A is the generator of a conformable $\mathcal{C}^0$-semigroup, $x_0 \in X$, and $f : [0, a] \times X \to X$. The remainder of this work is organized as follows:
Section 2 is reserved for preliminaries. In section 3, we present our new definition of conformable derivative and integral. Section 4 is devoted to the study of the new conformable $\mathcal{C}^0$-semigroup and its generator. Then, section 5 deals with the existence, uniqueness of the mild solution and the continuous dependence between mild solutions and initial data, for a conformable differential equation with conformable semigroup.

2. Preliminaries

Definition 2.1. *The conformable derivative of $f : [0, \infty[\to \mathbb{R}$ of order $\alpha \in (0, 1]$ at $t > 0$ is defined by $T_\alpha(f)(t) = \lim\limits_{\varepsilon\to 0} \frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon}$. If this limit exists, f is said α-differentiable.*

Definition 2.2. *([7]). Let $f : [0, \infty[\to \mathbb{R}$ n-differentiable. The conformable derivative of f of order $\alpha \in (n, n + 1]$ at $t > 0$ is defined by $T_\alpha(f)(t) = f^{(\alpha)}(t) = \lim\limits_{\varepsilon\to 0} \frac{f^{(\lceil\alpha\rceil-1)}(t+\varepsilon t^{(\lceil\alpha\rceil-\alpha)})-f^{(\lceil\alpha\rceil-1)}(t)}{\varepsilon}$, where $\lceil\alpha\rceil$ is the smallest integer greater than or equal to α. If this limit exists, we say that f is α-differentiable, and if $\lim\limits_{t\to 0^+} T_\alpha(f)(t)$ exists, we define $T_\alpha(f)(0) = \lim\limits_{t\to 0^+} T_\alpha(f)(t)$.*

Definition 2.3. *([7]). Define $I^a_\alpha(f)(t) = I^a_1(t^{\alpha-1}f) = \int_a^t \frac{f(x)}{x^{1-\alpha}}dx$, where the integral is the usual Riemann improper integral, and $\alpha \in (0, 1)$.*

Theorem 2.4. *([2] and [7]). If f is a continuously α-differentiable function in the domain of I_α then, for $t \geq a$: $T_\alpha I^a_\alpha(f)(t) = f(t)$ and $I^a_\alpha T_\alpha(f)(t) = f(t) - f(a)$.*

Definition 2.5. *([3]). A conformable strongly continuous α-semigroup on a Banach space X, is a family $\{T(t), t \geq 0\}$ of linear bounded operators from X into itself verifying:*

(i) $T(0) = I$, *(ii)* $T((s + t)^{\frac{1}{\alpha}}) = T(s^{\frac{1}{\alpha}})T(t^{\frac{1}{\alpha}})$ *for all* $s, t \geq 0$,

(iii) the map $g : t \mapsto T(t)x$ is continuous at $t = 0$, for all $x \in X$, i.e., $\lim\limits_{t\to 0^+} T(t)x = x$.

Definition 2.6. *([3]). The infinitesimal generator A of the fractional $\mathcal{C}^0 - \alpha$-semigroup $\{T(t), t \geq 0\}$ is defined by $A : x \mapsto Ax = \lim\limits_{t\to 0^+} T^{(\alpha)}(T(t)x)$ the conformable α-derivative of $T(t)$ at $t = 0$, with domain $D(A) = \{x \in X| \lim\limits_{t\to 0^+} T^{(\alpha)}(T(t)x)$ exists$\}$.*

Notations: For $a > 0$, we consider $\mathcal{C}_a = \mathcal{C}([0, a], X)$ the space of all continuous functions on $[0, a]$ into the Banach space X. For $x, y \in \mathcal{C}_a$, we define the metric:
$H_a(x, y) = \sup\limits_{t\in[0,a]} \|x(t) - y(t)\|$. Then, $(\mathcal{C}_a, H_a)$ is a complete metric space.
For real functions $g_1, g_2 : [0, \infty[\to \mathbb{R}$ and $\alpha \geq 0$, where $g_2(x)$ is nonzero, we will use the "Little o" notation: $g_1(x) = o(g_2(x))$ as $x \to \alpha$ if and only if $\lim\limits_{x\to\alpha} \frac{g_1(x)}{g_2(x)} = 0$.

3. A new definition of generalized fractional conformable derivative

In the sequel, let X be a real Banach space, and $\varphi : [0, \infty[\to \mathbb{R}$ be a fixed positive, increasing (or decreasing) and differentiable function.

Definition 3.1. *The (α,φ)-conformable fractional derivative of $f:[0,\infty[\to X$ of order $\alpha\in(0,1]$ at $t>0$ is defined by*

$$T_{\alpha,\varphi}(f)(t)=f_{\varphi}^{(\alpha)}(t)=\lim_{\varepsilon\to 0}\frac{f(t+\varepsilon(\varphi(t))^{1-\alpha})-f(t)}{\varepsilon}.$$

If this limit exists, we say that f is (α,φ)-differentiable. We extend this definition to $t=0$ by setting: $T_{\alpha,\varphi}(f)(0)=\lim\limits_{t\to 0^+}T_{\alpha,\varphi}(f)(t)$, if this limit exists.

Notice that for $\varphi(t)=t:\ T_{\alpha,t}(f)=T_{\alpha}(f)$ coincides with the definition in [7]. Moreover, one can consider $\varphi(t)=e^{-t}$ as a decreasing mapping.

Theorem 3.2. *If a function $f:[0,\infty[\to\mathbb{R}$ is (α,φ)-differentiable at $x_0>0$, for $\alpha\in(0,1]$, then f is continuous at x_0.*

Proof. Using $f(x_0+\varepsilon(\varphi(x_0))^{1-\alpha})=f(x_0)+\varepsilon.\frac{f(x_0+\varepsilon(\varphi(x_0))^{1-\alpha})-f(x_0)}{\varepsilon}$ and by tending $h=\varepsilon(\varphi(x_0))^{1-\alpha}$ to 0, we get $\lim\limits_{h\to 0}f(x_0+h)=f(x_0)$, that is f is continuous at x_0. □

Theorem 3.3. *Let $\alpha\in(0,1]$ and $f,g:[0,\infty[\to X$ be (α,φ)-differentiable. Then*

1. *$T_{\alpha,\varphi}(\lambda f+\mu g)=\lambda T_{\alpha,\varphi}(f)+\mu T_{\alpha,\varphi}(g)$ and $T_{\alpha,\varphi}(c)=0$, for all $c,\lambda,\mu\in\mathbb{R}$.*
2. *$T_{\alpha,\varphi}(gf)=gT_{\alpha,\varphi}(f)+T_{\alpha,\varphi}(g)f$ and $T_{\alpha,\varphi}(f/g)=\frac{gT_{\alpha,\varphi}(f)-T_{\alpha,\varphi}(g)f}{g^2}$, with $g:\mathbb{R}\to\mathbb{R}$.*
3. *If f is differentiable on $[0,\infty[$, then $T_{\alpha,\varphi}(f)(t)=(\varphi(t))^{1-\alpha}f'(t)$ and*

$$T_{\alpha,\varphi}\left(f\left(\frac{(\varphi(t))^{\alpha}}{\alpha}\right)\right)=\varphi'(t)f'\left(\frac{(\varphi(t))^{\alpha}}{\alpha}\right). \tag{1}$$

Proof. We prove the identity (1), because the other points are trivial. Using the Taylor series, we obtain for $\varepsilon>0$ very small:

$$\begin{aligned}
f\left(\frac{1}{\alpha}(\varphi(t+\varepsilon(\varphi(t))^{1-\alpha}))^{\alpha}\right)-f\left(\frac{(\varphi(t))^{\alpha}}{\alpha}\right) &= f\left(\frac{1}{\alpha}(\varphi(t)+\varepsilon(\varphi(t))^{1-\alpha}.\varphi'(t)+o(\varepsilon))^{\alpha}\right)-f\left(\frac{(\varphi(t))^{\alpha}}{\alpha}\right)\\
&= f\left(\frac{1}{\alpha}(\varphi(t))^{\alpha}\left(1+\varepsilon\varphi'(t)(\varphi(t))^{-\alpha}+o(\varepsilon)\right)^{\alpha}\right)-f\left(\frac{(\varphi(t))^{\alpha}}{\alpha}\right)\\
&= f\left(\frac{1}{\alpha}(\varphi(t))^{\alpha}\left(1+\alpha\varepsilon\varphi'(t)(\varphi(t))^{-\alpha}+o(\varepsilon)\right)\right)-f\left(\frac{(\varphi(t))^{\alpha}}{\alpha}\right)\\
&= f\left(\frac{1}{\alpha}(\varphi(t))^{\alpha}+\varepsilon\varphi'(t)+o(\varepsilon)\right)-f\left(\frac{(\varphi(t))^{\alpha}}{\alpha}\right)\\
&= f\left(\frac{1}{\alpha}(\varphi(t))^{\alpha}\right)+\varepsilon\varphi'(t)f'\left(\frac{1}{\alpha}(\varphi(t))^{\alpha}\right)+o(\varepsilon)-f\left(\frac{(\varphi(t))^{\alpha}}{\alpha}\right)\\
&= \varepsilon\varphi'(t)f'\left(\frac{(\varphi(t))^{\alpha}}{\alpha}\right)+o(\varepsilon)
\end{aligned}$$

Hence,

$$\lim_{\varepsilon\to 0}\frac{f\left(\frac{1}{\alpha}(\varphi(t+\varepsilon(\varphi(t))^{1-\alpha}))^{\alpha}\right)-f\left(\frac{1}{\alpha}(\varphi(t))^{\alpha}\right)}{\varepsilon}=\varphi'(t)f'\left(\frac{(\varphi(t))^{\alpha}}{\alpha}\right).$$

□

From the formula (1), we get generalization of some identities in [7]:

1. $T_{\alpha,\varphi}(e^{bx})=b(\varphi(x))^{1-\alpha}e^{bx},\ b\in\mathbb{R}$.
2. $T_{\alpha,\varphi}(\sin bx)=b(\varphi(x))^{1-\alpha}\cos bx$ and $T_{\alpha,\varphi}(\cos bx)=-b(\varphi(x))^{1-\alpha}\sin bx$.
3. $T_{\alpha,\varphi}\left(\frac{1}{\alpha}(\varphi(x))^{\alpha}\right)=\varphi'(x)$.

Let us extend the (α,φ)-conformable derivative for $\alpha\in(n,n+1]$ and $n\in\mathbb{N}$.

Definition 3.4. *Let $f : [0, \infty[\to X$ be n-differentiable at $t > 0$ and $\alpha \in (n, n+1]$. Then the (α, φ)-conformable fractional derivative of f of order α at $t > 0$ is defined by*

$$T_{\alpha,\varphi}(f)(t) = f_{\varphi}^{(\alpha)}(t) = \lim_{\varepsilon \to 0} \frac{f^{(\lceil\alpha\rceil - 1)}(t + \varepsilon(\varphi(t))^{(\lceil\alpha\rceil - \alpha)}) - f^{(\lceil\alpha\rceil-1)}(t)}{\varepsilon}.$$

where $\lceil\alpha\rceil$ is the smallest integer greater than or equal to α.
If this limit exists, we say that f is (α, φ)-differentiable at t. This definition can be extended to $t = 0$ by setting $T_{\alpha,\varphi}(f)(0) = \lim_{t \to 0^+} T_{\alpha,\varphi}(f)(t)$, if this limit exists.

Remark 3.5. If f is $(n+1)$-differentiable at $t > 0$, then we have:

$$T_{\alpha,\varphi}(f)(t) = (\varphi(t))^{(\lceil\alpha\rceil - \alpha)} T_{\lceil\alpha\rceil,\varphi}(f)(t) = (\varphi(t))^{(\lceil\alpha\rceil-\alpha)} f^{(\lceil\alpha\rceil)}(t).$$

Theorem 3.6. *((α, φ)-Conformable Rolle's Theorem). Let $b > a > 0$ and consider $f : [a, b] \to \mathbb{R}$ a continuous function on $[a, b]$, which is (α, φ)-differentiable on (a, b), for $\alpha \in (0, 1)$, verifying $f(a) = f(b)$. Then, there exists $c \in (a, b)$, such that $f_{\varphi}^{(\alpha)}(c) = 0$.*

Proof. If f is constant, then each $c \in (a, b)$ fits. Assume that f is not constant, then f admits at least one of its extremums at some $c \in (a, b)$, since f is continuous hence, it is bounded on the compact interval $[a, b]$, and $f(a) = f(b)$. Without loss of generality, assume c is a point of global maximum. Thus

$$T_{\alpha,\varphi}(f)(t) = \lim_{\varepsilon \to 0^+} \frac{f(t + \varepsilon(\varphi(t))^{1-\alpha}) - f(t)}{\varepsilon} = \lim_{\varepsilon \to 0^-} \frac{f(t + \varepsilon(\varphi(t))^{1-\alpha}) - f(t)}{\varepsilon}.$$

The first limit is non-positive, and the second is non-negative. So, $f_{\varphi}^{(\alpha)}(c) = 0$. □

Theorem 3.7. *((α, φ)-Conformable Mean Value Theorem). Let $b > a > 0$ and consider $f : [a, b] \to \mathbb{R}$ a continuous function on $[a, b]$, which is (α, φ)-differentiable for $\alpha \in (0, 1)$. Then, there exists $c \in (a, b)$, such that*

$$f_{\varphi}^{(\alpha)}(c) = \frac{f(b) - f(a)}{\frac{1}{\alpha}(\varphi(b))^{\alpha} - \frac{1}{\alpha}(\varphi(a))^{\alpha}} \varphi'(c).$$

Proof. Consider the following auxiliary function

$$h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{\frac{(\varphi(b))^{\alpha}}{\alpha} - \frac{(\varphi(a))^{\alpha}}{\alpha}} \left(\frac{(\varphi(x))^{\alpha}}{\alpha} - \frac{(\varphi(a))^{\alpha}}{\alpha} \right)$$

h is continuous on $[a, b]$, (α, φ)-differentiable such that $h(a) = h(b) = 0$. Then by Rolle's theorem, there exists $c \in (a, b)$, such that $h_{\varphi}^{(\alpha)}(c) = 0$. A simple calculation, taking into account $T_{\alpha,\varphi}\left(\frac{1}{\alpha}(\varphi(x))^{\alpha}\right) = \varphi'(x)$, leads to the desired result. □

Definition 3.8. *For $\alpha \in (0, 1)$, define $I_{\alpha,\varphi}^{a}(f)(t) = I_{1,\varphi}^{a}(t^{\alpha-1} f) = \int_a^t \frac{f(x)}{(\varphi(x))^{1-\alpha}} dx$, where we adopt the Bochner integral.*

Theorem 3.9. *If f is a continuous function in the domain of $I_{\alpha,\varphi}$ then:*

1. *$T_{\alpha,\varphi} I_{\alpha,\varphi}^{a}(f)(t) = f(t)$, for all $t \geq a$.*
2. *$I_{\alpha,\varphi}^{a} T_{\alpha,\varphi}(f)(t) = f(t) - f(a)$, provided that f is **differentiable**.*

Proof. 1. Since f is continuous, then $I_{\alpha,\varphi}^{a}(f)(t)$ is obviously differentiable. Thus,

$$\begin{aligned} T_{\alpha,\varphi}\left(I_{\alpha,\varphi}^{a}(f)\right)(t) &= (\varphi(t))^{1-\alpha} \frac{d}{dt}\left[I_{\alpha,\varphi}^{a}(f)(t)\right] = (\varphi(t))^{1-\alpha} \frac{d}{dt}\left[\int_a^t \frac{f(x)}{(\varphi(x))^{1-\alpha}} dx\right] \\ &= (\varphi(t))^{1-\alpha} \frac{f(t)}{(\varphi(t))^{1-\alpha}} = f(t). \end{aligned}$$

2. Since f is differentiable, then

$$I^a_{\alpha,\varphi}\left(T_{\alpha,\varphi}(f)\right)(t) = I^a_{1,\varphi}\left((\varphi(t))^{1-\alpha}f'(t)\right) = \int_a^t \frac{(\varphi(x))^{1-\alpha}f'(x)}{(\varphi(x))^{1-\alpha}}dx = \int_a^t f'(x)dx = f(t)-f(a).$$

So, $I^a_{\alpha,\varphi}T_{\alpha,\varphi}(f)(t) = f(t)-f(a)$. □

In the next result, we show that point 2 of Theorem 3.9 hold true if we assume only that f is an (α,φ)-**differentiable** function and $X=\mathbb{R}$.

Theorem 3.10. *If $f:[0,\infty[\to\mathbb{R}$ is (α,φ)-differentiable then, for all $t\geq 0$: $I^a_{\alpha,\varphi}T_{\alpha,\varphi}(f)(t) = f(t)-f(a)$.*

Proof.

- Step 1: If $f(x)=\sum\limits_{k=0}^{p}\lambda_k f_k$ is a polynomial mapping, where $f_k(x)=x^k, 0\leq k\leq p$. Then by the linearity of conformable derivative and integral, we get

$$\begin{aligned} I^a_{\alpha,\varphi}T_{\alpha,\varphi}(f)(t) &= \sum_{k=0}^{p}\lambda_k I^a_{\alpha,\varphi}T_{\alpha,\varphi}(f_k)(t) = \sum_{k=0}^{p}\lambda_k\int_a^t \frac{T_{\alpha,\varphi}(f_k)(t)}{(\varphi(x))^{1-\alpha}}dx \\ &= \sum_{k=0}^{p}\lambda_k\int_a^t \frac{kx^{k-1}(\varphi(x))^{1-\alpha}}{(\varphi(x))^{1-\alpha}}dx = \sum_{k=0}^{p}\lambda_k\int_a^t kx^{k-1}dx = \sum_{k=0}^{p}\lambda_k(t^k-a^k) \end{aligned}$$

  where we have used point 5 of Theorem 3.3. Thus

$$I^a_{\alpha,\varphi}T_{\alpha,\varphi}(f)(t) = f(t)-f(a).$$

- Step 2: General case. Let $t\geq 0$ and $[b,c]\subset[0,\infty[$ such that $t\in[b,c]$. Since f is (α,φ)-differentiable, then f is continuous. By using Weierstrass approximation Theorem, there exists a sequence (P_k) of polynomial functions, which is uniformly convergent to f on the compact interval $[b,c]$. Thus using the first case, we have

$$\begin{aligned} \left|I^a_{\alpha,\varphi}T_{\alpha,\varphi}(f)(t)-f(t)+f(a)\right| &= \left|I^a_{\alpha,\varphi}T_{\alpha,\varphi}(f)(t)-I^a_{\alpha,\varphi}T_{\alpha,\varphi}(P_k)(t)+P_k(t)-f(t)+f(a)-P_k(a)\right| \\ &\leq \left|I^a_{\alpha,\varphi}T_{\alpha,\varphi}(f-P_k)(t)\right|+|P_k(t)-f(t)|+|f(a)-P_k(a)|. \end{aligned}$$

Clearly, $|P_k(t)-f(t)|$ and $|f(a)-P_k(a)|$ converge to 0 as $k\to\infty$. Moreover, using the double limit theorem, one obtains

$$\begin{aligned} \lim_{k\to\infty}T_{\alpha,\varphi}(f-P_k)(t) &= \lim_{k\to\infty}\lim_{\varepsilon\to 0}\frac{f(t+\varepsilon(\varphi(t))^{1-\alpha})-P_k(t+\varepsilon(\varphi(t))^{1-\alpha})+P_k(t)-f(t)}{\varepsilon} \\ &= \lim_{\varepsilon\to 0}\lim_{k\to\infty}\frac{f(t+\varepsilon(\varphi(t))^{1-\alpha})-P_k(t+\varepsilon(\varphi(t))^{1-\alpha})+P_k(t)-f(t)}{\varepsilon} = 0. \end{aligned}$$

We deduce by the dominated convergence Theorem that

$$\lim_{k\to\infty}I^a_{\alpha,\varphi}T_{\alpha,\varphi}(f-P_k)(t) = 0.$$

Hence, $I^a_{\alpha,\varphi}T_{\alpha,\varphi}(f)(t) = f(t)-f(a)$.
So, the condition "f is differentiable" , added by the author of [2] to prove this result in Lemma 2.8, is superfluous and it is sufficient to assume that f is (α,φ)-differentiable. □

4. New fractional conformable strongly continuous semigroups

Remark 4.1. To generalize the notion of a conformable $C^0-\alpha$-semigroup on X, we assume from now on that $\varphi : [0,\infty[\to [0,\infty[$ is an increasing diffeomorphism of class $\mathcal{C}^1$, verifying $\varphi(0)=0$ and $\lim\limits_{t\to 0^+}\frac{\varphi(t)}{t}=C>0$.

Definition 4.2.
By a conformable fractional strongly continuous (α,φ)-semigroup on X, we mean a family $\{T(t), t\geq 0\}$ of linear bounded operators from X into itself verifying points (i), (iii) of Definition 2.5 and the following alternative condition:

(ii) $T((\varphi(s+t))^{\frac{1}{\alpha}}) = T((\varphi(s))^{\frac{1}{\alpha}})T((\varphi(t))^{\frac{1}{\alpha}})$ *for all* $s,t\geq 0$.

Definition 4.3.
The (α,φ)-infinitesimal generator A of the conformable $\mathcal{C}^0-(\alpha,\varphi)$-semigroup $\{T(t), t\geq 0\}$ is defined by $A : x\mapsto Ax = \lim\limits_{t\to 0^+} T^{(\alpha)}_{\varphi}(T(t)x)$ the conformable (α,φ)-derivative of $T(t)$ at $t=0$, with domain $D(A)=\{x\in X|\lim\limits_{t\to 0^+} T^{(\alpha)}_{\varphi}(T(t)x)$ exists$\}$.

For a family $\{T(t), t\geq 0\}$ of linear bounded operators from X into itself, define $S(t)=T((\varphi(t))^{\frac{1}{\alpha}})$ that is $T(t)=S(\varphi^{-1}(t^\alpha))$, for all $t\geq 0$. The following results describe the relationship between conformable fractional and classical semigroups and give the link between their generators. Some proofs will be omitted since they are obvious.

Proposition 4.4. *$\{T(t), t\geq 0\}$ is a $\mathcal{C}^0-(\alpha,\varphi)$-semigroup on X if and only if $\{S(t), t\geq 0\}$ is a $\mathcal{C}^0$-semigroup on X.*

Theorem 4.5. *Let $\{T(t), t\geq 0\}$ be a $\mathcal{C}^0-(\alpha,\varphi)$-semigroup on X. Then, there exist two constants $M\geq 1$ and $w\geq 0$ such that*

$$\|T(t)\|\leq Me^{w\varphi^{-1}(t^\alpha)}, \text{ for all } t\geq 0. \tag{2}$$

Proof. It is well known that: $\exists M\geq 1, w\geq 0, \forall t\geq 0,\ \|S(t)\|\leq Me^{wt}$.
Therefore, $\|T(t)\|=\|S(\varphi^{-1}(t^\alpha))\|\leq Me^{w\varphi^{-1}(t^\alpha)}$, for all $t\geq 0$. □

Proposition 4.6. *If $\{T(t), t\geq 0\}$ is a $\mathcal{C}^0-(\alpha,\varphi)$-semigroup on X then for all $x\in X$, the function $g : t\mapsto T(t)x$ is continuous all over $[0,\infty[$.*

Proof. Consider the continuous function $\psi : t\mapsto \varphi^{-1}(t^\alpha)$ and the map $h : t\mapsto S(t)x$, then $g(t)=T(t)x=S(\varphi^{-1}(t^\alpha))=h\circ\psi(t)$, for all $t\geq 0$. Since $\{S(t), t\geq 0\}$ is a classical $\mathcal{C}^0$-semigroup on X , then h is continuous over $[0,\infty[$ (see [10]). Thus, $g=h\circ\psi$ is also continuous over $[0,\infty[$. □

The major defect of Definition 2.6, due to Alhorani et al., is that it defines implicitly the generator of a conformable $\mathcal{C}^0-\alpha$-semigroup. Hence, the interest and the importance of our following Theorem 4.7, in which we will give an explicit expression for the α-infinitesimal generator A of a conformable fractional $\mathcal{C}^0-\alpha$-semigroup $\{T(t), t\geq 0\}$.

Theorem 4.7. *Let A be the infinitesimal generator of the conformable $\mathcal{C}^0-(\alpha,\varphi)$-semigroup $\{T(t), t\geq 0\}$ on X. Then, for all $x\in D(A)$:*

$$Ax=\frac{\alpha C^{1-\alpha}}{\varphi'(0)}\lim_{t\to 0^+}\frac{T((\varphi(t))^{\frac{1}{\alpha}})x-x}{t}. \tag{3}$$

Proof. Let $x\in D(A)$, we have

$$\begin{aligned}Ax &= \lim_{t\to 0^+} T^{(\alpha)}_{\varphi}(T(t)x) = \lim_{t\to 0^+}\lim_{\varepsilon\to 0^+}\frac{T(t+\varepsilon\varphi(t)^{1-\alpha})x-T(t)x}{\varepsilon}\\ &= \lim_{\varepsilon\to 0^+}\lim_{t\to 0^+}\frac{S\circ\varphi^{-1}((t+\varepsilon\varphi(t)^{1-\alpha})^\alpha)x-S\circ\varphi^{-1}(t^\alpha)x}{\varepsilon},\end{aligned}$$

where the exchange of the limits follows from the uniform continuity of the mapping $(t,\varepsilon) \to \frac{T(t+\varepsilon\varphi(t)^{1-\alpha})x-T(t)x}{t}$ on the compact $[0,1]\times[0,1]$, since it is continuous therein. Therefore

$$\begin{aligned}
Ax &= \lim_{\varepsilon\to0^+}\lim_{t\to0^+}\frac{S\circ\varphi^{-1}(t^\alpha(1+\varepsilon\varphi(t)^{1-\alpha}/t)^\alpha)x - S\circ\varphi^{-1}(t^\alpha)x}{\varepsilon}\\
&= \lim_{\varepsilon\to0^+}\lim_{t\to0^+}\frac{S\circ\varphi^{-1}(t^\alpha(1+\alpha\varepsilon\varphi(t)^{1-\alpha}/t+o(\varepsilon))x - S\circ\varphi^{-1}(t^\alpha)x}{\varepsilon}\\
&= \lim_{\varepsilon\to0^+}\lim_{t\to0^+}\frac{S\circ\varphi^{-1}(t^\alpha+\alpha\varepsilon(\varphi(t)/t)^{1-\alpha}+o(\varepsilon))x - S\circ\varphi^{-1}(t^\alpha)x}{\varepsilon}\\
&= \lim_{\varepsilon\to0^+}\lim_{t\to0^+}\frac{S(\varphi^{-1}(t^\alpha)+\alpha\varepsilon(\varphi(t)/t)^{1-\alpha}(\varphi^{-1})'(t^\alpha)+o(\varepsilon))x - S\circ\varphi^{-1}(t^\alpha)x}{\varepsilon}\\
&= \lim_{\varepsilon\to0^+}\lim_{t\to0^+}S\circ\varphi^{-1}(t^\alpha)\left[\frac{S(\alpha\varepsilon(\varphi(t)/t)^{1-\alpha}(\varphi^{-1})'(t^\alpha)+o(\varepsilon))x-x}{\varepsilon}\right]\\
&= \lim_{\varepsilon\to0^+}S\circ\varphi^{-1}(0)\left[\frac{S(\alpha\varepsilon C^{1-\alpha}(\varphi^{-1})'(0)+o(\varepsilon))x-x}{\varepsilon}\right]\\
&= \lim_{\varepsilon\to0^+}S(0)\left[\frac{S(\alpha\varepsilon C^{1-\alpha}(\varphi^{-1})'(0))x-x}{\varepsilon}\right] = \lim_{\varepsilon\to0^+}\frac{S(\alpha\varepsilon C^{1-\alpha}(\varphi^{-1})'(0))x-x}{\varepsilon};\ S(0)=I\\
&= \alpha C^{1-\alpha}(\varphi^{-1})'(0)\lim_{\varepsilon\to0^+}\frac{S(\alpha\varepsilon C^{1-\alpha}(\varphi^{-1})'(0))x-x}{\alpha\varepsilon C^{1-\alpha}(\varphi^{-1})'(0)} = \frac{\alpha C^{1-\alpha}}{\varphi'(0)}\lim_{t\to0^+}\frac{S(t)x-x}{t}.
\end{aligned}$$

By consequence, $Ax = \frac{\alpha C^{1-\alpha}}{\varphi'(0)}\lim\limits_{t\to0^+}\frac{T((\varphi(t))^{\frac{1}{\alpha}})x-x}{t}$. □

Remark 4.8. In the particular case $\varphi(t)=t$, the expression (3) becomes

$$Ax = \alpha\lim_{t\to0^+}\frac{T(t^{\frac{1}{\alpha}})x-x}{t}. \tag{4}$$

Theorem 4.9. *Let A be the infinitesimal generator of the conformable $\mathcal{C}^0-(\alpha,\varphi)$-semigroup $\{T(t), t\geq 0\}$ and B be the infinitesimal generator of the $\mathcal{C}^0$-semigroup $\{S(t), t\geq0\}$on X. Then*

1. *$D(A)=D(B)$ and for all $x\in D(A)$, we have $Ax = \frac{\alpha C^{1-\alpha}}{\varphi'(0)}Bx$.*
2. *The domain $D(A)$ is dense in X and A is a closed linear operator.*

Proof.
1. let $x\in D(A)$, we have seen in the proof of Theorem 4.7 that $Ax = \frac{\alpha C^{1-\alpha}}{\varphi'(0)}\lim\limits_{t\to0^+}\frac{S(t)x-x}{t}$, so $x\in D(B)$ and $Ax = \frac{\alpha C^{1-\alpha}}{\varphi'(0)}Bx$. Now, let $x\in D(B)$ then by a similar way we show that $\lim\limits_{t\to0^+}T_\varphi^{(\alpha)}(T(t)x = \frac{\alpha C^{1-\alpha}}{\varphi'(0)}Bx$ exists. Hence, $x\in D(A)$. Therefore, $D(A)=D(B)$ and $Ax=\frac{\alpha C^{1-\alpha}}{\varphi'(0)}Bx$.
2. It is well known that the domain $D(B)$ of the $\mathcal{C}^0$-semigroup $\{S(t), t\geq0\}$ is dense in X and B is a closed linear operator (see [10]). Hence, the domain $D(A)$ is also dense in X and A is a closed linear operator, since $A=\frac{\alpha C^{1-\alpha}}{\varphi'(0)}B$.
□

Remark 4.10. By taking $\varphi(t)=t$, which leads to $\varphi'(t)=\varphi'(0)=C=1$,

- the identity (2) in Theorem 4.5 is reduced to $\|T(t)\|\leq Me^{wt^\alpha}$, for all $t\geq0$.
- The link between the two generators A and B turns into $A=\alpha B$.

Theorem 4.11. *Let A be the infinitesimal generator of the conformable $\mathcal{C}^0-(\alpha,\varphi)$-semigroup $\{T(t), t\geq 0\}$. Then, for all $x\in D(A)$, we have $T(t)x\in D(A)$, the function $g: t\mapsto T(t)x$ is (α,φ)-differentiable and*

$$\frac{d^\alpha}{dt^\alpha}(T(t)x)=\frac{\varphi'(0)}{\varphi'(\varphi^{-1}(t^\alpha))}\left(\frac{\varphi(t)}{C.t}\right)^{1-\alpha}AT(t)x=\frac{\varphi'(0)}{\varphi'(\varphi^{-1}(t^\alpha))}\left(\frac{\varphi(t)}{C.t}\right)^{1-\alpha}T(t)Ax. \tag{5}$$

Proof. In one hand, we have

$$\begin{aligned}
\frac{d^\alpha}{dt^\alpha}(T(t)x) &= T_\varphi^{(\alpha)}(T(t)x)=\lim_{\varepsilon\to 0^+}\frac{S\circ\varphi^{-1}(t^\alpha(1+\varepsilon\varphi(t)^{1-\alpha}/t)^\alpha)x-S\circ\varphi^{-1}(t^\alpha)x}{\varepsilon}\\
&= \lim_{\varepsilon\to 0^+}S\circ\varphi^{-1}(t^\alpha)\left[\frac{S(\alpha\varepsilon(\varphi(t)/t)^{1-\alpha}(\varphi^{-1})'(t^\alpha)+o(\varepsilon))x-x}{\varepsilon}\right]\\
&= T(t)\lim_{\varepsilon\to 0^+}\left[\frac{S(\alpha\varepsilon(\varphi(t)/t)^{1-\alpha}(\varphi^{-1})'(t^\alpha))x-x}{\varepsilon}\right]\\
&= \alpha(\varphi(t)/t)^{1-\alpha}(\varphi^{-1})'(t^\alpha)T(t)\lim_{h\to 0^+}\left[\frac{S(h)x-x}{h}\right]=\frac{\alpha(\varphi(t)/t)^{1-\alpha}}{\varphi'(\varphi^{-1}(t^\alpha))}T(t)Bx\\
&= \frac{\varphi'(0)}{\varphi'(\varphi^{-1}(t^\alpha))}\left(\frac{\varphi(t)}{C.t}\right)^{1-\alpha}T(t)Ax.
\end{aligned}$$

In the other hand, one can write

$$\begin{aligned}
T_\varphi^{(\alpha)}(T(t)x) &= \lim_{\varepsilon\to 0^+}\left[\frac{S(\alpha\varepsilon(\varphi(t)/t)^{1-\alpha}(\varphi^{-1})'(t^\alpha)+o(\varepsilon))S\circ\varphi^{-1}(t^\alpha)x-S\circ\varphi^{-1}(t^\alpha)x}{\varepsilon}\right]\\
&= \lim_{\varepsilon\to 0^+}\left[\frac{S(\alpha\varepsilon(\varphi(t)/t)^{1-\alpha}(\varphi^{-1})'(t^\alpha)+o(\varepsilon))T(t)x-T(t)x}{\varepsilon}\right]=\frac{\alpha(\varphi(t)/t)^{1-\alpha}}{\varphi'(\varphi^{-1}(t^\alpha))}BT(t)x\\
&= \frac{\varphi'(0)}{\varphi'(\varphi^{-1}(t^\alpha))}\left(\frac{\varphi(t)}{C.t}\right)^{1-\alpha}AT(t)x.
\end{aligned}$$

Finally, we have $T(t)x\in D(A)$ and

$$T_\varphi^{(\alpha)}(T(t)x)=\frac{\varphi'(0)}{\varphi'(\varphi^{-1}(t^\alpha))}\left(\frac{\varphi(t)}{C.t}\right)^{1-\alpha}T(t)Ax=\frac{\varphi'(0)}{\varphi'(\varphi^{-1}(t^\alpha))}\left(\frac{\varphi(t)}{C.t}\right)^{1-\alpha}AT(t)x.$$

□

Remark 4.12. The foregoing Theorem 4.11 was studied by Al Horani et al. [3], in the particular case $\varphi(t)=t$, but it was proved under the strong condition: "$T(t)$ is continuously α-differentiable". As it was shown above, this condition is superfluous, since it is sufficient to assume that $\{T(t), t\geq 0\}$ is a $\mathcal{C}^0-\alpha$-semigroup on X as in the classical case. Hence, for the generator of the conformable $\mathcal{C}^0-\alpha$-semigroup $\{T(t), t\geq 0\}$, the equality (5) is similar to usual formula and it becomes

$$\frac{d^\alpha}{dt^\alpha}(T(t)x)=AT(t)x=T(t)Ax. \tag{6}$$

Theorem 4.13. *Let A be the infinitesimal generator of the conformable $\mathcal{C}^0-(\alpha,\varphi)$-semigroup $\{T(t), t\geq 0\}$ and for fixed $x\in X$ let $g: s\mapsto T(s)x$. Then*

1. $\lim_{h\to 0}\frac{1}{h}\left(I^0_{\alpha,\varphi}(g)(t+h)-I^a_{\alpha,\varphi}(g)(t)\right)=(\varphi(t))^{\alpha-1}T(t)x.$
2. *For $x\in X$ let $k: s\mapsto \frac{\varphi'(0)}{\varphi'(\varphi^{-1}(s^\alpha))}\left(\frac{\varphi(s)}{C.s}\right)^{1-\alpha}T(s)x$, then $I^0_{\alpha,\varphi}(k)(t)\in D(A)$ and one has $A\left(I^0_{\alpha,\varphi}(k)(t)\right)=T(t)x-x$.*

3. *For fixed $x \in D(A)$, we consider* $h_1 : \tau \mapsto \frac{\varphi'(0)}{\varphi'(\varphi^{-1}(\tau^\alpha))}\left(\frac{\varphi(\tau)}{C.\tau}\right)^{1-\alpha} T(\tau)Ax$ *and* $h_2 : \tau \mapsto \frac{\varphi'(0)}{\varphi'(\varphi^{-1}(\tau^\alpha))}\left(\frac{\varphi(\tau)}{C.\tau}\right)^{1-\alpha} AT(\tau)x$, *then for $t \geq s \geq 0$, we have*

$$T(t)x - T(s)x = I^0_{\alpha,\varphi}(h_1)(t) - I^0_{\alpha,\varphi}(h_1)(s) = I^0_{\alpha,\varphi}(h_2)(t) - I^0_{\alpha,\varphi}(h_2)(s).$$

The proof of Theorem (4.13) is based on the application of the equality 5 and the well-known result $\lim\limits_{h\to 0}\frac{1}{h}\int_t^{t+h}\phi(s)ds = \phi(t)$, for continuous function $\phi : [0,\infty[\to X$.

5. Conformable differential equation with conformable semigroup

In this section, we study the following conformable fractional differential equation with initial condition

$$\begin{cases} \frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + f(t, x(t)),\ t \in [0,a]; \\ x(0) = x_0;\ x_0 \in X. \end{cases} \tag{7}$$

where $A : D(A) \to X$ and $f : [0,a] \times \mathcal{C}_a \to X$ satisfying the following assumptions:

(H_0) A is the infinitesimal generator of the conformable $\mathcal{C}^0 - (\alpha,\varphi)$-semigroup $\{T(t), t \geq 0\}$ on X such that $D(A) = X$ and there exists $M \geq 1$:

$$\|T(t)x - T(t)y\| \leq M\|x - y\|, \quad \text{for all } t \in [0,a], x, y \in X.$$

(H_1) $f : [0,a] \times \mathcal{C}_a \to X$ is a continuous function such that there exist $K > 0$, for all $t \in [0,a], x, y \in X$: $\|f(t,x) - f(t,y)\| \leq K\|x - y\|$.

(H_2) Suppose that the integral $\theta(t) = \frac{\alpha C^{1-\alpha}}{\varphi'(0)}\int_0^t (\varphi(s))^{\alpha-1}ds$ is convergent, for all $t \in [0,a]$.

Definition 5.1. *We say that x is a mild solution of equation* (7) *if*

(i) $x \in \mathcal{C}_a, x(t) \in D(A)$ *for all* $t \in [0,a]$, *such that* $x(0) = x_0$;

(ii) and for all $t \in [0,a]$:

$$x(t) = T\left((\varphi(\theta(t)))^{\frac{1}{\alpha}}\right)x_0 + \int_0^t (\varphi(s))^{\alpha-1}T\left((\varphi(\theta(t)-\theta(s)))^{\frac{1}{\alpha}}\right)f(s,x(s))ds.$$

Theorem 5.2. *Suppose that assumptions* $(H_0) - (H_2)$ *hold. Then equation* (7) *has a unique mild solution provided that* $L_0 = \frac{MK\varphi'(0)}{\alpha C^{1-\alpha}}\theta(a) < 1$.

Proof. We define a mapping $V : \mathcal{C}_a \to \mathcal{C}_a$, for all $x \in \mathcal{C}_a$, by

$$Vx(t) = T\left((\varphi(\theta(t)))^{\frac{1}{\alpha}}\right)x_0 + \int_0^t (\varphi(s))^{\alpha-1}T\left((\varphi(\theta(t)-\theta(s)))^{\frac{1}{\alpha}}\right)f(s,x(s))ds.$$

For $t \in [0,a]$, we have

$$Vx(t) = S(\theta(t))x_0 + \int_0^t (\varphi(s))^{\alpha-1}S(\theta(t)-\theta(s))f(s,x(s))ds.$$

(a) Step 1: Let $x \in \mathcal{C}_a$, we prove that $Vx \in \mathcal{C}_a$. For $t \in (0, a]$ and ξ very small such that $t + \xi \in (0, a]$:

$$\|Vx(t+\xi) - Vx(t)\| \leq R_1(\xi) + R_2(\xi) + R_3(\xi),$$

where using the assumption (H_0), one obtains

$$\begin{aligned} R_1(\xi) &= \|S(\theta(t+\xi))x_0 - S(\theta(t))x_0\| = \|S(\theta(t) + \xi\theta'(t) + o(\xi))x_0 - S(\theta(t))x_0\| \\ &= \|S(\theta(t))S(\xi\theta'(t) + o(\xi))x_0 - S(\theta(t))x_0\| \leq M\|S(\xi\theta'(t) + o(\xi))x_0 - x_0\|, \end{aligned}$$

Since, $\|S(\xi\theta'(t) + o(\xi))x_0 - x_0\| \to 0$, then $R_1(\xi) \to 0$ as $\xi \to 0^+$. Furthermore, we have

$$\begin{aligned} R_2(\xi) &= \left\|\int_0^\xi (\varphi(s))^{\alpha-1} S(\theta(t+\xi) - \theta(s)) f(s, x(s)) ds\right\| \leq \int_0^\xi (\varphi(s))^{\alpha-1} \left\|S(\theta(t+\xi) - \theta(s)) f(s, x(s))\right\| ds \\ &\leq M \int_0^\xi (\varphi(s))^{\alpha-1} \|f(s, x(s))\| ds \end{aligned}$$

It is clear that $\int_0^\xi (\varphi(s))^{\alpha-1}\|f(s, x(s))\|ds \to 0$ as $\xi \to 0^+$, thus $R_2(\xi) \to 0$ as $\xi \to 0^+$. Finally, we have using an affine change of variable

$$\begin{aligned} R_3(\xi) &= \left\|\int_\xi^{t+\xi} (\varphi(s))^{\alpha-1} S(\theta(t+\xi) - \theta(s)) f(s, x(s)) ds - \int_0^t (\varphi(s))^{\alpha-1} S(\theta(t) - \theta(s)) f(s, x(s)) ds\right\|. \\ &= \left\|\int_0^t \left[(\varphi(s+\xi))^{\alpha-1} S(\theta(t+\xi) - \theta(s+\xi)) f(s+\xi, x(s+\xi)) - (\varphi(s))^{\alpha-1} S(\theta(t) - \theta(s)) f(s, x(s))\right] ds\right\| \\ &\leq \int_0^t \left\|(\varphi(s+\xi))^{\alpha-1} S(\theta(t+\xi) - \theta(s+\xi)) f(s+\xi, x(s+\xi)) - (\varphi(s))^{\alpha-1} S(\theta(t) - \theta(s)) f(s, x(s))\right\| ds. \end{aligned}$$

And by the dominated convergence theorem, we get

$$\int_0^t \left\|(\varphi(s+\xi))^{\alpha-1} S(\theta(t+\xi) - \theta(s+\xi)) f(s+\xi, x(s+\xi)) - (\varphi(s))^{\alpha-1} S(\theta(t) - \theta(s)) f(s, x(s))\right\| ds$$

converges to 0 as $\xi \to 0^+$, so $R_3(\xi) \to 0$ as $\xi \to 0^+$. Hence, $\|Vx(t+\xi) - Vx(t)\| \to 0$ as $\xi \to 0^+$. Similarly, we prove that $\|Vx(t-\xi) - Vx(t)\| \to 0$ as $\xi \to 0^+$. So, Vx is continuous at each $t \neq 0$ and obviously at $t = 0$. Therefore, $Vx \in \mathcal{C}_a$ i.e., V maps $\mathcal{C}_a$ into itself.

(b) Step 2: Claim: V is a contraction on $\mathcal{C}_a$. For $x, y \in \mathcal{C}_a$ and $t \in [0, a]$, we have

$$\begin{aligned} \|Vx(t) - Vy(t)\| &= \left\|\int_0^t (\varphi(s))^{\alpha-1} S\left(\theta(t) - \theta(s)\right) \left[f(s, x(s)) - f(s, y(s))\right] ds\right\| \\ &\leq \int_0^t (\varphi(s))^{\alpha-1} \left\|S\left(\theta(t) - \theta(s)\right) \left[f(s, x(s)) - f(s, y(s))\right]\right\| ds \\ &\leq MK \int_0^t (\varphi(s))^{\alpha-1} \|x(s) - y(s)\| ds \\ &\leq \frac{MK\varphi'(0)}{\alpha C^{1-\alpha}} \theta(a) H_a(x, y) = L_0 H_a(x, y). \end{aligned}$$

Thus, $H_a(Vx, Vy) \leq L_0 H_a(x, y)$. Since $L_0 < 1$, then V is a contraction on the complete metric space $(\mathcal{C}_a, H_a)$. Hence, there exists a unique $x \in \mathcal{C}_a$ such that $Vx = x$. So, x is the unique mild solution of equation (7).

□

Theorem 5.3. *(Continuous dependence on initial data). Suppose that assumptions* $(H_0)-(H_2)$ *and the condition* $L_0 = \frac{MK\varphi'(0)}{\alpha C^{1-\alpha}}\theta(a) < 1$ *hold true. Let* $x = x(t, x_0)$ *and* $y = y(t, y_0)$ *be mild solutions of equation* (7) *corresponding to* x_0 *and* y_0 *respectively. Then*

$$H_a(x, y) \le M\|x_0 - y_0\| \exp\left(\frac{MK\varphi'(0)}{\alpha C^{1-\alpha}}\theta(a)\right). \tag{8}$$

Proof. Let $t \in [0, a]$, for all $u \in [0, t]$ we have

$$\begin{aligned}
\|x(u) - y(u)\| &\le \|T\left((\varphi(\theta(t)))^{\frac{1}{\alpha}}\right)x_0 - T\left((\varphi(\theta(t)))^{\frac{1}{\alpha}}\right)y_0\| \\
&\quad + \int_0^u (\varphi(s))^{\alpha-1}\left\|T\left((\varphi(\theta(u)-\theta(s)))^{\frac{1}{\alpha}}\right)[f(s, x(s)) - f(s, y(s))]\right\| ds \\
&\le M\|x_0 - y_0\| + M\int_0^u (\varphi(s))^{\alpha-1}\left\|f(s, x(s)) - f(s, y(s))\right\| ds \\
&\le M\|x_0 - y_0\| + MK\int_0^u (\varphi(s))^{\alpha-1}\|x(s)) - y(s)\| ds
\end{aligned}$$

Passing to the supremum on $[0, t]$, we obtain

$$H_t(x, y) \le M\|x_0 - y_0\| + MK\int_0^t (\varphi(s))^{\alpha-1}H_s(x, y)ds$$

Then by Gronwall's inequality, we have for all $t \in [0, a]$

$$H_t(x, y) \le M\|x_0 - y_0\| \exp\left(\frac{MK\varphi'(0)}{\alpha C^{1-\alpha}}\theta(a)\right)$$

By consequence

$$H_a(x, y) \le M\|x_0 - y_0\| \exp\left(\frac{MK\varphi'(0)}{\alpha C^{1-\alpha}}\theta(a)\right)$$

□

Remark 5.4. In the particular case $\varphi : t \mapsto \varphi(t) = t$, the expression of the mild solution for equation (7) becomes, for all $t \in [0, a]$: $x(t) = T(t)x_0 + \int_0^t s^{\alpha-1}T((t^\alpha - s^\alpha)^{\frac{1}{\alpha}})f(s, x(s))ds$. The value of L_0 in Theorem 5.2 is $L_0 = MK\frac{a^\alpha}{\alpha} < 1$. And inequation (8) in Theorem 5.3 is simplified to

$$H_a(x, y) \le M\|x_0 - y_0\| \exp\left(MK\frac{a^\alpha}{\alpha}\right)$$

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