Filomat 39:26 (2025), 9135–9155
https://doi.org/10.2298/FIL2526135S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Existence of solutions to the Steklov $p(x)$-Laplacian problem involving the critical exponent

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Abstract. In this paper, we consider a class of Steklov $p(x)$-Laplacian problems with a critical exponent, given by the following equation:

$$\begin{cases} (-\Delta)_{p(x)}u = |u|^{r(x)-2}u + f(x,u), & \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial v} = |u|^{s(x)-2}u, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded domain with Lipschitz boundary $\partial\Omega$. Here, $\frac{\partial}{\partial v}$ denotes the outer unit normal derivative, the function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying appropriate assumptions, and the functions p and r are continuous in $\overline{\Omega}$, such that $1 < r(x) \leq p^*(x)$ for all $x \in \Omega$, where $p^*(x)$ represents the critical Sobolev exponent. To establish the existence and multiplicity of solutions, we employ variational methods, including the mountain pass theorem and symmetric mountain pass theorem, combined with the concentration-compactness principle. These techniques enable us to find solutions that satisfy the given boundary conditions and exhibit interesting properties related to the critical exponent.

1. Introduction

In this paper, we focus on a class of Steklov $p(x)$-Laplacian problems with a critical exponent. The problem is formulated as follows:

$$\begin{cases} (-\Delta)_{p(x)}u = |u|^{r(x)-2}u + f(x,u) & \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial v} = |u|^{s(x)-2}u & \text{on } \partial\Omega, \end{cases} \tag{1}$$

2020 *Mathematics Subject Classification*. Primary 35J15; Secondary 35J62, 35P30, 47B92, 47H05.
Keywords. The $p(x)$-laplace operator, Variational methods, Symmetric mountain pass Theorem, Critical Sobolev exponent, Concentration-compactness principle
Received: 07 May 2024; Revised: 21 July 2025; Accepted: 01 August 2025
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where Ω is a bounded domain in $\mathbb{R}^N$, with Lipschitz boundary denoted by $\partial\Omega$. The symbol $\frac{\partial}{\partial \nu}$ represents the outer unit normal derivative and $\Delta_{p(x)}u = div\left(|\nabla u|^{p(x)-2}\ \nabla u\right)$.
The function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function that satisfies appropriate assumptions

We assume that both $p(x)$ and $r(x)$ are continuous functions in $\overline{\Omega}$, meaning they are defined and continuous on the closure of Ω. Moreover, we consider the condition $1 < p(x) < r(x) \leq p^*(x)$ for all $x \in \Omega$, where $p^*(x)$ represents the critical Sobolev exponent. Additionally, we assume that the set $A = \{x \in \Omega : r(x) = p^*(x)\}$ is non-empty.

In nonlinear elliptic equations with critical growth, the concentration-compactness principle introduced by Lions (see [23]) has been widely recognized as a fundamental tool for establishing the existence of solutions. This principle is particularly crucial when considering equations involving Sobolev embeddings, which capture the critical growth behavior. For a more comprehensive understanding of this topic, we suggest referring to the references [3, 4, 6, 12, 17–22, 27, 30–32] and the additional sources mentioned therein.

In their work [8], Bonder and al. extend Lions' well-known concentration-compactness principle to the setting of variable exponents. This extension is stated as Theorem 3.8. By considering equations with variable exponents, they allow for more flexibility in modeling various physical phenomena and capturing the heterogeneity of the problem domain.

To apply the extended concentration-compactness principle, The authors in [8] consider the following nonlinear elliptic equation:

$$\begin{cases} (-\Delta)_{p(x)}u = |u|^{r(x)-2}u + a(x)|u|^{q(x)-2}u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

here, p, q, and r are functions belonging to the space $C(\overline{\Omega})$, where Ω represents the domain of the problem. These functions are subject to the condition $1 < p(x) < r(x) \leq p^*(x)$ for all $x \in \Omega$, where $p(x)$ denotes the critical Sobolev exponent associated with $p(x)$. The set $A = \{x \in \Omega : r(x) = p^*(x)\}$ is assumed to be non-empty, indicating the presence of the critical growth behavior. To establish the existence of solutions, Bonder and al. [8] employ variational methods and make use of the mountain pass theorem. These techniques allow them to construct a suitable functional and apply critical point theory to find nontrivial solutions to the problem.

The study of problems with variable exponents has received significant attention in recent years. These problems have proven to be interesting and relevant in various applications, such as the modeling of electro-rheological fluids [13, 26, 28] and image processing [11]. Additionally, they give rise to challenging mathematical problems that require careful investigation. Also, we note that in reference [36], a concise summary of recent advancements and references in the study of problems with variable exponents is provided. This resource offers valuable insights into the progress made in this field, serving as a useful reference for further exploration. Furthermore, the study of variational problems with nonstandard growth is a new and intriguing topic, as mentioned in the references cited as [1, 2, 5, 7, 10, 14, 24, 25, 29, 34, 35]. These works contribute to understanding the behavior and properties of variational problems with nonstandard growth, expanding the knowledge and research in this field.

In their publication [9], the authors extensively investigated the following nonlinear Steklov problem with Neumann boundary value conditions:

$$\begin{cases} (-\Delta)_{p(x)}u + a(x)|u|^{p(x)-2}u = f(x,u) \quad \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} + b(x)|u|^{q(x)-2}u = g(x,u) \quad \text{on } \partial\Omega. \end{cases} \qquad (P)$$

Under suitable conditions on the functions a, b, p, q, f, and g, the authors employed variational methods, the mountain pass lemma, and the Ekeland principle to establish the existence and multiplicity of solutions for problem (P). Notably, they imposed the condition $f(x,u) \leq v_1(x)|u|^{\alpha(x)-1}$, where $\alpha(x) < p^*(x)$.

Motivated by the results presented in reference [9], our paper aims to contribute further by studying the critical case of the aforementioned problem. To this end, we utilize a recent concentration-compactness principle for Sobolev spaces with variable exponents to investigate the weighted Steklov problem (1). Our study provides a generalization, improvement, and extension of the aforementioned references under additional, appropriate conditions. Consequently, this research project holds significant importance and offers valuable insights.

In this paper, we consider the problem (1) in two distinct cases. Firstly, we examine the scenario where $f(x,u) = v_1(x)h_1(u)$. Under specific hypotheses, we employ the variational method, the mountain pass theorem, and the symmetric mountain pass theorem to establish the existence and multiplicity of nontrivial weak solutions for problem (1). This rigorous approach ensures the robustness and reliability of our results.

Secondly, we investigate the case where $f(x,u) = v_1(x)h_1(u) + \lambda|u|^{\gamma(x)-2}u$. By imposing suitable assumptions, we utilize the mountain pass theorem and the symmetric mountain pass theorem to prove the existence and multiplicity of nontrivial solutions for problem (1). This analysis provides valuable insights into the behavior and properties of the solutions in this particular case, further enriching the understanding of the problem.

In summary, our research significantly contributes to the existing literature by exploring the critical case of the Steklov problem with Neumann boundary conditions. Through rigorous mathematical techniques and the utilization of recent concentration-compactness principles, we establish the existence and multiplicity of solutions for problem (1) under different scenarios, enhancing the overall understanding of this important topic.

This paper is organized as follows: In Section 2, we present some necessary preliminary knowledge on variable exponent Lebesgue and Sobolev spaces. Section 3 contains our first main result, where we present and prove the existence and multiplicity of solutions for the weighted Steklov problem. In Section 4, we present and prove our second result, further extending our study of the weighted Steklov problem.

2. Preliminaries

In this section, we provide an overview of some important properties of variable exponent spaces. For more detailed information, we recommend referring to the works [2, 9, 10, 15, 16, 33] and the references therein. Let Ω be a bounded domain in $\mathbb{R}^N$, where $N \geq 2$. We consider the set

$$C_+(\overline{\Omega}) = \left\{ p \in C(\overline{\Omega}), p(x) > 1, \forall, x \in \overline{\Omega} \right\}.$$

For all $p \in C_+(\overline{\Omega})$, we define,

$$p^- = \inf_{\overline{\Omega}} p(x), \qquad \text{and} \qquad p^+ = \sup_{\overline{\Omega}} p(x).$$

Additionally, we define

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R}, \text{ measurable} : \int_\Omega |u(x)|^{p(x)} dx < \infty \right\},$$

with the norm on $L^{p(x)}(\Omega)$ defined as

$$|u|_{L^{p(x)}(\Omega)} = \inf \left\{ \mu > 0 : \int_\Omega |\frac{u(x)}{\mu}|^{p(x)} dx \leq 1 \right\}.$$

Also, we define

$$L^{p(x)}(\partial\Omega) = \left\{ u : \Omega \to \mathbb{R}, \text{ measurable} : \int_{\partial\Omega} |u(x)|^{p(x)} dx < \infty \right\},$$

with the norm on $L^{p(x)}(\partial\Omega)$ defined as

$$|u|_{L^{p(x)}(\partial\Omega)} = \inf\left\{\mu > 0 : \int_{\partial\Omega} |\frac{u(x)}{\mu}|^{p(x)}dx \leq 1\right\},$$

The space $\left(L^{p(x)}(\Omega), |.|_{L^{p(x)}(\Omega)}\right)$ and $\left(L^{p(x)}(\partial\Omega), |.|_{L^{p(x)}(\partial\Omega)}\right)$ are a Banach spaces, which we refer to as variable exponent Lebesgue spaces. Now, we introduce the space

$$W^{1,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\right\},$$

equipped with the norm

$$\|u\| = \|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{L^{p(x)}(\Omega)} + \|\nabla u\|_{L^{p(x)}(\Omega)}.$$

Finally, we denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$.
The following proposition provides important properties of variable exponent spaces.

Proposition 2.1. (*see [9, 15]*) *The following statements hold:*

1. *The space $\left(L^{p(x)}(\Omega), |.|_{L^{p(x)}(\Omega)}\right)$ is a separable and uniformly convex Banach space, and its conjugate space is $L^{p'(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. Moreover, the Hölder inequality holds, that is, for any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have*
$$\Big|\int_\Omega uvdx\Big| \leq (\frac{1}{p^-} + \frac{1}{(p')^-})|u|_{p(x)}|v|_{p'(x)}.$$
2. *If $p_1, p_2 \in C_+(\overline{\Omega})$ such that $p_1(x) \leq p_2(x)$ for all $x \in \overline{\Omega}$, then the embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$ is continuous.*

Note that $\|u\|$ and $\|\nabla u\|_{L^{p(x)}(\Omega)}$ are equivalent in the space $W_0^{1,p(x)}(\Omega)$, see [9, 10], so, for simplicity let's use $\|u\| = \|\nabla u\|_{L^{p(x)}(\Omega)}$.

The following proposition highlights the properties of the variable exponent Sobolev spaces.

Proposition 2.2. (*see [15, 16]*)*The following statements hold:*

1. *The spaces $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.*
2. *If $q \in C_+(\overline{\Omega})$ with $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$, then the embedding from $W^{1,p(x)}(\Omega)$ into $L^{q(x)}(\Omega)$ is compact and continuous. Here, $p^*(x)$ is defined as follows:*
$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & if \ \ p(x) < N, \\ \infty, & if \ \ p(x) \geq N, \end{cases}$$
3. *If $q \in C_+(\partial\Omega)$ with $q(x) < p_*(x)$ for all $x \in \partial\Omega$, then the embedding from $W^{1,p(x)}(\Omega)$ into $L^{q(x)}(\partial\Omega)$ is compact and continuous. Here, $p_*(x)$ is defined as follows:*
$$p_*(x) = \begin{cases} \frac{(N-1)p(x)}{N-p(x)}, & if \ \ p(x) < N, \\ \infty, & if \ \ p(x) \geq N, \end{cases}$$
here N is the dimension of the space.

For simplicity, let us denote

$$\Gamma(u) = \int_\Omega |\nabla u|^{p(x)}dx.$$

The following proposition provides important properties of the functional Γ.

Proposition 2.3. *(see [9, 36]) There exist positive constants ξ_1 and ξ_2 such that*

1. *If $\Gamma(u) \geq 1$, then $\xi_1\|u\|^{p^-} \leq \Gamma(u) \leq \xi_2\|u\|^{p^+}$.*
2. *If $\Gamma(u) \leq 1$, then $\xi_1\|u\|^{p^+} \leq \Gamma(u) \leq \xi_2\|u\|^{p^-}$.*
3. $\Gamma(u) \geq 1 (= 1, \leq 1) \Leftrightarrow \|u\| \geq 1 (= 1, \leq 1)$.

Let us define

$$\rho(u) = \int_\Omega |u(x)|^{p(x)} dx.$$

The next proposition provides properties of the functional ρ.

Proposition 2.4. *(see [9, 15, 16]) For all $u \in L^{p(x)}(\Omega)$, we have*

1. $|u|L^{p(x)}(\Omega) < 1; (resp = 1, > 1) \Leftrightarrow \rho(u) < 1; (resp = 1, > 1)$.
2. $|u|_{L^{p(x)}(\Omega)} > 1 \Rightarrow |u|^{p^-}_{L^{p(x)}(\Omega)} \leq \rho(u) \leq |u|^{p^+}_{L^{p(x)}(\Omega)}$.
3. $|u|_{L^{p(x)}(\Omega)} < 1 \Rightarrow |u|^{p^+}_{L^{p(x)}(\Omega)} \leq \rho(u) \leq |u|^{p^-}_{L^{p(x)}(\Omega)}$.

The next proposition relates the norms of a function in variable exponent Lebesgue spaces with its pointwise behavior:

Proposition 2.5. *(see [9, 15, 16]) If p and q are measurable functions such that $p \in L^\infty(\mathbb{R}^N)$ and $1 \leq p(x)q(x) \leq \infty$ for all $x \in \mathbb{R}^N$, then for all $u \in L^{q(x)}(\mathbb{R}^N)$ with $u \neq 0$, we have*

1. $|u|_{p(x)q(x)} \leq 1 \Rightarrow |u|^{q^+}_{p(x)q(x)} \leq \||u|^{p(x)}|_{q(x)} \leq |u|^{q^-}_{p(x)q(x)}$.
2. $|u|_{p(x)q(x)} \geq 1 \Rightarrow |u|^{q^-}_{p(x)q(x)} \leq \||u|^{p(x)}|_{q(x)} \leq |u|^{q^+}_{p(x)q(x)}$.

Denote for $u \in L^{p(\xi)}(\partial\Omega)$,

$$\rho_\partial(u) = \int_{\partial\Omega} |u(x)|^{p(\xi)} d\sigma.$$

Proposition 2.6. [9, 10] *For all $u \in L^{p(\xi)}(\partial\Omega)$, we have,*
(1) $|u|_{L^{p(\xi)}(\partial\Omega)} > 1 \Rightarrow |u|^{p^-}_{L^{p(\xi)}(\partial\Omega)} \leq \rho_\partial(u) \leq |u|^{p^+}_{L^{p(\xi)}(\partial\Omega)}$,
(2) $|u|_{L^{p(\xi)}(\partial\Omega)} < 1 \Rightarrow |u|^{p^+}_{L^{p(\xi)}(\partial\Omega)} \leq \rho_\partial(u) \leq |u|^{p^-}_{L^{p(\xi)}(\partial\Omega)}$.

Finally, let's define the Palais-Smale (PS) condition at a given level c for $\varphi \in C^1(X, \mathbb{R})$:

Definition 2.7. *Let X be a Banach space and $\varphi \in C^1(X, \mathbb{R})$, where $c \in \mathbb{R}$. We say that φ satisfies the (PS) condition at level c if any sequence $u_n \subset X$, such that*

$$\varphi(u_n) \to c, \text{ and } \varphi'(u_n) \to 0, \text{ in } X^*, \text{ as } n \to \infty,$$

contains a convergent subsequence.

In the context of our analysis and proofs, we will now introduce and recall several important theorems: the Mountain Pass Theorem, its symmetric version for even functions. These theorems play a crucial role in establishing our results. Here are the statements of the theorems:

Theorem 2.8. *(Mountain pass theorem)(see [2]) Let X be a Banach space. Consider a functional $\varphi \in C^1(X, \mathbb{R})$ satisfying the following conditions:*

1. $\varphi(0) = 0$,
2. *φ satisfies the Palais-Smale condition,*
3. *There exist positive constants η and ρ such that if $\|u\| = \eta$, then $\varphi(u) \geq \rho$,*

4. *There exists $e \in X$ with $\|e\| > \eta$ such that $\varphi(e) \leq 0$. Then, φ possesses a critical value $c \geq \rho$ which can be characterized as*
$$c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\varphi(\gamma(t)),$$
where,
$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = e\}.$$

Theorem 2.9. *(Symmetric mountain pass theorem)(see [2]) Let X be an infinite dimensional real Banach space. Let $\varphi \in C^1(X,\mathbb{R})$, satisfying the following conditions:*

1. *φ is an even functional such that $\varphi(0) = 0$,*
2. *φ satisfies the (PS)-condition,*
3. *There exist positive constants η and ρ, such that if $\|u\| = \eta$, then, $\varphi(u) \geq \rho$,*
4. *For each finite dimensional subspace $X_1 \subset X$, the set $\{u \in X_1, \varphi(u) \geq 0\}$ is bounded in X. Then φ has an unbounded sequence of critical values.*

Throughout the rest of the paper, the constants mentioned in the theorems, such as $c_i, i = 1, 2, ...,$, are positive constants that may vary from line to line in the proofs.

3. First main results

In this section, we will present and demonstrate the first main result of this paper. Before proceeding, we assume the following hypotheses:

(A₁) The function $f(x,u)$ can be expressed as $v_1(x)h_1(u)$, where v_1 and h_1 are measurable functions satisfying the following conditions: there exists $c_1 > 0$, $\alpha, S \in C_+(\overline{\Omega})$ such that for all $(x,u) \in \Omega\times\mathbb{R}$, we have

$$v_1(x) \in L^{\frac{S(x)}{S(x)-\alpha(x)}}(\Omega),\ h_1(u) \leq c_1 |u|^{\alpha(x)-1},$$

and

$$p^+ < \alpha(x) < S(x) < p^*(x) \ \text{and}\ p^+ < N. \tag{2}$$

(A₂) There exist $M_1 > 0$, $p^+ < \theta < \min(r^-, s^-)$, such that for all $x \in \Omega$, we have

$$0 < \theta v_1(x)H_1(u) \leq v_1(x)h_1(u)u,\ |u| \geq M_1,$$

where $H_1(t) = \displaystyle\int_0^t h_1(s)ds.$

(A₃) We have $p^+ \leq s(x) < p_*(x)$.

(A₄) For all $x \in \overline{\Omega}$, we have $h_1(-u) = -h_1(u)$.

(A₅) We have $\alpha^- < \min(r^-, s^-)$.

Next, we define a weak solution for the problem (1) as follows:

Definition 3.1. *We say that $u \in X := W_0^{1,p(x)}(\Omega)$ is a weak solution for the problem* (1) *if, for any $v \in X$, we have*

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla v - \int_\Omega |u|^{r(x)-2}uvdx - \int_\Omega v_1(x)h_1(u)vdx - \int_{\partial\Omega} |u|^{s(x)-2}uvdx = 0.$$

Now, we are ready to state and prove the first main results.

Theorem 3.2. *Under the hypotheses* $(\mathbf{A_1}) - (\mathbf{A_3})$, *problem* (1) *has a nontrivial weak solution.*

Theorem 3.3. *Under the hypotheses* $(\mathbf{A_1}) - (\mathbf{A_4})$, *problem* (1) *has infinitely many solutions.*

Now, we introduce the functional $\chi(u)$ associated with problem (1), which characterizes the critical points and plays a key role in the existence of solutions.

$$\chi(u) = L(u) - J(u) - \phi(u) - T(u),$$

where

$$L(u) = \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx, \;\; J(u) = \int_\Omega \frac{|u|^{r(x)}}{r(x)}dx \;\; \phi(u) = \int_\Omega v_1(x)H_1(u)dx \;\; and \;\; T(u) = \int_{\partial\Omega} \frac{|u|^{s(x)}}{s(x)}dx.$$

We recall from [9], that $L \in C^1(X, \mathbb{R})$. Moreover, for all $u, v \in X$, we have

$$< L'(u), v >= \int_\Omega (|\nabla u|^{p(x)-2}\nabla u \nabla v dx.$$

The functional L' satisfies the following properties.

Proposition 3.4. *[9]*

1. $L' : X \to X^*$ *is a continuous, bounded, and strictly monotone operator.*
2. L' *is a mapping of* (S_+) *type, that is, if* $u_n \rightharpoonup u$ *in* X *and* $\limsup_{n\to\infty} < L'(u_n) - L'(u), u_n - u > \leq 0$, *then,* $u_n \to u$ *strongly in* X.

Remark 3.5. *It can be shown, using* (A_1), *Propositions 2.3, 2.5, and the Hölder inequality, that* $\phi \in C^1(X, \mathbb{R})$. *Furthermore, for all* $u, v \in X$, *we have*

$$< \phi'(u), v >= \int_\Omega v_1(x)h_1(u(x))v(x)dx.$$

From Proposition 3.4 and Remark 3.5, it follows that $\chi \in C^1(X, \mathbb{R})$. Moreover, for all $u, v \in X$, we obtain

$$\begin{aligned} < \chi'(u), v > &= \int_\Omega (|\nabla u|^{p(x)-2}\nabla u \nabla v dx - \int_\Omega |u|^{r(x)-2}uv dx \\ &- \int_\Omega v_1(x)h_1(u(x))v(x)dx - \int_{\partial\Omega} |u|^{s(x)-2}uv dx. \end{aligned}$$

Hence, the weak solutions of problem (1) correspond to the critical points of the functional χ.

Now, we establish a key result that provides a lower bound for the functional $\chi(u)$ associated with problem (1).

Lemma 3.6. *Assume that* $(\mathbf{A_1}) - (\mathbf{A_3})$ *is satisfied. Then, there exist* $m, \eta > 0$ *such that, for* $u \in X$,

$$if \;\; \|u\| = \eta, \; then, \; \chi(u) \geq m.$$

Proof. Let $u \in X$, with $\|u\| < 1$. Under the hypothesis $(\mathbf{A_1})$, we have for all $x \in \Omega$,

$$H_1(u) \leq \frac{c_1}{\alpha(x)}|u|^{\alpha(x)}. \tag{3}$$

Using propositions 2.1, 2.5, and the Hölder inequality, and according to proposition 2.2, we obtain the existence of $c_3, c_4\, c_5 > 0$, such that

$$|u|_{L^{s(x)}(\Omega)} \leq c_3 \|u\|, |u|_{L^{r(x)}(\Omega)} \leq c_4 \|u\| \text{ and } , |u|_{L^{s(x)}(\partial\Omega)} \leq c_5 \|u\| \tag{4}$$

Since $1 < S(x) < p^*(x)$, $1 < r(x) \leq p^*(x)$, $1 < s(x) < p_*(x)$ Then, using Proposition 2.3, we obtain

$$\begin{aligned}\chi(u) &\geq \frac{\xi_1}{p^+}\|u\|^{p^+} - \frac{c_3}{\alpha^-}|v_1|_{L^{\frac{S(x)}{S(x)-\alpha(x)}}(\Omega)}\|u\|^{\alpha^-} - \frac{c_4}{r^-}\|u\|^{r^-} - \frac{c_5}{s^-}\|u\|^{s^-} \\ &\geq \|u\|^{p^+}\left(\frac{\xi_1}{p^+} - \frac{c_3}{\alpha^-}|v_1|_{L^{\frac{S(x)}{S(x)-\alpha(x)}}(\Omega)}\|u\|^{\alpha^- - p^+} - \frac{c_4}{r^-}\|u\|^{r^- - p^+} - \frac{c_5}{s^-}\|u\|^{s^- - p^+}\right) \\ &\geq \|u\|^{p^+}\left(\frac{\xi_1}{p^+} - t\|u\|^{\min(\alpha^- - p^+, r^- - p^+, s^- - p^+)}\right),\end{aligned}$$

where

$$t = \frac{c_3}{\alpha^-}|v_1|_{L^{\frac{S(x)}{S(x)-\alpha(x)}}(\Omega)} + \frac{c_4}{r^-} + \frac{c_5}{s^-}.$$

Since α^-, s^- and r^- are both greater than p^+, we can choose $\|u\| = \eta$ to be sufficiently small such that

$$\frac{\xi_1}{p^+} - t\eta^{\min(\alpha^- - p^+, r^- - p^+, s^- - p^+)} > 0.$$

Finally, we conclude that

$$\chi(u) \geq \eta^{p^+}\left(\frac{\xi_1}{p^+} - t\,\eta^{\min(\alpha^- - p^+, r^- - p^+, s^- - p^+)}\right) := m > 0.$$

□

In the following lemma, we establish a result regarding the boundedness of a Palais-Smale sequence in X.

Lemma 3.7. *Suppose that conditions* $(\mathbf{A_1}) - (\mathbf{A_2})$ *are satisfied. Let* $\{u_n\}$ *be a Palais-Smale sequence in* X*. Then* $\{u_n\}$ *is bounded in* X*.*

Proof. Let $\{u_n\}$ be a sequence in X such that

$$\chi(u_n) \to c, \text{ and } \chi'(u_n) \to 0, \text{ in } X^*, \text{ as } n \to \infty,$$

where c is a positive constant.

Since $\chi(u_n) \to c$, there exists $M_1 > 0$, such that

$$|\chi(u_n)| \leq M_1. \tag{5}$$

On the other hand, the fact that $\chi'(u_n) \to 0$ in X^*, implies that $< \chi'(u_n), u_n > \to 0$. In particular, $< \chi'(u_n), u_n >$ is bounded. Thus, there exists $M_2 > 0$, such that

$$|< \chi'(u_n), u_n >| \leq M_2. \tag{6}$$

We claim that the sequence $\{u_n\}$ is bounded. If it is not true, by passing to a sub-sequence if necessary, we may assume that $\|u_n\| \to \infty$. Without loss of generality, we assume that $\|u_n\| \geq 1$.

From (5), (6) and using the fact that $p^+ < \theta < \min(r^-, s^-)$, we obtain

$$\begin{aligned}M_1 \geq \chi(u_n) &= I(u_n) - J(u_n) - \phi(u_n) - T(u_n) \\ &\geq \frac{1}{p^+}\Gamma(u_n) - \frac{1}{r^-}\int_\Omega |u_n|^{r(x)}dx - \phi(u_n) - \frac{1}{s^-}\int_{\partial\Omega}|u_n|^{s(x)}dx \\ &\geq \frac{1}{p^+}\Gamma(u_n) - \frac{1}{\theta}\int_\Omega |u_n|^{r(x)}dx - \phi(u_n) - \frac{1}{\theta}\int_{\partial\Omega}|u_n|^{s(x)}dx,\end{aligned} \tag{7}$$

and

$$M_2 \geq -<\chi'(u_n), u_n> = -\Gamma(u_n) + \int_\Omega |u_n|^{r(x)} dx + \int_{\partial\Omega} |u_n|^{s(x)} dx + <\phi'(u_n), (u_n)>, \tag{8}$$

By combining (7), (8), and using proposition 2.3, we obtain

$$\begin{aligned} \theta M_1 + M_2 &\geq (\frac{\theta}{p^+} - 1)\Gamma(u_n) - \theta\phi(u_n) + <\phi^{'}(u_n), (u_n)> \\ &\geq (\frac{\theta}{p^+} - 1)\xi_1 \|u_n\|^{p^-} + \int_\Omega v_1(x)(h_1(u_n)u_n - \theta H_1(u_n))dx. \end{aligned}$$

Hence, assumption ($\mathbf{A_2}$) implies

$$\theta M_1 + M_2 \geq (\frac{\theta}{p^+} - 1)\xi_1 \|u_n\|^{p^-}.$$

So, by letting n tend to infinity, we obtain a contradiction. Therefore, $\{u_n\}$ is bounded in X. □

Now, we introduce the nonempty set A define by $A = \{x \in \Omega : r(x) = p^*(x)\}$. Also, define the set $A_\delta = \{x \in \Omega : dist((x, A) < \delta\}$ for $\delta > 0$. We note $r_\delta^- = \inf_{\overline{A_\delta}} r(x)$, and $r_A^- = \inf_{\overline{A}} r(x)$.
We will now introduce and recall several important theorem.

Theorem 3.8. *(Concentration-compactness principle)(see [8]) Let $p(x)$ and $r(x)$ be two continuous functions such that*

$$p^- = \inf_{\overline{\Omega}} p(x) \leq p^+ = \sup_{\overline{\Omega}} p(x) < N \ \ and \ \ 1 < r(x) \leq p^*(x) \quad in \ \ \Omega$$

Let $\{u_j\}_{j\in\mathbb{N}}$ be a weakly convergent sequence in $W^{1,p(x)}(\Omega)$ with weak limit u and such that:
- *$|u_j|^{r(x)} \rightharpoonup \nu$ weakly in the sense of measures.*
- *$|\nabla u_j|^{p(x)} \rightharpoonup \mu$ weakly in the sense of measures.*

Also assume that $A = \{x \in \Omega : r(x) = p^*(x)\}$ is nonempty. Then, for some countable index set I, we have:

$$\nu = |u|^{r(x)} + \sum_{i\in I} \nu_i \delta_{x_i}, \qquad v_i > 0.$$

$$\mu \geq |\nabla u|^{p(x)} + \sum_{i\in I} \mu_i \delta_{x_i}, \qquad \mu_i > 0.$$

$$S\nu_i^{\frac{1}{p^*(x_i)}} \leq \mu_i^{\frac{1}{p(x_i)}}.$$

where $\{x_i\}_{i\in I} \subset A$ and S is the best constant in the Gagliardo-Nirenberg-Sobolev inequality for variable exponents, namely

$$S = S_r(\Omega) = \inf_{\phi\in C_0^\infty(\Omega)} \frac{\|\nabla\phi\|_{L^{p(x)}}}{\|\phi\|_{L^{r(x)}}}.$$

If $\{u_n\}$ is a Palais-Smale sequence with energy level c, then according to Theorem 3.8, we have the following convergence results:

$$|u_n|^{r(x)} \rightharpoonup \nu = |u|^{r(x)} + \sum_{i\in I} \nu_i \delta_{x_i}, \qquad v_i > 0. \tag{9}$$

$$|\nabla u_n|^{p(x)} \rightharpoonup \mu \geq |\nabla u|^{p(x)} + \sum_{i\in I} \mu_i \delta_{x_i}, \qquad \mu_i > 0. \tag{10}$$

$$S\nu_i^{\frac{1}{p^*(x_i)}} \leq \mu_i^{\frac{1}{p(x_i)}}. \tag{11}$$

If the set $I = \emptyset$, then $u_n \to u$ in $L^{r(x)}(\Omega)$. It should be noted that $\{x_i\}_{i\in A} \subset A$.

We aim to demonstrate that if $c < (\frac{1}{p^+} - \frac{1}{r_A^-})S^N$, then $I = \emptyset$, where S is defined in Theorem 3.8.

The following lemma establishes an important result regarding the behavior of Palais-Smale sequences under certain conditions.

Lemma 3.9. *If conditions* $(\mathbf{A_1}) - (\mathbf{A_3})$ *are satisfied. Let* $\{u_n\}$ *be a Palais-Smale sequence in* X *with energy level* c*. If* $c < (\frac{1}{p^+} - \frac{1}{r_A^-})S^N$*, then the index set* I *is empty.*

Proof. Suppose that $I \neq \emptyset$ and let $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $\varphi(0) \neq 0$. Now, we consider the functions $\varphi_{i,\epsilon}(x) = \varphi(\frac{x-x_i}{\epsilon})$.

We have $< \chi'(u_n), \varphi_{i,\epsilon}u_n >\longrightarrow 0$. Thus,

$$\begin{aligned} < \chi'(u_n), \varphi_{i,\epsilon}u_n > &= \int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \nabla(\varphi_{i,\epsilon}u_n)dx - \int_\Omega |u_n|^{r(x)}\varphi_{i,\epsilon}dx \\ &- \int_\Omega v_1(x)h_1(u_n(x)) - \int_{\partial\Omega} |u_n|^{s(x)}\varphi_{i,\epsilon}dx. \end{aligned}$$

Passing to the limit as $n \to \infty$, we obtain

$$\begin{aligned} 0 &= \lim_{n\to\infty}\Big(\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \nabla(\varphi_{i,\epsilon})u_n dx + \int_\Omega \varphi_{i,\epsilon}d\mu - \int_\Omega \varphi_{i,\epsilon}d\nu \\ &- \int_\Omega v_1(x)h_1(u_n(x))\varphi_{i,\epsilon}u_n dx - \int_{\partial\Omega} |u_n|^{s(x)}\varphi_{i,\epsilon}dx\Big). \end{aligned}$$

By Hölder's inequality, we can show that

$$\lim_{n\to\infty}\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \nabla(\varphi_{i,\epsilon})u_n dx = 0 \ \ and \ \ \lim_{n\to\infty}\int_{\partial\Omega} |u_n|^{s(x)}\varphi_{i,\epsilon}dx \ = 0.$$

On the other hand, we have

$$\lim_{\epsilon\to 0}\int_\Omega v_1(x)h_1(u_n(x))\varphi_{i,\epsilon}u_n dx = 0,$$

$$\lim_{\epsilon\to 0}\int_\Omega \varphi_{i,\epsilon}d\mu = \mu_i\varphi(0),$$

and

$$\lim_{\epsilon\to 0}\int_\Omega \varphi_{i,\epsilon}d\nu = \nu_i\varphi(0).$$

By combining the aboe informations, we get $\big(\mu_i - \nu_i\big)\varphi(0) = 0$, which implies that $\mu_i = \nu_i$. Consequently,

$$S\nu_i^{\frac{1}{p^*(x_i)}} \leq \nu_i^{\frac{1}{p(x_i)}}.$$

Thus, we conclude that $\nu_i = 0$ or $S^N \leq \nu_i$.

Now, by the fact that $\min(r^+, s^+, \theta) > p^+$ and by $(\mathbf{A_2})$, we have

$$\begin{aligned} c &= \lim_{n\to\infty}\chi(u_n) = \lim_{n\to\infty}\Big(\chi(u_n) - \frac{1}{p^+} < \chi'(u_n), u_n >\Big) \\ &= \lim_{n\to\infty}\Big(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}dx - \int_\Omega \frac{|u_n|^{r(x)}}{r(x)}dx - \int_\Omega v_1(x)H_1(u_n)dx - \int_{\partial\Omega}\frac{|u_n|^{s(x)}}{s(x)}dx \\ &- \frac{1}{p^+}\int_\Omega |\nabla u_n|^{p(x)}dx + \int_\Omega \frac{|u_n|^{r(x)}}{p^+}dx + \frac{1}{p^+}\int_\Omega v_1(x)h_1(u_n)u_n dx + \int_{\partial\Omega}\frac{|u_n|^{s(x)}}{p^+}dx\Big) \\ &= \lim_{n\to\infty}\Big(\int_\Omega (\frac{1}{p(x)} - \frac{1}{p^+})|\nabla u_n|^{p(x)}dx + \int_\Omega (\frac{1}{p^+} - \frac{1}{r(x)})|u_n|^{r(x)}dx + \int_{\partial\Omega}(\frac{1}{p^+} - \frac{1}{s(x)})|u_n|^{s(x)}dx \end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{1}{p^+}\int_\Omega v_1(x)h_1(u_n)u_n dx - \int_\Omega v_1(x)H_1(u_n)dx\Big) \\
&\geq \quad \lim_{n\to\infty}\Big(\int_\Omega (\frac{1}{p(x)} - \frac{1}{p^+})|\nabla u_n|^{p(x)}dx + \int_\Omega (\frac{1}{p^+} - \frac{1}{r(x)})|u_n|^{r(x)}dx + \int_{\partial\Omega}(\frac{1}{p^+} - \frac{1}{s(x)})|u_n|^{s(x)}dx \\
&+ \quad \frac{1}{\theta}\int_\Omega v_1(x)h_1(u_n)u_n dx - \int_\Omega v_1(x)H_1(u_n)dx\Big) \\
&\geq \quad \lim_{n\to\infty}\int_\Omega (\frac{1}{p^+} - \frac{1}{r(x)})|u_n|^{r(x)}dx \\
&\geq \quad \lim_{n\to\infty}\int_{A_\delta}(\frac{1}{p^+} - \frac{1}{r^-_{A_\delta}})|u_n|^{r(x)}dx.
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
\lim_{n\to\infty}\int_{A_\delta}(\frac{1}{p^+} - \frac{1}{r^-_{A_\delta}})|u_n|^{r(x)}dx &= (\frac{1}{p^+} - \frac{1}{r^-_{A_\delta}})(\int_{A_\delta}|u|^{r(x)} + \sum_{i\in I}\nu_i) \\
&\geq (\frac{1}{p^+} - \frac{1}{r^-_{A_\delta}})\nu_i \\
&\geq (\frac{1}{p^+} - \frac{1}{r^-_{A_\delta}})S^N. \qquad (12)
\end{aligned}
$$

Therefore, since δ is positive and arbitrary and r is continuous, we have

$$
c \geq (\frac{1}{p^+} - \frac{1}{r^-_A})S^N.
$$

Then if $c < (\frac{1}{p^+} - \frac{1}{r^-_A})S^N$, the index set I is empty. □

We now present the following lemma that establishes an important convergence result.

Lemma 3.10. *If conditions* $(\mathbf{A_1}) - (\mathbf{A_3})$ *are satisfied and let* $\{u_n\}$ *be a Palais-Smale sequence in* X*, with energy level* c*. If* $c < (\frac{1}{p^+} - \frac{1}{r^-_A})S^N$*, then there exists a subsequence of* $\{u_n\}$ *that converges strongly in* X*.*

Proof. Let $\{u_n\}$ be a Palais-Smale sequence in X, with an energy level c such that $c < (\frac{1}{p^+} - \frac{1}{r^-_A})S^N$. By Lemma 3.7, $\{u_n\}$ is bounded in X. Therefore, there exists a subsequence of $\{u_n\}$ that converges weakly to u in X.

Using Lemma 3.9, the fact that $S(x) < p^*(x)$ and $s(x) < p_*(x)$, we deduce by proposition 2.2 that

$$
\begin{cases}
u_n \to u, \text{ strongly in } L^{S(x)}(\Omega), \\
u_n \to u, \text{ strongly in } L^{r(x)}(\Omega), \\
u_n \to u, \text{ strongly in } L^{s(x)}(\partial\Omega).
\end{cases}
$$

To complete the proof, we need to show that $u_n \to u$ strongly in X. We start by considering the inner product

$$
\begin{aligned}
< \chi'(u_n), u_n - u > \quad &= \quad < L'(u_n), u_n - u > - \int_\Omega |u_n|^{r(x)-2}u_n(u_n - u)dx \\
&- \int_\Omega v_1(x)h(u_n)(u_n - u)dx - \int_{\partial\Omega}|u_n|^{s(x)-2}u_n(u_n - u)dx.
\end{aligned}
$$

By applying Hölder's inequality, Propositions 2.2 and 2.5, we can estimate the integral term as follows:

$$
\int_\Omega |u_n|^{r(x)-1}|u_n - u|dx \quad \leq \quad |u_n - u|_{L^{r(x)}}||u|^{r(x)-1}|_{L^{\frac{r(x)}{r(x)-1}}}
$$

$$\begin{aligned} &\leq |u_n - u|_{L^{r(x)}} \max\left(|u_n|^{r^+-1}_{L^{r(x)}}, |u_n|^{r^--1}_{L^{r(x)}}\right) \\ &\leq c_1 |u_n - u|_{L^{r(x)}} \max\left(\|u_n\|^{r^+-1}, \|u_n\|^{r^--1}\right), \end{aligned}$$

This leads to the conclusion

$$\lim_{n\to\infty} \int_\Omega |u_n|^{r(x)-2} u_n (u_n - u) dx = 0. \tag{13}$$

Similarly, we have

$$\lim_{n\to\infty} \int_{\partial\Omega} |u_n|^{s(x)-2} u_n (u_n - u) dx = 0. \tag{14}$$

Now, by using ($\mathbf{A_1}$), propositions 2.2, 2.5, and Hölder's inequality, we obtain

$$\begin{aligned} \int_\Omega v_1(x) h_1(u_n)(u_n - u) dx &\leq \int_\Omega c_1 |v_1(x)| |u_n|^{\alpha(x)-1} |u_n - u| dx \\ &\leq c_1 |u_n - u|_{L^{S(x)}} |v_1(x)|_{L^{\frac{S(x)}{S(x)-\alpha(x)}}} \| |u_n|^{\alpha(x)-1} |_{L^{\frac{S(x)}{\alpha(x)-1}}} \\ &\leq c_1 |u_n - u|_{L^{S(x)}} |v_1(x)|_{L^{\frac{S(x)}{S(x)-\alpha(x)}}} \max\left(|u_n|^{\alpha^+-1}|_{L^{S(x)}}, |u_n|^{\alpha^--1}|_{L^{S(x)}}\right) \\ &\leq c_1 |u_n - u|_{L^{S(x)}} |v_1(x)|_{L^{\frac{S(x)}{S(x)-\alpha(x)}}} \max\left(\|u_n\|^{\alpha^+-1}|, \|u_n\|^{\alpha^--1}|\right). \end{aligned}$$

Hence, we have

$$\lim_{n\to\infty} \int_\Omega v_1(x) h_1(u_n)(u_n - u) dx = 0. \tag{15}$$

By combining (13)- (15), and using the fact that $< \chi'(u_n), u_n - u >\to 0$, we conclude that

$$< L'(u_n), u_n - u >= \int_\Omega |\nabla u_n|^{p(x)-2} \nabla u_n \nabla(u_n - u) dx \to 0,$$

Passing to the limit as n tends to infinity and using the fact that u_n converges weakly to u, we get

$$< L'(u), u_n - u >\to 0.$$

Hence

$$\lim_{n\to\infty} < L'(u_n) - L'(u), u_n - u >= 0.$$

Since L' is of type (S_+) (see Proposition 2.1), we deduce that $u_n \to u$ strongly in X. □

In order to further investigate the properties of the functional χ and its critical points, we establish the following lemma.

Lemma 3.11. *If conditions* ($\mathbf{A_1}$), ($\mathbf{A_2}$) *hold. Then, there exists an element $e_0 \in X$ such that*

$$\|e_0\| > \eta, \ \ \textit{and} \ \ \chi(e_0) < 0,$$

where η is defined in Lemma 3.6.

Proof. At first, by ($\mathbf{A_2}$), there exists $\xi > 0$, such that for all $(x, t) \in \Omega \times \mathbb{R}$, we have

$$v_1(x) H_1(t) \geq \xi |t|^\theta. \tag{16}$$

Let $e \in X$ be such that $\int_\Omega |e|^\theta dx > 0$ and let $t > 1$ be sufficiently large. Then, we have

$$\begin{aligned} \chi(te) &= \int_\Omega \frac{|\nabla(te)|^{p(x)}}{p(x)}dx - \int_\Omega \frac{|te|^{r(x)}}{r(x)}dx \\ &\quad - \int_\Omega v_1(x)H_1(te)dx - \int_{\partial\Omega} \frac{|te|^{s(x)}}{s(x)}dx. \end{aligned}$$

Using (16), we obtain

$$\begin{aligned} \chi(te) &\leq \frac{t^{p^+}}{p^-}\int_\Omega |\nabla(e)|^{p(x)}dx - \frac{t^{r^-}}{r^+}\int_\Omega |(e)|^{r(x)}dx \\ &\quad -\xi t^\theta \int_\Omega |e|^\theta dx. \end{aligned}$$

Since $\min(\theta, r^-) > p^+$, it follows that

$$\chi(te) \to -\infty, \text{ as } t \to \infty.$$

Therefore, we can choose $t_0 > 0$ and set $e_0 = t_0 e$ such that $\|e_0\| > \eta$ and $\chi(e_0) < 0$. This completes the proof. □

Now, we establish the following lemma that provides a key result regarding the boundedness of a set under certain hypotheses.

Lemma 3.12. *Under the hypotheses* $(\mathbf{A_1})$, $(\mathbf{A_2})$, *if F is a finite dimensional subspace of X, then the set*

$$T = \{u \in F, \text{ such that } \chi(u) \geq 0\},$$

is bounded in X.

Proof. Let $u \in T$. We have,

$$\chi(u) \leq \frac{1}{p_1^-}\int_\Omega |\nabla u|^{p(x)}dx - \frac{1}{r^+}\int_\Omega |u|^{r(x)}dx - \int_\Omega v_1(x)H_1(u)dx.$$

Using inequality (16) and proposition 2.3, we obtain

$$\begin{aligned} \chi(u) &\leq \frac{1}{p^-}\int_\Omega |\nabla u|^{p(x)}dx - \xi\int_\Omega |u|^\theta dx \\ &\leq \frac{\xi_2}{p^-}\left(\|u\|^{p^+} + \|u\|^{p^-}\right) - \xi|u|^\theta_{L^\theta}, \end{aligned}$$

where $|.|_{L^\theta}$ and $\|.\|$ are equivalent norms in F. Thus, there exists a positive constant k such that

$$\|u\|^\theta \leq k|u|^\theta_{L^\theta}.$$

Therefore, we have

$$\chi(u) \leq \frac{\xi_2}{p^-}\left(\|u\|^{p^+} + \|u\|^{p^-}\right) - \frac{\xi}{k}\|u\|^\theta.$$

Hence, since $p^- < p^+ < \theta$, we can conclude that the set T is bounded in X. □

Proof. [**Proof of Theorem 3.2**] Lemmas 3.6, 3.10, and 3.11 establish the fulfillment of all the conditions required by Theorem 2.8 (mountain pass theorem), ensuring the existence of a nontrivial solution to problem (1). With this, the proof of Theorem 3.2 is now concluded. □

Proof. [**Proof of Theorem 3.3**] We observe that $\chi(0) = 0$, and due to $(\mathbf{A_4})$, the functional χ is even. Furthermore, Lemmas 3.6, 3.10, and 3.12 establish the fulfillment of all the conditions stated in Theorem 2.9 (symmetric mountain pass theorem). Consequently, we can conclude that problem (1) possesses an unbounded sequence of nontrivial solutions. With this, the proof of Theorem 3.3 is now completed. □

4. Second main results

In this Section, we investigate the existence of solutions for problem (1) under small perturbations. Specifically, we consider a perturbation term in the form of $f(x,u) = v_1(x)h_1(u) + \lambda|u|^{\gamma(x)-2}u$, where λ is a positive parameter and $\gamma(x)$ belongs to the function space $C_+(\overline{\Omega})$. The parameter $\gamma(x)$ must satisfy the condition $1 < \gamma^- \leq \gamma^+ < p^-$, with p being another parameter. These choices allow us to examine the impact of perturbations on the problem and explore the existence of nontrivial solutions. We present two main theorems that establish conditions under which problem (1) has nontrivial solutions in the presence of these perturbations.

Theorem 4.1. *Assume that conditions* $(\mathbf{A_1}) - (\mathbf{A_3})$ *and* $(\mathbf{A_5})$ *hold. Under these assumptions, we prove the existence of a positive constant* λ_0 *such that for any* λ *in the interval* $(0, \lambda_0)$*, problem* (1) *has at least one nontrivial solution.*

Theorem 4.2. *Under the combined assumptions* $(\mathbf{A_1})$ *to* $(\mathbf{A_5})$*, we establish the existence of a positive constant* λ_0 *such that for any* λ *in the interval* $(0, \lambda_0)$*, problem* (1) *has at least one nontrivial solution.*

To prove the above theorems, we begin by noting that the funcional $\chi_\lambda : X \to \mathbb{R}$ associated with problem (1) is defined as

$$\chi_\lambda(u) = \chi(u) - \lambda \int_\Omega \frac{|u|^{\gamma(x)}}{\gamma(x)},$$

where X is the appropriate function space for the problem. This modified functional includes an additional integral term involving the parameter λ. The purpose of this term is to introduce a perturbation to the original functional $\chi(u)$, allowing us to study the behavior of critical points under small perturbations.

Remark 4.3. *Firstly, we remark that* χ_λ *is a* C^1 *function, meaning that it is continuously differentiable. This ensures the smoothness and well-behaved nature of the functional, which is important for variational analysis.*

Furthermore, the remark 4.3 highlights the connection between weak solutions of problem (1) and critical points of the modified functional χ_λ. In other words, solutions of the problem correspond to points in the function space X where the derivative of χ_λ vanishes. This observation establishes the variational nature of the problem, where finding solutions is equivalent to searching for critical points of the associated functional.

Now, to establish a lower bound for the functional $\chi_\lambda(u)$ and investigate the behavior of solutions under small perturbations, we present the following lemma:

Lemma 4.4. *Assume that conditions* $(\mathbf{A_1})$, $(\mathbf{A_3})$ *and* $(\mathbf{A_5})$ *hold. Then, there exist three positive constants* η, r *and* λ_0*, such that, for all* $u \in X$ *and all* $\lambda \in (0, \lambda_0)$*, the following statement holds:*

$$\text{if } \ \|u\| = \eta, \ \text{ then } \chi_\lambda(u) \geq m.$$

Proof. Let $u \in X$ with $\|u\| < 1$. By applying Hölder inequality and using proposition 2.5, we obtain the following inequalities:

$$\begin{aligned}
\chi_\lambda(u) \ &\geq \ \frac{1}{p^+}\Gamma(u) - \frac{c_1}{\alpha^-}\int_\Omega |v_1(x)||u|^{\alpha(x)}dx - \frac{1}{r^-}\int_\Omega |u|^{r(x)}dx - \frac{\lambda}{\gamma^-}\int_\Omega |u|^{\gamma(x)}dx - \frac{1}{s^-}\int_{\partial\Omega} |u|^{s(x)}dx \\
&\geq \ \frac{1}{p^+}\Gamma(u) - \frac{c_1}{\alpha^-}|v_1|_{L^{\frac{S(x)}{S(x)-\alpha(x)}}(\Omega)} \||u|^{\alpha(x)}|_{L^{\frac{S(x)}{\alpha(x)}}(\Omega)} - \frac{1}{r^-}\int_\Omega |u|^{r(x)}dx - \frac{\lambda}{\gamma^-}\int_\Omega |u|^{\gamma(x)}dx - \frac{1}{r^-}\int_{\partial\Omega} |u|^{s(x)}dx \\
&\geq \ \frac{1}{p^+}\Gamma(u) - \frac{c_1}{\alpha^-}|v_1|_{L^{\frac{S(x)}{S(x)-\alpha(x)}}(\Omega)} \max\left(|u|^{\alpha^-}_{L^{S(x)}(\Omega)}, |u|^{\alpha^+}_{L^{S(x)}(\Omega)}\right) - \frac{1}{r^-}\max\left(|u|^{r^-}_{L^{r(x)}(\Omega)}, |u|^{r^+}_{L^{r(x)}(\Omega)}\right) \\
&- \ \frac{\lambda}{\gamma^-}\max\left(|u|^{\gamma^-}_{L^{\gamma(x)}(\Omega)}, |u|^{\gamma^+}_{L^{\gamma(x)}(\Omega)}\right) - \frac{1}{s^-}\max\left(|u|^{s^-}_{L^{s(x)}(\Omega)}, |u|^{s^+}_{L^{s(x)}(\Omega)}\right).
\end{aligned}$$

Next, using Propositions 2.2, 2.3 and the fact that X is continuously embedded in $L^{S(x)}(\Omega)$, $L^{r(x)}(\Omega)$, $L^{s(x)}(\partial\Omega)$ and $L^{\gamma(x)}(\Omega)$, we have:

$$\begin{aligned}
\chi_\lambda(u) &\geq \frac{\xi_1}{p^+}\|u\|^{p^+} - \frac{c_1}{\alpha^-}|v_1|_{L^{\frac{S(x)}{S(x)-\alpha(x)}}(\Omega)}\|u\|^{\alpha^-} - \frac{1}{r^-}c_2\|u\|^{r^-} - \frac{\lambda}{\gamma^-}c_3\|u\|^{\gamma^-} - \frac{1}{s^-}c_4\|u\|^{s^-} \\
&\geq \|u\|^{\gamma^-}\left(\frac{\xi_1}{p^+}\|u\|^{p^+-\gamma^-} - (\frac{c_1}{\alpha^-}|v_1|_{L^{\frac{S(x)}{S(x)-\alpha(x)}}(\Omega)} + \frac{c_2}{r^-} + \frac{c_4}{s^-})\|u\|^{\alpha^- -\gamma^-} - \frac{\lambda}{\gamma^-}c_3\right) \\
&\geq \|u\|^{\gamma^-}\Big(\varphi(\|u\|) - \frac{\lambda}{\gamma^-}c_3\Big),
\end{aligned}$$

since $\alpha^- < \min(r^-, s^-)$, where φ is a continuous function on $[0,\infty)$ defined by:

$$\varphi(t) = \frac{\xi_1}{p^+}t^{p^+-\gamma^-} - \left(\frac{c_1}{\alpha^-}|v_1|_{L^{\frac{S(x)}{S(x)-\alpha(x)}}(\Omega)} + \frac{c_2}{r^-} + \frac{c_4}{s^-}\right)t^{\alpha^- -\gamma^-}.$$

Let us set

$$\eta = \left(\frac{\xi_1(p^+ - \gamma^-)}{(\frac{c_1}{\alpha^-}|v_1|_{L^{\frac{S(x)}{S(x)-\alpha(x)}}(\Omega)} + \frac{c_2}{r^-} + \frac{c_4}{s^-})p^+(\alpha^- - \gamma^-)}\right)^{\frac{1}{\alpha^- - p^+}}.$$

It is easy to see that

$$\max_{t\geq 0}\varphi(t) = \varphi(\eta) > 0.$$

Put

$$\lambda_0 = \frac{\gamma^-\varphi(\eta)}{c_3}, \text{ and } m = \eta^{\gamma^-}\Big(\varphi(\eta) - \frac{\lambda}{\gamma^-}c_3\Big). \tag{17}$$

Therefore, it is very simple to see that if $\|u\| = \eta$, then, for all $\lambda \in (0, \lambda_0)$, we have

$$\chi_\lambda(u) \geq m > 0.$$

□

We aim to establish the existence of a specific function satisfying certain properties. To accomplish this, we present the following lemma:

Lemma 4.5. *Assume that condition* $(\mathbf{A_3})$ *hold. Then, for all* $\lambda > 0$, *there exists* $e_0 \in X$, *such that*

$$\|e_0\| > \eta \quad \textit{and} \quad \chi_\lambda(e_0) < 0.$$

Proof. Let $e \in X$, such that $\int_\Omega |e|^{\theta_1}dx > 0$. Choose $t > 1$, sufficiently large. From (16), we have

$$\begin{aligned}
\chi_\lambda(te) &= \int_\Omega \frac{|\nabla(te)|^{p(x)}}{p(x)}dx - \int_\Omega v_1(x)H_1(te)dx - \int_\Omega \frac{|te|^{r(x)}}{r(x)}dx - \lambda\int_\Omega \frac{|te|^{\gamma(x)}}{\gamma(x)}dx - \int_{\partial\Omega}\frac{|te|^{s(x)}}{s(x)}dx \\
&\leq \int_\Omega \frac{|\nabla(te)|^{p(x)}}{p(x)}dx - \int_\Omega v_1(x)H_1(te)dx \\
&\leq \frac{t^{p^+}}{p^-}\int_\Omega |\nabla(e)|^{p(x)}dx - m_1 t^\theta \int_\Omega |e|^\theta dx.
\end{aligned}$$

Since $p^+ < \theta$, we can see that

$$\chi_\lambda(te) \to -\infty, \text{ as } t \to \infty.$$

Therefore, there exists $t_0 > 0$ sufficiently large such that

$$\|e_0\| > \eta, \quad \text{and} \quad \chi_\lambda(e_0) < 0,$$

where $e_0 = t_0 e$. □

In order to demonstrate an important property of a Palais-Smale sequence in X, we present the following Proposition:

Proposition 4.6. *Assuming that conditions* $(\mathbf{A_1})-(\mathbf{A_2})$ *hold, if* $\{u_n\}$ *is a Palais-Smale sequence in X, then* $\{u_n\}$ *is bounded in X.*

Proof. Let $\{u_n\}\subset X$, be a sequence such that

$$\chi_\lambda(u_n)\to c,\quad \text{and}\quad \chi'_\lambda(u_n)\to 0,\ \text{in } X^*,\ \text{as } n\to\infty,$$

where c is a positive constant.

As in the proof of Lemma 3.7, we can find two positive constants M_1 and M_2, such that

$$|\chi_\lambda(u_n)|\leq M_1, \tag{18}$$

and

$$|<\chi'_\lambda(u_n),u_n>|\leq M_2. \tag{19}$$

We claim that the sequence $\{u_n\}$is bounded. If this is not true, by passing to a subsequence if necessary, we may assume that $\|u_n\|\to\infty$. Without loss of generality, we assume that $\|u_n\|\geq 1$. Similar to the proof of Lemma 3.7, using assumptions $(\mathbf{A_2})$, and the fact that $\theta>p^+>\gamma^-$, we obtain

$$\begin{aligned}\theta M_1+M_2 &\geq (\frac{\theta}{p^+}-1)\xi_1\|u_n\|^{p^-}+\int_\Omega v_1(x)\Big(h_1(u_n)u_n-\theta H_1(u_n)\Big)dx+\lambda\int_\Omega(1-\frac{\theta}{\gamma(x)})|u_n|^{\gamma(x)}dx\\ &\geq (\frac{\theta}{p^+}-1)\xi_1\|u_n\|^{p^-}+\lambda\int_\Omega(1-\frac{\theta}{\gamma(x)})|u_n|^{\gamma(x)}dx,\\ &\geq (\frac{\theta}{p^+}-1)\xi_1\|u_n\|^{p^-}-c\lambda(\frac{\theta}{\gamma^-}-1)\|u_n\|^{\gamma^+}.\end{aligned}$$

Since $p^->\gamma^+$, letting n tend to infinity leads to a contradiction. Therefore, the sequence $\{u_n\}$ is bounded in X. □

Suppose $\{u_n\}$ is a Palais-Smale sequence with energy level c. By Theorem 3.8, we have:

$$|u_n|^{r(x)}\rightharpoonup\nu=|u|^{r(x)}+\sum_{i\in I}\nu_i\delta_{x_i},\qquad v_i>0.$$

$$|\nabla u_n|^{p(x)}\rightharpoonup\mu\geq|\nabla u|^{p(x)}+\sum_{i\in I}\mu_i\delta_{x_i},\qquad \mu_i>0.$$

$$S\nu_i^{\frac{1}{p^*(x_i)}}\leq\mu_i^{\frac{1}{p(x_i)}}.$$

If $I=\emptyset$, then $u_n\to u$ in $L^{r(x)}(\Omega)$. We know that $\{x_i\}_{i\in A}\subset A$.
We want to show that if $c<\left(\frac{1}{p^+}-\frac{1}{r_A^-}\right)S^n$, then $I=\emptyset$, where S is defined in Theorem 3.8.
In the study of Palais-Smale sequences and their properties, we present the following lemma.

Lemma 4.7. *If conditions* $(\mathbf{A_1})-(\mathbf{A_3})$ *are satisfied, let* u_n *be a Palais-Smale sequence in X with energy level c. There exists a positive constant* c_0 *such that if* $c<c_0$*, the index set I is empty.*

Proof. Suppose that $I\neq\emptyset$ and let $\varphi\in C_0^\infty(\mathbb{R}^N)$ be such that $\varphi(0)\neq 0$. We now consider the function

$$\varphi_{i,\epsilon}(x)=\varphi(\frac{x-x_i}{\epsilon}).$$

We have $<\chi'_\lambda(u_n),\varphi_{i,\epsilon}u_n>\to 0$, Therefore, we can write

$$\begin{aligned}<\chi'_\lambda(u_n),\varphi_{i,\epsilon}u_n> &= \int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n\nabla(\varphi_{i,\epsilon}u_n)dx-\int_\Omega |u_n|^{r(x)}\varphi_{i,\epsilon}dx-\int_{\partial\Omega}|u_n|^{s(x)}\varphi_{i,\epsilon}dx\\ &- \int_\Omega v_1(x)h_1(u_n(x))\varphi_{i,\epsilon}u_n dx-\lambda\int_\Omega |u_n|^{\gamma(x)}\varphi_{i,\epsilon}dx.\end{aligned}$$

Passing to the limit as $n\to\infty$, we have

$$\begin{aligned}0 &= \lim_{n\to\infty}\Big(\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n\nabla(\varphi_{i,\epsilon})u_n dx+\int_\Omega\varphi_{i,\epsilon}d\mu-\int_\Omega\varphi_{i,\epsilon}d\nu\\ &- \int_\Omega v_1(x)h_1(u_n(x))\varphi_{i,\epsilon}u_n dx-\lambda\int_\Omega|u_n|^{\gamma(x)}\varphi_{i,\epsilon}dx-\int_{\partial\Omega}|u_n|^{s(x)}\varphi_{i,\epsilon}dx\Big).\end{aligned}$$

By using the Hölder inequality, we can prove that

$$\lim_{n\to\infty}\int_\Omega|\nabla u_n|^{p(x)-2}\nabla u_n\nabla(\varphi_{i,\epsilon})u_n dx=0,\ \lim_{n\to\infty}\int_{\partial\Omega}|u_n|^{s(x)}\varphi_{i,\epsilon}dx=0\ \ and\ \ \lim_{n\to\infty}\int_\Omega|u_n|^{\gamma(x)}\varphi_{i,\epsilon}dx=0.$$

On the other hand, we have

$$\lim_{\epsilon\to0}\int_\Omega v_1(x)h_1(u_n(x))\varphi_{i,\epsilon}u_n dx=0$$

$$\lim_{\epsilon\to0}\int_\Omega\varphi_{i,\epsilon}d\mu=\mu_i\varphi(0),\qquad and\ \lim_{\epsilon\to0}\int_\Omega\varphi_{i,\epsilon}d\nu=\nu_i\varphi(0).$$

Thus, we obatain, $\big(\mu_i-\nu_i\big)\varphi(0)=0$, which implies that $\mu_i=\nu_i$. Consequently, we have

$$S\nu_i^{\frac{1}{p^*(x_i)}}\le\nu_i^{\frac{1}{p(x_i)}},$$

and we conclude that $\nu_i=0$ or $S^N\le\nu_i$.

Now, since $\min(r(x),s(x),\theta)>p^+$ and according to $(\mathbf{A_2})$, we have

$$\begin{aligned}c &= \lim_{n\to\infty}\chi_\lambda(u_n)=\lim_{n\to\infty}\Big(\chi_\lambda(u_n)-\frac{1}{p^+}<\chi'_\lambda(u_n),u_n>\Big)\\ &= \lim_{n\to\infty}\Big(\int_\Omega(\frac{1}{p(x)}-\frac{1}{p^+})|\nabla u_n|^{p(x)}dx+\int_\Omega(\frac{1}{p^+}-\frac{1}{r(x)})|u_n|^{r(x)}dx+\lambda\int_\Omega(\frac{1}{p^+}-\frac{1}{\gamma(x)})|u_n|^{\gamma(x)}dx\\ &+ \frac{1}{p^+}\int_\Omega v_1(x)h_1(u_n)u_n dx-\int_\Omega v_1(x)H_1(u_n)dx+\int_{\partial\Omega}(\frac{1}{p^+}-\frac{1}{s(x)})|u_n|^{s(x)}dx\Big)\\ &\ge \lim_{n\to\infty}\Big(\int_\Omega(\frac{1}{p(x)}-\frac{1}{p^+})|\nabla u_n|^{p(x)}dx+\int_\Omega(\frac{1}{p^+}-\frac{1}{r(x)})|u_n|^{r(x)}dx+\lambda\int_\Omega(\frac{1}{p^+}-\frac{1}{\gamma(x)})|u_n|^{\gamma(x)}dx\\ &+ \frac{1}{\theta}\int_\Omega v_1(x)h_1(u_n)u_n dx-\int_\Omega v_1(x)H_1(u_n)dx\Big)\\ &\ge \lim_{n\to\infty}\Big(\int_\Omega(\frac{1}{p(x)}-\frac{1}{p^+})|\nabla u_n|^{p(x)}dx+\int_\Omega(\frac{1}{p^+}-\frac{1}{r(x)})|u_n|^{r(x)}dx+\lambda\int_\Omega(\frac{1}{p^+}-\frac{1}{\gamma(x)})|u_n|^{\gamma(x)}dx\Big).\end{aligned}$$

On the other hand, since $p^+>\gamma^-$, and similarly to (12), we can prove that

$$(\frac{1}{p^+}-\frac{1}{\gamma^-})\int_\Omega|\nabla u|^{p(x)}dx\ge(\frac{1}{p^+}-\frac{1}{\gamma^-})S^N.$$

Then, using Proposition 2.1, we have,

$$c\ \ge\ (\frac{1}{p^+}-\frac{1}{\gamma^-})\int_\Omega|\nabla u|^{p(x)}dx+(\frac{1}{p^+}-\frac{1}{r^-})\int_\Omega|u|^{r(x)}dx+\lambda(\frac{1}{p^+}-\frac{1}{\gamma^-})\int_\Omega|u|^{\gamma(x)}dx$$

$$\geq (\frac{1}{p^+}-\frac{1}{\gamma^-})S^N+(\frac{1}{p^+}-\frac{1}{r^-})\int_\Omega |u|^{r(x)}dx+\lambda(\frac{1}{p^+}-\frac{1}{\gamma^-})\||u|^{\gamma(x)}|_{L^{\frac{r(x)}{q(x)}}(\Omega)}|\Omega|^{\frac{r^+}{r^--\gamma^+}}.$$

If $\||u|^{\gamma(x)}|_{L^{\frac{r(x)}{q(x)}}(\Omega)} \geq 1$, we have

$$c \geq (\frac{1}{p^+}-\frac{1}{\gamma^-})S^N+(\frac{1}{p^+}-\frac{1}{r^-})\||u|^{\gamma(x)}|^{(\frac{r}{\gamma})^-}_{L^{\frac{r(x)}{q(x)}}(\Omega)}-\lambda(\frac{1}{\gamma^-}-\frac{1}{p^+})\||u|^{\gamma(x)}|_{L^{\frac{r(x)}{q(x)}}(\Omega)}|\Omega|^{\frac{r^+}{r^--\gamma^+}}.$$

Now, let's consider the function $h(t)=ct^{(\frac{r}{\gamma})^-}-c'\lambda t$. This function attains its absolute minimum at

$$t_0=\Big(\frac{\lambda c'}{c(\frac{r}{\gamma})^-}\Big)^{\frac{1}{(\frac{r}{\gamma})^--1}}.$$

Moreover, if $\||u|^{\gamma(x)}|_{L^{\frac{r(x)}{q(x)}}(\Omega)} < 1$, we have

$$c \geq (\frac{1}{p^+}-\frac{1}{\gamma^-})S^N+(\frac{1}{p^+}-\frac{1}{r^-})\||u|^{\gamma(x)}|^{(\frac{r}{\gamma})^+}_{L^{\frac{r(x)}{q(x)}}(\Omega)}-\lambda(\frac{1}{\gamma^-}-\frac{1}{p^+})\||u|^{\gamma(x)}|_{L^{\frac{r(x)}{q(x)}}(\Omega)}|\Omega|^{\frac{r^+}{r^--\gamma^+}}.$$

Now, let's define $h_2(t)=ct^{(\frac{r}{\gamma})^+}-c'\lambda t$. This function attains its absolute minimum at

$$t_1=\Big(\frac{\lambda c'}{c(\frac{r}{\gamma})^+}\Big)^{\frac{1}{(\frac{r}{\gamma})^+-1}}.$$

Therefore, there exists a positive constant C such that

$$c \geq (\frac{1}{p^+}-\frac{1}{\gamma^-})S^N+C\min\{\lambda^{\frac{(\frac{r}{\gamma})^-}{(\frac{r}{\gamma})^--1}},\lambda^{\frac{(\frac{r}{\gamma})^+}{(\frac{r}{\gamma})^+-1}}\}.$$

Let's denote this constant as c_0, given by

$$c_0=(\frac{1}{p^+}-\frac{1}{\gamma^-})S^N+C\min\{\lambda^{\frac{(\frac{r}{\gamma})^-}{(\frac{r}{\gamma})^--1}},\lambda^{\frac{(\frac{r}{\gamma})^+}{(\frac{r}{\gamma})^+-1}}\}. \tag{20}$$

Therefore, if $c < c_0$, the index set I is empty. This completes the proof. □

In the following lemma, we establish the convergence of a Palais-Smale sequence under certain conditions.

Lemma 4.8. *Assuming that assumptions* $(\mathbf{A_1})-(\mathbf{A_3})$ *and* $(\mathbf{A_5})$ *are satisfied, consider a Palais-Smale sequence* $\{u_n\}$ *in* X *with energy level* c*, such that* $c < c_0$*, where* c_0 *is defined in* (20)*. Then, up to a subsequence,* $\{u_n\}$ *converges strongly in* X*.*

Proof. Let $\{u_n\}$ be a Palais-Smale sequence in X, with energy level c such that $c < c_0$, as defined in equation (20). By Proposition 4.6, $\{u_n\}$ is bounded in X. Thus, up to a subsequence, there exists $u \in X$ such that, $\{u_n\}$ converges weakly to u in X. Using Lemma 3.9, the fact that $S(x) < p^*(x)$ and $s(x) < p_*(x)$, we deduce from proposition 2.2 that

$$\begin{cases} u_n \to u, \text{ strongly in } L^{S(x)}(\Omega),\\ u_n \to u, \text{ strongly in } L^{r(x)}(\Omega)\\ u_n \to u, \text{ strongly in } L^{s(x)}(\partial\Omega).\end{cases}$$

To complete the proof, it remains to show that $u_n \to u$ strongly in X. For that, we have

$$<\chi'_\lambda(u_n), u_n-u> \;=\; <L'(u_n), u_n-u> - \int_\Omega |u_n|^{r(x)-2}u_n(u_n-u)dx$$

$$- \int_{\Omega} v_1(x)h(u_n)(u_n - u)dx - \lambda \int_{\Omega} |u_n|^{\gamma(x)-2}u_n(u_n - u)dx - \int_{\partial\Omega} |u_n|^{s(x)-2}u_n(u_n - u)dx.$$

Since $\gamma(x) < p^*(x)$, $r(x) < p^*(x)$ and using Hölder's inequality and propositions 2.2, 2.5, we obtain

$$\begin{aligned}\int_{\Omega} |u_n|^{r(x)-1}|u_n - u|dx &\leq |u_n - u|_{L^{r(x)}}||u|^{r(x)-1}|_{L^{\frac{r(x)}{r(x)-1}}} \\ &\leq |u_n - u|_{L^{r(x)}} \max(|u_n|^{r^+-1}_{L^{r(x)}}, |u_n|^{r^--1}_{L^{r(x)}}) \\ &\leq c_1|u_n - u|_{L^{r(x)}} \max(\|u_n\|^{r^+-1}, \|u_n\|^{r^--1}),\end{aligned} \tag{21}$$

$$\int_{\Omega} |u_n|^{\gamma(x)-1}|u_n - u|dx \leq c_1|u_n - u|_{L^{\gamma(x)}} \max(\|u_n\|^{\gamma^+-1}, \|u_n\|^{\gamma^--1}). \tag{22}$$

On the other hand, $\{u_n\}$ is bounded in X, $u_n \to u$, strongly in $L^{r(x)}(\Omega)$, $u_n \to u$, strongly in $L^{\gamma(x)}(\Omega)$ and using (21) and (22), we conclude that

$$\lim_{n\to\infty} \int_{\Omega} |u_n|^{r(x)-2}u_n(u_n - u)dx = 0 \ \ and \ \ \lim_{n\to\infty} \int_{\Omega} |u_n|^{\gamma(x)-2}u_n(u_n - u)dx = 0. \tag{23}$$

Similarly, since $s(x) < p_*(x)$ and using Hölder's inequality, propositions 2.5 and propositions 2.2, we obtain

$$\int_{\partial\Omega} |u_n|^{s(x)-1}|u_n - u|dx \leq c_1|u_n - u|_{L^{s(x)}(\partial\Omega)} \max(\|u_n\|^{s^+-1}, \|u_n\|^{s^--1}). \tag{24}$$

Moreover, $\{u_n\}$ is bounded in X, $u_n \to u$, strongly in $L^{s(x)}(\partial\Omega)$ and by (24), we conclude that

$$\lim_{n\to\infty} \int_{\partial\Omega} |u_n|^{s(x)-2}u_n(u_n - u)dx = 0. \tag{25}$$

On the other hand, using condition $(\mathbf{A_1})$, propositions 2.2, 2.5, and the Hölder's inequality, we have

$$\begin{aligned}\int_{\Omega} v_1(x)h_1(u_n)(u_n - u)dx &\leq \int_{\Omega} c_1|v_1(x)||u_n|^{\alpha(x)-1}|u_n - u|dx \\ &\leq c_1|u_n - u|_{L^{S(x)}}|v_1(x)|_{L^{\frac{S(x)}{S(x)-\alpha(x)}}} \||u_n|^{\alpha(x)-1}|_{L^{\frac{S(x)}{\alpha(x)-1}}} \\ &\leq c_1|u_n - u|_{L^{S(x)}}|v_1(x)|_{L^{\frac{S(x)}{S(x)-\alpha(x)}}} \max(|u_n|^{\alpha^+-1}|_{L^{S(x)}}, |u_n|^{\alpha^--1}|_{L^{S(x)}}) \\ &\leq c_1|u_n - u|_{L^{S(x)}}|v_1(x)|_{L^{\frac{S(x)}{S(x)-\alpha(x)}}} \max(\|u_n\|^{\alpha^+-1}|, \|u_n\|^{\alpha^--1}|).\end{aligned}$$

Thus, we obtain

$$\lim_{n\to\infty} \int_{\Omega} v_1(x)h_1(u_n)(u_n - u)dx = 0. \tag{26}$$

By combining (23)-(26), and using the fact that $< \chi'_\lambda(u_n), u_n - u > \to 0$, we conclude that

$$< L'(u_n), u_n - u >= \int_{\Omega} |\nabla u_n|^{p(x)-2}\nabla u_n \nabla(u_n - u)dx \to 0,$$

passing to the limit as n tends to infinity, and using the fact that u_n converges weakly to u, we get

$$< L'(u), u_n - u > \to 0.$$

Hence,

$$\lim_{n\to\infty} < L'(u_n) - L'(u), u_n - u >= 0.$$

Since L' is of type (S_+) (see Proposition 2.1), we deduce that $u_n \to u$ strongly in X. This completes the proof. □

In the following lemma, we establish a boundedness result for a set of functions under certain hypotheses.

Lemma 4.9. *Under hypotheses* $(\mathbf{A_1}) - (\mathbf{A_2})$, *if F is a finite-dimensional subspace of X, then the set*

$$T_\lambda = \{u \in F, \ \ such \ \ that \ \ \chi_\lambda(u) \geq 0\},$$

is bounded in X.

Proof. Let $u \in T_\lambda$. We have, $\chi_\lambda(u) \leq \chi(u)$, and by Lemma 3.12, it follows that

$$\chi(u) \leq \frac{\xi_2}{p^-}(\|u\|^{p^+} + \|u\|^{p^-}) - \frac{\xi}{k}\|u\|^{\theta}.$$

Consequently, we obtain

$$\chi_\lambda(u) \leq \frac{\xi_2}{p^-}(\|u\|^{p^+} + \|u\|^{p^-}) - \frac{\xi}{k}\|u\|^{\theta}.$$

Since $p^- < p^+ < \theta$, we conclude that the set T_λ is bounded in X. □

Proof. [**Proof of Theorem 4.1**] Lemmas 4.4, 4.5, and 4.8 establish the fulfillment of all the conditions required by Theorem 2.8 (mountain pass theorem), ensuring the existence of a nontrivial solution to problem (1). With this, the proof of Theorem 4.1 is now concluded. □

Proof. [**Proof of Theorem 4.2**] We observe that $\chi_\lambda(0) = 0$, and due to $(\mathbf{A_4})$, the functional χ_λ is even. Furthermore, Lemmas 4.4, 4.8, and 4.9 establish the fulfillment of all the conditions stated in Theorem 2.9 (symmetric mountain pass theorem). Consequently, we can conclude that problem (1) possesses an unbounded sequence of nontrivial solutions. With this, the proof of Theorem 4.2 is now completed. □

Acknowledgments

The authors would like to thank the anonymous referees for the valuable suggestions and comments which improved the quality of this paper.

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