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Hybrid differential problems with sequential fractional derivatives: Solvability and stability analysis

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Abstract. The subject of this paper is the existence, uniqueness and stability of solutions for fractional hybrid problem with two sequential fractional Caputo derivatives. The uniqueness of solutions for the proposed sequential hybrid problem is proved by applying Banach's fixed point theorem. Then, the existence of at least one is established by employing Leray-Schauder nonlinear alternative. The Gronwall's inequality is used to show the Ulam stability results. Example is constructed for the illustration of the main results.

1. Introduction and fractional calculus

The quadratic perturbations of nonlinear differential equations have attracted much attention. The equation is referred to as the hybrid differential equations. In recent years, many researchers have exposed attention in the theory of hybrid differential equations, see for example [2, 22, 23, 25]. Also, fractional-type hybrid differential equations have been extensively studied by several authors, for instance, see [9, 13, 15, 28] and the references cited therein. In recent times, many interesting results concerning the existence and stability of solutions for hybrid differential equations with fractional calculus were obtained, we recommend reading works [1, 5, 13, 14, 27]. The standard expression of the hybrid differential equation [6] is given by:

$$D^1\left[\frac{w(t)}{g(t,w(t))}\right] = h(t,w(t)),\ t\in\Omega,\ \Omega=[0,T],$$

under initial values $w(t_0)=w_0$, $w_0\in\mathbb{R}$, where $g\in C(\Omega\times\mathbb{R},\mathbb{R}-\{0\})$ and $h\in C(\Omega\times\mathbb{R}\to\mathbb{R})$. In quite recent times, some scientific researchers have studied fractional version of the above hybrid differential equation

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and some results have been obtained, for more details, see [10, 14, 16, 21, 29] and the references cited therein. In [29], the authors have examined the fractional hybrid differential equations of the following form:

$$D^{\vartheta}\left[\frac{w(t)}{g(t,w(t))}\right]=h(t,w(t)),\ t\in\Omega,$$

with initial value $w(0)=0$, where $g\in C(\Omega\times\mathbb{R},\mathbb{R}-\{0\}), h\in C(\Omega\times\mathbb{R},\mathbb{R})$ and $D^{\vartheta}, 0<\vartheta<1$, is the Riemann-Liouville fractional derivative. Also, in [10], the researchers considered the following fractional hybrid differential equations with boundary conditions involving Caputo's derivative:

$$D^{\vartheta}\left[\frac{w(t)}{g(t,w(t))}\right]=h(t,w(t)), 0<\vartheta<1,\ t\in\Omega,$$

under conditions

$$\varpi_1\frac{w(0)}{g(0,w(0))}+\varpi_2\frac{w(T)}{g(T,w(T))}=\varpi_3, \varpi_i\in\mathbb{R}, i=1,2,3,$$

where $g\in C(\Omega\times\mathbb{R},\mathbb{R}-\{0\}), h\in C(\Omega\times\mathbb{R},\mathbb{R})$ and D^{ϑ}, is the Caputo fractional derivative. Several scholars have recently studied sequential hybrid differential equations involving fractional calculus, for instance, see [4, 12, 17–19]. Also, some scholars have studied the existence, uniqueness and stability in Ulam sense of solutions of fractional sequential hybrid problems, we refer the reader to the papers [3, 11, 19, 24] and the references cited therein. So, in this present work, we discuss the existence, uniqueness and stability of solution for fractional sequential hybrid problem involving Caputo-type fractional derivative given by:

$$\begin{cases} D^{\vartheta}\left[D^{\mu}\left[\frac{w(t)}{g(t,w(t))}\right]\right]=h(t,w(t)),\ t\in\Omega, \\ w(0)=\beta\int_0^{\delta}w(s)\,ds,\ D^{\mu}\left[\frac{w(0)}{g(0,w(0))}\right]=0, \\ \quad 0<\vartheta,\mu<1,\ 0<\delta<T,\ \beta\in\mathbb{R}, \end{cases} \tag{1}$$

where, $D^{\upsilon}, \upsilon\in\{\vartheta,\mu\}$ are the Caputo fractional derivatives, $g:\Omega\times\mathbb{R}^2\to\mathbb{R}-\{0\}$ and $h:\Omega\times\mathbb{R}^2\to\mathbb{R}$, are continuous functions. The operator D^{υ} is the fractional derivative in the sense of Caputo [26], defined by

$$D^{\upsilon}[h(t)]=I^{n-\sigma}\left[h^{(n)}\right],\ n=[\upsilon]+1,$$

and the Riemann-Liouville fractional integral [26] of order $\mu>0$, defined by

$$I^{\epsilon}[h(t)]=\frac{1}{\Gamma(\epsilon)}\int_0^t(t-s)^{\epsilon-1}h(s)\,ds, t>0,$$

where $\Gamma(.)$ is the Euler gamma function given by

$$\Gamma(\epsilon)=\int_0^{+\infty}z^{\epsilon-1}e^{-z}dz, \epsilon>0.$$

We recall the following lemmas [20, 26].

Lemma 1.1. *Let $\upsilon,\epsilon>0$ and $h\in L^1([0,1])$. Then $I^{\upsilon}[I^{\epsilon}[h(t)]]=I^{\sigma+\epsilon}[h(t)]$ and $D^{\sigma}[I^{\sigma}[h(t)]]=h(t)$.*

Lemma 1.2. *Let $\upsilon>\epsilon>0$ and $h\in L^1([0,1])$. Then $D^{\epsilon}[I^{\upsilon}[h(t)]]=I^{\upsilon-\epsilon}[h(t)]$.*

To study the fractional sequential hybrid problem (1), we need the following lemmas.

Lemma 1.3. *[20] For $\upsilon > 0$, the general solution of the fractional differential equation $D^{\upsilon}[w(t)] = 0$ is given by*

$$w(t) = c_0 + c_1 t, + c_2 t^2 + ... + c_{n-1} t^{n-1},$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, ..., n-1, n-1 < \upsilon < n$.

Lemma 1.4. *[20] Let $\upsilon > 0$. Then*

$$I^{\upsilon}[D^{\upsilon}[w(t)]] = w(t) + c_0 + c_1 t, + c_2 t^2 + ... + c_{n-1} t^{n-1},$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, ..., n-1, n-1 < \upsilon < n$.

In what follow, we need the Gronwall inequality [8].

Lemma 1.5. *Let $z(t)$ and $\psi(t)$ be non-negative, continuous functions on $\Lambda = [0, \infty]$ for each inequality*

$$z(t) \leq z_0 + \int_0^t \psi(s) z(s)\, ds, \ t \in \Lambda,$$

hold, where z_0 is a non-negative constant. Then

$$z(t) \leq z_0 e^{\int_0^t \psi(s) ds}, \ t \in \Lambda,$$

Definition 1.6. *The sequential hybrid problem (1) is Ulam-Hyers stable if there exists a real number $\alpha_h > 0$ such that for each $\theta > 0$ and for each solution $y \in W$ of the inequality*

$$\left| D^{\vartheta}\left[D^{\mu}\left[\frac{y(t)}{g(t, y(t))} \right] \right] - h(t, y(t)) \right| \leq \theta, \ \ t \in \Omega. \tag{2}$$

there exists a solution $w \in W$ of the sequential hybrid problem (1) with

$$|y(t) - w(t)| \leq \alpha_h \theta, \quad t \in \Omega.$$

Definition 1.7. *The sequential hybrid problem (1) is Ulam-Hyers-Rassias stable with respect to $f \in C(\Lambda, \mathbb{R}_+)$ if there exists a real number $\alpha_h > 0$ such that for each $\theta > 0$ and for each solution $y \in W$ of the inequality*

$$\left| D^{\vartheta}\left[D^{\mu}\left[\frac{y(t)}{g(t, y(t))} \right] \right] - h(t, y(t)) \right| \leq \theta f(t), \ \ t \in \Omega, \tag{3}$$

there exists a solution $w \in W$ of the sequential hybrid problem (1) with

$$|y(t) - w(t)| \leq \alpha_h \theta f(t), \quad t \in \Omega.$$

We need the following auxiliary result.

Proposition 1.8. *Suppose that $\varphi \in C(\Omega, \mathbb{R})$ and $g \in C((\Omega \times \mathbb{R}, \mathbb{R} - \{0\})$. Then, the problem*

$$\begin{cases} D^{\vartheta}\left[D^{\mu}\left[\dfrac{w(t)}{g(t, w(t))} \right] \right] = \varphi(t), t \in \Omega, \\ w(0) = \beta \int_0^{\delta} w(s)\, ds, \ D^{\mu}\left[\dfrac{w(0)}{g(0, w(0))} \right] = 0, \end{cases} \tag{4}$$

where $\Omega := [0, T]$, $0 < \vartheta, \mu < 1, 0 < \delta < T$ and $\beta \in \mathbb{R}$, admits the following solution:

$$\begin{aligned} w(t) \ &= \ \frac{g(t, w(t))}{\Gamma(\vartheta + \mu)} \int_0^t (t-s)^{\vartheta+\mu-1} \varphi(s)\, ds \\ &\quad + \frac{\beta g(t, w(t))}{g(0, w(0)) - \beta \int_0^{\delta} g(s, w(s))\, ds} \int_0^{\delta} g(s, w(s))\, I^{\vartheta+\mu}[\varphi(s)]\, ds, \end{aligned} \tag{5}$$

Proof. Let w be a solution for the problem, by using Lemma 4 , we get

$$\frac{w(t)}{g(t,w(t))} = I^{\vartheta+\mu}[\varphi(t)] + I^{\mu}[e_1] + e_2, \tag{6}$$

where e_1, e_2 real constants. Using the boundary conditions, we can obtain

$$\begin{aligned} e_1 &= 0, \\ e_2 &= \frac{\beta}{g(0,w(0)) - \beta\int_0^{\delta} g(s,w(s))\,ds}\int_0^{\delta} g(s,w(s))\, I^{\vartheta+\mu}[\varphi(s)]\,ds, \end{aligned}$$

substituting the values of e_1 and e_2 in (6), we get (5). □

The rest of the work is organized as follows. In Section. 2, we discuss the existence and uniqueness of solutions for the proposed problem. The stability of solutions is fineded and studed in Section. 3. In Section. 4, we present an example to illustrate the main results. In the last section a conclusion is presented.

2. Existence results for fractional quantum sequential hybrid problem

We will use the standard fixed point theorems, to study the above fractional sequential hybrid problem.

Let $W = C(\Omega, \mathbb{R})$ denote the Banach space of all continuous functions from Ω to $\mathbb{R}$, with the norm defined by:

$$\|w\| = \sup\{|w(t)| : t \in \Omega\}.$$

In view of Proposition 8, we can define the operator $T : W \to W$ by:

$$\begin{aligned} Tw(t) &= \frac{g(t,w(t))}{\Gamma(\vartheta+\mu)}\int_0^t (t-s)^{\vartheta+\mu-1} h(s,w(s))\,ds \\ &\quad + \frac{\beta g(t,w(t))}{g(0,w(0)) - \beta\int_0^{\delta} g(s,w(s))\,ds} \\ &\quad \times \int_0^{\delta} g(s,w(s))\left[\int_0^{s} (s-\tau)^{\vartheta+\mu-1} h(\tau,w(\tau))\,d\tau\right] ds. \end{aligned} \tag{7}$$

We set the following

$$\Pi := \frac{\beta\eta}{(\vartheta+\mu+1)\left|g(0,w(0)) - \beta\int_0^{\delta} g(s,w(s))\,ds\right|}. \tag{8}$$

To prove the main results, we impose the following hypotheses:

(H_1) The function $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists constant $\lambda > 0$ such that for all $t \in \Omega$ and $w, y \in \mathbb{R}$, we have

$$\left|h(t,w) - h(t,y)\right| \le \lambda\left|w - y\right|,$$

(H_2) The function $g : \Omega \times \mathbb{R} \to \mathbb{R} - \{0\}$ is continuous and there exist constants $\eta > 0$ such that for each $t \in \Omega$ and $w \in \mathbb{R}$,

$$g(t,w) \le \eta.$$

In the following, we prove the existence and uniqueness of solutions of the fractional sequential hybrid problem (1) by applying Banach's fixed point theorem.

Theorem 2.1. *If* (H_1) *and* (H_2) *are satisfied and if*

$$\frac{T^{\vartheta+\mu}+\Pi\delta^{\vartheta+\mu}}{\Gamma(\vartheta+\mu+1)}<\frac{1}{\lambda\eta}, \tag{9}$$

where Π *is given by (8). Then, the considered hybrid problem admits a unique solution.*

Proof. Define $\sup_{t\in\Omega}|h(t,0)|=M<\infty$ where

$$\frac{\eta\left(T^{\vartheta+\mu}+\Pi\delta^{\vartheta+\mu}\right)}{\Gamma(\vartheta+\mu+1)-\eta\lambda\left(T^{\vartheta+\mu}+\Pi\delta^{\vartheta+\mu}\right)}\leq\epsilon.$$

We proceed to prove that $TB_\epsilon\subset B_\epsilon$, where $B_\epsilon=\{w\in W:\|w\|\leq\epsilon\}$. For $w\in B_\epsilon$, we have

$$\begin{aligned}|Tw(t)| &\leq \frac{\left|g(t,w(t))\right|}{\Gamma(\vartheta+\mu)}\int_0^t(t-s)^{\vartheta+\mu-1}|h(s,w(s))|\,ds+\frac{\beta\left|g(t,w(t))\right|}{\left|g(0,w(0))-\beta\int_0^\delta g(s,w(s))\,ds\right|}\\
&\quad\times\int_0^\delta\left|g(s,w(s))\right|\left[\int_0^s(s-\tau)^{\vartheta+\mu-1}|h(\tau,w(\tau))|\,d\tau\right]ds\\
&\leq \frac{\left|g(t,w(t))\right|}{\Gamma(\vartheta+\mu)}\int_0^t(t-s)^{\vartheta+\mu-1}\left[|h(s,w(s))-h(s,0)|+|h(s,0)|\right]ds\\
&\quad+\frac{\beta\left|g(t,w(t))\right|}{\left|g(0,w(0))-\beta\int_0^\delta g(s,w(s))\,ds\right|}\\
&\quad\times\int_0^\delta\left|g(s,w(s))\right|\left[\int_0^s(s-\tau)^{\vartheta+\mu-1}\left[|h(\tau,w(\tau))-h(\tau,0)|+|h(\tau,0)|\right]d\tau\right]ds.\end{aligned}$$

Now, using (H_1) and (H_2), we get

$$\begin{aligned}&|Tw(t)|\\
&\leq \frac{\eta T^{\vartheta+\mu}}{\Gamma(\vartheta+\mu+1)}(\lambda\|w\|+M)+\frac{\beta\eta^2\delta^{\vartheta+\mu}}{\Gamma(\vartheta+\mu+2)\left|g(0,w(0))-\beta\int_0^\delta g(s,w(s))\,ds\right|}(\lambda\|w\|+M)\\
&= \lambda\eta\left[\frac{T^{\vartheta+\mu}}{\Gamma(\vartheta+\mu+1)}+\frac{\beta\eta\delta^{\vartheta+\mu}}{\Gamma(\vartheta+\mu+2)\left|g(0,w(0))-\beta\int_0^\delta g(s,w(s))\,ds\right|}\right]\|w\|\\
&\quad+\eta\left[\frac{T^{\vartheta+\mu}}{\Gamma(\vartheta+\mu+1)}+\frac{\beta\eta\delta^{\vartheta+\mu}}{\Gamma(\vartheta+\mu+2)\left|g(0,w(0))-\beta\int_0^\delta g(s,w(s))\,ds\right|}\right]M,\end{aligned}$$

which implies that

$$\|T(w)\|\leq\frac{\eta\lambda\left(T^{\vartheta+\mu}+\Pi\delta^{\vartheta+\mu}\right)}{\Gamma(\vartheta+\mu+1)}\epsilon+\frac{\eta\left(T^{\vartheta+\mu}+\Pi\delta^{\vartheta+\mu}\right)}{\Gamma(\vartheta+\mu+1)}M\leq\epsilon.$$

which implies that $TB_\epsilon\subset B_\epsilon$.

Now, we shall show that T is a contraction. For $w, y \in B_\epsilon$ and for each $t \in \Lambda$, we have

$$\begin{aligned} & \left|Tw(t) - Ty(t)\right| \\ \leq \;& \eta \sup_{t\in\Omega}\left[\frac{1}{\Gamma(\vartheta+\mu)}\int_0^t (t-s)^{\vartheta+\mu-1}\left|h(s,w(s)) - h(s,y(s))\right| ds \right. \\ & + \frac{\beta}{\left|g(0,w(0)) - \beta\int_0^\delta g(s,w(s))\,ds\right|} \\ & \left. \times \int_0^\delta \left|g(s,w(s))\right| \left[\int_0^s (s-\tau)^{\vartheta+\mu-1}\left|h(\tau,w(\tau)) - h(\tau,y(\tau))\right| d\tau\right] ds\right\}. \end{aligned}$$

By (H_1), we can write

$$\begin{aligned} & \left\|T(w) - T(y)\right\| \\ \leq \;& \lambda\eta\left[\frac{T^{\vartheta+\mu}}{\Gamma(\vartheta+\mu+1)} + \frac{\beta\eta\delta^{\vartheta+\mu}}{\Gamma(\vartheta+\mu+2)\left|g(0,w(0)) - \beta\int_0^\delta g(s,w(s))\,ds\right|}\right]\left\|w-y\right\|, \end{aligned}$$

which implies that

$$\left\|T(w) - T(y)\right\| \leq \frac{\lambda\eta\left(T^{\vartheta+\mu} + \Pi\delta^{\vartheta+\mu}\right)}{\Gamma(\vartheta+\mu+1)}\left\|w-y\right\|,$$

Therefore, it follows from condition $\dfrac{\lambda\eta\left(T^{\vartheta+\mu} + \Pi\delta^{\vartheta+\mu}\right)}{\Gamma(\vartheta+\mu+1)} < 1$, we conclude that T is a contraction mapping. Hence, by the Banach fixed point theorem, there exists a unique fixed point which is a solution of problem (1). This completes the proof.
□

Now, we present the existence of solutions of the fractional sequential hybrid problem (1) by using Leray-Schauder nonlinear alternative [7].

Lemma 2.2. *(Leray-Schauder alternative). Let $T : E \to E$ be a completely continuous operator (i.e.,a map that restricted to any bounded set in E is compact). Let*

$$\Psi(T) = \{w \in T : w = \nu T(w) \text{ for some } 0 < \nu < 1\}.$$

Then either the set $\Psi(T)$ is unbounded, or T has at least one fixed point.

Now we are in a position to prove the next existence theorem for the studied fractional sequential hybrid problem. We introduce the following hypothesis:

(H_3) : The function $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous and there exist constants $\kappa_1 \geq 0$ and $\kappa_0 > 0$ such that for all $t \in \Lambda$ and $w \in \mathbb{R}$, we have:

$$h(t,w) \leq \kappa_0 + \kappa_1 |w|.$$

Theorem 2.3. *Suppose that hypotheses (H_2) and (H_3) are satisfied. If*

$$\Gamma(\vartheta+\mu+1) - \left(T^{\vartheta+\mu} + \Pi\delta^{\vartheta+\mu}\right)\eta\kappa_1 < 1, \tag{10}$$

where Π is given by (8). Then the hybrid problem (1) has at least one solution on Ω.

Proof. We begin by showing that $T : W \to W$ is completely continuous. By continuity of the functions g, h, it follows that the operator T is continuous.

Let $\Sigma \subset W$ be bounded. Then we can find positive constant N such that

$$|h(t, w(t))| \leq N, \text{ for any } w \in \Sigma.$$

Then for all $w \in \Sigma$, we have

$$\begin{aligned} |Tw(t)| \leq\ & \frac{\left|g(t, w(t))\right|}{\Gamma(\vartheta + \mu)} \int_0^t (t-s)^{\vartheta+\mu-1} |h(s, w(s))|\, ds \\ & + \frac{\beta \left|g(t, w(t))\right|}{\left|g(0, w(0)) - \beta \int_0^\delta g(s, w(s))\, ds\right|} \\ & \times \int_0^\delta \left|g(s, w(s))\right| \left[\int_0^s (s-\tau)^{\vartheta+\mu-1} |h(\tau, w(\tau))|\, d\tau\right] ds. \end{aligned}$$

Using (H_2) and (H_3), we can obtain

$$\|T(w)\| \leq \frac{T^{\vartheta+\mu}\eta N}{\Gamma(\vartheta+\mu+1)} + \frac{\Pi N \eta \delta^{\vartheta+\mu}}{\Gamma(\vartheta+\mu+1)},$$

which yields

$$\|T(w)\| \leq \frac{\left(T^{\vartheta+\mu} + \Pi\delta^{\vartheta+\mu}\right)}{\Gamma(\vartheta+\mu+1)} \eta N < +\infty,$$

Hence, we deduce that T is uniformly bounded.

Next, let $t_1, t_2 \in \Omega$, where $t_1 < t_2$, then for any $w \in \Sigma$. We demonstrate that T is equicontinuous

$$\begin{aligned} & |Tw(t_1) - Tw(t_2)| \\ \leq\ & \frac{\eta N}{\Gamma(\vartheta+\mu+1)} \left(\left[(t_1 - t_2)^{\vartheta+\mu} + \left|t_1^{\vartheta+\mu} - t_2^{\vartheta+\mu}\right|\right]\right), \end{aligned}$$

which does not depend on w and tends to 0 as $t_1 \to t_2$. Thus, T is equicontinuous. Thus, by using the Arzela-Ascoli theorem, $T : W \to W$ is completely continuous.

Lastly, it will be demonstrate that the set $\Phi = \{w \in W, w = \xi T(w), 0 < \xi < 1\}$ is bounded. Let $w \in \Phi$. Then, for each $t \in \Omega$, we can write

$$w(t) = \xi T(w)(t).$$

Then

$$|w(t)| \leq \left[\frac{\eta T^{\vartheta+\mu}}{\Gamma(\vartheta+\mu+1)} + \frac{\Pi\eta\delta^{\vartheta+\mu}}{\Gamma(\vartheta+\mu+1)}\right] (\kappa_0 + \kappa_1 \|w\|),$$

which imply that

$$\begin{aligned} \|w\| \leq\ & \frac{\left(T^{\vartheta+\mu} + \Pi\delta^{\vartheta+\mu}\right)}{\Gamma(\vartheta+\mu+1)} \eta (\kappa_0 + \kappa_1 \|w\|) \\ =\ & \frac{\left(T^{\vartheta+\mu} + \Pi\delta^{\vartheta+\mu}\right)}{\Gamma(\vartheta+\mu+1)} \eta\kappa_0 + \frac{\left(T^{\vartheta+\mu} + \Pi\delta^{\vartheta+\mu}\right)}{\Gamma(\vartheta+\mu+1)} \eta\kappa_1 \|w\|. \end{aligned}$$

Consequently,

$$\|w\| \leq \frac{\left(T^{\vartheta+\mu} + \Pi\delta^{\vartheta+\mu}\right)\eta\kappa_0}{\Gamma(\vartheta+\mu+1) - \left(T^{\vartheta+\mu} + \Pi\delta^{\vartheta+\mu}\right)\eta\kappa_1}.$$

This implies that Φ is bounded. According to Lemma 10, this indicates that the operator T contains at least one fixed point. The sequential hybrid problem (1) on Ω has hence at least one solution in this case. Hence the proof is completed. □

3. Stability analysis

In the following section, we will prove the Ulam-Hyers stability and Ulam-Hyers-Rassias stability of the fractional sequential hybrid problem (1).

Theorem 3.1. *If (H_1) and (H_2) are satisfied. Then the fractional sequential hybrid problem (1) is Ulam-Hyers stable.*

Proof. For $\theta > 0$ and all $y \in W$ solution of the inequality

$$\left| D^{\vartheta}\left[D^{\mu}\left[\frac{y(t)}{g(t,y(t))}\right]\right] - h(t,y(t))\right| \leq \theta, \quad t \in \Omega,$$

let $w \in W$ be unique solution of the problem

$$\begin{cases} D^{\vartheta}\left[D^{\mu}\left[\dfrac{w(t)}{g(t,w(t))}\right]\right] = h(t,w(t)), 0 < \vartheta, \mu < 1, t \in \Omega, \\ w(0) = y(0),\ D^{\mu}\left[\dfrac{w(0)}{g(0,w(0))}\right] = D^{\mu}\left[\dfrac{y(0)}{g(0,y(0))}\right]. \end{cases} \tag{11}$$

Then $w(t)$ is given by

$$w(t) = g(t,w(t))\left(I^{\vartheta+\mu}\left[\varphi_w(t)\right] + I^{\mu}[e_1] + e_2\right),\ e_1, e_2 \in \mathbb{R},$$

such that

$$\varphi_w(t) = h(t,w(t)).$$

Then, we have

$$\begin{aligned} & \left| y(t) - g(t,w(t))\left(I^{\vartheta+\mu}\left[\varphi_w(t)\right] + I^{\mu}[e_3] + e_4\right)\right| \\ \leq\ & \frac{\theta t^{\vartheta+\mu}}{\Gamma(\vartheta+\mu+1)} \leq \frac{\theta T^{\vartheta+\mu}}{\Gamma(\vartheta+\mu+1)}. \end{aligned}$$

By using (H_1) and (H_2), we can write

$$\begin{aligned} & \left| y(t) - w(t)\right| \\ \leq\ & \left| y(t) - g(t,w(t))\left(I^{\vartheta+\mu}\left[\varphi_w(t)\right] + I^{\mu}[e_3] + e_4\right)\right| \\ & + \left| g(t,w(t))\right| I^{\vartheta+\mu}\left[\left|\varphi_y(t) - \varphi_w(t)\right|\right] \\ \leq\ & \frac{\theta T^{\vartheta+\mu}}{\Gamma(\vartheta+\mu+1)} + \frac{\eta\lambda}{\Gamma(\vartheta+\mu)}\int_0^t (t-s)^{\vartheta+\mu-1}\left| y(s) - w(s)\right| ds. \end{aligned}$$

by applying Lemma 5, it becomes

$$\left|y(t)-w(t)\right| \leq \frac{\theta T^{\vartheta+\mu}}{\Gamma(\vartheta+\mu+1)} e^{\dfrac{\eta\lambda}{\Gamma(\vartheta+\mu)}\int_0^t (t-s)^{\vartheta+\mu-1} ds}.$$

So, we obtain

$$\left|y(t)-w(t)\right| \leq \frac{T^{\vartheta+\mu}}{\Gamma(\vartheta+\mu+1)} e^{\dfrac{\eta\lambda T^{\vartheta+\mu}}{\Gamma(\vartheta+\mu)\Gamma(\vartheta+\mu+1)}}\theta,$$

if we set

$$\alpha_h := \frac{T^{\vartheta+\mu}}{\Gamma(\vartheta+\mu+1)} e^{\dfrac{\eta\lambda T^{\vartheta+\mu}}{\Gamma(\vartheta+\mu)\Gamma(\vartheta+\mu+1)}}.$$

The inequality

$$\left|y(t)-w(t)\right| \leq \alpha_h\theta, \quad t \in \Omega.$$

hold, then the fractional sequential hybrid problem (1) is Ulam-Hyers stable. □

Theorem 3.2. *Suppose that* $(H_1),(H_2)$ *and*

(H_4) *The function* $f \in C(\Omega, \mathbb{R}_+)$ *is increasing and there exist* $\xi_f > 0$ *such that, for each* $t \in \Omega$*, we have*

$$I^{\vartheta+\mu}\left[f(t)\right] \leq \xi_f f(t), \tag{12}$$

valid, then the fractional sequential hybrid system (1) is Ulam-Hyers-Rassias stable with respect to f.

Proof. Let $y \in W$ be a solution of the inequality

$$\left|D^{\vartheta}\left[D^{\mu}\left[\frac{y(t)}{g(t,y(t))}\right]\right]-h(t,y(t))\right| \leq \theta f(t), \quad t \in \Omega.$$

Denote by $w \in W$ be the unique solution of the problem (11), that is

$$w(t) = g(t,w(t))\left(I^{\vartheta+\mu}\left[\varphi_w(t)\right]+I^{\mu}\left[e_1\right]+e_2\right), \; e_1,e_2 \in \mathbb{R},$$

On the other hand, for each $t \in \Omega$, we have

$$\begin{aligned}
&\left|y(t)-g(t,w(t))\left(I^{\vartheta+\mu}\left[\varphi_w(t)\right]+I^{\mu}\left[e_3\right]+e_4\right)\right| \\
\leq\;& \theta I^{\vartheta+\mu}\left[f(t)\right] \leq \theta\frac{T^{\vartheta+\mu}}{\Gamma(\vartheta+\mu+1)}f(t) = \theta\xi_f f(t),
\end{aligned}$$

Also, we have

$$\begin{aligned}
&\left|y(t)-w(t)\right| \\
\leq\;& \left|y(t)-g(t,w(t))\left(I^{\vartheta+\mu}\left[\varphi_w(t)\right]+I^{\mu}\left[e_3\right]+e_4\right)\right| \\
&+\left|g(t,w(t))\right| I^{\vartheta+\mu}\left[\left|\varphi_y(t)-\varphi_w(t)\right|\right] \\
\leq\;& \theta\xi_f f(t)+\frac{\eta\lambda}{\Gamma(\vartheta+\mu)}\int_0^t (t-s)^{\vartheta+\mu-1}\left|y(s)-w(s)\right| ds.
\end{aligned}$$

Now, using Lemme 5, we get

$$\left|y(t)-w(t)\right| \leq \theta\xi_f f(t)\, e^{\overline{\frac{\eta\lambda T^{\vartheta+\mu}}{\Gamma(\vartheta+\mu)\Gamma(\vartheta+\mu+1)}}},$$

which yields that

$$\left|y(t)-w(t)\right| \leq \theta\xi_f e^{\overline{\frac{\eta\lambda T^{\vartheta+\mu}}{\Gamma(\vartheta+\mu)\Gamma(\vartheta+\mu+1)}}} f(t).$$

Consequently,

$$\left|y(t)-w(t)\right| \leq \theta\alpha_h f(t), \quad t\in\Omega.$$

Then the fractional sequential hybrid problem (1) is stable in Ulam-Hyers-Rassias sense. □

4. Application

To illustrate our main results, we treat the following example.

Example 4.1. *Consider the following fractional sequential hybrid problem:*

$$\begin{cases} D^{\frac{e}{4}}\left[D^{\frac{\ln 2}{3}}\left[\dfrac{w(t)}{\dfrac{e^{-t}\tan^{-1}w(t)}{5}+\dfrac{1}{13}}\right]\right] = \dfrac{\cos|w(t)|}{35e^{t+!}} + \dfrac{2+e^{2t}}{3}, t\in[0,1], \\ w(0) = \frac{2e}{15}\int_0^{\frac{\sqrt{3}}{3}} w(s)\,ds, \quad D^{\frac{\ln 2}{3}}\left[\dfrac{w(0)}{g(0,w(0))}\right] = 0, \end{cases} \tag{13}$$

and the following inequalitiesand

$$\left|D^{\frac{e}{4}}\left[D^{\frac{\ln 2}{3}}\left[\dfrac{y(t)}{\dfrac{e^{-t}\tan^{-1}y(t)}{5}+\dfrac{1}{13}}\right]\right] - \dfrac{\cos\left|y(t)\right|}{35e^{t+!}} - \dfrac{2+e^{2t}}{3}\right| \leq \theta, \theta>0, t\in[0,1],$$

and

$$\left|D^{\frac{e}{4}}\left[D^{\frac{\ln 2}{3}}\left[\dfrac{y(t)}{\dfrac{e^{-t}\sin y(t)}{3}+\dfrac{1}{5}}\right]\right] - \dfrac{\cos\left|y(t)\right|}{5e^{t+!}} - \dfrac{2+e^{2t}}{3}\right| \leq \theta f(t)\,\theta>0, t\in[0,1].$$

For $w, y \in \mathbb{R}$ and $t\in[0,1]$, we have

$$\left|h(t,y)-h(t,w)\right| \leq \frac{1}{35e}\left|y-w\right|, \quad \left|g(t,w)\right| \leq \frac{8}{15}.$$

So, we take $\lambda = \dfrac{1}{5e}$ and $\Pi \simeq 3519\,5\times10^{-2}$. Hence, we obtain

$$\frac{T^{\vartheta+\mu}+\Pi\delta^{\vartheta+\mu}}{\Gamma(\vartheta+\mu+1)} \simeq 2.868\,2 < \frac{1}{\lambda\eta} = 25.484,$$

by Theorem 9, we conclude that the hybrid problem (13) has a unique solution. And from Theorem 12 we deduce that (13) is Ulam-Hyers stable with

$$\left|y(t) - w(t)\right| \leq 1.0763\theta, t \in [0,1],\ \theta > 0.$$

Let $f(t) = t^{\frac{e}{3}}$, *then*

$$I^{\frac{e}{4}+\frac{\ln 2}{3}}\left[f(t)\right] = I^{\frac{e}{4}+\frac{\ln 2}{3}}\left[t^{\frac{e}{3}}\right] \leq \frac{\Gamma\left(\frac{e+3}{3}\right)}{\Gamma\left(\frac{21e+12\ln 2+36}{36}\right)} t^{\frac{e}{3}} = \xi_f t^{\frac{e}{3}}.$$

Thus, the condition (12) of Theorem 13 is satisfied with $f(t) = t^{\frac{e}{3}}$ *and* $\xi_f = 0.56688$. *Hence from Theorem 13 the problem (13) is Ulam-Hyers-Rassias stable with*

$$\left|y(t) - w(t)\right| \leq 0.589\,03\theta t^{\frac{e}{3}}, t \in [0,1],\ \theta > 0.$$

5. Conclusion

In this paper, the existence and Ulam stability type of solutions have been established for fractional sequential hybrid problem with two sequential fractional Caputo derivatives. We have proved the existence and uniqueness results by using the Banach contraction principle. Also, the existence of at least one solutions has been demonstrated via Leray-Schauder's alternative. Moreover, the Ulam stability have been established by employing Gronwall's inequality. We have given an example to illustrate our main results. In the future, we will continue to study the above mentioned hybrid problem by using fractional quantum calculus to obtain new results, also we will continue to study the Ulam-Hyers-Mittag-Leffler stability for the above proposed problem by using generalized singular Gronwall's inequality.

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