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# The possible orders of growth of solutions to linear fractional differential equations with polynomial coefficients

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**Abstract.** In this paper, we study the possible orders of growth of solutions to certain class of linear fractional differential equations with polynomial coefficients. For that, we use the Nevanlinna theory in complex domain, the generalized Wiman-Valiron theorem in the fractional calculus and the Caputo fractional derivatives. Several illustrative examples are given.

## 1. Introduction

For an entire function $f(z)$, the order of growth is defined by

$$\sigma(f) = \limsup_{r\to+\infty} \frac{\log^+ m(r,f)}{\log r},$$

where

$$m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \ln^+ \left|f\left(re^{i\varphi}\right)\right| d\varphi;$$

and we have also

$$\sigma(f) = \limsup_{r\to+\infty} \frac{\log^+ \log^+ M(r,f)}{\log r},$$

where $M(r,f) = \max\left\{\left|f(z)\right| : |z| = r\right\}$; for more details see [8, 12, 19]. If $f(z)$ is given by $f(z) = \sum\limits_{n=0}^{+\infty} a_n z^n$, the order of growth is equal to

$$\sigma(f) = \limsup_{n\to+\infty} \frac{n\log n}{-\log|a_n|};$$

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see [3]. For example, $\sigma\left(e^{z^n}\right)=n,\ n\in\mathbb{N},\ \sigma\left(\exp\left\{e^z\right\}\right)=+\infty,\ \sigma\left(P\left(z\right)\right)=0$ where $P\left(z\right)$ is any polynomial.

We know that every solution $f$ of the differential equation

$$f^{(n)}+P_{n-1}\left(z\right)f^{(n-1)}+...+P_1\left(z\right)f'+P_0\left(z\right)f=0, \tag{1}$$

where $P_0\left(z\right)\not\equiv 0, P_1\left(z\right),...,P_{n-1}\left(z\right)$ are polynomials, is an entire function of finite rational order $\sigma\left(f\right)$ satisfying

$$\sigma\left(f\right)\leq 1+\max_{0\leq k\leq n-1}\frac{\deg P_k}{n-k}; \tag{2}$$

see [9, 12, 17, 18]. In 1998, Gundersen et al investigated the possible orders of solutions of (1), see [6]. We can ask the following question: how about the linear fractional differential equations with polynomial coefficients?

Recently, extensive research is being published about fractional differential equations and this is due to the importance of this theory for modeling diffusion phenomena and anomalous relaxation in many various fields of science and engineering; (see, for example, Kilbas et al. [10]). There are many definitions of fractional derivatives and many discussions for their properties notably similarities and differences of them, see [1, 2, 4, 11, 14, 15]. In this work we will use the Caputo fractional derivative operator which is defined as follows:

**Definition 1.1.** *[10, 15, 16] Suppose that $\alpha>0, r\geq 0$ and $g\left(r\right)$ is a real function defined on $[0,+\infty)$ and $n$ time continuously differentiable on $(0,+\infty)$. The fractional operator*

$$\mathcal{D}^\alpha g\left(r\right)=\begin{cases}\frac{1}{\Gamma(n-\alpha)}\int\limits_0^r\frac{g^{(n)}(t)}{(r-t)^{\alpha+1-n}}dt, & n-1<\alpha<n\\ \frac{d^n}{dr^n}g\left(r\right), & \alpha=n\in\mathbb{N}\setminus\{0\}\end{cases}$$

*is called the Caputo derivative.*

We recall that $\Gamma$ denotes the gamma function which is defined by the integrale

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}dt,$$

where $z$ is a complex number. It is well known that $\Gamma$ generalizes the factorial, i.e. $\Gamma(n)=n!$.

Consider the function $f\left(z\right)=\sum\limits_{j=0}^{+\infty}a_jz^j$, where $z=re^{i\theta}$. By using the properties of the Caputo operator derivative, for $n-1<\alpha<n$, we have

$$\mathcal{D}^\alpha f\left(re^{i\theta}\right)=\sum_{j=n}^{+\infty}\frac{\Gamma\left(j+1\right)}{\Gamma\left(j-\alpha+1\right)}a_jr^{j-\alpha}e^{ji\theta}, \tag{3}$$

$$r^\alpha\mathcal{D}^\alpha f\left(re^{i\theta}\right)=\sum_{j=n}^{+\infty}\frac{\Gamma\left(j+1\right)}{\Gamma\left(j-\alpha+1\right)}a_jz^j.$$

For $\alpha=n\in\mathbb{N}\setminus\{0\}$, we have

$$\frac{r^\alpha}{z^\alpha}\mathcal{D}^\alpha f\left(z\right)=\frac{d^n}{dz^n}f\left(z\right).$$

**Proposition 1.2.** *[7, 13] The two power series $f\left(z\right)=\sum\limits_{j=0}^{+\infty}a_jz^j$ and $r^\alpha D^\alpha f\left(z\right)$ have the same radius of convergence. Consequently, if $f\left(z\right)$ is an entire function, then $r^\alpha D^\alpha f\left(z\right)$ is also an entire function.*

Recently, the authors investigated the growth of solutions of the following linear fractional differential equations

$$\frac{r^{q_n}}{z^{[q_n]}}\mathcal{D}^{q_n} f(z) + P_{n-1}(z)\frac{r^{q_{n-1}}}{z^{[q_{n-1}]}}\mathcal{D}^{q_{n-1}} f(z) + ... + P_1(z)\frac{r^{q_1}}{z^{[q_1]}}\mathcal{D}^{q_1} f(z) + P_0(z) f(z) = 0, \tag{4}$$

where $P_0(z) \not\equiv 0, P_1(z), ..., P_{n-1}(z)$ are polynomials such that $P_0(0) = 0$ and $0 = q_0 < q_1 < q_2 < ... < q_n$. They proved that all solutions are entire functions of order of growth $\sigma(f)$ satisfying

$$\sigma(f) \leq \max_{0\leq k\leq n-1}\left\{\frac{d_k + [q_n] - [q_k]}{q_n - q_k}\right\}, \tag{5}$$

where $d_k = \deg P_k(z)$ and $[x]$ is the integer part of $x$. They proved also the following statements:

(i) If

$$\frac{d_0 + [q_n]}{q_n} = \max_{0\leq k\leq n-1}\frac{d_k + [q_n] - [q_k]}{q_n - q_k}, \tag{6}$$

holds for all $k = 1, ..., n-1$, then every solution $f \not\equiv 0$ of (4) is an entire function of order of growth

$$\sigma(f) = \frac{d_0 + [q_n]}{q_n}.$$

(ii) If

$$d_k - [q_k] < d_{n-1} - [q_{n-1}] \tag{7}$$

holds for all $k = 0, 1, ..., n-2$, then every solution $f \not\equiv 0$ of (4) is an entire function of order of growth

$$\sigma(f) = \frac{d_{n-1} + [q_n] - [q_{n-1}]}{q_n - q_{n-1}}.$$

The purpose of this paper is to determine all possible orders of growth of solutions of (4).

## 2. Main results

We will follow the same method used in [6] with necessary modifications. Set $d_j = \deg P_j$. We define a strictly decreasing finite sequence of non-negative integers

$$s_1 > s_2 > ... > s_p \geq 0,$$

in the following manner; we choose $s_1$ to be the unique integer satisfying

$$\begin{aligned} \frac{d_{s_1} + [q_n] - [q_{s_1}]}{q_n - q_{s_1}} &= \max_{0\leq k\leq n-1}\frac{d_k + [q_n] - [q_k]}{q_n - q_k}; \text{ and} \\ \frac{d_{s_1} + [q_n] - [q_{s_1}]}{q_n - q_{s_1}} &> \frac{d_k + [q_n] - [q_k]}{q_n - q_k} \text{ for all } 0 \leq k < s_1 \end{aligned} \tag{8}$$

Then given $s_j$, $j \geq 1$, we define $s_{j+1}$ to be the unique integer satisfying

$$\begin{aligned} \frac{d_{s_{j+1}} - d_{s_j} + \left[q_{s_j}\right] - \left[q_{s_{j+1}}\right]}{q_{s_j} - q_{s_{j+1}}} &= \max_{0\leq k<s_j}\frac{d_k - d_{s_j} + \left[q_{s_j}\right] - [q_k]}{q_{s_j} - q_k} > 0; \text{ and} \\ \frac{d_{s_{j+1}} - d_{s_j} + \left[q_{s_j}\right] - \left[q_{s_{j+1}}\right]}{q_{s_j} - q_{s_{j+1}}} &> \frac{d_k - d_{s_j} + \left[q_{s_j}\right] - [q_k]}{q_{s_j} - q_k} \text{ for all } 0 \leq k < s_{j+1}. \end{aligned} \tag{9}$$

For a certain $p$, the integer $s_p$ will exist, but the integer $s_{p+1}$ will not exist, and then the sequence $s_1, s_2, ..., s_p$ terminates with $s_p$. Obviously, $p \le n$.

Correspondingly, for $j = 1, 2, ..., p$, define the following values

$$\alpha_j = \frac{d_{s_j} - d_{s_{j-1}} + \left[q_{s_{j-1}}\right] - \left[q_{s_j}\right]}{q_{s_{j-1}} - q_{s_j}}, \tag{10}$$

where we set

$$s_0 = n \text{ and } d_{s_0} = d_n = 0.$$

We mention that we can define the integers $s_1, s_2, ..., s_p$ in (8) and (9) as the following manner:

$$s_1 = \min\left\{ i : \frac{d_i + [q_n] - [q_i]}{q_n - q_i} = \max_{0 \le k \le n-1} \frac{d_k + [q_n] - [q_k]}{q_n - q_k} \right\}; \tag{11}$$

given $s_j,\ j \ge 1$, we have

$$s_{j+1} = \min\left\{ i : \frac{d_i - d_{s_j} + \left[q_{s_j}\right] - [q_i]}{q_{s_j} - q_i} = \max_{0 \le k < s_j} \frac{d_k - d_{s_j} + \left[q_{s_j}\right] - [q_k]}{q_{s_j} - q_k} > 0 \right\}. \tag{12}$$

**Proposition 2.1.** *If $s_1 \ge 1$ and $p \ge 2$, then the following inequalities hold*

$$\alpha_1 > \alpha_2 > ... > \alpha_p \ge \frac{1}{q_{s_{p-1}} - q_{s_p}} \ge \frac{1}{q_{s_1} - q_{s_p}} \ge \frac{1}{q_{s_1}}.$$

*Proof.* First we prove $\alpha_1 > \alpha_2 > ... > \alpha_p$. From the definitions of $s_j$ and $\alpha_j$ we obtain for $j = 1, 2, ..., p-1$,

$$s_j > s_{j+1} \text{ and } \frac{d_{s_j} - d_{s_{j-1}} + \left[q_{s_{j-1}}\right] - \left[q_{s_j}\right]}{q_{s_{j-1}} - q_{s_j}} > \frac{d_{s_{j+1}} - d_{s_{j-1}} + \left[q_{s_{j-1}}\right] - \left[q_{s_{j+1}}\right]}{q_{s_{j-1}} - q_{s_{j+1}}},$$

which yields

$$-\left(d_{s_j} - \left[q_{s_j}\right]\right) q_{s_{j+1}} - \left(d_{s_{j-1}} - \left[q_{s_{j-1}}\right]\right)\left(q_{s_j} - q_{s_{j+1}}\right) > \left(d_{s_{j+1}} - \left[q_{s_{j+1}}\right]\right)\left(q_{s_{j-1}} - q_{s_j}\right) - \left(d_{s_j} - \left[q_{s_j}\right]\right) q_{s_{j-1}} \tag{13}$$

Adding $\left(d_{s_j} - \left[q_{s_j}\right]\right) q_{s_j}$ to both sides of (13) gives

$$\left(d_{s_j} - d_{s_{j-1}} + \left[q_{s_{j-1}}\right] - \left[q_{s_j}\right]\right)\left(q_{s_j} - q_{s_{j+1}}\right) > \left(d_{s_{j+1}} - d_{s_j} + \left[q_{s_j}\right] - \left[q_{s_{j+1}}\right]\right)\left(q_{s_{j-1}} - q_{s_j}\right)$$

we obtain immediately $\alpha_j > \alpha_{j+1}$ . This proves that

$$\alpha_1 > \alpha_2 > ... > \alpha_p.$$

From the definition of $s_j$ we have $s_j > s_{j+1}$ and then $q_{s_j} > q_{s_{j+1}}$ which implies

$$\frac{1}{q_{s_{p-1}} - q_{s_p}} \ge \frac{1}{q_{s_1} - q_{s_p}} \ge \frac{1}{q_{s_1}}$$

and so to complete the proof, we need only to prove that

$$\alpha_p \ge \frac{1}{q_{s_{p-1}} - q_{s_p}}.$$

We have

$$\frac{d_{s_p} - d_{s_{p-1}} + \left[q_{s_{p-1}}\right] - \left[q_{s_p}\right]}{q_{s_{p-1}} - q_{s_p}} > 0;$$

then

$$d_{s_p} - d_{s_{p-1}} + \left[q_{s_{p-1}}\right] - \left[q_{s_p}\right] \geq 1$$

because

$$q_{s_{p-1}} - q_{s_p} > 0 \text{ and } d_{s_p} - d_{s_{p-1}} + \left[q_{s_{p-1}}\right] - \left[q_{s_p}\right] \in \mathbb{N}^*;$$

therefore

$$\alpha_p = \frac{d_{s_p} - d_{s_{p-1}} + \left[q_{s_{p-1}}\right] - \left[q_{s_p}\right]}{q_{s_{p-1}} - q_{s_p}} \geq \frac{1}{q_{s_{p-1}} - q_{s_p}}.$$

□

**Theorem 2.2.** *Consider the equation (4) with $P_0(0) = 0$. Then every transcendental solution $f$ of (4) is of order of growth $\rho(f) = \alpha_j$ for some $j$, $1 \leq j \leq p$; where $\alpha_j$ is defined in (10).*

**Example 2.3.** *Consider the fractional differential equation*

$$\frac{r^{\alpha}}{z^{[\alpha]}}\mathcal{D}^{\alpha} f(z) + z f(z) = 0, \tag{14}$$

*where $n-1 < \alpha < n$ $(n \in \mathbb{N}^*)$. In this case $s_1 = s_p = 0$, and then there exists only one value $\alpha_1 = \frac{d_0 - d_1 + [\alpha] - [0]}{\alpha - 0} = \frac{n}{\alpha}$. So, by Thoerem 2.2 every transcendental solution $f$ of (14) is an entire function of order of growth $\sigma(f) = \alpha_1 = \frac{n}{\alpha}$. The two cases $n = 1$ and $n = 2$ are confirmed in [7] by the power series method.*

**Example 2.4.** *Consider the fractional differential equation*

$$\frac{r^{2.3}}{z^2}\mathcal{D}^{2.3} f(z) + z^2 \frac{r^{1.8}}{z}\mathcal{D}^{1.8} f(z) + z^2 r^{0.5}\mathcal{D}^{0.5} f(z) + z\mathcal{D} f(z) = 0. \tag{15}$$

*We have $s_1 = 2$ and $s_2 = 1 = s_p$. So $\alpha_1 = 6$ and $\alpha_2 = \frac{1}{1.3} = \frac{10}{13}$. By Theorem 2.2, every transcendental solution $f$ of (15) is an entire function of order of growth $\sigma(f) = 6$ or $\sigma(f) = \frac{10}{13}$.*

**Corollary 2.5.** *Consider the equation (4) with $P_0(0) = 0$. Then every transcendental solution $f$ of (4) is of order of growth $\rho(f)$ satisfying*

$$\rho(f) \geq \frac{1}{q_n}. \tag{16}$$

In fact, if $s_1 \geq 1$ then by Proposition 2.1, we have $\rho(f) \geq \frac{1}{q_{s_1}} \geq \frac{1}{q_n}$; and if $s_1 = 0$, then by theorem 2.2 there exists only one value $\alpha_1 = \frac{d_0 + [q_n]}{q_n}$ such that $\rho(f) = \frac{d_0 + [q_n]}{q_n}$ and since $P_0(0) = 0$, we have $d_0 \geq 1$ and then (16) holds.

**Remark 2.6.** *By Corollary 2.5, we deduce that there is no transcendental solution of (4) of order zero.*

**Theorem 2.7.** *Consider the equation (4) with $P_0(0) = 0$. Let $C = \max\limits_{0 \leq k \leq n} \left\{d_j - \left[q_j\right]\right\}$ and $A = \{i : d_i - [q_i] = C\}$, where $d_n = q_0 = 0$. Then, the following conclusions hold:*
*(a) If $Card(A) = 1$ then there is no polynomial solution for (4).*
*(b) If $Card(A) \geq 2$ then the equation (4) can admit a polynomial solutions;*
*where $Card(A)$ designates the number of elements of the set $A$.*

**Corollary 2.8.** *Consider the equation (4) with $P_0(0) = 0$. If $d_0 \geq d_k$ for all $k = 1, ..., n-1$, then there is no polynomial solution for (4).*

**Example 2.9.** *Consider the fractional differential equation*

$$\frac{r^{\frac{7}{2}}}{z^3}\mathcal{D}^{\frac{7}{2}}f(z) + \left(\frac{-16}{15\sqrt{\pi}}z^4 + \frac{24}{15\pi}z^2 + \frac{8}{3\sqrt{\pi}}z\right)r^{\frac{3}{2}}\mathcal{D}^{\frac{3}{2}}f(z) - \left(\frac{4}{\sqrt{\pi}}z^4 + \frac{8}{\sqrt{\pi}}z^2 + \frac{4}{\sqrt{\pi}}\right)r^{\frac{1}{2}}\mathcal{D}^{\frac{1}{2}}f(z) + \frac{224}{15\pi}z^4 f = 0 \quad (17)$$

*we have $C = \max\limits_{0\leq k\leq 2}\left(d_j - \left[q_j\right]\right) = 4$ and $A = \{0,1,2\}$, $Card(A) = 3$. So by Theorem 2.7 the equation (17) possibly has polynomial solutions. In fact, the two polynomials*

$$f_1(z) = z^2 + 1 \text{ and } f_2(z) = z^3 - 1$$

*are solutions to (17).*

**Example 2.10.** *Consider the fractional differential equation*

$$\frac{r^{3.2}}{z^3}D^{3.2}f(z) + \left(\frac{8}{5\Gamma(\frac{5}{3})}z^2 - \frac{2}{\Gamma(\frac{5}{3})}z + \frac{2}{\Gamma(\frac{5}{3})}\right)r^{1.6}D^{1.6}f(z) + \left(\frac{2}{\Gamma(1.4)}z^2\right)r^{1.3}D^{1.3}f(z) - \frac{4}{\Gamma(1.4)\Gamma(\frac{5}{3})}z^2 f = 0 \quad (18)$$

*we have $C = \sup\limits_{0\leq k\leq 2}\left(d_j - \left[q_j\right]\right) = 2$ and $A = \{0,1,2\}$, $Card(A) = 3$. So by Theorem 2.7 the equation possibly has polynomial solutions. In fact, the two polynomials*

$$f_1(z) = z \text{ and } f_2(z) = z^2 - 2z + 1$$

*are solutions to (18).*

By combining Theorem 2.2 and Theorem 2.7 we can obtain the following result:

**Corollary 2.11.** *Consider the equation (4) with $P_0(0) = 0$. If $s_1 = 0$ then every solution $f \not\equiv 0$ of (4) is an entire function of order of growth*

$$\sigma(f) = \frac{d_0 + [q_n]}{q_n}.$$

In fact, if $s_1 = 0$ then by theorem 2.2, there exists only one value $\alpha_1 = \frac{d_0+[q_n]}{q_n}$ as order of growth to transcendental solutions. It remains to prove that there is no polynomial solution for (4). By definition, $s_1 = 0$ means that we have

$$\frac{d_0 + [q_n]}{q_n} \geq \frac{d_k + [q_n] - [q_k]}{q_n - q_k}, \quad (19)$$

holds for all $k = 1, ..., n-1$; which yields $d_0 > d_k - [q_k]$ for all $k = 1, ..., n-1$ : because if we suppose to the contrary that there exists $k \in \{1, ..., n-1\}$ such that $d_0 \leq d_k - [q_k]$; then combining this with (19) gives

$$\frac{1}{q_n} \geq \frac{1}{q_n - q_k};$$

a contradiction. Now the statement $d_0 > d_k - [q_k]$ for all $k = 1, ..., n-1$ yields $A = \{0\}$, $Card(A) = 1$; and by Theorem 2.7 there is no polynomial solution for (4).

We signal here that the result of Corollary 2.11 is proved in [7] by another method.

## 3. Preliminary lemmas

In order to prove our results we state the following lemmas.

**Lemma 3.1.** *For any fixed $j = 0, 1, ..., p-1$, let $\alpha$ be any real number satisfying $\alpha > \alpha_{j+1}$, and let $k$ be any integer satisfying $0 \le k < s_j$. Then*

$$q_n + d_k + \alpha q_k - [q_k] < q_n + d_{s_j} + \alpha q_{s_j} - \left[q_{s_j}\right]. \tag{20}$$

*Proof.* Since

$$q_n + d_k + \alpha q_k - [q_k] = (q_n + d_{s_j} + \alpha q_{s_j} - \left[q_{s_j}\right]) + \alpha(q_k - q_{s_j}) + d_k - d_{s_j} - [q_k] + \left[q_{s_j}\right]$$

we get

$$q_n + d_k + \alpha q_k - [q_k] < (q_n + d_{s_j} + \alpha q_{s_j} - \left[q_{s_j}\right]) + \alpha_{j+1}(q_k - q_{s_j}) + d_k - d_{s_j} - [q_k] + \left[q_{s_j}\right]. \tag{21}$$

Now from the definition of $\alpha_{j+1}$, we obtain

$$\alpha_{j+1}(q_k - q_{s_j}) + d_k - d_{s_j} - [q_k] + \left[q_{s_j}\right] = \left(q_k - q_{s_j}\right)\left(\alpha_{j+1} - \frac{d_k - d_{s_j} + \left[q_{s_j}\right] - [q_k]}{q_{s_j} - q_k}\right). \tag{22}$$

Since $0 \le k < s_j$, it follows from the definition of $s_{j+1}$ that

$$\left(q_k - q_{s_j}\right)\left(\alpha_{j+1} - \frac{d_k - d_{s_j} + \left[q_{s_j}\right] - [q_k]}{q_{s_j} - q_k}\right) \le 0. \tag{23}$$

Then, (20) follows from (21)- (23). □

**Lemma 3.2.** *For any fixed $j = 1, 2, ..., p$ let $\alpha$ be any real number satisfying $\alpha < \alpha_j$ and let $k$ be any integer satisfying $s_j < k \le n$. Then*

$$q_n + d_k + \alpha q_k - [q_k] < q_n + d_{s_j} + \alpha q_{s_j} - \left[q_{s_j}\right]. \tag{24}$$

*Proof.* We consider two separate cases.
**Case (i).** Suppose that $s_j < k \le s_{j-1}$. As in the proof of Lemma 3.1, we have

$$q_n + d_k + \alpha q_k - [q_k] < (q_n + d_{s_j} + \alpha q_{s_j} - \left[q_{s_j}\right]) + \alpha_j(q_k - q_{s_j}) + d_k - d_{s_j} + \left[q_{s_j}\right] - [q_k].$$

If $k = s_{j-1}$, then

$$\alpha_j(q_k - q_{s_j}) + d_k - d_{s_j} + \left[q_{s_j}\right] - [q_k] = 0;$$

and then (24) follows.

On the other hand, if $s_j < k < s_{j-1}$, then from the definition of $s_j$ and $\alpha_j$ we obtain

$$\alpha_j(q_k - q_{s_{j-1}} + q_{s_{j-1}} - q_{s_j}) + d_k - d_{s_j} + \left[q_{s_j}\right] - [q_k] = (q_k - q_{s_{j-1}})\left(\alpha_j - \frac{d_k - d_{s_{j-1}} + \left[q_{s_{j-1}}\right] - [q_k]}{q_{s_{j-1}} - q_k}\right) \le 0$$

This proves Lemma 3.2 for Case (i).

**Case (ii).** Suppose that $s_{j-1} < k \leq n$. Since $s_j < s_{j-1} < ... < s_1 < s_0 = n$ and $s_{j-1} < k \leq n$, it follows that $j \geq 2$ and there exists an integer $m, 1 \leq m \leq j-1$, such that $s_{j-m} < k \leq s_{j-m-1}$. Also, from Proposition 2.1, we have

$$\alpha_j < \alpha_{j-1} < ... < \alpha_{j-m}.$$

Since $\alpha < \alpha_j$, we have $\alpha < \alpha_{j-m}$; hence we can apply Case (i) to obtain that

$$q_n + d_k + \alpha q_k - [q_k] < q_n + d_{s_{j-m}} + \alpha q_{s_{j-m}} - \left[q_{s_{j-m}}\right]$$

Now from successive applications of Case (i), we obtain the following inequalities:

$$\begin{array}{rcl} q_n + d_{s_{j-1}} + \alpha q_{s_{j-1}} - \left[q_{s_{j-1}}\right] & < & q_n + d_{s_j} + \alpha q_{s_j} - \left[q_{s_j}\right], \ \alpha < \alpha_j \\ q_n + d_{s_{j-2}} + \alpha q_{s_{j-2}} - \left[q_{s_{j-2}}\right] & < & q_n + d_{s_{j-1}} + \alpha q_{s_{j-1}} - \left[q_{s_{j-1}}\right], \ \alpha < \alpha_{j-1} \\ \cdots\cdots\cdots & < & \cdots\cdots\cdots \\ q_n + d_{s_{j-m}} + \alpha q_{s_{j-m}} - \left[q_{s_{j-m}}\right] & < & q_n + d_{s_{j-m+1}} + \alpha q_{s_{j-m+1}} - \left[q_{s_{j-m+1}}\right], \ \alpha < \alpha_{j-m+1} \end{array}$$

then we obtain

$$q_n + d_k + \alpha q_k - [q_k] < q_n + d_{s_j} + \alpha q_{s_j} - \left[q_{s_j}\right], \ \alpha < \alpha_j, \ s_j < k \leq n$$

□

**Lemma 3.3.** *Let $\alpha > 0$. Then for any integer k satisfying $0 \leq k < s_p$, we have*

$$q_n + d_k + \alpha q_k - [q_k] < q_n + d_{s_p} + \alpha q_{s_p} - \left[q_{s_p}\right] \tag{25}$$

*Proof.* Since $s_p$ is the last element in the sequence $s_1, s_2, ..., s_p$ it follows from the construction of $s_p$ that for any $k < s_p$, we obtain

$$\frac{d_k - d_{s_p} + \left[q_{s_p}\right] - [q_k]}{q_{s_p} - q_k} \leq 0.$$

This gives $d_k - [q_k] \leq d_{s_p} - \left[q_{s_p}\right]$. Since $q_k < q_{s_p}$, for any $k < s_p$, (25) holds. □

**Lemma 3.4.** *[5] Let $f(z)$ be a transcendental entire function, $\alpha > 0,\ 0 < \delta < \frac{1}{4}$ and z be such that $|z| = r$ and that*

$$\left|f(z)\right| > M(r, f)\, \nu(r)^{-\frac{1}{4}+\delta}$$

*holds; where $\nu(r)$ is the central index of f. Then there exists a set $E \subset (0, +\infty)$ of finite logarithmic measure, that is $\int_E \frac{dt}{t} < +\infty$, such that*

$$\frac{r^\alpha \mathcal{D}^\alpha f(z)}{f(z)} = (\nu(r))^\alpha (1 + o(1)) \tag{26}$$

*holds for $r \to +\infty$ and $r \notin E$.*

**Remark 3.5.** *We signal here that the authors of [5] have used the Riemann-Liouville operator derivative in the proof of Lemma 3.4; and for an entire function $f(z) = \sum_{j=0}^{+\infty} a_j z^j$ we have*

$$\mathcal{D}^\alpha_{RL} f(z) = \sum_{j=0}^{+\infty} \frac{\Gamma(j+1)}{\Gamma(j-\alpha+1)} a_j r^{j-\alpha} e^{ji\theta}. \tag{27}$$

*By (3) and (27), we confirm that the proof of Lemma 3.4 is valid also for the Caputo fractional derivative operator.*

**Remark 3.6.** *For a non-constant polynomial $P(z) = a_n z^n + ... + a_0$ of degree $n$ and $0 < \alpha \le n$, it is easy to get*

$$\left| \frac{r^{\alpha} \mathcal{D}^{\alpha} P(z)}{P(z)} \right| = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} (1 + o(1)) \text{ as } r \to \infty.$$

*In fact, by taking the limit as $r \to \infty$, the leading term of $|r^{\alpha} \mathcal{D}^{\alpha} P(z)|$ is $\frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} |a_n| r^n$.*

**Lemma 3.7.** *[12] Let $P(z) = a_n z^n + ... + a_0$ be a polynomial of degree $n$. Then, for any given $\varepsilon > 0$ there exists $r_0 > 0$ such that for all $r = |z| > r_0$ the inequalities*

$$(1 - \varepsilon) |a_n| r^n \le |P(z)| \le (1 + \varepsilon) |a_n| r^n$$

*hold.*

**Lemma 3.8.** *[7] Let $P_0(z) \not\equiv 0, P_1(z), ..., P_{n-1}(z)$ be polynomials such that $P_0(0) = 0$; let $0 < q_1 < q_2 < ... < q_n$ be real constants. Then, all solutions of (4) are entire functions.*

## 4. Proof of theorems

*Proof.* **[Proof of Theorem 2.2]** By Lemma 3.8, all solutions of (4) are entire functions. Let $f$ be a transcendental solution of (4) with order $\rho(f) = \alpha$. By (5) and Remark 2.6, we have $0 < \alpha < \infty$. By Lemma 3.7, as $r \to \infty$ we have

$$\left| P_j(z) \right| = b_j r^{d_j} (1 + o(1)). \tag{28}$$

let $b_j$ denote the leading coefficient of the polynomial $P_j$. By Lemma 3.4, there exists a set $E \subset (0, +\infty)$ of finite logarithmic measure, such that for $r \to +\infty$ and $r \notin E$, we have

$$\left| \frac{r^{q_k} \mathcal{D}^{q_k} f(z)}{f(z)} \right| = (\nu(r))^{q_k} (1 + o(1)), \ (j = 1, ..., n). \tag{29}$$

As $0 < \alpha < \infty$ it is well known that

$$\nu(r) = (1 + o(1)) C r^{\alpha} \tag{30}$$

as $r \to \infty$, where $\nu(r)$ is the central index of $f$ and $C$ is a positive constant; see [9]. Set $a_j = C^j \left| b_j \right|$. We now divide equation (4) by $f$ and then substitute (28)-(30) into (4). This yields an equation whose right side is zero and whose left side consists of a sum of $n+1$ terms whose absolute values are asymptotic (as $r \to \infty$; $r \notin E$) to the following $n+1$ terms:

$$a_n r^{\alpha q_n - [q_n]}, a_{n-1} r^{q_n + d_{n-1} + \alpha q_{n-1} - [q_{n-1}]}, ..., a_k r^{q_n + d_k + \alpha q_k - [q_k]}, ..., a_0 r^{q_n + d_0}. \tag{31}$$

From(5) and (10), the order of any solution of (4) is at most $\alpha_1$ i.e. $\alpha \le \alpha_1$. Now suppose that $\alpha_{j+1} < \alpha < \alpha_j$ for some $j = 1, 2, ..., p-1$. Then from Lemma 3.1 and Lemma 3.2, we obtain

$$q_n + d_k + \alpha q_k - [q_k] < q_n + d_{s_j} + \alpha q_{s_j} - \left[ q_{s_j} \right] \text{ for any } k \ne s_j \tag{32}$$

which implies that there will exist exactly one dominant term (as $r \to \infty$; $r \notin E$) in (31) : there exists exactly one term in (31) with exponent $q_n + d_{s_j} + \alpha q_{s_j} - \left[ q_{s_j} \right]$, where $a_{s_j} \ne 0$, which is greater than all the other exponents of the terms. This is impossible.

Now, suppose that $\alpha < \alpha_p$. Then from Lemma 3.2 and Lemma 3.3, we obtain

$$q_n + d_k + \alpha q_k - [q_k] < q_n + d_{s_p} + \alpha q_{s_p} - \left[ q_{s_p} \right] \text{ for any } k \ne s_p \tag{33}$$

Again, there exists exactly one term in (31) with exponent $q_n + d_{s_p} + \alpha q_{s_p} - \left[ q_{s_p} \right]$, where $a_{s_p} \ne 0$, which is greater than all the other exponents of the terms. This is impossible.

Therefore, the only admissible values for $\alpha$, the order of $f$, are $\alpha_1, \alpha_2, ..., \alpha_p$. □

*Proof.* **[Proof of Theorem 2.7]** (i) Assume that $f(z) = a_n z^n + ... + a_0$ $(a_n \neq 0)$ is a polynomial solution of (4). By Remark 3.6, we have

$$\left|\frac{\mathcal{D}^{\alpha} f(z)}{f(z)}\right| = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}(1+o(1)) \text{ as } r \to \infty. \tag{34}$$

Dividing equation (4) by $f$ and using (34), we get an equation whose left side is a sum of $n+1$ terms whose absolute values are asymptotic (as $r \to \infty$) to the following $n+1$ terms:

$$b_n r^{-[q_n]}, b_{n-1} r^{d_{n-1}-[q_{n-1}]}, ..., b_k r^{d_k-[q_k]}, ..., b_0 r^{d_0} \tag{35}$$

where $b_0, ..., b_n$ are positive constants. If $Card(A) = 1$, then there exists exactly one dominant term in (35) (as $r \to \infty$); which is impossible. Therefore, if $Card(A) = 1$ there is no polynomial solution for (4).
(ii) If $Card(A) \geq 2$ there does not exist one dominant term in (35), then it is possible that there exists a polynomial solution as it is shown in Example 2.9 and Example 2.10. □

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