Filomat 39:26 (2025), 9179–9189
https://doi.org/10.2298/FIL2526179Z

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A mathematical insight of fractional logistic equation of variable order on finite intervals

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Abstract. The logistic equation, which is usually represented as a differential equation, is used to simulate the population expansion or the spread or evolution of phenomena within a confined area. However, it is frequently discussed in terms of the evolution of solutions in unbounded time and the longterm behavior of the population size. In this paper, we analyze it by using variable order fractional calculus within the constraint of a finite time frame, allowing for a more realistic and applicable assessment of real-world scenarios with finite resources or boundaries.

1. Introduction

The theory of dynamical systems is a field of mathematics and physics that investigates the mathematical behavior and the corresponding classification concerning how systems evolve over time by providing valuable insights into the complexities that surround the development of real-world processes in response to their initial conditions and the governing equations that describe the dynamics, which may vary from simple deterministic systems to extremely complicated and chaotic ones. This theory possesses wide-ranging applications in physics, engineering, biology, economics, and even in the social sciences, allowing researchers to examine the stability, periodicity, and long-term behavior of systems, making it an indispensable tool for predicting and modeling the behavior of different events in the natural and social world.

As a result of its obvious importance and vast range of applications, fractional calculus in mathematical modeling has grown in popularity and significance, thus gaining prominence in diverse scientific

2020 *Mathematics Subject Classification*. Primary 26A33; Secondary 34A08.
Keywords. variable order fractional operators, logistic equation, finite interval, uniform stability, finite differences.
Received: 11 September 2024; Revised: 06 August 2025; Accepted: 06 August 2025
Communicated by Maria Alessandra Ragusa
The research of R. Rodríguez-López was partially supported by the Agencia Estatal de Investigación (AEI) of Spain, co-financed by the European Fund for Regional Development (FEDER) corresponding to the 2021–2024 multiyear financial framework, project PID2020-113275GB-I00, and ED431C 2023/12 (GRC Xunta de Galicia).
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and engineering domains such as science, engineering, finance, and social sciences, by extending the traditional concepts of differentiation and integration to non-integer orders rather than being confined to integer numbers, and consequently allowing to examine derivatives and integrals with fractional indexes, revolutionizing our understanding of complicated structures thanks to its ability for capturing intricate behaviors involving memory, heredity, and complex dynamics. However, recent research work has shown that constant fractional order calculus is not the ultimate tool for modeling every natural occurrence. As a result, variable order fractional calculus is presented and extensively investigated for its capacity to offer a better description for problems with local and nonlocal circumstances (see [1, 4, 5, 12–15, 20, 22] and the references therein).

The logistic equation is a fundamental mathematical model that is utilized to predict the population growth or the spread of phenomena in a constrained context. It was initially presented in the nineteenth century by Pierre François Verhulst and has since then been a cornerstone in many domains, including ecology, epidemiology, economics, and even the study of social trends.

The logistic equation's continuous form is expressed as a nonlinear ordinary differential equation of the type

$$\begin{cases} \dfrac{d}{dt}X(t) = \kappa X(t)\left(1 - \dfrac{X(t)}{K}\right), \\ X(0) = X_0. \end{cases} \tag{1}$$

This equation takes two essential aspects into account: the Malthusian parameter $\kappa > 0$, expressing the intrinsic growth rate of the species, and K, representing the carrying capacity of the environment.

If we take $\vartheta = \dfrac{X}{K}$, then equation (1) is reduced to the nonlinear differential equation written as

$$\begin{cases} \dfrac{d}{dt}\vartheta(t) = \kappa\vartheta(t)\left(1 - \vartheta(t)\right), \\ \vartheta(0) = \vartheta_0, \end{cases} \tag{2}$$

where $\vartheta_0 = \dfrac{X_0}{K}$.

This simple but powerful model captures the idea that growth is initially exponential but eventually flattens as the resources become scarce, making it especially useful for predicting and understanding population dynamics, disease outbreaks, market saturation, and other scenarios where the growth is constrained by the available resources. Because of its wide range of uses and adaptability, the logistic equation has become a vital tool for academics as well as decision-makers across many fields.

In [19], B. J. West studied a more generalized version of the logistic equation by incorporating a memory term through the use of fractional derivatives in continuous time

$$\begin{cases} {}^{C}\mathcal{D}^{\varpi}_{0^+}\vartheta(t) = \kappa^{\varpi}\vartheta(t)(1 - \vartheta(t)), \\ \vartheta(0) = \vartheta_0, \end{cases} \tag{3}$$

where $0 < \varpi < 1$, $\vartheta_0 \in \mathbb{R}$, and ${}^{C}D^{\varpi}_{0^+}$ denotes the Caputo fractional derivative operator of order ϖ.

The author of [19] provided the exact solution to this extension of equation, which has been denominated as West function, given by

$$\vartheta(t) = \sum_{n=0}^{\infty}\left(\frac{\vartheta - 1}{\vartheta}\right)^n E_{\varpi}(-n\kappa^{\varpi}t^{\varpi}), \tag{4}$$

where E_{ϖ} denotes the so–called one parameter Mittag–Leffler function, denoted by

$$E_{\varpi}(\Lambda) = \sum_{n=0}^{\infty}\frac{\Lambda^n}{\Gamma(\varpi n + 1)}, \quad \varpi > 0,\ \Lambda \in \mathbb{C}, \tag{5}$$

which was first proposed by G. M. Mittag-Leffler and may be thought of as a generalization of the exponential function. In the previous expression, Γ represents the Gamma function, defined as:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt, \quad \Re(z) > 0.$$

Shortly after, I. Area et al., in a short note [3], showed that the real function (4) proposed by B. J. West [19] is not an exact solution for the fractional logistic equation.

In summary, finding exact solutions to the fractional logistic equation explicitly can be challenging. For this purpose, researchers typically resort to numerical methods to explore the system's behavior and dynamics, such as finite difference methods or spectral methods. These methods can provide valuable insights into the behavior of the system described by the fractional logistic equation (for more details, see [8–11] and the references therein).

Recently, in [7], K. Devendra et al. analyzed the logistic equation with the novel fractional derivative given by Caputo and Fabrizio

$$\begin{cases} {}^{CF}\mathcal{D}^{\varpi}_{0;t}\vartheta(t) = \kappa\vartheta(t)(1-\vartheta(t)), \\ \vartheta(0) = \vartheta_0. \end{cases} \tag{6}$$

Motivated by the aforementioned works, in this paper, we study the dynamical properties of the following fractional logistic equation involving the variable order Caputo fractional derivative

$$\begin{cases} {}^{C}\mathcal{D}^{\varpi(t)}_{0^+}\vartheta(t) = \kappa\vartheta(t)(1-\vartheta(t)), \quad 0 \le t \le T < +\infty, \\ \vartheta(0) = \vartheta_0, \end{cases} \tag{VOFLE}$$

where $0 < \varpi(t) < 1$, $\vartheta_0 \in \mathbb{R}^+$, $\kappa > 0$, and ${}^{C}D^{\varpi(t)}_{0^+}$ denotes the Caputo fractional derivative operator of variable order $\varpi(t)$ for the function $\vartheta(t)$ defined by [2, Definition 30]

$${}^{C}\mathcal{D}^{\varpi(t)}_{a^+}\vartheta(t) = \frac{1}{\Gamma(1-\varpi(t))}\int_a^t (t-s)^{-\varpi(t)}\vartheta'(s)ds, \quad t > a,$$

and the Riemann-Liouville integral of variable order $\varpi(t)$ for ϑ is given by [18]

$$I^{\varpi(t)}_{a^+}\vartheta(t) = \frac{1}{\Gamma(\varpi(t))}\int_a^t (t-s)^{\varpi(t)-1}\vartheta(s)ds, \quad t > a.$$

It is known that, when the order $\varpi(t)$ is a constant ϖ, then the variable order fractional integral and derivative operators coincide with its constant order counterparts. Therefore, due to the property of semi-group, we obtain the following properties

$$\begin{aligned} I^{\varpi_1}_{a^+}\, I^{\varpi_2}_{a^+} &= I^{\varpi_2}_{a^+}\, I^{\varpi_1}_{a^+} \\ &= I^{\varpi_1+\varpi_2}_{a^+}. \end{aligned}$$

However, some recent studies have proved that such properties do not hold for variable order fractional operators. Indeed,

$$\begin{aligned} I^{\varpi_1(t)}_{a^+}\, I^{\varpi_2(t)}_{a^+} &\neq I^{\varpi_2(t)}_{a^+}\, I^{\varpi_1(t)}_{a^+} \\ &\neq I^{\varpi_1(t)+\varpi_2(t)}_{a^+}, \end{aligned}$$

where $\varpi_1(t)$ and $\varpi_2(t)$ are general non negative functions. We shall give an example to prove these claimed arguments.

Example 1.1. *Let* $\varpi_1(t) = \dfrac{t+1}{4}$, $\varpi_2(t) = \dfrac{3-t}{4}$, $\vartheta(t) = t$, *for* $0 \le t \le 1$. *Then*

$$\begin{aligned}
I_{0^+}^{\varpi_1(t)} I_{0^+}^{\varpi_2(t)} \vartheta(t) &= \int_0^t \frac{(t-s)^{\frac{t+1}{4}-1}}{\Gamma\left(\frac{t+1}{4}\right)} \left(\int_0^s \frac{(s-h)^{\frac{3-s}{4}-1}}{\Gamma\left(\frac{3-s}{4}\right)} h \, dh \right) ds \\
&= \int_0^t \frac{(t-s)^{\frac{t-3}{4}} s^{\frac{7-s}{4}}}{\Gamma\left(\frac{t+1}{4}\right)\Gamma\left(\frac{11-s}{4}\right)} ds, \\
I_{0^+}^{\varpi_1(t)} I_{0^+}^{\varpi_2(t)} \vartheta(t)\Big|_{t=\frac{1}{2}} &= \int_0^{\frac{1}{2}} \frac{\left(\frac{1}{2}-s\right)^{-\frac{5}{8}} s^{\frac{7-s}{4}}}{\Gamma\left(\frac{3}{8}\right)\Gamma\left(\frac{11-s}{4}\right)} ds \\
&\approx 0.12138, \\
I_{0^+}^{\varpi_2(t)} I_{0^+}^{\varpi_1(t)} \vartheta(t) &= \int_0^t \frac{(t-s)^{\frac{3-t}{4}-1}}{\Gamma\left(\frac{3-t}{4}\right)} \left(\int_0^s \frac{(s-h)^{\frac{s+1}{4}-1}}{\Gamma\left(\frac{s+1}{4}\right)} h \, dh \right) ds \\
&= \int_0^t \frac{(t-s)^{\frac{-1-t}{4}} s^{\frac{5+s}{4}}}{\Gamma\left(\frac{3-t}{4}\right)\Gamma\left(\frac{7+s}{4}\right)} ds, \\
I_{0^+}^{\varpi_2(t)} I_{0^+}^{\varpi_1(t)} \vartheta(t)\Big|_{t=\frac{1}{2}} &= \int_0^{\frac{1}{2}} \frac{\left(\frac{1}{2}-s\right)^{-\frac{3}{8}} s^{\frac{5+s}{4}}}{\Gamma\left(\frac{5}{8}\right)\Gamma\left(\frac{7+s}{4}\right)} ds \\
&\approx 0.19956, \\
I_{0^+}^{\varpi_1(t)+\varpi_2(t)} \vartheta(t)\Big|_{t=\frac{1}{2}} &= \int_0^{\frac{1}{2}} s \, ds = \frac{s^2}{2}\Big|_0^{\frac{1}{2}} \\
&= 0.125.
\end{aligned}$$

Here, we have written the approximations with several decimal positions in order to clearly illustrate that these quantities are different. Therefore,

$$\begin{aligned}
I_{0^+}^{\varpi_1(t)} I_{0^+}^{\varpi_2(t)} \vartheta(t)\Big|_{t=\frac{1}{2}} &\neq I_{0^+}^{\varpi_2(t)} I_{0^+}^{\varpi_1(t)} \vartheta(t)\Big|_{t=\frac{1}{2}} \\
&\neq I_{0^+}^{\varpi_1(t)+\varpi_2(t)} \vartheta(t)\Big|_{t=\frac{1}{2}}.
\end{aligned}$$

This paper is organized as follows. In Section 2, we present some definitions and necessary lemmas associated with the variable order logistic equation. In Section 3, we establish some existence and uniqueness results for the solutions of the problem (*VOFLE*). In Section 4, the uniform stability of initial value problem (*VOFLE*) associated with variable order Caputo fractional derivative is discussed, and we complete the paper with some numerical approximations to illustrate the obtained results.

2. Preliminaries

This section introduces certain essential concepts and lemmas that will be important to present the main results in the next sections.

Definition 2.1. *Let* $T > 0$, *and* $[0, T]$ *be a closed interval of* $\mathbb{R}$. *We denote by:*

i) $C([0, T], \mathbb{R})$ *the Banach space of continuous functions* $\vartheta : [0, T] \longrightarrow \mathbb{R}$, *with the usual supremum norm*

$$\|\vartheta\|_\infty := \sup\{|\vartheta(t)|, \quad t \in [0, T]\}.$$

ii) $L^1([0,T],\mathbb{R})$ *the Banach space of measurable functions* $\vartheta : [0,T] \longrightarrow \mathbb{R}$ *that are Lebesgue integrable, equipped with the norm*

$$\|\vartheta\|_{L^1} = \int_0^T |\vartheta(s)|\, ds.$$

Definition 2.2. *[21, Definitions 2.1 − 2.3] Let S be a subset of the real space* $\mathbb{R}$*. We introduce the following notation:*

i) *S is called a generalized interval if it is either a standard interval, a point, or the empty set* $\emptyset$*.*

ii) *Assuming that S is a generalized interval, the finite set* $\mathcal{P}$ *consisting of all the generalized intervals contained in S is termed as a partition of S provided that every* $x \in S$ *lies in exactly one of the generalized intervals in the finite set* $\mathcal{P}$*.*

iii) *Evidently, the function* $\varpi : t \mapsto \mathbb{R}$ *is piecewise constant with respect to the partition* $\mathcal{P}$ *of S, if, for any* $I \in S$*,* ϖ *is constant on I.*

Lemma 2.3. *[14] Let* $\varpi_1,\ \varpi_2 > 0$*,* $0 < a < b$*, and* $\vartheta \in L^1(a,b)$ *with* ${}^C\mathcal{D}_{a^+}^{\varpi_1}\vartheta \in L^1(a,b)$*. Then the unique solution of the following equation*

$${}^C\mathcal{D}_{a^+}^{\varpi_1}\vartheta(t) = 0$$

is given by

$$\vartheta(t) = \varrho_0 + \varrho_1(t-a) + \varrho_2(t-a)^2 + \cdots + \varrho_{l-1}(t-a)^{l-1},$$

and

$$I_{a^+}^{\varpi_1}\, {}^C\mathcal{D}_{a^+}^{\varpi_1}\vartheta(t) = \vartheta(t) + \varrho_0 + \varrho_1(t-a) + \varrho_2(t-a)^2 + \cdots + \varrho_{l-1}(t-a)^{l-1},$$

with $l = [\varpi_1] + 1$*,* $\varrho_k \in \mathbb{R}$*,* $k = 0, 1, \ldots, l-1$*.*

Furthermore,

$${}^C\mathcal{D}_{a^+}^{\varpi_1}\, I_{a^+}^{\varpi_1}\vartheta(t) = \vartheta(t),$$

and

$$I_{a^+}^{\varpi_1} I_{a^+}^{\varpi_2}\vartheta(t) = I_{a^+}^{\varpi_2} I_{a^+}^{\varpi_1}\vartheta(t) = I_{a^+}^{\varpi_1+\varpi_2}\vartheta(t).$$

3. Existence of solutions

Based on the previous discussion, in this section, we present our main results.

Let $P = \{[0,T_1],\ (T_1,T_2],\ (T_2,T_3],\ldots,\ (T_{n-1},T]\}$ be a partition of the finite interval $[0,T]$, and let $\varpi(t) : [0,T] \longrightarrow (0,1]$ be a piecewise constant function with respect to P given by

$$\varpi(t) = \sum_{l=1}^{n} \varpi_l \mathbb{I}_l(t) = \begin{cases} \varpi_1, & t \in [0,T_1],\\ \varpi_2, & t \in (T_1,T_2],\\ \vdots & \\ \varpi_n, & t \in (T_{n-1},T], \end{cases}$$

where $0 < \varpi_l < 1$, $l \in \{1,2,\ldots,n\}$ are constants, and $\mathbb{I}_l$ is the characteristic function for the interval $[T_{l-1},T_l]$, $l \in \{1,2,\ldots,n\}$, i.e.,

$$\mathbb{I}_l(t) = \begin{cases} 1, & \text{if } t \in [T_{l-1},T_l],\\ 0, & \text{elsewhere.} \end{cases}$$

To reach our primary conclusions, we first do some basic analysis on the equation of problem (*VOFLE*). Indeed, since

$$\varpi(t) = \sum_{l=1}^{n} \varpi_l \mathbb{I}_l(t),$$

we get

$$^{C}\mathcal{D}_{0^+}^{\varpi(t)}\vartheta(t) = \int_0^t \frac{(t-s)^{-\varpi(t)}}{\Gamma(1-\varpi(t))}\vartheta'(s)\,ds = \sum_{l=1}^{n}\mathbb{I}_l(t)\int_0^t \frac{(t-s)^{-\varpi_l}}{\Gamma(1-\varpi_l)}\vartheta'(s)\,ds. \tag{7}$$

So, the equation of the problem (*VOFLE*) can be written as the following

$$^{C}\mathcal{D}_{0^+}^{\varpi(t)}\vartheta(t) = \sum_{l=1}^{n}\mathbb{I}_l(t)\int_0^t \frac{(t-s)^{-\varpi_l}}{\Gamma(1-\varpi_l)}\vartheta'(s)\,ds = \kappa\vartheta(t)(1-\vartheta(t)), \quad 0 \le t \le T < +\infty. \tag{8}$$

Therefore, in the interval $[0, T_1]$, it can be written as

$$^{C}\mathcal{D}_{0^+}^{\varpi_1}\vartheta(t) = \int_0^t \frac{(t-s)^{-\varpi_1}}{\Gamma(1-\varpi_1)}\vartheta'(s)\,ds = \kappa\vartheta(t)(1-\vartheta(t)), \quad 0 < t \le T_1. \tag{9}$$

Again, in the interval $(T_1, T_2]$, it can be written as

$$^{C}\mathcal{D}_{0^+}^{\varpi_2}\vartheta(t) = \int_0^t \frac{(t-s)^{-\varpi_2}}{\Gamma(1-\varpi_2)}\vartheta'(s)\,ds = \kappa\vartheta(t)(1-\vartheta(t)), \quad T_1 < t \le T_2. \tag{10}$$

In the same way, in the interval $(T_{l-1}, T_l]$, it can be written as

$$^{C}\mathcal{D}_{0^+}^{\varpi_l}\vartheta(t) = \int_0^t \frac{(t-s)^{-\varpi_l}}{\Gamma(1-\varpi_l)}\vartheta'(s)\,ds = \kappa\vartheta(t)(1-\vartheta(t)), \quad T_{l-1} < t \le T_l. \tag{11}$$

We denote by $E_l = (C([0, T_l], \mathbb{R}), \|\cdot\|_{E_l})$ the class of functions that form a Banach space with the equivalent norm

$$\|\cdot\|_{E_l} = \sup_{t\in[0,T_l]} e^{-Nt}|\cdot(t)|, \quad N > 0, \quad l \in \{1, 2, \ldots, n\}.$$

Thus, we may consider the following auxiliary initial value problems of constant order defined on the intervals of the type $[T_{l-1}, T_l]$, $l \in \{1, 2, \ldots, n\}$, as follows

$$\begin{cases} ^{C}\mathcal{D}_{0^+}^{\varpi_l}\vartheta_l(t) = \kappa\vartheta_l(t)(1-\vartheta_l(t)), \quad T_{l-1} < t \le T_l, \\ \vartheta_l(0) = \vartheta_0. \end{cases} \tag{12}$$

Definition 3.1. *For $l \in \{1, 2, \ldots, n\}$, we say that ϑ_l is a solution of the initial value problem* (12) *if $\vartheta_l \in C[0, T_l]$ and it satisfies* (12)*. Moreover, we say that ϑ_l is a solution of the initial value problem* (12) *in the set*

$$\mathcal{B}_l := \{\vartheta_l \in C([0, T_l], \mathbb{R}) : |\vartheta_l(t)| \le R_l, \forall t \in [0, T_l]\},$$

with $R_l > 0$, if:

1) *For every $t \in [0, T_l]$, $(t, \vartheta_l(t)) \in D$, where $D := [0, T_l] \times B_{R_l}$, being $B_{R_l} := \{\vartheta_l \in \mathbb{R} : |\vartheta_l| \le R_l\}$.*
2) *ϑ_l satisfies* (12).

Definition 3.2. *We say that the problem* (*VOFLE*) *has a solution ϑ, if there exist functions ϑ_l, $l \in \{1, 2, \ldots, n\}$, such that $\vartheta_1 \in C[0, T_1]$ satisfying equation* (9)*, and $\vartheta_1(0) = \vartheta_0$; $\vartheta_2 \in C[0, T_2]$ satisfying equation* (10)*, and $\vartheta_2(0) = \vartheta_0$; $\vartheta_l \in C[0, T_l]$ satisfying equation* (11)*, and $\vartheta_l(0) = \vartheta_0$, for all $l \in \{3, \ldots, n\}$.*

Remark 3.3. *We say that the problem (VOFLE) has a unique solution, if the functions ϑ_l in Definition 3.2 are unique.*

Theorem 3.4. *Let $R_l > 0$, for $l \in \{1,\dots,n\}$ be such that*

$$|\vartheta_0| + H_l \frac{\kappa}{\Gamma(\varpi_l+1)} t^{\varpi_l} < R_l,$$

where $H_l := \max\left\{\frac{1}{4}, |R_l(1-R_l)|, |-R_l(1+R_l)|\right\}$. Then the auxiliary initial value problem (12) *has a unique solution with $\vartheta_l \in \mathcal{B}_l$, for all $l \in \{1,\dots,n\}$.*

Proof. For all $l \in \{1,\dots,n\}$, and from the properties of fractional calculus, the fractional order differential equation in (12) can be written as

$$I^{1-\varpi_l} \frac{d}{dt} \vartheta_l(t) = \kappa \vartheta_l(t)(1-\vartheta_l(t)),$$

using Lemma 2.3, we integrate the above equation ϖ_l−times. Therefore, we obtain

$$\vartheta_l(t) = \vartheta_0 + I_{0^+}^{\varpi_l}\left(\kappa\vartheta_l(1-\vartheta_l)\right)(t). \tag{13}$$

Define the family of operators $\mathcal{N}_l : \mathcal{B}_l \longrightarrow E_l$, $l \in \{1,\dots,n\}$, by

$$(\mathcal{N}\vartheta_l)(t) = \vartheta_0 + I_{0^+}^{\varpi_l}\left(\kappa\vartheta_l(1-\vartheta_l)\right)(t).$$

We prove that $\mathcal{N}_l(\mathcal{B}_l) \subseteq \mathcal{B}_l$, $l \in \{1,\dots,n\}$. Indeed, for each $l \in \{1,\dots,n\}$, we have

$$\begin{aligned}
|\mathcal{N}_l(\vartheta_l)(t)| &= \left|\vartheta_0 + I_{0^+}^{\varpi_l}\left(\kappa\vartheta_l(1-\vartheta_l)\right)(t)\right| \\
&= \left|\vartheta_0 + \frac{\kappa}{\Gamma(\varpi_l)} \int_0^t (t-s)^{\varpi_l-1} \vartheta_l(s)(1-\vartheta_l(s))ds\right| \\
&\le |\vartheta_0| + H_l \frac{\kappa}{\Gamma(\varpi_l)} \frac{1}{\varpi_l} t^{\varpi_l} = |\vartheta_0| + H_l \frac{\kappa}{\Gamma(\varpi_l+1)} t^{\varpi_l} < R_l.
\end{aligned}$$

This way, $\mathcal{N}_l : \mathcal{B}_l \longrightarrow \mathcal{B}_l$, for $l \in \{1,\dots,n\}$. Moreover, for ϑ_l, $\tilde{\vartheta}_l \in \mathcal{B}_l$,

$$\begin{aligned}
e^{-Nt}|\mathcal{N}_l(\vartheta_l)(t) - \mathcal{N}_l(\tilde{\vartheta}_l)(t)| &= \kappa \left| e^{-Nt} I_{0^+}^{\varpi_l}\left[(\vartheta_l(t) - \tilde{\vartheta}_l(t)) - (\vartheta_l^2(t) - \tilde{\vartheta}_l^2(t))\right]\right| \\
&\le \kappa \int_0^t \frac{(t-s)^{\varpi_l-1}}{\Gamma(\varpi_l)} e^{-Nt} |\vartheta_l(s) - \tilde{\vartheta}_l(s)|(1 + |\vartheta_l(s)| + |\tilde{\vartheta}_l(s)|)\, ds \\
&\le \kappa \int_0^t \frac{(t-s)^{\varpi_l-1}}{\Gamma(\varpi_l)} e^{-N(t-s)} e^{-Ns} |\vartheta_l(s) - \tilde{\vartheta}_l(s)|(1 + |\vartheta_l(s)| + |\tilde{\vartheta}_l(s)|)\, ds \\
&\le \kappa(1+2R_l)\|\vartheta_l - \tilde{\vartheta}_l\|_{E_l} \int_0^t \frac{(t-s)^{\varpi_l-1}}{\Gamma(\varpi_l)} e^{-N(t-s)}\, ds \\
&\le \kappa(1+2R_l)\|\vartheta_l - \tilde{\vartheta}_l\|_{E_l} \int_0^t \frac{\tau^{\varpi_l-1} e^{-N\tau}}{\Gamma(\varpi_l)}\, d\tau.
\end{aligned}$$

This implies that

$$\|\mathcal{N}_l(\vartheta_l) - \mathcal{N}_l(\tilde{\vartheta}_l)\|_{E_l} \le \kappa(1+2R_l) \int_0^{T_l} \frac{\tau^{\varpi_l-1} e^{-N\tau}}{\Gamma(\varpi_l)}\, d\tau\, \|\vartheta_l - \tilde{\vartheta}_l\|_{E_l},$$

and it can be proved, by virtue of the Banach contraction principle, that, if we choose $N > 0$ large enough such that $\kappa(1+2R_l) \int_0^{T_l} \frac{\tau^{\varpi_l-1} e^{-N\tau}}{\Gamma(\varpi_l)}\, d\tau < 1$, then we obtain that the operator $\mathcal{N}_l$ has a unique fixed point for all $l \in \{1,2,\dots,n\}$.

Conversely, from Eq (13), we get

$$\begin{aligned}
\frac{d}{dt}\vartheta_l(t) &= \kappa\frac{d}{dt}I_{0^+}^{\varpi_l}(\vartheta_l(t)-\vartheta_l^2(t))\\
I_{0^+}^{1-\varpi_l}\frac{d}{dt}\vartheta_l(t) &= \kappa I_{0^+}^{1-\varpi_l}\frac{d}{dt}I_{0^+}^{\varpi_l}(\vartheta_l(t)-\vartheta_l^2(t))\\
{}^C\mathcal{D}_{0^+}^{\varpi_l}\vartheta_l(t) &= \kappa\frac{d}{dt}\, I_{0^+}^{1-\varpi_l}\, I_{0^+}^{\varpi_l}(\vartheta_l(t)-\vartheta_l^2(t))\\
{}^C\mathcal{D}_{0^+}^{\varpi_l}\vartheta_l(t) &= \kappa(\vartheta_l(t)-\vartheta_l^2(t)).
\end{aligned}$$

Also, from the continuity of the solution, we deduce that

$$\begin{aligned}
\vartheta_l(0) &= \vartheta_0 + \kappa\, I_{0^+}^{\varpi_l}(\vartheta_l(t)-\vartheta_l^2(t))\Big|_{t=0}\\
&= \vartheta_0.
\end{aligned}$$

In view of Remark 3.3, we have the uniqueness of solution to problem (*VOFLE*). □

4. Uniform stability

Definition 4.1. *Under uniqueness hypotheses, for a given $l \in \{1, 2, \ldots, n\}$, we say that the solution ϑ_l of the initial value problem* (12) *is uniformly stable if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$|\vartheta_0 - \tilde{\vartheta}_0| \leq \delta \implies \|\vartheta_l - \tilde{\vartheta}_l\|_{E_l} \leq \varepsilon,$$

where $\tilde{\vartheta}_l$ is the solution to the initial value problem (12) *with the initial condition*

$$\tilde{\vartheta}_l(0) = \tilde{\vartheta}_0.$$

Definition 4.2. *Under uniqueness hypotheses, we say that the solution to the problem* (*VOFLE*) *is uniformly stable if all the functions ϑ_l, $l \in \{1, 2, \ldots, n\}$, are uniformly stable.*

Theorem 4.3. *Under uniqueness hypotheses (see the statement of Theorem 3.4), for all $l \in \{1, 2, \ldots, n\}$, the solution ϑ_l of the initial value problem* (12) *is uniformly stable, that is, problem* (*VOFLE*) *is uniformly stable.*

Proof. Direct computation gives

$$\|\vartheta_l - \tilde{\vartheta}_l\|_{E_l} \leq |\vartheta_0 - \tilde{\vartheta}_0| + \kappa(1 + 2R_l)\int_0^{T_l} \frac{\tau^{\varpi_l - 1}e^{-N\tau}}{\Gamma(\varpi_l)}\, d\tau\, \|\vartheta_l - \tilde{\vartheta}_l\|_{E_l},$$

which implies that, given $\varepsilon > 0$,

$$\|\vartheta_l - \tilde{\vartheta}_l\|_{E_l} \leq \left(1 - \kappa(1 + 2R_l)\int_0^{T_l} \frac{\tau^{\varpi_l - 1}e^{-N\tau}}{\Gamma(\varpi_l)}\, d\tau\right)^{-1} |\vartheta_0 - \tilde{\vartheta}_0| \leq \varepsilon,$$

provided that $|\vartheta_0 - \tilde{\vartheta}_0| \leq \delta$, where the relation between $\varepsilon > 0$ and $\delta > 0$ is given by

$$\varepsilon = \left(1 - \kappa(1 + 2R_l)\int_0^{T_l} \frac{\tau^{\varpi_l - 1}e^{-N\tau}}{\Gamma(\varpi_l)}\, d\tau\right)^{-1} \delta.$$

In view of Definition 4.2, we have proved the uniform stability of the solution to problem (*VOFLE*). □

5. Numerical methods and results

We recall the problem of interest, which is the following fractional logistic equation involving the variable order Caputo fractional derivative

$$\begin{cases} {}^{C}\mathcal{D}_{0^+}^{\varpi(t)}\vartheta(t) = \kappa\vartheta(t)(1-\vartheta(t)), \quad 0 \le t \le T < +\infty, \\ \vartheta(0) = \vartheta_0, \end{cases} \tag{VOFLE}$$

where $0 < \varpi(t) < 1$, $\vartheta_0 \in \mathbb{R}^+$, and $\kappa > 0$.

For a numerical study of this problem, we choose the finite difference method with different space steps h [6, 17].

The first step

Since the exact solution to this problem with $\varpi(t) = 1$ is

$$\vartheta_1(t) = \frac{\vartheta_0}{\vartheta_0 + (1-\vartheta_0)\exp(-\kappa t)},$$

we apply the finite difference method in this problem with size 0.001 and $T = 10$ for different expressions of $\varpi(t)$ and ϑ_0, and we present the results obtained in the following images:

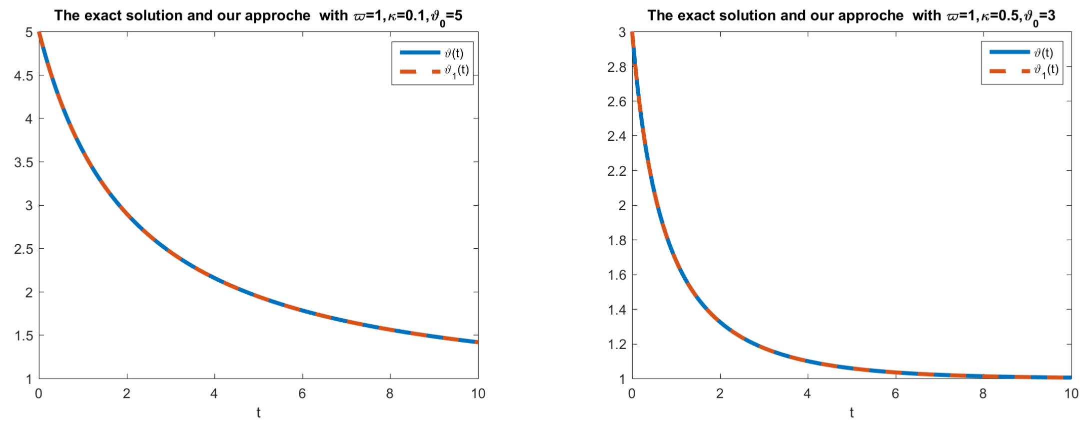

Figure 1: The exact solution ϑ_1 and our solution with $\varpi(t) = 1$ and $\kappa < 1$.

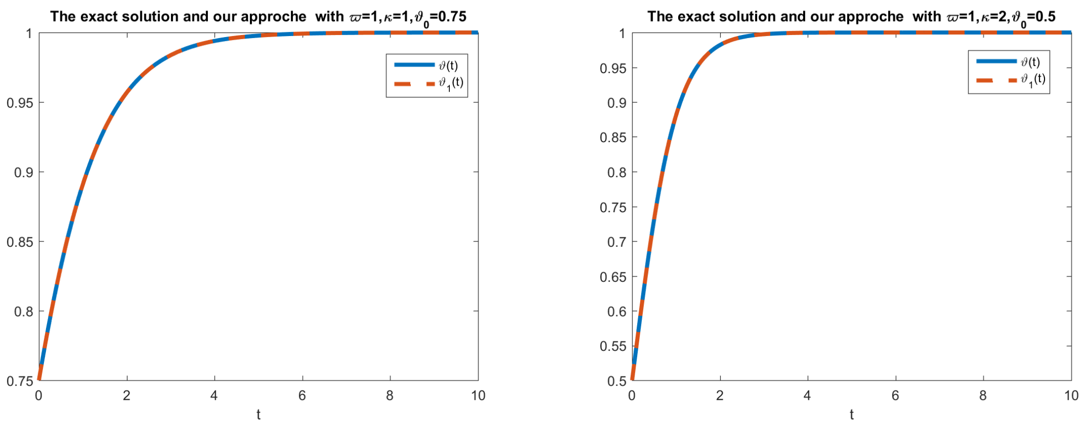

Figure 2: The exact solution ϑ_1 and our solution with $\varpi(t) = 1$ and $\kappa \ge 1$.

We observe that the solution obtained with this method is the same as the exact solution.

The second step

Now, we calculate the solution of the problem ($VOFLE$) with the variable order $\varpi(t)$ in two cases (one for the increasing case and other for the decreasing case) in the interval $]0, 1]$ for different ϑ_0 and size 0.001, and the results are presented in the following figures:

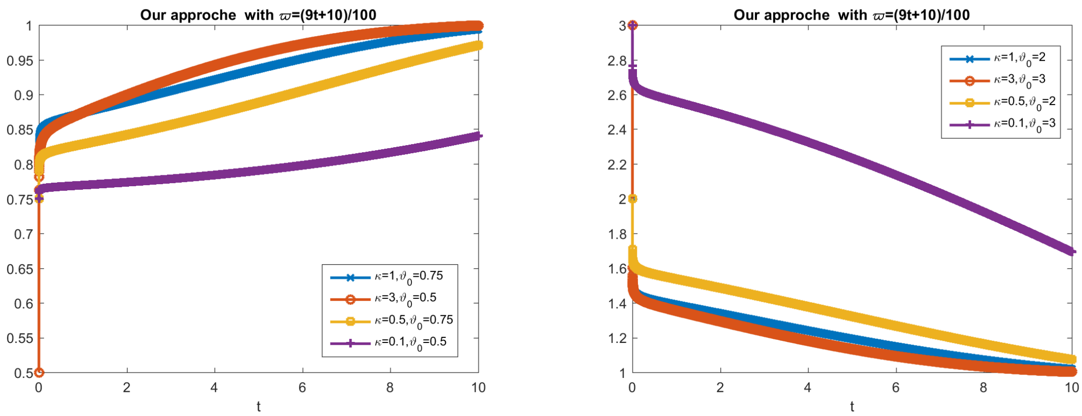

Figure 3: The solution ϑ for different values of κ, ϑ_0, and $\varpi(t) = \frac{9t+10}{100}$.

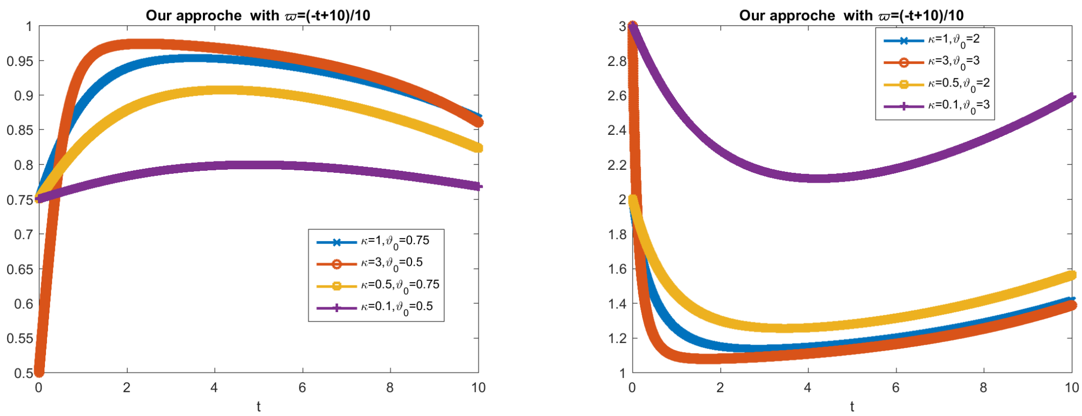

Figure 4: The solution ϑ for different values of κ, ϑ_0, and $\varpi(t) = \frac{-t+10}{10}$.

Statements and Declarations

Competing Interests: The authors do not have financial or non-financial interests that are directly or indirectly related to the work submitted for publication.

Acknowledgements

We are grateful to the Editor and the anonymous Referees for their comments and suggestions that helped to improve the paper.

The research of R. Rodríguez-López was partially supported by the Agencia Estatal de Investigación (AEI) of Spain, co-financed by the European Fund for Regional Development (FEDER) corresponding to the 2021–2024 multiyear financial framework, project PID2020-113275GB-I00, and ED431C 2023/12 (GRC Xunta de Galicia).

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