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Existence and blow up of solutions for a singular higher-order viscoelastic parabolic equation with logarithmic nonlinearity

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Abstract. In this work, we obtain the singular higher-order viscoelastic parabolic type equation with logarithmic nonlinearity. We establish the local existence, global existence and blow up of solutions. By employing the cut-off technique and the Faedo-Galerkin approximation method, the local existence of a weak solution is established. The global existence of the weak solution is then derived using the potential well method. Moreover, we demonstrate that the blow up in finite time through the application of concavity method.

1. Introduction

In this work, we investigate the following the initial-boundary value problem of singular higher-order viscoelastic parabolic equation with logarithmic nonlinearity

$$\begin{cases} \frac{z_t}{|x|^s} + \mathcal{A}z - \int_0^t g(t-\tau)\mathcal{A}z(\tau)\,d\tau = |z|^{r-2} z \ln z, & (x,t) \in \Omega \times (0,T), \\ \frac{\partial^i z(x,t)}{\partial v^i} = 0, \; i = 0,1,...,m-1, & (x,t) \in \partial\Omega \times (0,T), \\ z(x,0) = z_0(x) & x \in \Omega, \end{cases} \tag{1}$$

here $\mathcal{A} = (-\Delta)^m$, $m \geq 1$ is naturel number, $\Omega \subset R^N$ $(N > 2m)$ be open bounded Lipschitz domain with a smooth boundary $\partial\Omega$, $z_0(x) \in X = \left(H_0^m(\Omega) \cap L^{r+1}(\Omega)\right) \setminus \{0\}$, $2 < r < 2\left(1 + \frac{2m}{N}\right)$ and $0 \leq s \leq 2$ is a constant. $T > 0$ and a unit outher normal ν, $x = (x_1, x_2, ..., x_n)$, $|x| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$.

In 2021, Han [17] proved the following the equation of the form

$$z_t + \Delta^2 z = k(t) f(z).$$

He established the explosion in finite time using differential inequalities. Furthermore, he derived both upper and lower limits for the time at which the explosion happens.

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Han [16] studied the following the equation of the form

$$\frac{z_t}{|x|^2} - \Delta z = k(t)|z|^{p-1} z.$$

He proved the upper and lower bounds on the blow-up time of weak solutions.

In 2021, Thanh et al. [26] considered the reaction-diffussion parabolic problem with time dependent coefficients

$$\frac{z_t}{|x|^4} + \Delta^2 z = k(t)|z|^{p-1} z.$$

They proved the upper and lower bound for blow-up time. Problems with variable coefficients have been handled carefully in several papers, some results relating the local existence, global existence, blow up and stability have been found [2, 9, 11, 13, 16, 22, 23].

Heat equation with singular potential and logarithmic nonlinearity which can be used to describe many phenomena in the viscoelastic mechanics, quantum mechanics theory [3–8, 10, 12, 14, 15, 19, 28].

In 2023, Wu et al. [30] investigated the following fourth-order parabolic equation

$$\frac{z_t}{|x|^4} + \Delta^2 z - \Delta z_t = |z|^{p-2} z \ln |z|.$$

They obtained finite-time blowup results of weak solutions using the Galerkin method and determined upper and lower bounds for the blowup time.

In 2020, Deng and Zhou [6] considered the following of singular and nonlinear parabolic equations with logarithmic source term

$$\frac{z_t}{|x|^s} + \Delta z = z \ln |z|.$$

They obtained infinite time blow-up of the solutions and the global existence.

In 2024, Yang [32] considered the following $p-$Laplacian type pseudo-parabolic equation with singular potential and logarithmic nonlinearity

$$\frac{z_t}{|x|^s} + \Delta_p z - \Delta z_t = |z|^{q-2} z \ln |z|.$$

He has established a new criterion for solutions to blow up in finite time using Gagliardo-Nirenberg's interpolation inequality and inverse Sobolev inequality.

In [27], Thanh et al. proved the higher-order version $\Delta\left(|\Delta|^{m-2}\Delta\right)$ of the $p-$Laplacian and the function $k(t)$ non- newtonian filtration equation and obtained the blow-up result with lower and upper bounded.

In 2024 [15] Gao et al. studied the the following of singular and viscoelastic nonlinear parabolic equations with logarithmic source

$$\frac{z_t}{|x|^s} - \Delta z + \int_0^t g(t-\tau)\Delta z(\tau)\, d\tau = |z|^{q-2} z \ln z.$$

They obtained global existence and blow up of solutions.

This work is organized as follows:

- In part 2, we give some assumptions needed in this work.
- In Part 3, we obtain the local existence of a weak solution with the cut-off technique and the Faedo-Galerkin approximation method.
- In part 4, we obtain the global existence of solutions using the potential well method.
- In Part 5, we prove the blow up in finite time through the application of concavity analysis of the weak solutions.

2. Preliminaries

In this part, we present certain lemmas and assumptions required for the formulation and proof of our results. Let $\|.\|, \|.\|_r$ and $\|.\|_{W^{m,r}(\Omega)}$ indicate the typical $L^2(\Omega)$, $L^r(\Omega)$ $(1 \le r \le \infty)$ and $W^{m,r}(\Omega)$ norms (see [1, 24]). We denote the inner product by $\langle .,. \rangle$. By problem (1), assume that r and $g(.)$ satisfy the following conditions:

$(A1)$ $2 < r < 2\left(1 + \frac{2m}{N}\right), N > 2m,$

$(A2)$ $g \in C^1(R^+, R^+)$ satisfiying $g(s) \ge 0$, $g'(s) \le 0$, $l = 1 - \int_0^\infty g(s)\,ds > 0.$

Multiplying equation (1) by z_t and integrating over $\Omega \times [0,t)$, we have

$$\begin{aligned}
&\int_0^t \left\| \frac{z_\tau}{|x|^{s/2}} \right\|^2 d\tau + \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\left\|\mathcal{A}^{\frac{1}{2}} z\right\|^2 \\
&+\frac{1}{2}\int_0^t g(\tau)\left\|\mathcal{A}^{\frac{1}{2}} z\right\|^2 d\tau - \frac{1}{2}\int_0^t \left(g' \circ \mathcal{A}^{\frac{1}{2}} z\right)(\tau)\,d\tau + \frac{1}{2}\left(g \circ \mathcal{A}^{\frac{1}{2}} z\right)(t) \\
= \;& \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\left\|\mathcal{A}^{\frac{1}{2}} z_0\right\|^2 + \frac{1}{2}\left(g \circ \mathcal{A}^{\frac{1}{2}} z_0\right)(t) \\
&+\frac{1}{r}\int_\Omega |z|^r \ln z\,dx - \frac{1}{r}\int_\Omega |z_0|^r \ln|z_0|\,dx - \frac{1}{r^2}\|z\|_r^r + \frac{1}{r^2}\|z_0\|_r^r.
\end{aligned} \tag{2}$$

For each $z \in H_0^m(\Omega) \cap L^r(\Omega)$ and $t \in [0,\infty)$ define the functinals of the problem (1) following:

$$\begin{aligned}
J(z) = \;& \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\left\|\mathcal{A}^{\frac{1}{2}} z\right\|^2 \\
&-\frac{1}{r}\int_\Omega |z|^r \ln z\,dx + \frac{1}{2}\left(g \circ \mathcal{A}^{\frac{1}{2}} z\right)(t) + \frac{1}{r^2}\|z\|_r^r,
\end{aligned} \tag{3}$$

and Nehari functional is as follows:

$$\begin{aligned}
I(z) = \;& \left(1 - \int_0^t g(s)\,ds\right)\left\|\mathcal{A}^{\frac{1}{2}} z\right\|^2 \\
&-\int_\Omega |z|^r \ln z\,dx + \left(g \circ \mathcal{A}^{\frac{1}{2}} z\right)(t),
\end{aligned} \tag{4}$$

$$\begin{aligned}
E(t) = \;& \int_0^t \left\| \frac{z_\tau}{|x|^{s/2}} \right\|^2 d\tau + \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\left\|\mathcal{A}^{\frac{1}{2}} z\right\|^2 \\
&-\frac{1}{r}\int_\Omega |z|^r \ln z\,dx + \frac{1}{2}\left(g \circ \mathcal{A}^{\frac{1}{2}} z\right)(t) + \frac{1}{r^2}\|z\|_r^r,
\end{aligned} \tag{5}$$

where $\left(g \circ \mathcal{A}^{\frac{1}{2}} z\right)(t) = \int_0^t g(t-s)\left\|\mathcal{A}^{\frac{1}{2}} z(t) - \mathcal{A}^{\frac{1}{2}} z(s)\right\|^2 ds.$

Then it follows from (3) and (4) that

$$\begin{aligned}
J(z) = \;& \frac{1}{r} I(z) + \frac{r-2}{2r}\left(1 - \int_0^t g(s)\,ds\right)\left\|\mathcal{A}^{\frac{1}{2}} z\right\|^2 \\
&+\frac{r-2}{2r}\left(g \circ \mathcal{A}^{\frac{1}{2}} z\right)(t) + \frac{1}{r^2}\|z\|_r^r.
\end{aligned} \tag{6}$$

Furthermore, we introduce the Nehari manifold

$$\mathcal{N} = \{z \in X : I(z) = 0\} \tag{7}$$

and the following sets:

$$\mathcal{W} = \{z \in X : J(z) < d, I(z) > 0\},$$

$$\mathcal{V} = \{z \in X : J(z) < d, I(z) < 0\}.$$

The depth of potential well is defined as follows:

$$d = \inf_{z \in \mathcal{N}} J(z).$$

Now, we give some definitions.

Definition 2.1. *(Weak solution) A function z is called a weak solution to equation (1) if* $z \in L^{\infty}\left(0, T; H_0^m(\Omega) \cap L^r(\Omega)\right)$ *and* $\frac{z_t}{|x|^{s/2}} \in L^2\left(0, T; L^2(\Omega)\right)$ *where z satisfies the following equation:*
(i) For any $\varphi \in H_0^m(\Omega)$ *and* $t \in [0, T)$, *so that*

$$\left\langle \frac{z_t}{|x|^s}, \varphi \right\rangle + \left\langle \mathcal{A}^{\frac{1}{2}} z, \mathcal{A}^{\frac{1}{2}} \varphi \right\rangle - \int_0^t g(t-s) \left\langle \mathcal{A}^{\frac{1}{2}} z, \mathcal{A}^{\frac{1}{2}} \varphi \right\rangle ds = \left\langle |z|^{r-2} z \ln z, \varphi \right\rangle, \tag{8}$$

(ii) $z(x, 0) = z_0(x)$ *in* $H^m(\Omega) \cap L^{r+1}(\Omega)$.

Definition 2.2. *(see [31])(Finite time blow-up) Assume that* $z(t)$ *is a weak solution to (1). If* $z(t)$ *exists for all t in the interval* $[0, T^*)$, *and the limit as to blow up at a finite time* T^* *if* $z(t)$ *exists for all* $t \in [0, T^*)$ *and*

$$\lim_{t \to T^{*-}} \left\| \frac{z(x, t)}{|x|^{s/2}} \right\|^2 = +\infty. \tag{9}$$

Where T^* *is called the maximal existence time of* $z(t)$ *and also the blow time. If (4) does not happen for any finite time* T^*, *then* $z(t)$ *is called a global solution and the maximal existence time of* $z(t)$ *is* ∞.

Definition 2.3. *(Maximal existence time [29, 31]) Suppose that* $z(x, t)$ *is a weak solution of problem (1), we determine the maximal existence time* $T_{\max}$ *as follows*

$$T_{\max} = \sup\{T > 0; z(x, t) \text{ exists on } [0, T]\}.$$

(i) If $T_{\max} = +\infty$, *then* $z(t)$ *is global;*
(ii) If $T_{\max} < +\infty$, *we say that the solution* $z(t)$ *is blow up in finite time where* $T_{\max}$ *is the blow-up time.*

After that, in Lemma 2.4, we outline some fundamental properties of the fiber mapping $J(\lambda z)$ that can be verified directly.

Lemma 2.4. *Assume that* $z \in X$, *then*
(i) $\lim_{\lambda \to 0^+} J(\lambda z) = 0$, $\lim_{\lambda \to +\infty} J(\lambda z) = -\infty$.
(ii) There exists a unique $\lambda^* = \lambda^*(z) > 0$ *so that* $\frac{d}{d\lambda} J(\lambda z)|_{\lambda = \lambda^*} = 0$; $J(\lambda z)$ *is increasing on* $0 < \lambda < +\infty$, *and attains the maximum at* $\lambda = \lambda^*$.
(iii) $I(\lambda z) > 0$ *for* $0 < \lambda < \lambda^*$, $I(\lambda z) < 0$ *for* $\lambda^* < \lambda < +\infty$, *and* $I(\lambda^* z) = 0$.

Proof. (i) By the definition of $J(u)$, we obtain

$$\begin{aligned} J(\lambda z) &= \frac{1}{2}\lambda^2\left(1-\int_0^t g(s)\,ds\right)\left\|\mathcal{A}^{\frac{1}{2}}z\right\|^2-\frac{\lambda^r}{r}\ln\lambda\,\|z\|_r^r \\ &-\frac{\lambda^r}{r}\int_\Omega |z|^r\ln z dx+\frac{1}{2}\lambda^2\left(g\circ\mathcal{A}^{\frac{1}{2}}z\right)(t)+\frac{\lambda^r}{r^2}\|z\|_r^r, \end{aligned} \tag{10}$$

where $\lambda > 0$. Therefore, it is evident that the conclusion of (i) is valid

(ii) By differentiating $J(\lambda z)$ at λ we get:

$$\begin{aligned} \frac{d}{d\lambda}J(\lambda z) &= \lambda\left(1-\int_0^t g(s)\,ds\right)\left\|\mathcal{A}^{\frac{1}{2}}z\right\|^2-\lambda^{r-1}\ln\lambda\,\|z\|_r^r \\ &-\lambda^{r-1}\int_\Omega |z|^r\ln z dx+\lambda\left(g\circ\mathcal{A}^{\frac{1}{2}}z\right)(t), \\ &= \lambda\left[\begin{array}{c}\left(1-\int_0^t g(s)\,ds\right)\left\|\mathcal{A}^{\frac{1}{2}}z\right\|^2-\lambda^{r-1}\ln\lambda\,\|z\|_r^r \\ -\lambda^{r-2}\int_\Omega |z|^r\ln z dx+\left(g\circ\mathcal{A}^{\frac{1}{2}}z\right)(t)\end{array}\right]. \end{aligned} \tag{11}$$

Let $\mathcal{A}(\lambda z)=\frac{1}{\lambda}\frac{d}{d\lambda}J(\lambda z)$, then

$$\begin{aligned} &\frac{d}{d\lambda}\mathcal{A}(\lambda z) = -(r-2)\lambda^{r-3}\ln\lambda\,\|z\|_r^r-\lambda^{r-3}\|z\|_r^r \\ &-(r-2)\lambda^{r-3}\int_\Omega |z|^r\ln z dx \\ &= -\lambda^{r-3}\left[(r-2)\ln\lambda\,\|z\|_r^r+\|z\|_r^r+(r-2)\int_\Omega |z|^r\ln z dx\right] \end{aligned} \tag{12}$$

Hence, by taking

$$\lambda_1=\exp\left[\frac{\|z\|_r^r+(r-2)\int_\Omega |z|^r\ln z dx}{(2-r)\|z\|_r^r}\right]>0, \tag{13}$$

so that

$$\begin{aligned} \frac{d}{d\lambda}\mathcal{A}(\lambda z) &> 0 \text{ on } \lambda\in(0,\lambda_1), \\ \frac{d}{d\lambda}\mathcal{A}(\lambda z) &< 0 \text{ on } \lambda\in(\lambda_1,+\infty) \text{ and} \\ \frac{d}{d\lambda}\mathcal{A}(\lambda_1 z) &= 0. \end{aligned}$$

Since $\mathcal{A}(\lambda z)|_{\lambda=0}=\left(1-\int_0^t g(s)\,ds\right)\left\|\mathcal{A}^{\frac{1}{2}}z\right\|^2+\left(g\circ\mathcal{A}^{\frac{1}{2}}z\right)(t)>0$ and $\lim_{\lambda\to+\infty}\mathcal{A}(\lambda z)=-\infty$, there exists

$$\begin{aligned} \lambda^* &> 0 \text{ so that } \mathcal{A}(\lambda^* z)=0, \\ \mathcal{A}(\lambda z) &> 0 \text{ on } \lambda\in(0,\lambda^*) \text{ and} \\ \mathcal{A}(\lambda z) &< 0 \text{ on } \lambda\in(\lambda^*,+\infty). \end{aligned}$$

So,

$$\begin{aligned} \frac{d}{d\lambda}J(\lambda z) &> 0 \text{ is positive on } (0,\lambda^*), \\ \frac{d}{d\lambda}J(\lambda z) &< 0 \text{ is negative on } (\lambda^*,+\infty) \text{ and} \\ \frac{d}{d\lambda}J(\lambda^* z) &= 0. \end{aligned}$$

Therefore (ii) is valid.

(iii) From the definition of $I(z)$, we get

$$\begin{aligned} I(\lambda z) &= \lambda^2\left(1-\int_0^t g(s)\,ds\right)\left\|\mathcal{A}^{\frac{1}{2}}z\right\|^2-\lambda^r\ln\lambda\,\|z\|_r^r \\ &\quad -\lambda^r\int_\Omega |z|^r\ln z dx+\lambda^2\left(g\circ\mathcal{A}^{\frac{1}{2}}z\right)(t) \\ &= \lambda\left[\begin{array}{c} \lambda\left(1-\int_0^t g(s)\,ds\right)\left\|\mathcal{A}^{\frac{1}{2}}z\right\|^2-\lambda^{r-1}\ln\lambda\,\|z\|_r^r \\ -\lambda^{r-1}\int_\Omega |z|^r\ln z dx+\lambda\left(g\circ\mathcal{A}^{\frac{1}{2}}z\right)(t)\end{array}\right] \\ &= \lambda\frac{d}{d\lambda}J(\lambda z) \end{aligned} \tag{14}$$

here $\lambda>0$. When combined with (ii), result (iii) holds. □

Lemma 2.5. *Let (A1) and (A2) hold and $z\in X$ satisfy $I(z)<0$. Later, there exists a $\lambda^*\in(0,1)$ such that $I(\lambda^* z)=0$.*

Proof. For $\forall\lambda>0$, we get

$$I(\lambda z)=\lambda^2\left[\left(1-\int_0^t g(s)\,ds\right)\left\|\mathcal{A}^{\frac{1}{2}}z\right\|^2+\left(g\circ\mathcal{A}^{\frac{1}{2}}z\right)(t)-\varphi(\lambda)\right], \tag{15}$$

here

$$\varphi(\lambda)=\lambda^{r-2}\int_\Omega |z|^r\ln z dx+\lambda^{r-2}\ln\lambda\,\|z\|_r^r. \tag{16}$$

By $I(z)<0$, we obtain

$$\int_\Omega |z|^r\ln z dx>\left(1-\int_0^t g(s)\,ds\right)\left\|\mathcal{A}^{\frac{1}{2}}z\right\|^2+\left(g\circ\mathcal{A}^{\frac{1}{2}}z\right)(t). \tag{17}$$

By (15) and (17), we get

$$\begin{aligned} \varphi(1) &= \int_\Omega |z|^r\ln z dx \\ &> \left(1-\int_0^t g(s)\,ds\right)\left\|\mathcal{A}^{\frac{1}{2}}z\right\|^2+\left(g\circ\mathcal{A}^{\frac{1}{2}}z\right)(t) \\ &> 0, \end{aligned} \tag{18}$$

$$\varphi(\lambda)=\lambda^{r-2}\int_\Omega |z|^r\ln z dx+\lambda^{r-2}\ln\lambda\,\|z\|_r^r\to 0 \text{ as } \lambda\to 0^+. \tag{19}$$

Combining (15),(18) and from the above equation, we can deduce that there is $\lambda^*\in(0,1)$ so that

$$\varphi(\lambda^*)=\left(1-\int_0^t g(s)\,ds\right)\left\|\mathcal{A}^{\frac{1}{2}}z\right\|^2+\left(g\circ\mathcal{A}^{\frac{1}{2}}z\right)(t)$$

and $I(\lambda^* z)=0$. The proof is completed. □

Lemma 2.6. *Suppose that (A1) and (A2) hold and $z(x,t)$ be a weak solution of problem (1). Then, $E(t)$ is nonincreasing function, that is*

$$E'(t)\le 0. \tag{20}$$

Proof. Multiplying the equation (1) with z_t and integrating with respect to x over the domain Ω, we obtain

$$\begin{aligned}
&\left\|\frac{z_\tau}{|x|^{s/2}}\right\|^2+\frac{1}{2}\frac{d}{dt}\left\|\mathcal{A}^{\frac{1}{2}}z\right\|^2-\int_0^t g(t-\tau)\,d\tau\int_\Omega \mathcal{A}^{\frac{1}{2}}z(s)\,\mathcal{A}^{\frac{1}{2}}z_t dxds\\
=\;&\int_\Omega |z|^{r-2}zz_t\ln z dx. \qquad (21)
\end{aligned}$$

Through direct calculation, for the third term from the left it can be seen that

$$\begin{aligned}
\int_0^t g(t-s)\int_\Omega \mathcal{A}^{\frac{1}{2}}z(s)\,\mathcal{A}^{\frac{1}{2}}z_t dxds \;=\;& \frac{d}{dt}\left[-\frac{1}{2}\left(g\circ\mathcal{A}^{\frac{1}{2}}z\right)(t)+\frac{1}{2}\int_0^t g(s)\,ds\left\|\mathcal{A}^{\frac{1}{2}}z\right\|^2\right]\\
&+\left[\frac{1}{2}\left(g'\circ\mathcal{A}^{\frac{1}{2}}z\right)(t)-\frac{1}{2}g(t)\left\|\mathcal{A}^{\frac{1}{2}}z\right\|^2\right]. \qquad (22)
\end{aligned}$$

If similar operations are performed on the left side of the equation,

$$\int_\Omega |z|^{r-2}zz_t\ln z dx=\frac{1}{r}\frac{d}{dt}\int_\Omega |z|^r\ln z dx-\frac{1}{r^2}\frac{d}{dt}\|z\|_r^r. \qquad (23)$$

Inserting (22) and (23) into (21), we get

$$\begin{aligned}
&\left\|\frac{z_t}{|x|^{s/2}}\right\|^2+\frac{d}{dt}\left[\begin{array}{c}\frac{1}{2}\left(1-\int_0^t g(s)\,ds\right)\left\|\mathcal{A}^{\frac{1}{2}}z\right\|^2+\frac{1}{2}\left(g\circ\mathcal{A}^{\frac{1}{2}}z\right)(t)\\ -\frac{1}{r}\int_\Omega |z|^r\ln z dx+\frac{1}{r^2}\|z\|_r^r\end{array}\right]\\
=\;&\frac{1}{2}\left(g'\circ\mathcal{A}^{\frac{1}{2}}z\right)(t)-\frac{1}{2}g(t)\left\|\mathcal{A}^{\frac{1}{2}}z\right\|^2\le 0. \qquad (24)
\end{aligned}$$

The proof is completed. □

Lemma 2.7. *Assume that (A1) and (A2) hold and $z_0\in X$. Later,*
(*i*) *the solution z of problem (1) with $z_0\in\mathcal{W}$ satisfies that $z(t)\in\mathcal{W}$ for all $t\in[0,T^*]$.*
(*ii*) *the solution z of problem (1) with $z_0\in\mathcal{V}$ satisfies that $z(t)\in\mathcal{V}$ for all $t\in[0,T^*]$.*

Proof. (*i*) Suppose that $z(t)$ be the weak solution by problem (1) with $z_0\in\mathcal{W}$, The meaning is that $J(z_0)<d$, $I(z_0)>0$. The time variable on $(0,t)$ is integrated on both sides with respect to t (21), we have

$$J(z(t))+\int_0^t\left\|\frac{z_\tau}{|x|^{s/2}}\right\|^2 d\tau+\frac{1}{2}\int_0^t g(\tau)\left\|\mathcal{A}^{\frac{1}{2}}z\right\|^2 d\tau-\frac{1}{2}\int_0^t\left(g'\circ\mathcal{A}^{\frac{1}{2}}z\right)(\tau)\,d\tau=J(z_0). \qquad (25)$$

By (25), we can get

$$J(z)<J(z_0)<d,\ \forall t\in[0,T^*]. \qquad (26)$$

Next, we assert that $I(z(t))>0$ for all $t\in[0,T^*]$, which, combined with equation (26), implies that $z(x,t)\in\mathcal{W}$. Otherwise, by the continuity of $I(z)$, there would exist a time $t_0\in(0,T^*)$ such that $I(z(t))>0$ for $t\in[0,t_0)$ and $I(z(t_0))=0$ while $z(t_0)\neq 0$. This would imply that $z(t_0)\in\mathcal{N}$. Referring to the definition of d, it is clear that $d\le J(z(t_0))$ which contradiction with (26). Therefore, $z(t)\in\mathcal{W}$ for all $t\in[0,T^*]$.

(*ii*) Since the proof is similar to part (*i*), so we omit it. □

Lemma 2.8. (*see [21]*) *(Hardy-Sobolev Inequality). Suppose that $R^N: R^k\times R^{N-k}$, $2\le k<N$ and $x=(y,z)\in R^k\times R^{N-k}$. For specific values of γ and s, there exists a range where $1<\gamma<N$, $0\le s\le\gamma$, and $s<k$, such that $m(s,N,\gamma)$ equals γ, and the ratio $\frac{N-s}{N-\gamma}$ is constant. Additionally, $H=H(s,N,\gamma,k)$ is positive*

$$\int_{R^N}|y|^{-s}|z(x)|^m dx\le H\left(\int_{R^N}\left|\mathcal{A}^{\frac{1}{2}}z(x)\right|^\gamma dx\right)^{\frac{N-s}{N-\gamma}},\ \forall z\in W_0^{1,\gamma}(\Omega). \qquad (27)$$

Remark 2.9. *When $m = 2$ is set, the above inequality becomes*

$$\int_{\Omega} |x|^{-s} |z(x)|^2 dx \leq H \left(\int_{\Omega} \left| \mathcal{A}^{\frac{1}{2}} z(x) \right|^{\frac{2N}{N-s+2}} dx \right)^{\frac{N-s+2}{N}}. \tag{28}$$

From $0 \leq s \leq 2$ and $N > 2$, we can obtain by Hölder's inequality

$$\begin{aligned} \int_{\Omega} |x|^{-s} |z(x)|^2 dx &\leq H \left(\int_{\Omega} \left| \mathcal{A}^{\frac{1}{2}} z(x) \right|^{\frac{2N}{N-s+2}} dx \right)^{\frac{N-s+2}{N}} \\ &\leq H |\Omega|^{\frac{N-s+2}{N}-1} \left\| \mathcal{A}^{\frac{1}{2}} z \right\|^2 \\ &= H_N \left\| \mathcal{A}^{\frac{1}{2}} z \right\|^2. \end{aligned} \tag{29}$$

We introduce the following inequality to address the logarithmic nonlinearity.

Lemma 2.10. *[18] Assume that μ is a positive number. Then we have the following inequalities:*

$$s^r \ln s \leq (e\mu)^{-1} s^{r+\mu}, \quad \textit{for all } s \geq 1,$$

and

$$|s^r \ln s| \leq (er)^{-1}, \quad \textit{for all } 0 < s < 1.$$

Lemma 2.11. *(see [1, 18, 24]).*
(i) For any given function $z \in W_0^{m,p}(\Omega)$, we have the inequality

$$\|z\|_r \leq B \left\| \mathcal{A}^{\frac{1}{2}} z \right\|_p,$$

for everyone $1 \leq r \leq p^*$, here

$$p^* = \begin{cases} \frac{Np}{N-p} & \textit{if } N > p \\ \infty & \textit{if } N \leq p. \end{cases}$$

The optimal constant B depends only on Ω, N, p and r.
(ii) For any $z \in W_0^{m,p}(\Omega)$, $p \geq 1, q \geq 1$, the inequality

$$\|z\|_r \leq C \left\| \mathcal{A}^{\frac{1}{2}} z \right\|_p^{\alpha} \|z\|_q^{1-\alpha}$$

is valid, here

$$\alpha = \left(\frac{1}{q} - \frac{1}{r} \right) \left(\frac{1}{N} - \frac{1}{r} + \frac{1}{q} \right)^{-1},$$

$$\begin{array}{lll} \bullet p \geq N = 1 & q \leq r \leq \infty & \\ \bullet N > 1 \textit{ and } p < N & \begin{array}{ll} r \in [q, p^*] & \textit{if } q \leq p^* \\ r \in [p^*, q] & \textit{if } q \geq p^* \end{array} & \textit{and} \\ \bullet p = N > 1, & q \leq r < \infty & \\ \bullet p > N > 1 & q \leq r \leq \infty & \end{array}$$

where, the constant C is determined by N, p, r and q.

Our next result is known as the concavity argument, which is widely used in the literature and is used for the sufficient condition of blow-up.

Lemma 2.12. *[20] Let $F(t)$ be a nonincreasing function defined on the $[t_0, \infty)$ that satisfies the inequality*

$$F'(t)^2 \geq a + bF(t)^{2+(1/\delta)},\ t \geq t_{0,}$$

where $a > 0$and $b \in R$. Then, there exists a finite positive time T^* so that

$$\lim_{t\to T^*} F(t) = 0.$$

The upper bound of T^* can be estimated in the following cases:

(i) When $b < 0$ and $F(t_0) < \min\left\{1, \left(-\frac{a}{b}\right)^{\frac{1}{2}}\right\}$,

$$T^* \leq t_0 + \sqrt{\left(\frac{1}{-b}\right)} \ln\left(\frac{\sqrt{\left(-\frac{a}{b}\right)}}{\sqrt{\left(-\frac{a}{b}\right)} - F(t_0)}\right)$$

(ii) When $b = 0$,

$$T^* \leq t_0 + \frac{F(t_0)}{\sqrt{a}}.$$

(iii) When $b > 0$,

$$T^* \leq t_0 + 2^{3\delta + \frac{1}{2\delta}} \left(\frac{\delta h}{\sqrt{a}}\right)\left\{1 - [1 + hF(t_0)]^{-\frac{1}{2\delta}}\right\}$$

here $h = \left(\frac{a}{b}\right)^{2+\frac{1}{2\delta}}$.

3. Local existence

We will show the local existence of weak solutions to the problem (1).

Theorem 3.1. *Assume that (A1) and (A2) hold and $z_0(x) \in H_0^m(\Omega) \cap L^{r+1}(\Omega)$. Therefore, there is a positive constant T and a unique weak solution*

$$\begin{aligned} z(x,t) &\in L^\infty\left(0,T; H_0^m(\Omega) \cap L^{r+1}(\Omega)\right) \text{ of problem (1)}, \\ \frac{z_t}{|x|^{\frac{s}{2}}} &\in L^2\left(0,T; L^2(\Omega)\right). \end{aligned} \tag{30}$$

Proof. The proof of Theorem 13 into 4 steps:

Step 1. Approximate problem.

$\forall n \in Z^+$, problem (1) has a corresponding solution z_{nt} satisfying

$$\begin{cases} N(x) z_{nt} + \mathcal{A} z_n - \int_0^t g(t-s) \mathcal{A} z_n(s)\, ds = |z_n|^{r-2} z_n \ln z_n, & (x,t) \in \Omega \times (0,T), \\ \frac{\partial^i z_n(x,t)}{\partial v^i} = 0,\ i = 0,1,...,m-1, & (x,t) \in \partial\Omega \times (0,T), \\ z_n(x,0) = z_{n0}(x) \in X & x \in \Omega, \end{cases} \tag{31}$$

where

$$N(x) = \min\{|x|^{-s}, n\} \, \forall n \in Z^+. \tag{32}$$

Let $\{w_j\}_{j=1}^{\infty}$ denote an orthogonal basis for the space $H_0^m(\Omega)$ which is a complete orthogonal system in $L^2(\Omega)$. Set

$$\begin{aligned} \mathcal{A}w_j &= \lambda_j w_j, \\ (w_j, w_j) &= \delta_{ij}, \end{aligned} \tag{33}$$

for $\forall i, j \in Z^+$, here $\lambda_j \in R$ and δ_{ij} is the Kronecker's delta. We know that $z_{n0} \in C_0^{\infty}(\Omega)$ so that

$$z_{n0} \to z_0(x) \text{ strongly in } H_0^m(\Omega) \cap L^{r+1}(\Omega).$$

We define the finite-dimensional space

$$W_h = span\{w_1, w_2, ..., w_h\}, \quad h \in Z^+$$

and create the approximate solution

$$z_n^h(x,t) = \sum_{j=1}^{h} \xi_{nj}^h(t) w_j(x), \quad \xi_{nj}^h \in C^1([0,T]), \tag{34}$$

solving the problem

$$\begin{aligned} &\left\langle N(x) z_{nt}^h, w_j \right\rangle + \left\langle \mathcal{A}^{\frac{1}{2}} z_n, \mathcal{A}^{\frac{1}{2}} w_j \right\rangle - \int_0^t g(t-s) \left\langle \mathcal{A}^{\frac{1}{2}} z_n^h(s), \mathcal{A}^{\frac{1}{2}} w_j \right\rangle ds \\ = \; &\left\langle \left|z_n^h\right|^{r-2} z_n^h \ln\left|z_n^h\right|, w_j \right\rangle. \end{aligned} \tag{35}$$

$$z_n^h(x,0) = \sum_{j=1}^{h} \xi_{nj}^h(0) w_j(x) = z_{n0}^h \to z_0(x) \text{ in } H_0^m(\Omega) \cap L^{r+1}(\Omega), \tag{36}$$

as $h \to +\infty$, $n \to +\infty$. We get

$$\left\langle N(x) z_{nt}^h, w_j \right\rangle = \sum_{j=1}^{h} \left(\int_\Omega N(x) w_j(x) w_j dx \right) \left[\xi_{nj}^h(t)\right]_t = \sum_{j=1}^{h} a_{ij} \left[\xi_{nj}^h(t)\right]_t. \tag{37}$$

Furthermore, one has

$$\left\langle \mathcal{A}^{\frac{1}{2}} z_n, \mathcal{A}^{\frac{1}{2}} w_j \right\rangle = \left\langle \sum_{j=1}^{h} \xi_{nj}^h(t) \lambda_j w_j(x), w_j \right\rangle = \lambda_j \xi_{nj}^h(t),$$

$$\begin{aligned} &\left\langle \left|z_n^h\right|^{r-2} z_n^h \ln\left|z_n^h\right|, w_j \right\rangle + \int_0^t g(t-s) \left\langle \mathcal{A}^{\frac{1}{2}} z_n^h(s), \mathcal{A}^{\frac{1}{2}} w_j \right\rangle ds \\ = \; &\left\langle \left|\sum_{j=1}^{h} \xi_{nj}^h(t) w_j(x)\right|^{r-2} \sum_{j=1}^{h} \xi_{nj}^h(t) w_j(x) \ln\left|\sum_{j=1}^{h} \xi_{nj}^h(t) w_j(x)\right|, w_j \right\rangle \\ &+\lambda_j \int_0^t g(t-s) \xi_{nj}^h(s) ds \end{aligned}$$

$$= \quad F_{nj}^{h}(t). \tag{38}$$

Hence, $\left\{\xi_{nj}^{h}\right\}_{j=1}^{h}$ is determined by the following Cauchy problem:

$$\begin{aligned} \sum_{j=1}^{h} a_{ij}\left[\xi_{nj}^{h}(t)\right]_{t} + \lambda_{j}\xi_{nj}^{h}(t) &= F_{nj}^{h}(t), \\ \sum_{j=1}^{h} \xi_{nj}^{h}(0) &= \int_{\Omega} z_{n0}^{h} w_{j} dx. \end{aligned} \tag{39}$$

A standard result for systems of ODEs guarantees the existence of a unique solution $\xi_{nj}^{h} \in C^{1}([0,T])$ to (39) and consequently, $z_{n}^{h}(x,t) \in C^{1}\left(0,T;H_{0}^{m}(\Omega)\cap L^{r+1}(\Omega)\right)$.

Step 2. A Priori estimates.

By multiplying equation (35) by $\xi_{nj}^{h}(t)$ and summing over $j = 1, 2, ..., h$, we obtain

$$\begin{aligned} &\left\langle N(x) z_{nt}^{h}, z_{n}^{h}\right\rangle + \left\langle \mathcal{A}^{\frac{1}{2}} z_{n}, \mathcal{A}^{\frac{1}{2}} z_{n}^{h}\right\rangle - \int_{0}^{t} g(t-s)\left\langle \mathcal{A}^{\frac{1}{2}} z_{n}^{h}(s), \mathcal{A}^{\frac{1}{2}} z_{n}^{h}\right\rangle ds \\ = \; &\left\langle \left|z_{n}^{h}\right|^{r-2} z_{n}^{h} \ln\left|z_{n}^{h}\right|, z_{n}^{h}\right\rangle. \end{aligned} \tag{40}$$

If we integrate from 0 to t in (40),we get

$$\begin{aligned} S_{n}^{h}(t) \leq\; & S_{n}^{h}(0) + \int_{0}^{t}\int_{0}^{\tau}\int_{\Omega} g(\tau - s)\mathcal{A}^{\frac{1}{2}} z_{n}^{h}(x,s)\mathcal{A}^{\frac{1}{2}} z_{n}^{h}(x,\tau)\,dxdsd\tau \\ &+ \int_{0}^{t}\int_{\Omega}\left|z_{n}^{h}(x,\tau)\right|^{r}\ln\left|z_{n}^{h}(x,\tau)\right| dxd\tau, \end{aligned} \tag{41}$$

where

$$S_{n}^{h}(t) = \frac{1}{2}\left\| |N(x)|^{\frac{1}{2}} z_{n}^{h}(t)\right\|^{2} + \int_{0}^{t}\left\|\mathcal{A}^{\frac{1}{2}} z_{n}^{h}(\tau)\right\|^{2} d\tau. \tag{42}$$

Morever, by applying Hölder's inequality and Young's inequality, we get

$$\begin{aligned} &\int_{0}^{t}\int_{0}^{\tau}\int_{\Omega} g(\tau - s)\mathcal{A}^{\frac{1}{2}} z_{n}^{h}(x,s)\mathcal{A}^{\frac{1}{2}} z_{n}^{h}(x,\tau)\,dxdsd\tau \\ \leq\; & \frac{1}{2}\int_{0}^{t}\left\|\mathcal{A}^{\frac{1}{2}} z_{n}^{h}(\tau)\right\|^{2} d\tau + \frac{1}{2}\int_{0}^{t}\left(\int_{0}^{\tau} g(\tau - s)\left\|\mathcal{A}^{\frac{1}{2}} z_{n}^{h}(s)\right\|^{2} ds\right)^{2} d\tau \\ <\; & \frac{1}{2}\int_{0}^{t}\left\|\mathcal{A}^{\frac{1}{2}} z_{n}^{h}(\tau)\right\|^{2} d\tau + \frac{1}{2}(1-l)\int_{0}^{t}\int_{0}^{\tau} g(\tau - s)\left\|\mathcal{A}^{\frac{1}{2}} z_{n}^{h}(s)\right\|^{2} dsd\tau \\ <\; & \left(1 - \frac{l}{2}\right)\int_{0}^{\tau}\left\|\mathcal{A}^{\frac{1}{2}} z_{n}^{h}(\tau)\right\|^{2} d\tau \\ \leq\; & \left(1 - \frac{l}{2}\right) S_{n}^{h}(t). \end{aligned} \tag{43}$$

Furthermore, by Lemma 2.10, we have

$$\int_{\Omega}\left|z_{n}^{h}\right|^{r}\ln\left|z_{n}^{h}\right| dx = \int_{\Omega_{1}=\{x\in\Omega;|z_{n}(x)|\geq 1\}}\left|z_{n}^{h}\right|^{r}\ln\left|z_{n}^{h}\right| dx$$

$$
\begin{aligned}
&+\int_{\Omega_2=\{x\in\Omega;|z_n(x)|<1\}}\left|z_n^h\right|^r\ln\left|z_n^h\right|dx\\
\leq &\int_{\Omega_1=\{x\in\Omega;|z_n(x)|\geq1\}}\left|z_n^h\right|^r\ln\left|z_n^h\right|dx\\
\leq &\ (e\mu)^{-1}\left\|z_n\right\|_{r+\mu}^{r+\mu}.
\end{aligned}
\tag{44}
$$

Furthermore, by Lemma 2.11, Young's inequality and we can choose $0<\mu<2\left(1+\frac{2m}{N}\right)-r$, we get

$$
\begin{aligned}
\int_\Omega\left|z_n^h\right|^r\ln\left|z_n^h\right|dx &\leq (e\mu)^{-1}\left\|z_n^h\right\|_{r+\mu}^{r+\mu}\\
&\leq (e\mu)^{-1}C_0\left\|\mathcal{A}^{\frac{1}{2}}z_n^h\right\|^{\beta(r+\mu)}\left\|z_n^h\right\|^{(1-\beta)(r+\mu)}\\
&\leq \varepsilon\left\|\mathcal{A}^{\frac{1}{2}}z_n^h\right\|^2+C(\varepsilon)\left\|z_n^h\right\|^{\frac{2(1-\beta)(r+\mu)}{2-\beta(r+\mu)}},
\end{aligned}
\tag{45}
$$

here $\varepsilon\in\left(0,\frac{l}{2}\right)$,

$$
\alpha=\frac{(1-\beta)(r+\mu)}{2-\beta(r+\mu)},
\tag{46}
$$

then $\alpha>1$, since $2<r<2\left(1+\frac{2m}{N}\right)$

Since Ω is a bounded domain in R^N, we can conclude that

$$
\begin{aligned}
\int_\Omega\left|z_n^h(t)\right|^2dx &= \int_\Omega\frac{1}{|N(x)|}|N(x)|\left|z_n^h(t)\right|^2dx\\
&\leq C(\Omega)\left\||N(x)|^{\frac{1}{2}}z_n^h(t)\right\|^2,
\end{aligned}
\tag{47}
$$

here $C(\Omega)$ is related to Ω. Thus, by (45) and (47), we get

$$
\begin{aligned}
&\int_0^t\int_\Omega\left|z_n^h(x,\tau)\right|^r\ln\left|z_n^h(x,\tau)\right|dxd\tau\\
\leq &\ \varepsilon\int_0^t\left\|\mathcal{A}^{\frac{1}{2}}z_n^h(\tau)\right\|^2d\tau+C(\varepsilon)\int_0^t\left\|z_n^h(\tau)\right\|^{2\alpha}d\tau\\
\leq &\ \varepsilon S_n^h(t)+C(\varepsilon)\int_0^t\left(S_n^h(\tau)\right)^\alpha d\tau.
\end{aligned}
\tag{48}
$$

Combining with (41), (43) and (48), we have

$$
S_n^h(t)\leq C_1+C_2\int_0^t\left[S_n^h(\tau)\right]^\alpha d\tau,
\tag{49}
$$

here $C_1=\frac{S_n^h(0)}{l-2\varepsilon}$, $C_2=\frac{2C(\varepsilon)}{l-2\varepsilon}$.

We get

$$
S_n^h(t)\leq C_T,
\tag{50}
$$

here C_T (constant dependent on T and) is independent of n and h, that is,

$$
\frac{1}{2}\left\||N(x)|^{\frac{1}{2}}z_n^h(t)\right\|^2+\int_0^t\left\|\mathcal{A}^{\frac{1}{2}}z_n^h(\tau)\right\|^2d\tau\leq C_T,\ \ \forall h,n\in Z^+.
\tag{51}
$$

By multiplying the first equation of problem (35) by $\left\{\xi_{nj}^{h}\right\}_{j=1}^{h}$ and integrating on $\Omega\times(0,t)$, we get

$$\begin{aligned}
&\int_0^t \left\||N(x)|^{\frac{1}{2}} z_{n\tau}^h\right\|^2 d\tau + \frac{1}{2}\left(1-\int_0^t g(s)\,ds\right)\left\|\mathcal{A}^{\frac{1}{2}} z_n^h\right\|^2 \\
&+\frac{1}{2}\left(g\circ\mathcal{A}^{\frac{1}{2}} z_n^h\right)(t) + \frac{1}{r^2}\left\|z_n^h\right\|_r^r \\
&+\frac{1}{2}\int_0^t g(\tau)\left\|\mathcal{A}^{\frac{1}{2}} z_n^h\right\|^2 d\tau - \frac{1}{2}\int_0^t \left(g'\circ\mathcal{A}^{\frac{1}{2}} z_n^h\right)(\tau)\,d\tau \\
&-\frac{1}{r}\int_\Omega \left|z_n^h\right|_r^r \ln\left|z_n^h\right| dx \\
=\;& J\left(z_{n0}^h\right),\;\; 0\le t\le T. \qquad (52)
\end{aligned}$$

Given the continuity of the functional $J(z)$ in $H_0^m(\Omega)$ there exists a constant $C>0$ such that

$$J\left(z_{n0}^h\right)\le C,\text{ for every positive integer and } \forall h,n\in Z^+. \qquad (53)$$

From (48), (50), (52) and (53), we obtain

$$\begin{aligned}
C \;\ge\;& J\left(z_{n0}^h\right) \ge \int_0^t \left\||N(x)|^{\frac{1}{2}} z_{n\tau}^h\right\|^2 d\tau \\
&+\frac{1}{2}\left(1-\int_0^t g(s)\,ds\right)\left\|\mathcal{A}^{\frac{1}{2}} z_n^h\right\|^2 + \frac{1}{2}\left(g\circ\mathcal{A}^{\frac{1}{2}} z_n^h\right)(t) \\
&+\frac{1}{r^2}\left\|z_n^h\right\|_r^r - \frac{1}{r}\int_\Omega \left|z_n^h(t)\right|^r \ln\left|z_n^h(t)\right| dx \\
\ge\;& \int_0^t \left\||N(x)|^{\frac{1}{2}} z_{n\tau}^h\right\|^2 d\tau + \left(\frac{l}{2}-\frac{\varepsilon}{r}\right)\left\|\mathcal{A}^{\frac{1}{2}} z_n^h\right\|^2 \\
&-\frac{C(\varepsilon)}{r}\left\|z_n^h(t)\right\|^{2\alpha} + \frac{1}{2}\left(g\circ\mathcal{A}^{\frac{1}{2}} z_n^h\right)(t) + \frac{1}{r^2}\left\|z_n^h\right\|_r^r \\
\ge\;& \int_0^t \left\||N(x)|^{\frac{1}{2}} z_{n\tau}^h\right\|^2 d\tau + \left(\frac{l}{2}-\frac{\varepsilon}{r}\right)\left\|\mathcal{A}^{\frac{1}{2}} z_n^h\right\|^2 \\
&-\frac{C(\varepsilon)}{r}\left(2S_n(t)\right)^{\alpha} + \frac{1}{2}\left(g\circ\mathcal{A}^{\frac{1}{2}} z_n^h\right)(t) + \frac{1}{r^2}\left\|z_n^h\right\|_r^r. \qquad (54)
\end{aligned}$$

Subsequently, we get

$$\begin{aligned}
&\int_0^t \left\||N(x)|^{\frac{1}{2}} z_{n\tau}^h\right\|^2 d\tau + \left(\frac{l}{2}-\frac{\varepsilon}{r}\right)\left\|\mathcal{A}^{\frac{1}{2}} z_n^h\right\|^2 \\
&+\frac{1}{2}\left(g\circ\mathcal{A}^{\frac{1}{2}} z_n^h\right)(t) + \frac{1}{r^2}\left\|z_n^h\right\|_r^r \\
\le\;& C_{T},\;\; \forall h,n\in Z^+. \qquad (55)
\end{aligned}$$

Therefore by (55), we obtain

$$\begin{aligned}
\int_0^t\int_\Omega (z_{n\tau})^2\,dxd\tau \;\le\;& \int_0^t \int\limits_{\Omega_1=\left\{x\in\Omega;|x|^{-s}\ge n\right\}} \frac{1}{N(x)} N(x)\,(z_{n\tau})^2\,dxd\tau \\
&+\int_0^t \int\limits_{\Omega_2=\left\{x\in\Omega;|x|^{-s}< n\right\}} \frac{1}{N(x)} N(x)\,(z_{n\tau})^2\,dxd\tau
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{n}\int_0^t \int_\Omega N(x)(z_{n\tau})^2\,dxd\tau + diam(\Omega)^s \int_0^t \int_\Omega N(x)(z_{n\tau})^2\,dxd\tau \\
&\leq (1+diam(\Omega)^s)\int_0^t \int_\Omega N(x)(z_{n\tau})^2\,dxd\tau \leq C. \qquad (56)
\end{aligned}
$$

Step 3. Pass to the limit

From (51), (55) and (56), there exists a subsequence of $\{z_n^h\}_{h,n=1}^\infty$, which we still state with this $\{z_n^h\}_{h,n=1}^\infty$ for simplicity. As $h\to+\infty, n\to+\infty$, we get

$$z_n^h \to z \quad \text{weakly star in } L^\infty\left(0,T;H_0^m(\Omega)\right), \tag{57}$$

$$z_n^h \to z \quad \text{in } L^2\left(0,T;H_0^m(\Omega)\right), \tag{58}$$

$$|N(x)|^{\frac{1}{2}} z_{nt}^h \to \frac{z_t}{|x|} \ \text{ in } L^2\left(0,T;L^2(\Omega)\right), \tag{59}$$

$$z_{nt}^h \to z_t \quad \text{in } L^2\left(0,T;L^2(\Omega)\right). \tag{60}$$

Since (57) and (60), from the Aubin-Lions Lemma it follows that(see [25], Corollary 4)

$$z_n^h \to z \quad \text{in } C\left(0,T;L^2(\Omega)\right), \tag{61}$$

as $h\to+\infty, n\to+\infty$. Thus, we obtain $z_n^h \to z$, a.e. $(x,t)\in\Omega\times(0,T)$, which implies

$$\left|z_n^h\right|^{r-2} z_n^h \ln\left|z_n^h\right| \to |z|^{r-2} z\ln|z| \quad \text{a.e. } (x,t)\in\Omega\times(0,T). \tag{62}$$

Furthermore, by Lemma 2.10 and Lemma 2.11, we get

$$
\begin{aligned}
\int_\Omega \left|\left|z_n^h\right|^{r-2} z_n \ln\left|z_n^h\right|\right|^2 dx &= \int\limits_{\Omega_1=\{x\in\Omega;|z_n(x)|\geq 1\}} \left|\left|z_n^h\right|^{r-2} z_n^h \ln\left|z_n^h\right|\right|^2 dx \\
&\quad + \int\limits_{\Omega_2=\{x\in\Omega;|z_n(x)|<1\}} \left|\left|z_n^h\right|^{r-2} z_n^h \ln\left|z_n^h\right|\right|^2 dx \\
&\leq \int\limits_{\Omega_1=\{x\in\Omega;|z_n(x)|\geq 1\}} \left|\left|z_n^h\right|^{r-1} \ln\left|z_n^h\right|\right|^2 dx \\
&\quad + \int\limits_{\Omega_2=\{x\in\Omega;|z_n(x)|<1\}} \left|\left|\left|z_n^h\right|^{-\mu} \ln\left|z_n^h\right|\left|z_n^h\right|\right|^{r-1+\mu}\right|^2 dx \\
&\leq (e\mu)^{-2}\left\|z_n^h\right\|_{2(r-1+\mu)}^{2(r-1+\mu)} + [e(r-1)]^{-2}|\Omega| \\
&\leq (e\mu)^{-2}\mathcal{B}_2\left\|\mathcal{A}^{\frac{1}{2}} z_n^h\right\|^{2(r-1+\mu)} + [e(r-1)]^{-2}|\Omega| \leq C,
\end{aligned}
\tag{63}
$$

here $\mathcal{B}_2$ is the optimal constant for the Sobolev embedding $H_0^m(\Omega)\hookrightarrow L^{2(r-1+\mu)}(\Omega)$. Where we choose $0<\mu\leq 2\left(1+\frac{2m}{N}\right)+1-r$, $r-1<2\left(1+\frac{2m}{N}\right)$, we know that by (61) and (63), we obtain

$$\left|z_n^h\right|^{r-2} z_n^h \ln\left|z_n^h\right| \to |z|^{r-2} z\ln|z| \quad \text{weakly star in } L^\infty\left(0,T;L^2(\Omega)\right). \tag{64}$$

From (61), we get $z_n^h(x,0)\to z(x,0)$ in $L^2(\Omega)$. By combining (36) with $z_{n0}^h \to z_0(x)$ in $H_0^m(\Omega)\cap L^{r+1}(\Omega)$, we observe this $z(x,0)=z_0$ in $H_0^m(\Omega)\cap L^{r+1}(\Omega)$. From (57), (59) and (64) passing to the limit in (35) as $h\to+\infty$, $n\to+\infty$, we see this z satisfies

$$\langle |x|^{-s} z_t,\varphi\rangle + \left\langle \mathcal{A}^{\frac{1}{2}} z, \mathcal{A}^{\frac{1}{2}}\varphi\right\rangle - \int_0^t g(t-s)\left\langle \mathcal{A}^{\frac{1}{2}} z(s), \mathcal{A}^{\frac{1}{2}}\varphi\right\rangle ds = \left\langle |z|^{r-2} z\ln|z|, \varphi\right\rangle, \tag{65}$$

for all $\varphi \in H_0^m(\Omega)$, and for a.e. $t \in [0,T)$.

Step 4. Uniqueness

Suppose there are two solutions z_1 and z_2 to the problem (1) with identical initial condition $z_1(x,0) = z_2(x,0) = z_0 \in H_0^2(\Omega)$, we obtain

$$\langle |x|^{-s} z_{1t}, v\rangle + \left\langle \mathcal{A}^{\frac{1}{2}} z_1, \mathcal{A}^{\frac{1}{2}} v\right\rangle - \int_0^t g(t-s)\left\langle \mathcal{A}^{\frac{1}{2}} z_1(s), \mathcal{A}^{\frac{1}{2}} v\right\rangle ds = \left\langle |z_1|^{r-2} z_1 \ln|z_1|, v\right\rangle, \tag{66}$$

and

$$\langle |x|^{-s} z_{2t}, v\rangle + \left\langle \mathcal{A}^{\frac{1}{2}} z_2, \mathcal{A}^{\frac{1}{2}} v\right\rangle - \int_0^t g(t-s)\left\langle \mathcal{A}^{\frac{1}{2}} z_2(s), \mathcal{A}^{\frac{1}{2}} v\right\rangle ds = \left\langle |z_2|^{r-2} z_2 \ln|z_2|, v\right\rangle. \tag{67}$$

Suppose that $\omega = z_1 - z_2$ and $\omega(x,0) = 0$,then by subtracting the (66) and (67), we can derive

$$\langle |x|^{-s} \omega_t, v\rangle + \left\langle \mathcal{A}^{\frac{1}{2}} \omega, \mathcal{A}^{\frac{1}{2}} v\right\rangle - \int_0^t g(\tau-s)\left\langle \mathcal{A}^{\frac{1}{2}} \omega(s), \mathcal{A}^{\frac{1}{2}} v\right\rangle ds = \left\langle |z_1|^{r-2} z_1 \ln|z_1| - |z_2|^{r-2} z_2 \ln|z_2|, v\right\rangle. \tag{68}$$

Let $v = \omega$ integrating it on $[0,t]$, we get

$$\begin{aligned} &\frac{1}{2}\left\||x|^{-\frac{s}{2}}\omega\right\|^2 + \int_0^t \left\|\mathcal{A}^{\frac{1}{2}}\omega\right\|^2 d\tau - \int_0^t\int_0^\tau g(\tau-s)\int_\Omega \mathcal{A}^{\frac{1}{2}}\omega(s)\mathcal{A}^{\frac{1}{2}}\omega(\tau)\,dxdsd\tau \\ = &\int_0^t\int_\Omega \left(|z_1|^{r-2} z_1 \ln|z_1| - |z_2|^{r-2} z_2 \ln|z_2|\right)\omega dxd\tau. \end{aligned} \tag{69}$$

Then (43) into the (69), we have

$$\begin{aligned} &\int_0^t\int_\Omega \left(|z_1|^{r-2} z_1 \ln|z_1| - |z_2|^{r-2} z_2 \ln|z_2|\right)\omega dxd\tau \\ = &\frac{1}{2}\left\||x|^{-\frac{s}{2}}\omega\right\|^2 + \int_0^t \left\|\mathcal{A}^{\frac{1}{2}}\omega\right\|^2 d\tau - \int_0^t\int_0^\tau g(\tau-s)\int_\Omega \mathcal{A}^{\frac{1}{2}}\omega(s)\mathcal{A}^{\frac{1}{2}}\omega(\tau)\,dxdsd\tau \\ \geq &\frac{1}{2}\left\||x|^{-\frac{s}{2}}\omega\right\|^2 + \int_0^t \left\|\mathcal{A}^{\frac{1}{2}}\omega\right\|^2 d\tau - \left(1-\frac{l}{2}\right)\int_0^t \left\|\mathcal{A}^{\frac{1}{2}}z\right\|^2 d\tau \\ = &\frac{1}{2}\left\||x|^{-\frac{s}{2}}\omega\right\|^2 + \frac{l}{2}\int_0^t \left\|\mathcal{A}^{\frac{1}{2}}z\right\|^2 d\tau \\ \geq &\frac{1}{2}\left\||x|^{-\frac{s}{2}}\omega\right\|^2. \end{aligned} \tag{70}$$

Furthermore

$$\frac{1}{2}\left\||x|^{-\frac{s}{2}}\omega\right\|^2 \leq \int_0^t\int_\Omega \left(|z_1|^{r-2} z_1 \ln|z_1| - |z_2|^{r-2} z_2 \ln|z_2|\right)\omega dxd\tau. \tag{71}$$

We define $G(z): R^+ \to R^+$ and $G(z) = |z|^{r-2} z \ln|z|$. Which implies $G(z)$ is locally Lipscitz continue, so we get

$$\int_0^t\int_\Omega [G(z_1) - G(z_2)]\,\omega dxd\tau \leq C_T \int_0^t \|\omega\|^2 d\tau. \tag{72}$$

Combining with (47), (71) and (72), we have

$$\|\omega\|^2 \leq \frac{2}{C_\Omega} C_T \int_0^t \|\omega\|^2 d\tau. \tag{73}$$

Using Gronwall's inequality, the inequality above implies that $\|\omega\|^2 = 0$. Consequently, we have $\omega = 0$ almost everywhere in $\Omega \times (0,T)$. Moreover, the uniqueness of problem (1) can be inferred.

The proof of Theorem 3.1 is completed. □

4. Global existence

In this section, we will show the global existence of problem (1).

Theorem 4.1. *Let $z_0(x) \in H_0^m(\Omega) \cap L^{r+1}(\Omega)$, (A1) and (A2) hold. If $J(z_0) \le d$, $I(z_0) > 0$, then problem (1) admits a global solution $z \in L^\infty\left(0, T; H_0^m(\Omega) \cap L^{r+1}(\Omega)\right)$ with $\frac{z_t}{|x|^{s/2}} \in L^2\left(0, T; L^2(\Omega)\right)$.*

Proof. To demonstrate the existence of global solutions, we consider the following two cases:

Case 1. $J(z_0) < d$ and $I(z_0) > 0$.

By (27), then we get

$$\begin{aligned} & J(z(t)) + \int_0^t \left\| \frac{z(\tau)}{|x|^{s/2}} \right\|^2 d\tau + \frac{1}{2}\int_0^t g(\tau) \left\| \mathcal{A}^{\frac{1}{2}} z \right\|^2 d\tau \\ & -\frac{1}{2}\int_0^t \left(g' \circ \mathcal{A}^{\frac{1}{2}} z\right)(\tau)\, d\tau \\ = \ & J(z_0) < d, \ 0 \le t \le T_{\max}. \end{aligned} \tag{74}$$

here $T_{\max}$ is the maximal existence time of solution $z(t)$, we will show that $T_{\max} = +\infty$.

After, we will demonstrate that

$$z(x,t) \in \mathcal{W} \text{ for all } 0 \le t \le T_{\max}. \tag{75}$$

Essentially, suppose that (75) doesn't hold and let t_* be the smallest time for which $z(t_*) \notin \mathcal{W}_1^+$.

Then, due to the continuity of $z(t)$, we have $z(t_*) \in \partial \mathcal{W}_1^+$. Therefore the following result emerges

$$J(z(t_*)) = d, \tag{76}$$

or

$$J(z(t_*)) = 0. \tag{77}$$

However, it is clear that (76) is invalid compared to (74). Conversely, assuming (77) is true, according to the definition of d we get

$$J(z(t_*)) \ge \inf_{z \in \mathcal{N}} J(z) = d,$$

which also contradicts with (74). From here we obtain $z(x,t) \in \mathcal{W}$, i.e. $I(z(t)) > 0$.

Next, we demostrate that $T^* = +\infty$. By Lemma 2.7, we obtain $z \in \mathcal{W}$, for all $t \in [0, T^*]$. By combining equations (6) and (74), we derive

$$\begin{aligned} & \int_0^t \left\| \tfrac{z_\tau}{|x|^{s/2}} \right\|^2 d\tau + \tfrac{r-2}{2r}\left(1 - \int_0^t g(s)\, ds\right) \left\| \mathcal{A}^{\frac{1}{2}} z \right\|^2 \\ & + \tfrac{r-2}{2r}\left(g \circ \mathcal{A}^{\frac{1}{2}} z\right)(t) + \tfrac{1}{r^2}\|z\|_r^r < d, \end{aligned} \tag{78}$$

which implies

$$\begin{aligned} \int_0^t \left\| \frac{z_\tau}{|x|^{s/2}} \right\|^2 d\tau &< d, \\ \left\| \mathcal{A}^{\frac{1}{2}} z \right\|^2 &< \frac{2rd}{(r-2)\, l}, \\ (g \circ \nabla z)(t) &< \frac{2rd}{r-2}, \\ \|z\|_r^r &< r^2 d. \end{aligned} \tag{79}$$

It is clear that the constants on the right-hand side of equations (78)-(79) are independent of T. This estimation enables us to set $T_{\max} = +\infty$. Consequently, we can infer that there is a singular global solution, denoted as $z(t) \in \mathcal{W}$, for problem (1).

Case 2. $J(z_0) = d$ and $I(z_0) > 0$.

For $k = 2, 3, ...,$ we define $\theta_k = 1 - \left(\frac{1}{k}\right)$ and $z_{k0} = \theta_k z_0$. Then we take into account the following problem

$$\begin{cases} \frac{z_t}{|x|^s} + \mathcal{A}z - \int_0^t g(t-s)\mathcal{A}z(s)\,ds = |z|^{r-2} z \ln z, & (x,t) \in \Omega \times (0,T), \\ \frac{\partial^i z(x,t)}{\partial v^i} = 0, \ i = 0, 1, ..., k-1, & (x,t) \in \partial\Omega \times (0,T), \\ z(x,0) = z_{0k}(x) & x \in \Omega. \end{cases} \tag{80}$$

From $I(z_0) > 0$, By Lemma 2.10, it follows that $\lambda^* \geq 1 > \theta_m$.

From here, from $\theta_k < 1 < \lambda^*$ we can deduce that $I(z_{0k}) = I(\theta_k z_0) > 0$ and $J(z_{0k}) = J(\theta_k z_0) < J(z_0) = d$, which means $z_{0k} \in \mathcal{W}$. Similar to Case 1. We see that the (80) problem admits of the global weak solution $z_k \in L^\infty\left(0, T; H_0^m(\Omega) \cap L^{r+1}(\Omega)\right)$ with $\frac{z_{kt}}{|x|^{\frac{s}{2}}} \in L^2\left(0, T; L^2(\Omega)\right)$ and $z_k \in \mathcal{W}$. The remainder of the proof can be processed similarly to the previous subsection. The proof of Theorem 4.1 is completed. □

5. Blow up

In this part, we consider with the finite time blow-up results.

Lemma 5.1. *Suppose that (A1) and (A2) hold and $z_0 \in \mathcal{V}$, then we get*

$$(r-2)\left(1 - \int_0^t g(s)\,ds\right)\left\|\mathcal{A}^{\frac{1}{2}} z\right\|^2 + (r-2)\left(g \circ \mathcal{A}^{\frac{1}{2}} z\right)(t) + \frac{2}{r}\|z\|_r^r \geq 2rd. \tag{81}$$

Proof. Since $z_0 \in \mathcal{V}$, from Lemma 2.7, we get $z \in \mathcal{V}$, i.e. $J(z(t)) < d$, $I(z(t)) < 0$. By Lemma 2.5, we know that there exists $\lambda^* \in (0,1)$, such that $I(\lambda^* z) = 0$. Using the definition of d, we can get

$$\begin{aligned} d \quad \leq \quad & J(\lambda^* z) = \frac{1}{r} I(\lambda^* z) + \frac{r-2}{2r}\left(1 - \int_0^t g(s)\,ds\right)\left\|\mathcal{A}^{\frac{1}{2}}\lambda^* z\right\|^2 \\ & + \frac{r-2}{2r}\left(g \circ \mathcal{A}^{\frac{1}{2}}\lambda^* z\right)(t) + \frac{1}{r^2}\|\lambda^* z\|_r^r. \end{aligned} \tag{82}$$

Then

$$(r-2)\left(1 - \int_0^t g(s)\,ds\right)\left\|\mathcal{A}^{\frac{1}{2}} z\right\|^2 + (r-2)\left(g \circ \mathcal{A}^{\frac{1}{2}} z\right)(t) + \frac{2}{r}\|z\|_r^r \geq 2rd. \tag{83}$$

We will write it for convenience

$$\mathcal{L}(t) = \int_0^t \left\|\frac{z(\tau)}{|x|^{s/2}}\right\|^2 d\tau + (T-t)\left\|\frac{z_0}{|x|^{s/2}}\right\|^2, \tag{84}$$

for each $t \in [0,T)$. Then

$$\mathcal{L}'(t) = \left\|\frac{z(t)}{|x|^{s/2}}\right\|^2 - \left\|\frac{z_0}{|x|^{s/2}}\right\|^2,$$

$$\mathcal{L}''(t) \quad = \quad 2\int_\Omega \frac{z(t)\,z_t}{|x|^s}dx$$

$$
\begin{aligned}
&= -2\left\|\mathcal{A}^{\frac{1}{2}}z\right\|^{2}+2\int_{0}^{t}g(t-s)\left(\mathcal{A}^{\frac{1}{2}}z(s),\mathcal{A}^{\frac{1}{2}}z(t)\right)ds \\
&\quad +2\left(|z|^{r-2}z\ln z,z\right) \\
&= -2\left\|\mathcal{A}^{\frac{1}{2}}z\right\|^{2}+2\int_{0}^{t}g(t-s)\left(\mathcal{A}^{\frac{1}{2}}z(s),\mathcal{A}^{\frac{1}{2}}(t)\right)ds \\
&\quad +2\int_{0}^{t}|z|^{r}\ln z dx. \qquad (85)
\end{aligned}
$$

□

Lemma 5.2. *Assume that* $\int_0^t g(s)\,ds \leq \frac{r-3}{r-2}$, *then*

$$
\begin{aligned}
&\mathcal{L}''(t)-2r\int_{0}^{t}\left\|\frac{z_{\tau}}{|x|^{s/2}}\right\|^{2}d\tau \\
&\geq -2rE(0) \\
&+\alpha\left[(r-2)\left(1-\int_{0}^{t}g(\tau)d\tau\right)\left\|\mathcal{A}^{\frac{1}{2}}z\right\|^{2}+(r-2)\left(g\circ\mathcal{A}^{\frac{1}{2}}z\right)(t)+\frac{2}{r}\int_{\Omega}|z|^{r}dx\right] \qquad (86)
\end{aligned}
$$

where $\alpha = 1-\frac{1}{(r-2)l}$.

Proof. By applying Young's inequality and Lemma 2.6, we have

$$
\begin{aligned}
&\mathcal{L}''(t)-2r\int_{\Omega}\left\|\frac{z_{\tau}}{|x|^{s/2}}\right\|^{2}d\tau \\
&= -2\left\|\mathcal{A}^{\frac{1}{2}}z\right\|^{2}+2\int_{0}^{t}g(t-s)\left(\mathcal{A}^{\frac{1}{2}}z(s),\mathcal{A}^{\frac{1}{2}}z(t)\right)ds \\
&\quad +2\int_{\Omega}|z|^{r}\ln z dx-2r\int_{0}^{t}\left\|\frac{z_{\tau}}{|x|^{s/2}}\right\|^{2}d\tau \\
&\geq -2r\int_{0}^{t}\left\|\frac{z_{\tau}}{|x|^{s/2}}\right\|^{2}d\tau-2\left(1-\int_{0}^{t}g(t-s)ds\right)\left\|\mathcal{A}^{\frac{1}{2}}z\right\|^{2}+2\int_{\Omega}|z|^{r}\ln z dx \\
&\quad -2\left[\left(g\circ\mathcal{A}^{\frac{1}{2}}z\right)(t)+\frac{1}{4}\int_{0}^{t}g(t-s)ds\left\|\mathcal{A}^{\frac{1}{2}}z\right\|^{2}\right] \\
&\geq -2rE(0)+(r-2)\left(1-\int_{0}^{t}g(t-s)ds\right)\left\|\mathcal{A}^{\frac{1}{2}}z\right\|^{2} \\
&\quad -\frac{1}{2}\int_{0}^{t}g(t-s)ds\left\|\mathcal{A}^{\frac{1}{2}}z\right\|^{2}+\frac{2}{r}\int_{\Omega}|z|^{r}dx+(r-2)\left(g\circ\mathcal{A}^{\frac{1}{2}}z\right)(t) \\
&\geq -2rE(0) \\
&\quad +\left(1-\frac{1}{(r-2)l}\right)\left[\begin{array}{c}(r-2)\left(1-\int_0^t g(\tau)d\tau\right)\left\|\mathcal{A}^{\frac{1}{2}}z\right\|^{2} \\ +(r-2)\left(g\circ\mathcal{A}^{\frac{1}{2}}z\right)(t)+\frac{2}{r}\int_{\Omega}|z|^{r}dx\end{array}\right]. \qquad (87)
\end{aligned}
$$

The proof of Lemma 5.2 is completed. □

Lemma 5.3. *[15] Suppose that (A1) and (A2) hold and* $z_0 \in \mathcal{V}$, $z(x,t)$ *is the solution of problem (1), if one of the following conditions is true*

$$
\begin{aligned}
E(0) &< 0, \\
E(0) &= 0,
\end{aligned}
$$

$$0 \quad < \quad E(0) < \alpha d. \tag{88}$$

Then, $\mathcal{L}'(t) > 0$ *for* $t > 0$.

Theorem 5.4. *Assume that conditions (A1) and (A2) are satisfied and that* $z_0 \in \mathcal{V}$. *Let* z *be the weak solution of problem (1), if one of the following conditions is true*

Case 1 $E(0) < 0$, $T^* \leq t_* - \left(\frac{\mathcal{F}(t_*)}{\mathcal{F}'(t_*)}\right)$ *and if* $\mathcal{F}(t_*) < \min\left\{1, \sqrt{\frac{\alpha}{-\beta}}\right\}$, *then*

$$T^* \leq t_* + \sqrt{\frac{\alpha}{-\beta}} \ln\left(\frac{\sqrt{\frac{\alpha}{-\beta}}}{\sqrt{\frac{\alpha}{-\beta}} - \mathcal{F}(t_*)}\right).$$

Case 2 $E(0) = 0$, $T^* \leq t_* + \frac{\mathcal{F}(t_*)}{\sqrt{\alpha}}$.

Case 3 $0 < E(0) < \alpha d$, $T^* \leq t_* - \left(\frac{\mathcal{F}(t_*)}{\mathcal{F}'(t_*)}\right)$, *and if* $\mathcal{F}(t_*) < \min\left\{1, \sqrt{\frac{\alpha_1}{-\beta_1}}\right\}$, *then*

$$T^* \leq t_* + \sqrt{\frac{1}{-\beta_1}} \ln\left(\frac{\sqrt{\frac{\alpha_1}{-\beta_1}}}{\sqrt{\frac{\alpha_1}{-\beta_1}} - \mathcal{F}(t_*)}\right).$$

Proof. Let

$$\mathcal{F}(t) = \mathcal{L}(t)^{-\left(\frac{r-2}{2}\right)}. \tag{89}$$

Then

$$\begin{aligned} \mathcal{F}'(t) &= -\frac{r-2}{2}\mathcal{L}(t)^{-\left(\frac{r-2}{2}\right)-1}\mathcal{L}'(t) \\ &= -\frac{r-2}{2}\mathcal{L}(t)^{-\frac{r}{2}}\mathcal{L}'(t), \end{aligned} \tag{90}$$

$$\mathcal{F}''(t) = -\frac{r-2}{2}\mathcal{L}(t)^{1+\left(\frac{4}{r}-2\right)}\left[\mathcal{L}''(t)\mathcal{L}(t) - \frac{r}{2}\left(\mathcal{L}'(t)\right)^2\right]. \tag{91}$$

Applying Lemma 5.1, Lemma 5.2 and Hölder's inequality, we get

$$\begin{aligned} & \mathcal{F}''(t)\mathcal{F}(t) - \frac{r}{2}\left(\mathcal{F}'(t)\right)^2 \\ \geq & \left\{ \begin{array}{c} 2r\int_0^t \left\|\frac{z_\tau}{|x|^{s/2}}\right\|^2 d\tau - 2rE(0) \\ +\alpha\left[(r-2)\left(1-\int_0^t g(s)\,ds\right)\left\|\mathcal{A}^{\frac{1}{2}}z\right\|^2 + \frac{2}{r}\|z\|_r^r + (r-2)\left(g\circ\mathcal{A}^{\frac{1}{2}}z\right)(t)\right] \end{array} \right\}\mathcal{F}(t) \\ & -\frac{r}{2}\left[4\mathcal{F}(t)\int_0^t \left\|\frac{z_\tau}{|x|^{s/2}}\right\|^2 d\tau\right] \\ \geq & \left\{ \begin{array}{c} -2rE(0) \\ +\alpha\left[(r-2)\left(1-\int_0^t g(s)\,ds\right)\left\|\mathcal{A}^{\frac{1}{2}}z\right\|^2 + \frac{2}{r}\|z\|_r^r + (r-2)\left(g\circ\mathcal{A}^{\frac{1}{2}}z\right)(t)\right] \end{array} \right\}\mathcal{F}(t) \\ = & \left\{ \begin{array}{c} -2rE(0) \\ +\alpha\left[(r-2)\left(1-\int_0^t g(s)\,ds\right)\left\|\mathcal{A}^{\frac{1}{2}}z\right\|^2 + \frac{2}{r}\|z\|_r^r + (r-2)\left(g\circ\mathcal{A}^{\frac{1}{2}}z\right)(t)\right] \end{array} \right\}\mathcal{F}(t)^{-\frac{2}{r-2}} \\ \geq & \left[2r\alpha d - 2rE(0)\right]\mathcal{F}(t)^{-\frac{2}{r-2}} \end{aligned} \tag{92}$$

Next, we substitute equation (92) into equation (91) to obtain

$$\mathcal{F}''(t) \leq r(r-2)(E(0)-\alpha d)\mathcal{F}(t)^{1+\frac{2}{r-2}}. \tag{93}$$

If **Case 1** or **Case 2** holds, then from equation (93), we obtain

$$\mathcal{F}''(t) \leq r(r-2)E(0)\mathcal{F}(t)^{1+\frac{2}{r-2}}. \tag{94}$$

From Lemma 5.3, multiplying equation (94) by $\mathcal{F}'(t)$ and integrating over the interval $[t_*, t]$, we obtain

$$(\mathcal{F}'(t))^2 \geq \alpha + \beta\mathcal{F}(t)^{2+\frac{2}{r-2}}, \quad t \geq t_* \tag{95}$$

here

$$\begin{aligned} \alpha &= \mathcal{F}'(t_*)^2 - \frac{r(r-2)^2}{r-1}E(0)\mathcal{F}(t_*)^{2+\frac{2}{r-2}}, \\ \beta &= \frac{r(r-2)^2}{r-1}E(0). \end{aligned} \tag{96}$$

If the **Case 3** holds, we can get

$$\mathcal{F}''(t) \leq -r(r-2)(\alpha d - E(0))\mathcal{F}(t)^{1+\frac{2}{r-2}}. \tag{97}$$

By applying the same reasoning as in equation (93), we know that

$$(\mathcal{F}'(t))^2 \geq \alpha_1 + \beta_1\mathcal{F}(t)^{2+\frac{2}{r-2}}, \quad t \geq t_*, \tag{98}$$

here

$$\begin{aligned} \alpha_1 &= \mathcal{F}'(t_*)^2 - \frac{r(r-2)^2}{r-1}(E(0)-\alpha d)\mathcal{F}(t_*)^{2+\frac{2}{r-2}}, \\ \beta &= \frac{r(r-2)^2}{r-1}(E(0)-\alpha d). \end{aligned} \tag{99}$$

Therefore, when $\gamma = \frac{r-2}{2}$ and $t_0 = t_* > 0$, by Lemma 2.12, there exists a finite time T^* so that

$$\lim_{t\to T^{*-}} \mathcal{F}(t) = 0, \tag{100}$$

i.e.,

$$\lim_{t\to T^{*-}} \left\| \frac{z(x,t)}{|x|^{s/2}} \right\|^2 = +\infty. \tag{101}$$

This finished the proof of Theorem 5.4. □

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