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Some numerical radius inequalities via maximization

Amit Kumar[a], S. K. Kaushik[b,*], Khole Timothy Poumai[c]

[a]*Department of Mathematics, University of Delhi, Delhi-110007, India*
[b]*Department of Mathematics, Kirori mal College, University of Delhi, Delhi-110007, India*
[c]*Department of Mathematics, Motilal Nehru College, University of Delhi, Delhi-110021, India*

Abstract. Dilation theory offers various structural characterizations of numerical contractions, i.e. operators whose numerical radius does not exceed one. Among these, a recent factorization result establishes that a matrix $A \in \mathcal{M}_{n\times n}(\mathbb{C})$ is a numerical contraction if and only if it can be written as $A = 2X^*Y$, where $X, Y \in \mathcal{M}_{n\times n}(\mathbb{C})$ satisfy $X^*X + Y^*Y = I_n$. In this article, we show how an equivalent formulation of this factorization may be employed to derive new bounds for the numerical radius. The principal contributions include generalizations of several known numerical radius inequalities for Hilbert space operators, thereby extending and refining existing results in operator theory.

1. Introduction

Let $\mathcal{M}_{m\times n}(\mathbb{C})$ denote the C^*-algebra of all complex $m \times n$ matrices, equipped with the standard inner product $\langle\cdot,\cdot\rangle$. For $A \in \mathcal{M}_{n\times n}(\mathbb{C})$, the numerical radius $\omega(A)$ is a norm, defined as

$$\omega(A) = \sup\{|\langle Ax, x\rangle| : x \in \mathbb{C}^n, \|x\| = 1\}.$$

The numerical radius plays a crucial role in understanding the behavior of operators. Several important characterizations of operators with numerical radius bounded by one have been discovered by various researchers, including Berger[6], Sz-Nagy and Foias [16], as well as T. Ando [1]. Recently, Bhatia and Jain [3] established a crucial factorization theorem for such operators: A matrix $A \in \mathcal{M}_{n\times n}(\mathbb{C})$ satisfies $\omega(A) \le 1$ if and only if there exist matrices $X, Y \in \mathcal{M}_{n\times n}(\mathbb{C})$ such that $A = 2X^*Y$ and $X^*X + Y^*Y = I_n$. Their proof employs an operator-theoretic adaptation of the classical Fejer-Riesz theorem from complex analysis, which provides conditions under which a Laurent polynomial is positive on the unit circle.

The primary objective of this article is to demonstrate how an equivalent version of the factorization theorem by Bhatia and Jain can be used to derive new inequalities for the numerical radius. This is achieved by optimizing unitary invariant norms of matrix functions on the set $\{A \in \mathcal{M}_{n\times n}(\mathbb{C}) : \omega(A) \le 1\}$.

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* Corresponding author: S. K. Kaushik
Email addresses: amitpunfermiran@gmail.com (Amit Kumar), shikk2003@yahoo.co.in (S. K. Kaushik), kholetim@yahoo.co.in (Khole Timothy Poumai)
ORCID iDs: https://orcid.org/0009-0006-2610-4029 (Amit Kumar), https://orcid.org/0000-0002-2850-0432 (S. K. Kaushik), https://orcid.org/0009-0006-0457-5807 (Khole Timothy Poumai)

A norm $\|\cdot\|_u$ on $\mathcal{M}_{n\times n}(\mathbb{C})$ is called a *unitarily invariant norm* if

$$\|A\|_u = \|UAV\|_u \quad \text{for all } U, V \in \mathcal{U}(n),$$

where $\mathcal{U}(n)$ denotes the set of unitary matrices in $\mathcal{M}_{n\times n}(\mathbb{C})$. The unitarily invariant norm of a matrix solely depends on its singular values. Among the class of unitarily invariant norms, the *Ky Fan k-norm* [4] is of particular interest. For a matrix $A \in \mathcal{M}_{m\times n}(\mathbb{C})$, it is defined by

$$\|A\|_k = \sum_{i=1}^{k} \sigma_i(A), \quad \text{for } 1 \le k \le n,$$

where $\sigma_1(A) \ge \sigma_2(A) \ge \cdots \ge \sigma_n(A)$ denote the singular values of A, arranged in non-increasing order. In particular, when $k = 1$, the Ky Fan 1-norm coincides with the usual operator norm, which we denote by $\|A\|$. It is well known that the numerical radius $\omega(A)$ and the operator norm $\|A\|$ of a matrix $A \in \mathcal{M}_{n\times n}(\mathbb{C})$ are related by the inequality

$$\frac{1}{2}\|A\| \le \omega(A) \le \|A\|. \tag{1}$$

A proof of this result can be found in [4]. A significant refinement of these inequalities was later obtained by Kittaneh [14], who established that

$$\frac{1}{4}\|AA^* + A^*A\| \le \omega^2(A) \le \frac{1}{2}\|AA^* + A^*A\|. \tag{2}$$

Subsequently, El-Haddad and Kittaneh[8] generalized the second inequality in (2) by employing a special case of Schlömilch's inequality for weighted means of non-negative real numbers, along with various norm inequalities, including a generalized version of the mixed Schwarz inequality. They established that for arbitrary matrix $A \in \mathcal{M}_{n\times n}(\mathbb{C})$, for $0 < \alpha < 1$ and $r \ge 1$, we have

$$\omega^r(A) \le \frac{1}{2}\|(A^*A)^{r\alpha} + (AA^*)^{r(1-\alpha)}\| \tag{3}$$

and

$$\omega^{2r}(A) \le \|\alpha(A^*A)^r + (1-\alpha)(AA^*)^r\|. \tag{4}$$

Several operator-theoretic generalizations of the Cauchy–Schwarz inequality have been established and effectively employed to derive numerical radius inequalities analogous to the second inequality in (2)[9, 10, 15]. However, finding an alternative approach to Kittaneh's elegant proof of the first inequality in (2), which can lead to its generalized versions, remains a significant challenge.

The inequality $\omega(AB) \le \|A\| \cdot \omega(B)$ does not hold in general, even when A and B are commuting matrices in $\mathcal{M}_{n\times n}(\mathbb{C})$. Nevertheless, two notable positive results in this direction were established in [7], which state that for matrices $A, B \in \mathcal{M}_{n\times n}(\mathbb{C})$, the following inequalities hold:

$$\omega(AB + BA) \le 2\sqrt{2}\,\omega(A)\|B\|, \tag{5}$$

and

$$\omega(AB + B^*A) \le 2\,\omega(A)\|B\|. \tag{6}$$

Throughout this article, by $\mathrm{tr}(A)$, $\rho(A)$ and $\sigma(A)$, we denote the trace, rank and spectrum of a matrix A, respectively. The Hadamard product of matrices $A, B \in \mathcal{M}_{n\times n}(\mathbb{C})$ is denoted by $A \circ B$.

Now, we state a result due to Bhatia and Jain [3], which will be used in the subsequent work.

Theorem 1.1. *A matrix $A \in \mathcal{M}_{n\times n}(\mathbb{C})$ is a numerical contraction if and only if there exists $X, Y \in \mathcal{M}_{n\times n}(\mathbb{C})$ such that $A = 2X^*Y$, where $X^*X + Y^*Y = I_n$.*

In this paper, we present several generalizations of the first inequality in (2), including analogues of inequality (4) for the case where the exponent $r \geq 1$ is restricted to be a natural number. For the special case $\alpha = \frac{1}{2}$, we further extend these results to the class of unitarily invariant norms. As an application of our generalizations, we derive mean-type extensions of inequalities (5) and (6) and also obtain a generalization of the first inequality in (1), for arbitrary unitarily invariant norm. Moreover, we show that the methods developed in this paper can be effectively employed to derive a range of new numerical radius inequalities. Finally, we establish bounds for the unitarily invariant norm of the Hadamard product $A \circ B$, under the assumption that B is a matrix with numerical radius at most one.

2. Main Results

To achieve our goal we need to recall the following lemma. This lemma presents the generalized version of the Von Neumann trace inequality and has been proved in [11].

Lemma 2.1. *Let $A_1, A_2, \dots, A_p$ be $n \times n$ complex matrices. Then*

$$\left|\mathrm{tr}(A_1A_2\cdots A_p)\right| \leq \sum_{i=1}^{n} \sigma_i(A_1)\sigma_i(A_2)\cdots\sigma_i(A_p),$$

where $\sigma_i(A_j)$ denotes the i-th singular value of A_j, for all $j \in \{1, 2, \dots, p\}$ and $i \in \{1, 2, \dots, n\}$.

As an application of a version of the Von Neumann trace inequality for rectangular matrices[12], we establish the following lemma, which will be useful in the proofs of several theorems presented in this article.

Lemma 2.2. *For an arbitrary matrix $A \in \mathcal{M}_{n\times m}(\mathbb{C})$,*

$$\|A\|_k = \max\{|tr(AX)| : X \in \mathcal{M}_{m\times n}(\mathbb{C}) \text{ satisfies } \|X\| \leq 1 \text{ and } \rho(X) \leq k\}.$$

Proof. Let $A \in \mathcal{M}_{n\times n}(\mathbb{C})$ admit a singular value decomposition of the form $A = W_1^*\Sigma W_2$, where $\Sigma \in \mathcal{M}_{n\times m}(\mathbb{C})$ is a rectangular diagonal matrix whose diagonal entries are the singular values of A, arranged in non-increasing order: $\sigma_1(A) \geq \sigma_2(A) \geq \cdots \geq \sigma_{\min\{n,m\}}(A)$, and the matrices $W_1 \in \mathcal{U}(n)$ and $W_2 \in \mathcal{U}(m)$ are unitary. Define $\Delta \in \mathcal{M}_{m\times n}(\mathbb{C})$ to be the rectangular diagonal matrix whose first k diagonal entries are equal to 1, with all remaining entries equal to zero. Then the matrix $Y_k = W_2^*\Delta W_1$ is a partial isometry of rank k. We have

$$tr(AY_k) = tr(W_1^*\Sigma\Delta W_1) = \sum_{i=1}^{k} \sigma_i(A) = \|A\|_k. \tag{7}$$

Also, by rectangular version of Von Neumann trace inequality, we can deduce that

$$\max\{|tr(AX)| : \|X\| \leq 1, \rho(X) \leq k\} \leq \sum_{i=1}^{k} \sigma_i(A) = \|A\|_k. \tag{8}$$

Inequalities (7) and (8) complete the proof. □

Remark 2.3. *By Lemma 2.1 and Lemma 2.2, it can be deduced that for $1 \leq k \leq n$,*

$$\|A_1A_2\cdots A_p\|_k \leq \sum_{i=1}^{k} \sigma_i(A_1)\sigma_i(A_2)\cdots\sigma_i(A_p). \tag{9}$$

Next, as an application of Theorem 1.1, we present an alternative necessary and sufficient condition for a matrix to be a numerical contraction, which will be useful in proving main results of this article. Before proceeding, we recall some relevant terminology from the functional calculus of matrices.

Let $C \in \mathcal{M}_{n\times n}(\mathbb{C})$ be a Hermitian matrix, and let $f : \sigma(C) \subseteq I \to \mathbb{R}$ be a continuous function, where $\sigma(C)$ denotes the spectrum of C and $I \subseteq \mathbb{R}$ is an interval containing $\sigma(C)$. If $C = U^*\Delta U$ is the spectral decomposition of C, then

$$f(C) = U^* f(\Delta) U,$$

where $f(\Delta)$ denotes the diagonal matrix obtained by applying f to each diagonal entry of Δ.

Lemma 2.4. *For an arbitrary matrix $A \in \mathcal{M}_{n\times m}(\mathbb{C})$, the following statements are equivalent:*

(i) $\omega(A) \leq 1$.

(ii) There exists a Hermitian matrix $C \in \mathcal{M}_{n\times m}(\mathbb{C})$ and a unitary matrix $W \in \mathcal{M}_{n\times n}(\mathbb{C})$ such that A admits a factorization of the form

$$A = 2cos(C)Wsin(C).$$

Proof. $(i) \implies (ii)$ Suppose $\omega(A) \leq 1$. Then by Theorem 1.1, there exists matrices $X \in \mathcal{M}_{m\times n}(\mathbb{C})$, and $Y \in \mathcal{M}_{n\times m}(\mathbb{C})$ such that

$$A = 2X^*Y, \quad \text{where } X^*X + Y^*Y = I_n. \tag{10}$$

Let $X = U^*\Sigma V$ be the singular value decomposition of the matrix X. Then, there exists a diagonal matrix Δ such that

$$\begin{aligned} Y^*Y &= I_n - X^*X \\ &= V^*(I_n - \Sigma^2)V \\ &= V^* sin^2(\Delta)V \quad (\text{ since } \sigma(I_n - \Sigma^2) \subseteq [0,\ 1]). \end{aligned} \tag{11}$$

Equation (11) yields that there exists an unitary matrix $U_1 \in \mathcal{U}(n)$ such that $Y = U_1 sin(\Delta)V$. Moreover, since $sin^2(\Delta) = I_n - \Sigma^2$, $X = U^*\Sigma V = U^*cos(\Delta)V$. Substituting these factorizations of X and Y into equation (10), we obtain

$$\begin{aligned} A &= 2X^*Y \\ &= 2V^*cos(\Delta)V \cdot (V^*UU_1V) \cdot V^*sin(\Delta)V. \end{aligned} \tag{12}$$

Consider the Hermitian matrix $C = V^*\Delta V$. Then, equation (12) implies that

$$A = 2cos(C)Wsin(C),$$

where $W = V^*UU_1V$ is an unitary matrix.

Conversely, suppose that there exists a Hermitian matrix $C \in \mathcal{M}_{n\times m}(\mathbb{C})$ and a unitary matrix $W \in \mathcal{M}_{n\times n}(\mathbb{C})$ such that

$$A = 2cos(C)Wsin(C). \tag{13}$$

Let $X = cos(C)$ and $Y = Wsin(C)$. Then, by equation (13), we have $A = 2X^*Y$ and $X^*X + Y^*Y = cos^2(C) + sin^2(C) = I_n$. Therefore, by Theorem 1.1, we can conclude that A is a numerical contraction. □

We are now in a position to present the first main result of this section. By a *normalized unitarily invariant norm*, we mean a unitarily invariant norm $\|\cdot\|_u$ satisfying $\|E_{11}\|_u = 1$, where $E_{11} \in \mathcal{M}_{n\times n}(\mathbb{C})$ denotes the matrix with entry 1 in the $(1, 1)$-position and zeros elsewhere.

Proposition 2.5. *Let $A \in \mathcal{M}_{n\times n}(\mathbb{C})$. Then there exists a matrix $A_0 \in \mathcal{M}_{n\times n}(\mathbb{C})$ with $\omega(A_0) = 1$ such that, for any normalized unitarily invariant norm on $\mathcal{M}_{n\times n}(\mathbb{C})$, the following inequality holds:*

$$\|A\|_u \leq \|A_0\|_u \cdot \omega(A).$$

Proof. Suppose $A \in \mathcal{M}_{n\times n}(\mathbb{C})$ is an arbitrary matrix satisfying $\omega(A) \leq 1$. Then, by Lemma 2.4, there exists an unitary matrix W and a Hermitian matrix C such that $A = 2\cos(C)W\sin(C)$. For $1 \leq k \leq n$, we have

$$\begin{aligned}
\|A\|_k &\leq 2\ \max\{\|cos(C)Wsin(C)\|_k : C^* = C,\ W \in U(n)\} \\
&= 2\ \max\{\|U^*\Sigma U W U^* \sqrt{I-\Sigma^2}U\|_k : O \leq \Sigma \leq I,\ U \in U(n),\ W \in U(n)\} \\
&\quad \text{(by using spectral decomposition theorem for Hermitian matrix } cos(C) \text{)} \\
&= 2\ \max\{\|\Sigma W \sqrt{I-\Sigma^2}W^*\|_k : O \leq \Sigma \leq I,\ W \in U(n)\}. \qquad (14)
\end{aligned}$$

Let $P_n \in \mathcal{M}_{n\times n}(\mathbb{C})$ denote a permutation matrix such that diagonal entries of matrix $P_n^* \sqrt{(I-\Sigma^2)}P_n$ are arranged in increasing order. Then, using inequality (9) and inequality (14), we obtain

$$\begin{aligned}
\|A\|_k &\leq 2\ \max\{\|\Sigma P_n \sqrt{I-\Sigma^2}P_n^*\|_k : O \leq \Sigma \leq I\} \\
&= 2\max\left\{\sum_{i=1}^{k} \sigma_i \sqrt{(1-\sigma_{n+1-i}^2)} : 0 \leq \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_n \leq 1\right\} \\
&= \begin{cases} 2k & : k \leq n/2, \\ n & : else. \end{cases} \qquad (15)
\end{aligned}$$

Consider matrix A_0 defined as,

$$A_0 = \begin{cases} 2(E_{1,2l} + E_{2,2l-1} + \cdots + E_{l,l+1}) & : n = 2l \\ 2(E_{1,2l} + E_{2,2l-1} + \cdots + E_{l,l+1}) + E_{2l+1,2l+1} & : n = 2l+1 \end{cases}, \qquad (16)$$

where $E_{i,j}$ denote the $n \times n$ matrix with a 1 in the (i, j)-th position and zeros elsewhere. It is straightforward to verify that $\omega(A_0) = 1$, and the singular values of A_0 are as follows: if $n = 2l$, then the singular values are 0 and 1, each with multiplicity l; if $n = 2l + 1$, then the singular values are 0, 1, and 2, with multiplicities l, 1, and l, respectively. Therefore, by equation (15), it follows that $\|A\|_k \leq \|A_0\|_k$. The result now follows using the Ky Fan dominance theorem[12]. □

In view of Proposition 2.5, it follows that A_0 is a numerical contraction attaining the maximal value with respect to any normalized unitarily invariant norm.

The following result gives a convex functional application of Proposition 2.5.

Corollary 2.6. *Let $A \in \mathcal{M}_{n\times n}(\mathbb{C})$ be a numerical contraction satisfying $tr(|A|) = tr(|A_0|)$, and let $\|\cdot\|_u$ denote a normalized unitarily invariant norm on $\mathcal{M}_{n\times n}(\mathbb{C})$. Then, for any convex function ϕ on $[0,\infty)$, we have*

$$tr(\phi(|A|)) \leq tr(\phi(|A_0|)), \qquad (17)$$

and if ϕ is strictly convex function, then equality holds in (17) *if and only if $\sigma_i(A) = \sigma_i(A_0)$, for all $1 \leq i \leq n$.*

Proof. Since A is a numerical contraction, by Proposition 2.5, we have

$$\sum_{i=1}^{k} \sigma_i(A) \leq \sum_{i=1}^{k} \sigma_i(A_0) \quad \forall\ 1 \leq k \leq n-1 \qquad (18)$$

and so $tr(|A|) = tr(|A_0|)$ implies that

$$\sum_{i=1}^{n} \sigma_i(A) = \sum_{i=1}^{n} \sigma_i(A_0). \tag{19}$$

By equations (18) and (19), singular value tuple $(\sigma_1(A), \sigma_2(A), \ldots, \sigma_n(A))$ is majorized by $(\sigma_1(A_0), \sigma_2(A_0), \ldots, \sigma_n(A_0))$. Now the conclusion follows directly from an application of Karamata's inequality[13]. □

Remark 2.7. *(i) Observe that the largest singular value of A_0 is 2. Consequently, by Proposition 2.5, we have $\|A\| \leq 2\,\omega(A)$. Thus, Proposition 2.5 provides a generalization of the first inequality in equation (1) for unitarily invariant norms.*

(ii) Since $\omega(A_0) = 1$, it follows that for any normalized unitarily invariant norm, the upper bound $\|A_0\|_u$ in the inequality is optimal.

(iii) The proof of Proposition 2.5 can be refined to establish that for arbitrary matrix $B \in \mathcal{M}_{n\times n}(\mathbb{C})$ and normalized unitarily invariant norm $\|.\|_u$, following inequality holds,

$$\max_{\omega(A)\leq 1} \|AB\|_u \leq \|\mathrm{Diag}(\sigma_1(A_0)\sigma_1(B), \sigma_2(A_0)\sigma_2(B), \ldots, \sigma_n(A_0)\sigma_n(B))\|_u. \tag{20}$$

The corresponding version of this inequality for the Hadamard product $A \circ B$ will be discussed at the end of this section.

The following result provides a generalization of the first inequality in (2), analogues to inequality (4).

Theorem 2.8. *Let $A \in \mathcal{M}_{n\times n}(C)$, and let $m \in \mathbb{N}$, $\alpha \in [0, 1]$. Suppose $A_0 \in \mathcal{M}_{n\times n}(\mathbb{C})$ is the matrix as defined in (16). Then the following inequality holds:*

$$\|\alpha|A|^{2m} + (1-\alpha)|A^*|^{2m}\| \leq \|\alpha|A_0|^{2m} + (1-\alpha)|A_0^*|^{2m}\|\ \omega^{2m}(A). \tag{21}$$

Proof. Suppose that $\omega(A) \leq 1$. Then, by Lemma 2.4, there exists a Hermitian matrix C and a unitary matrix W such that

$$A = 2cos(C)Wsin(C). \tag{22}$$

Let $X = Wsin^2(C)W^*cos^2(C)$. Note that X is a contraction. For $m \in \mathbb{N}$, we have

$$(AA^*)^m = 4^m cos(C)X^{m-1}Wsin^2(C)W^*cos(C)$$

and similarly

$$(A^*A)^m = 4^m sin(C)Y^{m-1}W^*cos^2(C)Wsin(C),$$

for some contraction Y. The operator matrix $\begin{bmatrix}\alpha(AA^*)^m & (1-\alpha)(A^*A)^m\end{bmatrix}$ can be factorized as the product of contractions as

$$4^m \begin{bmatrix} cos(C) & sin(C)\end{bmatrix} \begin{bmatrix} \alpha X^{m-1}Wsin^2(C)W^* & O \\ O & (1-\alpha)Y^{m-1}W^*cos^2(C)W\end{bmatrix} \begin{bmatrix} cos(C) & O \\ O & sin(C)\end{bmatrix}. \tag{23}$$

Since $X^{m-1}Wsin^2(C)W^*$ and $Y^{m-1}W^*cos^2(C)W$ are also contractions, from equation (23), one can conclude that

$$\|\begin{bmatrix}\alpha(A^*A)^m & (1-\alpha)(AA^*)^m\end{bmatrix}\| \leq 4^m max\{\alpha,\ 1-\alpha\}.$$

Therefore, for $m \geq 2$, we have

$$\begin{aligned}\|\alpha|A|^{2m} + (1-\alpha)|A^*|^{2m}\| &\leq \|\begin{bmatrix}\alpha(A^*A)^{m-1} & (1-\alpha)(AA^*)^{m-1}\end{bmatrix}\| \left\|\begin{bmatrix}A^*A \\ AA^*\end{bmatrix}\right\| \\ &\leq 4^m\ max\{\alpha, (1-\alpha)\}. \end{aligned} \tag{24}$$

Next we prove inequality (24), for $m = 1$. Using equation (22), the operator matrix $\begin{bmatrix} \alpha A^* & (1-\alpha)A \end{bmatrix}$ can be factorized as the product of contractions as

$$2\begin{bmatrix} sin(C) & cos(C) \end{bmatrix}\begin{bmatrix} \alpha W^* & O \\ O & (1-\alpha)W \end{bmatrix}\begin{bmatrix} sin(C) & O \\ O & cos(C) \end{bmatrix}.$$

This implies that

$$\| \begin{bmatrix} \alpha A^* & (1-\alpha)A \end{bmatrix} \| \leq 2\max\{\alpha, 1-\alpha\}. \tag{25}$$

We obtain,

$$\begin{aligned} \|\alpha A^*A + (1-\alpha)AA^*\| &\leq \| \begin{bmatrix} \alpha A^* & (1-\alpha)A \end{bmatrix} \| \left\| \begin{bmatrix} A \\ A^* \end{bmatrix} \right\| \\ &\leq 4\ max\{\alpha, 1-\alpha\}. \end{aligned} \tag{26}$$

Let $E_{i,j}$ denote the $n \times n$ matrix with a 1 in the (i, j)-th position and zeros elsewhere. Then

$$A_0A_0^* = \begin{cases} 4\displaystyle\sum_{k=1}^{l} E_{k,k}, & \text{if } n = 2l, \\ 4\displaystyle\sum_{k=1}^{l} E_{k,k} + E_{2l+1,2l+1}, & \text{if } n = 2l+1, \end{cases} \tag{27}$$

and

$$A_0^*A_0 = \begin{cases} 4\displaystyle\sum_{k=l+1}^{2l} E_{k,k}, & \text{if } n = 2l, \\ 4\displaystyle\sum_{k=1}^{l} E_{l+k,l+k} + E_{2l+1,2l+1}, & \text{if } n = 2l+1. \end{cases} \tag{28}$$

From equations (27) and (28), it can be concluded that,

$$\|\alpha|A_0|^{2m} + (1-\alpha)|A_0^*|^{2m}\| = 4^m\ max\{\alpha, (1-\alpha)\}. \tag{29}$$

Equation (29), with inequalities (24) and (26) completes the proof. □

Substituting $\alpha = 1$ and $m = 1$ into inequality (21), we obtain

$$\||A|^2\| \leq \||A_0|^2\|\omega^2(A).$$

Since $\||A|\|^2 = \|A\|^2$ and $\|A_0\| = 2$, it follows that

$$\|A\| \leq 2\omega(A),$$

which coincides with the first inequality in (1).

Similarly, setting $\alpha = \frac{1}{2}$ and $m = 1$ in inequality (21), we deduce that

$$\|AA^* + A^*A\| \leq \|A_0A_0^* + A_0^*A_0\|\omega^2(A) = 4\omega^2(A),$$

thus recovering the first inequality in (2).

Hence, inequality (21) serves as a unifying generalization of the first inequalities in both (1) and (2). Moreover, Proposition 2.5 extends the first inequality in (1) to the broader class of normalized unitarily invariant norms. This naturally raises the question: can inequality (21) be similarly generalized to all unitarily invariant norms?

The following result discusses the versions of inequality (21) for some special unitarily invariant norms.

Corollary 2.9. *Let $A \in \mathcal{M}_{n\times n}(C)$. Suppose $m \in \mathbb{N}$, $\alpha \in [0,1]$, and $r \geq \frac{1}{2}$. Also, let $A_0 \in \mathcal{M}_{n\times n}(\mathbb{C})$ be the matrix as defined in* (16). *Then, the following holds*

(i) For the Ky Fan $n-$norm,

$$\|\alpha|A|^{2r} + (1-\alpha)|A^*|^{2r}\|_n \leq \|\alpha|A_0|^{2r} + (1-\alpha)|A_0^*|^{2r}\|_n \ \omega^{2r}(A). \tag{30}$$

(ii) For any normalized unitarily invariant norm,

$$\||A|^{2m} + |A^*|^{2m}\|_u \leq \||A_0|^{2m} + |A_0^*|^{2m}\|_u \ \omega^{2m}(A). \tag{31}$$

Proof. (i) Suppose A is a numerical contraction. It follows from Proposition 2.5, that

$$\sum_{i=1}^{k} \sigma_i(A) \leq \sum_{i=1}^{k} \sigma_i(A_0) \quad \forall\, 1 \leq k \leq n.$$

Given that t^{2r} is a monotonically increasing and convex function on $[0,\infty)$ for all $r \geq \frac{1}{2}$, using Exercise II.3.2 in [4], we deduce that

$$\||A|^{2r}\|_n \leq \||A_0|^{2r}\|_n.$$

Now observation that $\||A|^{2r}\|_n = tr(|A|^{2r}) = \|\alpha|A|^{2r} + (1-\alpha)|A^*|^{2r}\|_n$, holds for arbitrary matrices $A \in \mathcal{M}_{n\times n}(\mathbb{C})$. This completes the proof.

(ii) For $m \in \mathbb{N}$, using equations (27) and (28), we obtain

$$|A|^{2m} + |A^*|^{2m} = \begin{cases} 4^m I_n, & \text{if } n \text{ is even,} \\ 4^m I_{n-1} \oplus 2I_1, & \text{if } n \text{ is odd.} \end{cases}. \tag{32}$$

If n is even, then applying inequalities (24), (26), and (32), we deduce that for all $1 \leq k \leq n$,

$$\left\||A|^{2m} + |A^*|^{2m}\right\|_k \leq 4^m k\, \omega^{2m}(A) = \left\||A_0|^{2m} + |A_0^*|^{2m}\right\|_k \ \omega^{2m}(A). \tag{33}$$

Similarly, if n is odd, then for all $1 \leq k \leq n-1$,

$$\left\||A|^{2m} + |A^*|^{2m}\right\|_k \leq 4^m k\, \omega^{2m}(A) = \left\||A_0|^{2m} + |A_0^*|^{2m}\right\|_k \ \omega^{2m}(A). \tag{34}$$

By inequalities (30), (33) and (34), we deduce that for any $n \in \mathbb{N}$, and $1 \leq k \leq n$,

$$\left\||A|^{2m} + |A^*|^{2m}\right\|_k \leq \left\||A_0|^{2m} + |A_0^*|^{2m}\right\|_k \ \omega^{2m}(A).$$

Now, Ky Fan dominance principle completes the proof. □

Remark 2.10. *(i) When $m = 1$ and $\|\cdot\|_u = \|\cdot\|$, inequality* (31) *reduces to*

$$\|AA^* + A^*A\| \leq \|A_0A_0^* + A_0^*A_0\|\omega(A) = 4\omega(A),$$

which coincides with the first inequality in (2). Consequently, inequality (31) *may be regarded as an extension of the first inequality in (2) to the broader class of unitarily invariant norms.*

(ii) By the power inequality for the numerical radius, namely $\omega(A^2) \leq \omega^2(A)$, and applying the first inequality in (2), we obtain

$$\|A^2\| \leq 2\omega^2(A). \tag{35}$$

Note that $A_0^2 = O$. Thus, in contrast to Proposition 2.5 and inequalities (21), (30), *and* (31), *the bound in this inequality* (35) *is not attained at A_0.*

Next, we demonstrate how the techniques employed in the proof of Theorem 2.8 can be further used to obtain generalizations of inequalities (5) and (6).

Note that

$$\left\|\begin{bmatrix} B & B^* \end{bmatrix}\right\| \leq \sqrt{2}\|B\|.$$

So, we can deduce from inequality (25) that

$$\|AB + A^*B^*\| \leq 2\sqrt{2}\,\omega(A)\,\|B\|.$$

In case A and B commute, this inequality reduces to

$$\|\operatorname{Re}(AB)\| \leq \sqrt{2}\,\omega(A)\,\|B\|. \tag{36}$$

Moreover, by substituting B with $e^{\iota\theta}B$ in inequality (36), we obtain

$$\omega(AB) = \sup_{\theta\in\mathbb{R}} \|\operatorname{Re}(e^{\iota\theta}AB)\| \leq \sqrt{2}\,\omega(A)\,\|B\|.$$

Halbrook and Fong [7] derived this inequality as a special case of a more general result:

$$\omega(AB + BA) \leq 2\sqrt{2}\,\omega(A)\,\|B\|. \tag{37}$$

The proof of above inequality (37) uses the fact that for any $A \in \mathcal{M}_{n\times n}(\mathbb{C})$ with $\omega(A) \leq 1$, one has

$$\|A^*x\|^2 + \|Ax\|^2 \leq 4\|x\|^2,$$

which is a result appearing in the work of T. Ando [1].

Suppose $\alpha, \beta \in \mathbb{R}$. If $\omega(A) \leq 1$, then for any unit vector $x \in \mathbb{C}^n$, by Cauchy-Schwarz inequality, we have

$$\alpha^2\|A^*x\|^2 + \beta^2\|Ax\|^2 = \langle(\alpha^2AA^* + \beta^2A^*A)x, x\rangle \leq \|\alpha^2AA^* + \beta^2A^*A\|. \tag{38}$$

Using a slight generalization of inequality (25), we further have

$$\|\alpha^2AA^* + \beta^2A^*A\| \leq \left\|\begin{bmatrix} \alpha A & \beta A^* \end{bmatrix}\right\| \cdot \left\|\begin{bmatrix} \alpha A^* \\ \beta A \end{bmatrix}\right\| \leq 4\, max\{\alpha^2, \beta^2\}.$$

Combining this inequality with (38), we obtain

$$\alpha^2\|A^*x\|^2 + \beta^2\|Ax\|^2 \leq 4\, max\{\alpha^2, \beta^2\}.$$

Consequently, for any numerical contraction A and unit vector $x \in \mathbb{C}^n$, we have

$$\alpha\|A^*x\| + \beta\|Ax\| \leq 2\sqrt{2}\, \max\{|\alpha|, |\beta|\}. \tag{39}$$

Thus, for any contraction matrix $B \in \mathcal{M}_{n\times n}(\mathbb{C})$, we have

$$\begin{aligned} |\langle \alpha AB + \beta BAx, x\rangle| &\leq |\alpha|\|Bx\|\|A^*x\| + |\beta|\|Ax\|\|B^*x\| \\ &\leq |\alpha|\|A^*x\| + |\beta|\|Ax\| \\ &\leq 2\sqrt{2}\, \max\{|\alpha|, |\beta|\}\|x\|. \end{aligned}$$

Hence

$$\omega(\alpha AB + \beta BA) \leq 2\sqrt{2}\, \max\{|\alpha|, |\beta|\}\omega(A)\|B\|,$$

which is a slight generalization of inequality (6).

Next, we provide corresponding generalization of inequality (5).

Corollary 2.11. *Suppose $A, B \in \mathcal{M}_{n\times n}(C)$ and $\alpha, \beta \in \mathbb{R}$. Then, following inequality holds*

$$\omega(\alpha AB + \beta B^* A) \leq 2 \ \max\{|\alpha|, |\beta|\}\omega(A)\|B\|,$$

Proof. Suppose $A = 2cos(C)Wsin(C)$, for some unitary matrix W and Hermitian matrix C. Then, for unit vector $x \in \mathbb{C}^n$, as an application of Cauchy Schwarz inequality, we have

$$\begin{aligned} |\langle(\alpha AB + \beta B^* A)x, x\rangle| &\leq |\langle \alpha ABx, x\rangle| + |\langle \beta B^* Ax, x\rangle| \\ &\leq 2|\alpha|\|sin(C)Bx\| \cdot \|W^* cos(C)x\| + 2|\beta|\|Wsin(C)x\| \cdot \|cos(C)Bx\| \\ &\leq \|B\|\langle(4|\alpha|^2 cos(C)WW^* cos(C) + 4|\beta|^2 sin(C)W^* Wsin(C))x, x\rangle^{1/2} \\ &\leq 2\|B\| \cdot \||\alpha|^2 cos^2(C) + |\beta|^2 sin^2(C)\|^{1/2}, \end{aligned} \tag{40}$$

and since matrix $|\alpha|^2 cos^2(C) + |\beta|^2 sin^2(C)$ can be factorized as

$$\begin{bmatrix} cos(C) & sin(C) \end{bmatrix} \begin{bmatrix} |\alpha|^2 & O \\ O & |\beta|^2 \end{bmatrix} \begin{bmatrix} cos(C) \\ sin(C) \end{bmatrix},$$

we have,

$$\||\alpha|^2 cos(C)WW^* cos(C) + |\beta|^2 sin(C)WW^* sin(C)\| \leq \left\| \begin{bmatrix} |\alpha|^2 & O \\ O & |\beta|^2 \end{bmatrix} \right\| \leq max\{|\alpha|^2, |\beta|^2\}. \tag{41}$$

Hence, inequalities (40) and (41) complete the proof. □

Next, we demonstrate that the techniques used in the proof of Theorem 2.8 can be effectively applied to obtain some new numerical radius inequalities.

Suppose $A, B \in \mathcal{M}_{n\times n}(\mathbb{C})$. Then, using inequality (25), we have

$$\|A^* B + AB^*\| \leq \| \begin{bmatrix} A^* & A \end{bmatrix} \| \cdot \left\| \begin{bmatrix} B \\ B^* \end{bmatrix} \right\| \leq 4 \ \omega(A) \cdot \omega(B). \tag{42}$$

Inequality (42) generalizes first inequality in (2) in the sense that it reduces to it in case $A = B$ and hence 4 is best possible constant in this case.

Now substituting, $A = A^l$ and $B = A^m$ in inequality (42), and using power inequality for numerical radius, we deduce the following inequality

$$\|A^{*l} A^m + A^l A^{*m}\| \leq 4 \ \omega^{l+m}(A).$$

Next, we show that a variant of this inequality also hold.

Corollary 2.12. *Suppose $A \in \mathcal{M}_{n\times n}(C)$. Then, the following inequality hold,*

$$\|A^{*l} A^m + A^m A^{*l}\| \leq 4 \ \omega^{l+m}(A). \tag{43}$$

Proof. Using similar ideas as used in Theorem 2.8 and Corollary 2.11, it can be proved that if $\omega(A) \leq 1$, then for $l, m \in \mathbb{N}$,

$$\| \begin{bmatrix} A^{*l} & A^m \end{bmatrix} \| \leq 2. \tag{44}$$

So, the required inequality follows easily. □

Note that for $l = m = 1$ inequality (43), reduces to the first inequality in (2). Thus, it presents another new generalization of the first inequality in (2).

The next result presents variants of the following inequality from [7],

$$\omega(A^* BA) \leq \|A\|^2 \omega(B).$$

Corollary 2.13. *Suppose $A \in \mathcal{M}_{n\times n}(C)$ and $n \geq 3$. Then, for all $l, m \in \mathbb{N}$ the following inequalities hold,*

$$\omega(A^l BA^m) \leq 2\, \omega^{m+l}(A)\|B\|,$$

and

$$\omega(A^l BA^{*m}) \leq 4\, \omega^{m+l}(A)\|B\|.$$

Proof. We prove only the first inequality as the second inequality can be proved similarly. For $\theta \in \mathbb{R}$, the matrix $Re(e^{\iota\theta}A^l BA^m)$ can be factorized as

$$\frac{1}{2}\begin{bmatrix} A^l & A^{*m} \end{bmatrix}\begin{bmatrix} e^{\iota\theta}B & O \\ O & e^{-\iota\theta}B^* \end{bmatrix}\begin{bmatrix} A^m \\ A^{*l} \end{bmatrix}.$$

Now the proof follows by using inequality (44). □

Our next objective is to establish an inequality analogous to inequality (20), for the case when the conventional matrix product is replaced by the *Hadamard product* of matrices. Recall that the Hadamard product of matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ in $\mathcal{M}_{n\times m}(\mathbb{C})$ is a matrix $C = [c_{ij}]$ in $\mathcal{M}_{n\times m}(\mathbb{C})$ such that $c_{ij} = a_{ij}b_{ij}$. It represents the element-wise product of matrices. To achieve our goal, we need to recall the following result from [2].

Lemma 2.14. *Every matrix $A \in \mathcal{M}_{n\times n}(C)$ with singular values $\sigma_1(A) \geq \sigma_1(A) \geq \cdots \geq \sigma_n(A)$ can be written as sum*

$$A = \sum_{i=1}^{n} \alpha_i K_i,$$

where K_i is partial isometry of rank i and $\alpha_i \geq 0$ are non negative real numbers such that

$$\sigma_j(A) = \sum_{i=j}^{n} \alpha_i.$$

Using Lemma 2.2, we derive the following result.

Lemma 2.15. *For any matrix $A \in \mathcal{M}_{n\times n}(\mathbb{C})$*

$$max\{|tr[(A \circ X)Y]| : \|X\| \leq 1, \|Y\| \leq 1, \rho(X) = k, \rho(Y) = l\} = \|A\|_{min\{k,l\}}.$$

Proof. Suppose Y is a contraction matrix having rank l. Then, by Lemma 2.2, and inequality (1) in [5], we have

$$|tr[(A \circ X)Y]| \leq \|A \circ X\|_l \leq \sum_{i=1}^{l} \sigma_i(A)\sigma_i(X) \leq \sum_{i=1}^{\min\{l,k\}} \sigma_i(A) = \|A\|_{\min\{k,l\}}.$$

□

Finally, we are ready to present our main result.

Theorem 2.16. *Suppose $A \in \mathcal{M}_{n\times n}(\mathbb{C})$ be an arbitrary matrix and $\|.\|_u$ be any unitarily invariant norm. Then*

$$max\{\|A \circ B\|_u : \omega(B) \leq 1\} \leq \|\mathrm{Diag}(\sigma_1(A_0)\sigma_1(B), \sigma_2(A_0)\sigma_2(B), \ldots, \sigma_n(A_0)\sigma_n(B))\|_u.$$

Proof. The initial steps of this proof are inspired from the proof of Theorem 2.8 in [2]. Let $A \in \mathcal{M}_{n\times n}(\mathbb{C})$ and $1 \leq k \leq n$ be given. Let $B \in \mathcal{M}_{n\times n}(\mathbb{C})$ be any matrix with $\omega(B) \leq 1$. Then, by Lemma 2.2, there exists a partial isometry C_k of rank k such that $\|A \circ B\|_k = tr[(A \circ B)C_k]$. Also, let

$$A = \sum_{i=1}^{n} \alpha_i K_i, \quad \text{where } \sigma_j(A) = \sum_{i=j}^{n} \alpha_i,$$

and $\rho(K_i) = i$, be the decomposition of A given by Lemma 2.14. Then

$$\begin{aligned}
\|A \circ B\|_k &= tr[(A \circ B)C_k] \\
&= \sum_{i=1}^{n} \alpha_i tr[(B \circ K_i)C_k] \\
&\leq \sum_{i=1}^{n} \alpha_i |tr[(B \circ K_i)C_k]| \\
&\leq \sum_{i=1}^{n} \alpha_i \max\{|tr[(B \circ K_i)C_k]| : \|K_i\| \leq 1, \|C_k\| \leq 1, \rho(C_k) \leq k, \rho(K_i) \leq i\} \\
&= \sum_{i=1}^{n} \alpha_i \|B\|_{\min\{i,k\}}. \qquad (45)
\end{aligned}$$

Note that the equation (45) follows from Lemma 2.15. Therefore, by Proposition 2.5 and Lemma 2.15, we obtain

$$\begin{aligned}
\max\{\|A \circ B\|_k : \omega(B) \leq 1\} &\leq \sum_{i=1}^{n} \alpha_i \max_{w(B)\leq 1} \|B\|_{\min\{i,k\}}. \\
&= \begin{cases} \sum_{i=1}^{k} 2i\alpha_i + \sum_{i=k+1}^{n} 2k\alpha_i & : k \leq \frac{n}{2} \\ \sum_{i=1}^{m} 2i\alpha_i + \sum_{i=m+1}^{2m} 2m\alpha_i & : k > \frac{n}{2},\ n = 2m \\ \sum_{i=1}^{m} 2i\alpha_i + \sum_{i=m+1}^{2m+1} (2m+1)\alpha_i & : k > \frac{n}{2},\ n = 2m+1 \end{cases} \\
&= \begin{cases} 2\sum_{i=1}^{m} \sigma_i(A) & : \ n = 2m \\ \sigma_{m+1}(A) + 2\sum_{i=1}^{m} \sigma_i(A) & : \ n = 2m+1 \end{cases} \\
&= \begin{cases} \|\mathrm{diag}(2\sigma_1(A), 2\sigma_2(A), \ldots, 2\sigma_m(A), \underbrace{0, 0, \ldots, 0}_{m})\|_k & : \ n = 2m \\ \|\mathrm{diag}(2\sigma_1(A), 2\sigma_2(A), \ldots, 2\sigma_m(A), \sigma_{m+1}(A), \underbrace{0, 0, \ldots, 0}_{m})\|_k & : \ n = 2m+1 \end{cases} \\
&= \|\mathrm{Diag}(\sigma_1(A_0)\sigma_1(B), \sigma_2(A_0)\sigma_2(B), \ldots, \sigma_n(A_0)\sigma_n(B))\|_k.
\end{aligned}$$

Hence the proof follows using Ky Fan Dominance Theorem. □

In particular, Theorem 2.16 yields the following inequality for the operator norm.

Corollary 2.17. *For arbitrary matrix $A \in \mathcal{M}_{n\times n}(\mathbb{C})$, the following inequality holds*

$$\|A \circ B\| \leq 2\omega(A)\|B\|.$$

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