



Zero quantum discord for Markovian bipartite systems

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Abstract. Quantum discord measures the total non-classical correlations in a composite quantum system. As a possible resource for the classically non-achievable tasks, amount of quantum discord is crucial for efficient performance of the quantum technology tasks. Zero discord means the absence of the required resources. For a wide class of the bipartite open quantum systems, initial nonzero discord is known practically never goes to zero, except asymptotically. However, if initial discord equals zero, there remains the possibility to remain zero for the finite time intervals for certain Markovian models of open systems—that we call Markovian classicality. In this paper we search for the models that allow for Markovian classicality. We point out a general model that represents a matter-of-principle formal proof, i.e. a sufficient condition for the, otherwise not obvious, existence of Markovian classicality. Physical relevance of the model is twofold. First, the model is in intimate relation to the topics of quantum information locality, quantum discord saturation and quantum decorrelation. Second, the model is of the general physical interest. It pertains to a specific structure (decomposition into parts/subsystems) of a composite system, not to a special kind of composite system. Being a characteristic of a structure, by definition, the model of Markovian classicality is not a model of sudden death of discord. We emphasize wide-range implications of our results.

1. Introduction

“Quantum discord” is a common term for different measures of non-classical correlations in composite (e.g. bipartite) quantum systems [1–6]. Historically the first and probably the best known is the so-called “one-way” discord [1, 2]. The closely related “two-way” discord is even a more stringent criterion for classicality of the correlations.

Current interest in quantum discord (QD) stems from the basic physics as well as from the fact that QD is recognized as a resource for performing certain tasks that cannot be efficiently performed on the classical setup [7–22] (we provide a small sample of some recent literature). That is, the nascent field of quantum technology directly employs the quantum (non-classical) correlations as the classically unknown characteristics of the composite (e.g. bipartite) quantum systems. To this end, existence of the quantum

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correlations is critical. That is, the temporary absence or dynamical reduction of the quantum correlations makes the system essentially classical and the basic resource for performing the tasks is lost.

It is known that “for almost all states of positive discord, the interaction with any (non-necessarily local) Markovian bath can never lead either to a sudden, permanent vanishing of discord, nor to one lasting a finite time-interval” [23]. In effect, not only sudden death of discord cannot be expected, but Markovian dynamics may only lead to *asymptotically close* a zero-discord state. From this result one may possibly expect the Markovian bipartite systems are practically deprived of zero discord states and the quantum resources are present all the time.

However, the analysis in [23] does not rule out that there can be zero discord *for all times*. That is, it is not excluded the possibility that, if a system starts with zero discord, it could have zero discord for all times. Thus, complementary to Ferraro et al [23], Markovian dynamics may probably provide *non-asymptotic* zero-discord for a bipartite system in a *long* time interval, that we call “*Markovian classicality*” due to the absence of the quantum correlations in a bipartite system; for an up-to-date analysis of quantum Markovianity see [24].

In this paper, our task is twofold. *First*, we are interested in answering the following questions: Which kind, if any, of the zero-discord states can provide Markovian classicality (MC)? Given an answer (or a guess) to the first question: is there a physical model that can justify such zero-discord states dynamics? What are the physical characteristics of such model(s)? *Second*, we are interested in linking such model(s) to the *realistic* physical systems and situations. While the first task bases itself on the existing knowledge about the discord dynamics, the second one welcomes a change in perspective to the composite quantum systems.

Regarding the first task: As two-way discord tends to be larger than one-way discord [25], we consider the zero *two-way-discord*, which applies only to the so-called classical-classical (the CC) quantum states [5]. Accordingly, Markovian classicality is defined by zero-discord as a constant of motion, that is by the open system’s dynamics as a dynamical map from one to another CC state (cf. Definition 2.1, Section 2). Then we *construct*, not deduce, such a model; the model satisfies both the C- and the P-criterion for classicality [26]. Our approach is a formal mathematical analysis that leads us to the simplest possible model of tensor-product state for the open system. Interestingly enough, *we were not able to find any alternative to the model*. The model reveals a number of physically interesting observations such as relations to the quantum information locality [27–29], quantum discord saturation [30] and quantum decorrelation [31, 32] topics (Section 3.1).

In regard of the second task, we emphasize importance of “structure” (decomposition into parts/subsystems) of a composite system [33, 34]. We emphasize (Section 3.2) the model of Markovian classicality is a matter of a special structure of a composite system. The composite systems not describable by such structure may be deprived of zero discord states. So, being a matter of structure, the model is of the general physical relevance. The following example will turn out to be paradigmatic for MC. Consider a three-qubit system, $C = 1 + 2 + 3$, and its bipartite structures, $1 + S_1$ and $S_2 + 3$, where the bipartite systems $S_1 = 2 + 3$ and $S_2 = 1 + 2$. As it is well known from quantum teleportation [35], the C ’s state $|\phi\rangle_1|\Phi^+\rangle_{S_1}$, where $|\Phi^+\rangle_{S_1} = (|0\rangle_2|0\rangle_3 + |1\rangle_2|1\rangle_3)/2^{-1/2}$, can be re-written as $\sum_i |\chi_i\rangle_{S_2}|i\rangle_3/2$, where the S_2 ’s states represent the Bell states [36] for the pair $1 + 2$. The point is that for the $1 + S_1$ structure, the state is tensor-product and therefore not carrying any correlations between the 1 and S_1 systems, while there is entanglement in the same state in regard of the $S_2 + 3$ structure. So, for the closed C system, the structure $1 + S_1$ bears Markovian classicality, which is not the case for the $S_2 + 3$ structure. Practically, *operating with the local observables* of the 1 and/or S_1 subsystems cannot give any “quantum advantage” via the use of the quantum correlations. Being a characteristic of a structure, the Markovian classicality model is not a model of sudden death of discord.

This paper is organized as follows. In Section 2, we give a precise formulation of the task and design the model supporting Markovian classicality. Information theoretic analysis of the model in Section 3 gives rise to a need to relax the definition of classicality. In Section 4, we introduce approximate classicality and recognize a model implementing such approximate classicality. Section 5 is discussion where we emphasize wide-range implications of our results. Section 6 is Conclusion.

2. The model

One-way quantum discord for the $S + S'$ system, $D^{\leftarrow}(S|S') = I(S : S') - J^{\leftarrow}(S|S') \geq 0$, and von Neumann entropy of a state ρ , $\mathcal{S} = -\text{tr} \rho \ln \rho$. Both the total mutual information, $I(S : S') = \mathcal{S}(S) + \mathcal{S}(S') - \mathcal{S}(S, S')$, and the classical correlations, $J^{\leftarrow}(S|S') = \mathcal{S}(S) - \inf_{\{\Pi_{S'i}\}} \sum_i |c_i|^2 \mathcal{S}(\rho_{S|\Pi_{S'i}})$ —where $\rho_{S|\Pi_{S'i}} = I_S \otimes \Pi_{S'i} \rho I_S \otimes \Pi_{S'i}$ is the state remaining after a selective quantum measurement defined by the orthogonal projectors $\Pi_{S'i}$ —are non-negative. The CC states are the only states fulfilling the condition $D^{\leftarrow}(S|S') = 0 = D^{\rightarrow}(S|S') \Leftrightarrow D^{\leftrightarrow}(S|S') = 0$.

Definition 2.1. An open quantum system, C , consisting of two subsystems, S and S' , is said to bear Markovian classicality if and only if it can be described by a classical-classical (CC) state in long time-interval. A CC state is of the form $\sum_{m,n} \omega_{mn} P_{Sm} \otimes \Pi_{S'n}$, where the real numbers $\omega_{mn} \geq 0$ and $\sum_{m,n} \omega_{mn} \text{tr}_S P_{Sm} \text{tr}_{S'} \Pi_{S'n} = 1$ for the orthogonal projectors P_{Sm} and $\Pi_{S'n}$ on the respective Hilbert spaces.

For separable $\omega_{mn} = p_m q_n$, $\forall m, n$, such that $\sum_m p_m \text{tr}_S P_{Sm} = 1 = \sum_n q_n \text{tr}_{S'} \Pi_{S'n}$, one obtains the tensor-product states, $\rho_S \otimes \rho_{S'}$, as a special kind of CC states. Physically, the composite system C may be e.g. a pair "object of measurement + apparatus" or "the internal + the center-of-mass" degrees of freedom of the Brownian particle [37].

As typical of open systems, we assume a coarse-grained time scale for the open system's dynamics [37]. On the other hand, the time scale characteristic for Markovian dynamics we are *exclusively interested in* is bounded also from the above [37]—zero discord is not required for arbitrary long time-interval either. This way understood classicality does both: permits non-classicality for the time intervals shorter than e.g. the "decoherence time", τ_D , for the open system, C , and still assumes the long time intervals for the possible thermal relaxation of the open system, as well as for the "recurrence time" regarding the closed system, $C + \mathcal{E}$, where $\mathcal{E}$ is the C 's environment [37].

Definition 2.1 directly sets the following constraint on constructing a Markovian classicality model: **Classicality Constraint:** Two-way quantum discord is exactly zero in every instant in time before eventual thermalization of the open system.

Getting into details, we detect the following obstacles to construct a model fulfilling the Classicality Constraint. First, initial non-zero discord in $S + S'$ system; Second, interaction between S and S' ; Third, the common environment, $\mathcal{E}$, for S and S' ; Fourth, non-completely positive dynamics for the S' system; Fifth, the initial non-tensor-product state for C and $\mathcal{E}$; Sixth, arbitrary initial zero-discord state for C .

The origin of these obstacles is respectively as follows: First, an initial non-zero discord state cannot fulfil the classicality condition. e.g. The dynamic transition

$$\sum_i \lambda_i \rho_{Si} \otimes \rho_{S'i} \rightarrow \sum_{m,n} \omega_{mn} |m\rangle_S \langle m| \otimes |n\rangle_{S'} \langle n| \quad (1)$$

is not allowed as long as the rhs of (1) refers to a continuous time interval. There are at least three ways for dynamically obtaining a non-zero-discord state: Interaction between S and S' , the common environment for S and S' , and non-completely positive dynamics for the open system S' . Markovian dynamics requires the tensor product initial state $\rho_C \otimes \rho_E$ [37]. Finally, in general, the external (e.g. experimentally uncontrollable) local influence can raise the initially zero discord [3, 38–40]. The local operations exerted on S and/or on S' , the rhs of (1), can give rise to non-zero-discord final state. The *only state immune* to this (yet for the completely positive dynamics) is actually the tensor-product state, $\rho_S \otimes \rho_{S'}$.

Bearing all this in mind, the *only* option we offer is the following model:

$$S + (S' + \mathcal{E}) \quad (2)$$

where the subsystem S does not interact with any other subsystem (S' and $\mathcal{E}$) while assuming Markovian and completely positive dynamics for the open system, S' , and the tensor-product initial state $\rho_S \otimes \rho_{S'} \otimes \rho_E$ for the total system, see Fig. 1. In principle, both S and S' can be composite systems themselves. Since the S system is isolated and S' Markovian, the total open system $C = S + S'$ is also Markovian.

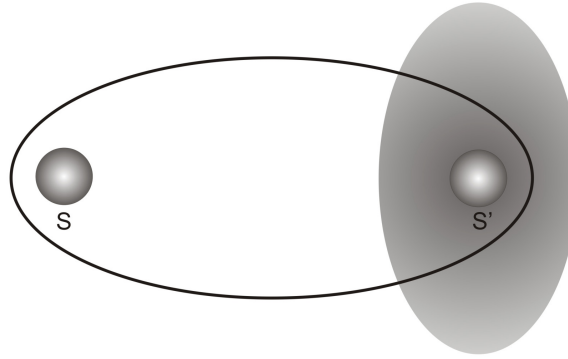


Figure 1: Schematic illustration of the model (2). The composite system $C = S + S'$ is distinguished by the elliptic line. The gray area designates the environment $\mathcal{E}$ in interaction with S' . The S system does not interact with both S' and $\mathcal{E}$. Physically, the S and S' systems can represent respectively e.g. the “relative (internal)”- and the center-of-mass-degrees of freedom of the Brownian particle C . The pair $S + S'$ is described by (5) and by the zero two-way discord, $D^{\leftarrow}(S|S') = 0 = D^{\rightarrow}(S'|S)$, in every instant in time.

Formally, the model (2) is defined by the Hilbert state space for the total system $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_{S'} \otimes \mathcal{H}_E$ and by the Hamiltonian of the total system:

$$H = H_S + H_{S'} + H_E + H_{S'E} \quad (3)$$

where the last term on the rhs of (3) represents interaction between S' and $\mathcal{E}$. Then the unitary operator for the total system separates as:

$$U(t) = U_S(t) \otimes U_{S'+E}(t) = \exp\{-itH_S/\hbar\} \otimes \exp[-it(H_{S'} + H_E + H_{S'E})/\hbar], \quad (4)$$

and provides unitary (the Schrödinger) dynamics for both the S system as well as for the $S' + \mathcal{E}$ system. Markovian and completely positive dynamics of S' does not introduce any additional correlation for S and S' . Then for the model Eq. (2), one can write for the open system's state:

$$\rho_S(t) \otimes \rho_{S'}(t) \quad (5)$$

in every instant in time, where $\rho_S(t) = U_S(t)\rho_S(0)U_S^\dagger(t)$ and $\rho_{S'}(t)$ is a solution to a Markovian-type master equation. The proof of (5) obviously follows from (4).

From Eq. (5) it easily follows: $\mathcal{S}(S, S') = \mathcal{S}(S) + \mathcal{S}(S')$ and therefore the equalities $D^{\leftarrow}(S|S') = 0 = D^{\rightarrow}(S|S')$ in every instant in time. So, we can say *we have designed a model that fulfills the very tight conditions for non-asymptotic zero-discord classicality of a Markovian bipartite system*, Definition 2.1: (i) the model Eq. (2)-(5) is distinguished, and (ii) the open system's dynamics is a completely positive map.

3. Analysis of the model

The model Eqs. (2)-(5) is *designed* so as to fulfill the Classicality Constraint, Section 2. For the tensor-product initial state, $\rho_S \otimes \rho_{S'}$, the subsystems S and S' remain mutually exactly uncorrelated in every instant in time, (5). In the terms of [23]: the composite system's state remains in the Ω_0 set of zero-discord states, all the time. As $[\rho_S \otimes I_{S'}, \rho_{S+S'}] = 0 = [I_S \otimes \rho_{S'}, \rho_{S+S'}], \forall t$, the state $\rho_{S+S'}$ (5) is a “doubly” lazy state [41]. Thus, we point out a ‘niche’ for the bipartite system's Markovian classicality. Some proofs of the presented statements are direct corollaries of the material provided in the Appendices section.

3.1. Quantum Information Locality and Classicality

If one assumes the pure initial states for both S' and $\mathcal{E}$, then Eqs. (2)-(4) directly give for the total system's instantaneous state:

$$\rho_S \otimes |\Psi\rangle_{S'+E}\langle\Psi| \quad (6)$$

and *vice versa*—given the above assumptions, Eq. (6) implies Eqs. (2)–(4). In (6), the S' and $\mathcal{E}$ systems are in entangled pure state; for $\rho_S^2 = \rho_S$, the ρ_S state is also pure. The entanglement is due to the interaction $H_{S'E}$, (3), i.e. due to the fact that the environment effectively monitors and purifies the S' system.

As we show next, the state (6) is in intimate relation to quantum information localization measured by “locally inaccessible information (LII)” flow [27], as well as with quantum discord saturation [30] and quantum decorrelation [4, 32].

Lemma 3.1. *The following are mutually equivalent statements: (i) the system $S + S' + \mathcal{E}$ is in the state (6), (ii) quantum discord $D^\leftarrow(S + S'|E) = \mathcal{S}(E)$ is saturated (maximal), (iii) there is total decorrelation of the S system from the system S' and (iv) there is quantum information localization in the $S' + \mathcal{E}$ system.*

We prove this lemma in a way supporting some intuition about the zero-discord classicality. The more formal and more simple proofs will be provided elsewhere.

Proof. Bearing in mind (i) is equivalent to (ii) (cf. Theorem 1 in [32]), the proof can be given by proving (i) is equivalent to (iii) and to (iv). That (i) implies (iv) is easy obtained. The “locally inaccessible information” flow [27], $\mathcal{L}^\leftarrow = D^\leftarrow(S'|S) + D^\leftarrow(E|S') + D^\leftarrow(S|E) = D^\leftarrow(E|S')$; there is only information flow in $S' + \mathcal{E}$ system. Now we prove the inverse to this implication. Due to non-negativity of discord, the above equality for $\mathcal{L}^\leftarrow$ directly implies $D^\leftarrow(S|S') = 0 = D^\leftarrow(S|E)$. As we know $D^\leftarrow(S'|E) \neq 0$, the condition $D^\leftarrow(S|S') = 0 = D^\leftarrow(S|E)$ can be satisfied only by the state (6); e.g., the alternative tripartite state, $\sum_i c_i |i\rangle_S |i\rangle_{S'} |i\rangle_E$, that satisfies $D^\leftarrow(S|S') = 0 = D^\leftarrow(S|E)$, does not satisfy $D^\leftarrow(S'|E) \neq 0$. Here (without loss of generality) we assume the total system $S + S' + E$ is subject to the Schrödinger law, cf. (4), and that the initial states of both S' and $\mathcal{E}$ are pure—thus the alternative mixed states are of no interest here. Finally, we prove equivalence of (i) and (iii). The decorrelation is defined [4, 32] as a difference of the two total correlations in the initial and the final state, $I_{\text{initial}}(S : S') - I_{\text{final}}(S : S')$. For every initial state, decorrelation is maximal if $I_{\text{final}}(S : S') = 0$. So, we prove that $I_{\text{final}}(S : S') = 0$ is equivalent to (6). From (6) it directly follows: $I(S : S') = \mathcal{S}(S) + \mathcal{S}(S') - \mathcal{S}(S, S') = 0$. The inverse is easily proved, as from $I(S : S') = 0$ follows $\mathcal{S}(S, S') = \mathcal{S}(S) + \mathcal{S}(S')$, which, in turn, is fulfilled only for the product states, (5). By purifying the product state, (5), one obtains the state (6). $\square$

The proof of Lemma 3.1 distinguishes the physical relevance of the model (2). Saturation of quantum discord (in $S' + \mathcal{E}$) is equivalent to locking information locally (in $S' + \mathcal{E}$), i.e. to decorrelation of the rest (S) of the composite system. So, Markovian classicality of $S + S'$ coincides with quantumness of $S' + \mathcal{E}$. Of course, external influence on $S' + \mathcal{E}$ leads to the loss of maximum discord. Bearing in mind the result of Ferraro et al [23], cf. Introduction, Lemma 3.1 suggests the locking of information [27], discord saturation [30] and quantum decorrelation [4, 32] are dynamically feasible only asymptotically.

The model Eqs. (2)–(5) is in accordance with the following logic of the decoherence theory [37, 42]: only certain degrees of freedom (S') of a composite system are subject to decoherence. The remaining degrees of freedom (the S system) can exhibit quantum mechanical behavior.

On the other hand, the total system, $S + S' + \mathcal{E}$, is not allowed to correlate with any outer system, denoted by W . This is a direct consequence of the discord saturation, $D^\leftarrow(S + S'|E) = \mathcal{S}(E)$, the point (ii) of Lemma 3.1. The discord saturation implies non-correlation of $\mathcal{E}$ with W [30]. Furthermore, both the S system and the S' system are uncorrelated with the outer W system, in every instant in time. This conclusion follows from the very construction of the model (2). Namely, the S system is closed, while interaction of the S' system with the W system would correlate $\mathcal{E}$ and W , in contradiction with the saturation of $D^\leftarrow(C|E)$. Of course, isolation of $S + S' + \mathcal{E}$ from the rest of the world, W , is physically crude and naive.

3.2. Quantum Structures

The following objection is in order: for the realistic particles that mutually interact, one can hardly expect isolation as presented by the S system. To answer, we need a switch in perspective to describing the composite systems.

Definition 3.2. *A set of subsystems of a composite system, C , is called a structure of C . Different structures are mutually related by the proper canonical transformations (CTs), which provide the different tensor-product forms for the system’s Hilbert space.*

The CTs induce a change in both the composite system's Hilbert-space tensor-product form as well as in the system's Hamiltonian form [33, 34]. Regarding Figs. 1 and 2, for the Hilbert state space of the composite system, $\mathcal{H}$, one can write: $\mathcal{H}_1 \otimes \mathcal{H}_2 = \mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_{S'}$. The Hamiltonian, H , takes the different forms for the different structures, $H_1 + H_2 + H_{12} = H = H_S + H_{S'}$; H_{12} is interaction term, while the analogous term is absent for $S + S'$ structure—compare to the three-qubit example for teleportation in Introduction.

While the composite system's Hilbert state-space, the Hamiltonian and quantum state is unique (in every instant in time), the correlations (for isolated or open system) are *not*. This *correlations relativity* formally means [43] (which has recently been rediscovered [18]): correlations (quantum or classical, for isolated or open system) are in general not invariants of the CTs. In other words: the amount of correlations in *instantaneous* state of C is *not* a matter of a composite system itself, *but* a matter of the composite system's structure. What can be told about the system depends on the choice of the measurements and other actions performed on the system.

Grouping subsystems (the "coarse graining" the composite system's structure) is formally a trivial kind of CTs. Entanglement swapping (see Introduction) is typical of this kind of CTs that the initially tensor-product form of a state transforms into entangled form of the state, for the same instant in time. Regarding the model (2), for instantaneous state (6), the two bipartite structures, $(S + S') + \mathcal{E}$ and $S + (S' + \mathcal{E})$, also bear the different discords, $D^-(S + S'|E) = S(E) \neq 0$ and $D^-(S|S' + E) = 0$, respectively.

Quantum correlations relativity is also implicit e.g. in the "entanglement renormalization" methods for the finite-dimensional many-body systems [44] (and the references therein). A specific decoupling (variables separation) procedure provides a bipartite structure for the system of interacting spins. The original 'microscopic' degrees of freedom are transformed to introduce a pair of noninteracting systems. Then the ground energy (pure) state, that carries entanglement for the 'microscopic' structure, obtains the tensor-product form. From the operational perspective, dealing with the observables of the S and S' systems cannot, in principle, utilize quantum correlations. However, dealing with observables of some alternative structure of the composite system C ($C = S + S'$), typically allows for the operational use of certain quantum correlations [33, 34].

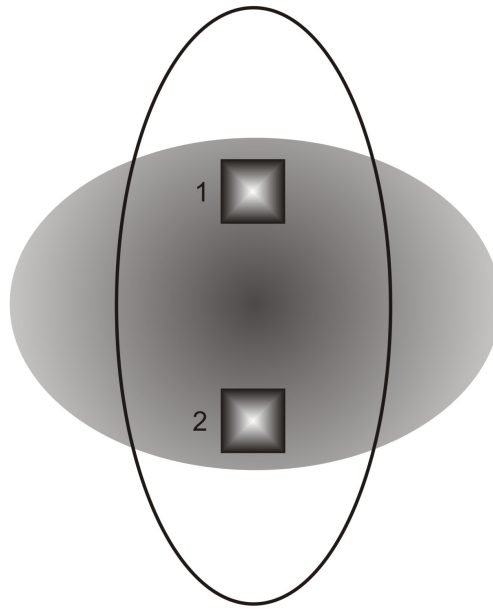


Figure 2: Schematic illustration of the $1 + 2$ structure of the composite system, C , is distinguished by the elliptic line. The gray area designates the environment $\mathcal{E}$ that is not in interaction with all the C 's degrees of freedom. The realistic particles, $1 + 2$, degrees of freedom are linked with the degrees of freedom of $S + S'$ (cf. Fig. 1) via the proper canonical transformations; $1 + 2 = C = S + S'$. As distinct from the $S + S'$ structure, the $1 + 2$ structure may be expected of non-zero discord.

For the continuous variable (CV) systems, the variables separation procedure is an open issue in intimate relation to the issue of (quantum) *integrability*. To this end, paradigmatic are the composite system's center-of mass (CM) and the "relative (internal)" degrees of freedom, cf. Figs. 1 and 2. Regarding the CV Gaussian states, the important results in [45] strongly support the model (2). As the only zero-discord Gaussian states are tensor product [45], the only Gaussian states dynamically supporting Markovian classicality—are of the form (5). Of course, for non-Gaussian states the things may generally look different.

So the physical relevance of the model $S + S'$, (2), follows also from its *universal* applicability—just transform the "original" structure 1 + 2 into a structure formally presented by (2). The systems S and S' , Fig. 1, can represent respectively the "relative (internal)" (R) and the center-of-mass (CM) degrees of freedom for the pair 1 + 2, Fig. 2, or the original spin-chain and a pair of noninteracting blocks [44]. The composite systems not describable by the structure (2) are practically deprived [23] of the zero discord states.

3.3. Summary

In support of the model, we distinguish: a. the model is in accordance with the general logic that only a subset of the open-system's degrees of freedom (S') is subject to decoherence; b. $S + S'$ resembles the classical-mechanics model-structure in the general use, $CM + R$, Section 3.2; c. regarding the Gaussian states, the results in [45] strongly support the model.

On the other hand, the model can be considered *too crude and idealized*, as: d. exact separation of the S system from the rest in (2) does not seem very realistic; e. in disagreement with the general logic of the open system and decoherence theory, the model does not allow approximate isolation of (i.e. the information flow from and to) the total system $S + S' + \mathcal{E}$.

In conclusion of this section we define a new task that is a subject of the next section: to search for a variation, i.e. approximation, of the model in order to avoid the objections 'd' and 'e', while saving its virtues, the above points 'a-c'.

4. Approximate Markovian Classicality

Definition 4.1. An open quantum system, C , consisting of two subsystems, S and S' , is said to bear approximate Markovian classicality if and only if it can be described by a approximate classical-classical (CC) state in a sufficiently long time interval.

"Approximate CC state" is a state that can be approximated by an CC state, Definition 2.1. "Sufficiently long time" emphasizes the time interval for validity of the approximate Markovian classicality (AMC) is long compared to the time intervals characteristic for certain physical processes of interest, but shorter than the open system's relaxation time, if it is defined for the model. Now we formulate:

Approximate Classicality Constraint: Two-way quantum discord is approximately zero in a sufficiently long time interval before eventual thermalization of the open system.

Prima facie, one could expect that nonzero discord will dynamically quickly become non-negligible [23, 38–40]. On the other hand, having in mind the obstacles emphasized in Section 2, it is not obvious where, and which kind of approximations can be made in order to provide AMC. Nevertheless, below we emphasize that a model of the quantum information locality [30] implements also the approximate Markovian classicality.

The dynamic model in [30] is formally a variant of the model Eq. (2): In a tripartite system, $S, S', \mathcal{E}$, the interaction between S' and $\mathcal{E}$ dominates the composite system's dynamics. The S system interacts with S' but not with the environment $\mathcal{E}$. For a special initial state, $|p\rangle_{S'}$, the total system's state can be presented (see Appendix Appendix D for some details) in the following simplified form [30, 46]:

$$|\Phi^p\rangle_{SS'E} = |\psi_p(t)\rangle_S \otimes |p\rangle_{S'} \otimes |\phi_p(t)\rangle_E + |O(\epsilon, p; t)\rangle_{S+S'+E}. \quad (7)$$

In (7): $\epsilon \equiv c/C \ll 1$, where c is the strength of interaction between S and S' , and C is the interaction strength for S' and $\mathcal{E}$. The first term in (7) is totally-tensor-product state in the time interval $\tau \sim \epsilon^{-1}$. In the limit $\epsilon \rightarrow 0$, Eq. (7) becomes a variant of (6).

Physically, the state (7) provides approximate separation of all subsystems and therefore very small discord for the open system $C = S + S'$ in the time interval $\tau \sim C/c$. This is a direct consequence of the following lemma.

Lemma 4.2. *Von Neumann entropy of every subsystem in (7) is proportional to ϵ , in the time interval $\tau \sim C/c$.*

Proof. The tripartite system can be decomposed as a bipartite system by grouping, e.g. $(S + S') + \mathcal{E}$. Then the (normalized) state (7) takes the form $\sqrt{1-\epsilon}|\phi\rangle_{SS'}|\chi\rangle_E + \sqrt{\epsilon}\sum_i \sqrt{p_i}|\phi\rangle_{SS'}|i\rangle_E$, $\sum_i p_i = 1, \forall t$. For $\rho_{SS'} = \text{tr}_E|\Phi\rangle_{SS'E}\langle\Phi|$, the $S+S'$ entropy, $\mathcal{S}(S+S') = -(1-\epsilon)\ln(1-\epsilon) - \sum_i \epsilon p_i \ln(\epsilon p_i) \sim \epsilon(1 - \ln \epsilon - \sum_i p_i \ln p_i) \leq \kappa\epsilon$, $\kappa \equiv 1 - \ln \epsilon - \ln p_{\max}$, where $p_{\max} = \max\{p_i\}$. As $\mathcal{S}(E) = \mathcal{S}(S + S')$ and the analogous result follows for the other bipartite decompositions, $S + (S' + \mathcal{E})$ and $S' + (S + \mathcal{E})$, Lemma 4.1 is proved. $\square$

Now it is straightforward to see the total mutual information in the $S + S'$ system, $I(S : S') = \mathcal{S}(S) + \mathcal{S}(S') - \mathcal{S}(S, S')$, is proportional to ϵ . Due to nonnegativity of the discord and of the classical correlations (cf. Section 2), it is clear that the discord is also proportional to ϵ and is very small. The only exception is the case of the maximum entanglement in the small term in $|\Phi\rangle_{SS'E}$. i.e. For $|O(\epsilon, p; t)\rangle_{S+S'+E} = \sum_{i=1}^N p_i |i\rangle_{S+S'} |i\rangle_E$ and for $p_i = N^{-1}, \forall i$, one obtains $\mathcal{S}(S + S') \approx \epsilon \ln N$ [47], when, in principle, for given ϵ there may exist N such that $\epsilon \ln N \sim 1$. The "locally inaccessible information (LII)" flow [27] is also negligible, $\mathcal{L}^{\leftrightarrow} = D^{\leftrightarrow}(S'|S) + D^{\leftrightarrow}(E|S') + D^{\leftrightarrow}(S|E) \propto \epsilon$. Like in Ref. [39], we can hope that the model (7) may provide both the discord and the LII flow are zero for some *practical purposes* (even though not rigorously null), not only for the already known purposes of combating decoherence [46, 48] and providing identity of micro-particles in a solution [49].

While respecting the points 'a-c', Section 3.3, the following are the virtues of the model (7): 1. the model does not require yet supports Markovian environment $\mathcal{E}$; 2. the model is generally applicable—it equally targets the finite dimensional as well as the continuous variable systems; 3. the model allows interaction between S and S' , (2). This way the model resolves the above point 'd', Section 3.3; 4. Due to the non-zero second term in (7), the discord $D^-(C|E)$ is not saturated (cf. Theorem 1 in Ref. [32]). Therefore the above point 'e', Section 3.3, is also resolved: the environment $\mathcal{E}$ is allowed to correlate with the external system W .

5. Discussion

We are not aware of any constraints on discord dynamics for the non-Markovian open systems. In principle, every subsystem $(1, 2, S, S')$ in the model (2) may be a composite system itself. Related multi-partitions of the total system C require separate analysis to be presented elsewhere.

Essential for our model of Markovian classicality are the assumptions on the special initial state (the tensor-product state) and the completely positive dynamics of the open system. Regarding the initial state, our assumption is of general use for noninteracting systems—from the hydrogen atom and the "ideal gas" to the macroscopic bodies typically modelled by their center of mass and the internal degrees of freedom (the $CM + R$ structure).

The model (2) is already in use. An ancilla qubit, appearing in a number of the quantum information protocols and algorithms, is easily recognized as the S system in the model (2), which, in turn, has been proposed as a testbed for investigating non-Markovian dynamics of open systems [50]. "Entanglement renormalization" provides decoupling (the variables separation) for a spin-chain system [44]. In general, nonexistence of the model (2) for a concrete physical system suggests the composite system is practically deprived of the zero discord states [23]. To this end, existence of an alternative to the model Eq. (2) may vary the conclusions. However, bearing in mind the tight conditions for Markovian classicality, Section 2, we are free to *conjecture nonexistence of alternate model* that would *rigorously* provide Markovian classicality. Nevertheless, alternate models of approximate Markovian classicality can be expected.

The approximate Markovian classicality model, Section 4, suggests physically there is not ideal Markovian classicality. Worse, approximate Markovian classicality can last for only a finite time interval. To this

end, the details are case sensitive and establishing approximate Markovian classicality for some practical purposes can hardly be formulated in full generality.

Our results have wide-range implications. First, from a fundamental perspective, they imply that only-classically correlated states are a matter of the open system's structure. Bearing in mind Lemma 3.1, we realize that non-asymptotic information locality [23], discord saturation [30] and quantum decorrelation [4, 32] are also a matter of the composite system's structure. Second, completely positive dynamics is a necessary condition for the models (6), (7). So, a change in structure of the composite system removes the conflict [23] between the completely positive map and the rarity of zero discord states. Third, by Definition 3.1 (Definition 4.1) the model (6) ((7)) is not a model of the sudden death of discord—discord is zero (or approximately zero) in a long time interval without the sudden change or sudden death. Fourth, the model (6) directly sets a basis for the task of "local broadcasting" [51].

A final comment about experimental implications. Very much like the classical systems, the structure $S + S'$ described by the model (6), or (7), is not capable of performing a useful quantum information processing. So, instead of experimentally testing discord (that is not feasible [23] yet), one can try to perform quantum information processing. The failure of every possible quantum protocol, e.g. of the discord-based quantum computation [5, 6] (and the references therein), reveals, at least approximate, Markovian classicality of the composite system's structure. Thereby, avoiding Markovian classicality now appears basic to performing efficient information processing on bipartitions of the quantum information hardware: just avoid operations and manipulations of the observables of the subsystems that support Markovian classicality (as it is the case in the general model Eq. (2)).

6. Conclusion

Complementary to Ferraro et al [23], we construct a model of a Markovian bipartite system that provides zero two-way discord in a long time interval (that we name 'Markovian classicality'). The model is a sufficient condition for Markovian classicality and is in close relation to the topics of quantum information locality, discord saturation and quantum decorrelation. We emphasize Markovian classicality is not a matter of the open system itself, but of the open system's structure (decomposition into parts/subsystems). Bearing this in mind, the model is of general interest and is not a model of quantum discord sudden death. We finally conjecture about the absence of alternate model, which would rigorously meet the criteria for Markovian classicality.

Appendix A. Calculating discord for different structures

We are concerned with the two possible decompositions of the total system, $S + S' + \mathcal{E}$. We calculate the states, their entropies and the related quantum discords for both structures separately. For simplicity, by $S(A)$ we denote von Neumann entropy of the A system's state, ρ_A , $S(A) \equiv S(\rho_A)$.

Quantum state (6) can be written as:

$$\rho \equiv \rho_S \otimes \sum_{i,j} c_i c_j^* |i\rangle_{S'} \langle j| \otimes |i\rangle_E \langle j|. \quad (\text{A.1})$$

1) Structure $(S + S') + \mathcal{E}$, denoted by $C + \mathcal{E}$, is a bipartite system and the subsystems' states are:

$$\begin{aligned} \rho_{S+S'} &= \text{tr}_E \rho = \rho_S \otimes \sum_i |c_i|^2 |i\rangle_S \langle i|; & \rho_E &= \text{tr}_{S+S'} \rho = \sum_i |c_i|^2 |i\rangle_E \langle i| \\ \rho_{S'} &= \text{tr}_{S+E} \rho = \text{tr}_S \rho_{S+S'} = \sum_i |c_i|^2 |i\rangle_{S'} \langle i|. \end{aligned} \quad (\text{A.2})$$

Then von Neumann entropies, $S(S, S') = S(S) + S(S')$ and $S(S') = S(E) = \sum_i |c_i|^2 \log |c_i|^2$. From Eq. (A.2) we directly obtain $\rho_{S+S'}|_i \equiv I_{S+S'} \otimes |i\rangle_E \langle i| \rho_{S+S'} \otimes |i\rangle_E \langle i| = \rho_S \otimes |i\rangle_{S'} \langle i|$, assuming the measurement in the $\{|i\rangle_E\}$

basis is performed, as well as $\mathcal{S}(\rho_{S+S'}|_i) = \mathcal{S}(S) + \mathcal{S}(S') = \mathcal{S}(S)$. For the total system's entropy, Eq. (A.1) gives for, in general, mixed state $\rho_S = \sum_\alpha \omega_\alpha |\alpha\rangle_S \langle \alpha|$:

$$\begin{aligned} \mathcal{S}(S, S', E) &= -\text{tr}_{S+S'+E} \rho \ln \rho = -\text{tr}_{S+S'+E} \sum_{i,j,\alpha,k,l\beta} c_i c_j^* \omega_\alpha [\omega_\beta \ln c_k c_l^* \\ &+ c_k c_l^* \ln \omega_\beta] |i\rangle_{S+S'} \langle j| \otimes |i\rangle_E \langle j| = -\sum_\alpha \omega_\alpha \ln \omega_\alpha = \mathcal{S}(S), \end{aligned} \quad (\text{A.3})$$

where we made use of $\text{tr}_{S+S'+E} = \text{tr}_S \text{tr}_{S'} \text{tr}_E$ and the basis independence of the tracing out operation, while $|i\rangle_{S+S'} \equiv |i\rangle_S |\alpha\rangle_{S'}$.

Then the total correlations, $I(S, S' : E) = \mathcal{S}(S, S') + \mathcal{S}(E) - \mathcal{S}(S, S', E)$ and the classical correlations $J^-(S, S'|E) = \mathcal{S}(S, S') - \inf_{\{\Pi_E\}} \sum_i |c_i|^2 \mathcal{S}(\rho_{S+S'}|_{\Pi_E})$. With the use of the above calculated entropies, we obtain for the one-way discord (cf. Section 2):

$$\begin{aligned} D^-(S, S'|E) &= \mathcal{S}(S, S') + \mathcal{S}(E) - \mathcal{S}(S, S', E) - \mathcal{S}(S, S') + \sum_i |c_i|^2 \mathcal{S}(\rho_{S+S'}|_i) \\ &= \mathcal{S}(E) - \mathcal{S}(S, S', E) + \sum_i |c_i|^2 \mathcal{S}(\rho_{S+S'}|_i) = \mathcal{S}(E), \end{aligned} \quad (\text{A.4})$$

that is the discord saturation discussed in Section 3.1.

2) Alternative structure $S + (S' + \mathcal{E})$ is more easy to handle. Then (6) is of direct use and the results follow:

$$\begin{aligned} \rho_S &= \text{tr}_{S'+E} \rho; \quad \rho_{S'+E} = \text{tr}_S \rho = |\Psi\rangle_{S'+E} \langle \Psi| \\ \rho_{S'} &= \sum_i |c_i|^2 |i\rangle_{S'} \langle i|; \quad \rho_E = \sum_i |c_i|^2 |i\rangle_E \langle i|. \end{aligned} \quad (\text{A.5})$$

From Eq. (A.5) it easily follows: $\mathcal{S}(S, S', E) = \mathcal{S}(S) + \mathcal{S}(S', E)$, while $\mathcal{S}(S', E) = 0$ and $\mathcal{S}(S') = \mathcal{S}(E) = -\sum_i |c_i|^2 \ln |c_i|^2$.

Then the total correlations, $I(S : S', E) = \mathcal{S}(S) + \mathcal{S}(S', E) - \mathcal{S}(S, S', E) = 0$. As both quantum discord and the classical correlations are non-negative, the one-way discord $D^-(S|S' + E) = 0$ as well as $D^-(S|S' + E) = 0$ —the state (6) is a CC state, Definition 2.1 in the main text. The third structure, $(S + \mathcal{E}) + S'$, can be alternatively managed with the conclusion that $D^-(S + E|S') = \mathcal{S}(S') = \mathcal{S}(E) \neq 0$.

It is interesting that even the trivial change of structure, by simply grouping the constituent subsystems, S , S' and $\mathcal{E}$, exhibits the general notion of quantum correlations relativity [43]: quantum discord is a matter of structure, and is here zero only for the $S + (S' + \mathcal{E})$ bipartite structure. The different structures reveal the different facets of the total system.

Finally, we show that the tensor product state (5), considered as a P -classical state [26], satisfies the C -criterion for classicality. The generic P -classical state (1) in Ref. [26] reduces to the tensor-product state (considered in Section 3.2) for the separability condition [in their notation] $P(\alpha, \beta) = P(\alpha)P(\beta)$. For this choice, one obtains for the states considered, (6) in Ref. [23]: $\rho_A \equiv \text{tr}_B \rho_A \otimes \rho_B = \rho_A$, and (after normalization) $\rho_\circ \equiv \text{tr}_B \rho_A \otimes \rho_B |0\rangle_B \langle 0| = \rho_A$. So, one obtains $[\rho_A, \rho_\circ] = 0$, that is the criterion for the C -classicality, which, in turn, is already well-known. Our proof is given in terms of P -classical states in order to match the considerations in Ref. [26].

Appendix B. Constructing the Markovian classicality model

Consider a composite system C consisting of N physical particles, $1, 2, 3, \dots, N$. Then the set $C_1 = \{1, 2, 3, \dots, N\}$ is a structure describing C as a multiparticle system. The set of the C 's degrees of freedom, $\{x_{i\alpha}, i = 1, 2, \dots, N\}$, can be transformed to provide a new structure of C ; the index α enumerates the individual particles degrees of freedom. E.g. by grouping the particles into two sets described by their degrees of freedom, $A = \{x_{i\alpha}, i = 1, 2, \dots, M\}$ and $B = \{x_{i\alpha}, i = M + 1, M + 2, \dots, N\}$, we obtain a bipartite structure of C , presented formally as $C_2 = A + B$. This grouping the particles is kind of formally trivial canonical transformations (CTs). Formally nontrivial kind of the CTs assume non-local symplectic transformations

that introduce the new degrees of freedom, $\{\xi_{p\beta}, p = 1, 2, \dots, N\}$. To this end, paradigmatic are the CTs introducing the C 's center-of-mass (CM) and the relative positions (R) degrees of freedom. Then C can be described by another bipartite structure, $C_3 = CM + R$.

Below we briefly discuss the task of constructing the structure $S + S'$.

In general, the proper canonical transformations convert a bipartite-system's 'fundamental' structure $1 + 2$ (Fig. 2) into the structure $S + S'$ (Fig. 1). While the system's Hamiltonian, H_C , is unique, it obtains different forms for the different structures: $H_1 + H_2 + H_{12} = H_C = H_S + H_{S'}$.

The structure $S + S'$ follows from the variables separation for the original $1 + 2$ structure. This is closely related to the general mathematical topic of integrability of quantum mechanical models. Regarding the "mixed" states (described by the density matrix), this is an instance of the task of the Quantum Separability Problem (QUSEP). QUSEP is investigated in the literature for the finite-dimensional composite systems and is computationally a "strongly NP-Hard" problem [52].

On the other hand, separation of variables is not much more easier even for the pure states. The task is to obtain the equality

$$\sum_i c_i |i\rangle_1 |i\rangle_2 = |\Psi\rangle_C = |\phi\rangle_S |\chi\rangle_{S'} \quad (\text{B.1})$$

for an instantaneous state, $|\Psi\rangle_C$, of the composite system C . Physically, Eq. (B.1) assumes there are interactions and therefore entanglement for the 'fundamental' structure $1 + 2$ while mutually noninteracting systems S and S' are described by a tensor-product state. We believe these easily formulated tasks are largely intact in the present quantum theory.

Appendix C. In support of the Conjecture

This short appendix is devoted to the Conjecture made in Section 5. Most of the classical physics deals with the collective variables of the macroscopic bodies, CM and R . "Classicality" of the macroscopic bodies is tacitly assumed for *this* kind of structure of the classical-physics systems. So, the $S + S'$ structure, Section 2, naturally resembles the macroscopic-systems structure $CM + R$.

In addition, we want to emphasize that the model Eqs. (2)-(5) reflects the general experience with atoms and molecules [53]. Their "relative positions" degrees of freedom are monitored by the quantum vacuum fluctuations [37] (and the references therein), while the center-of-mass (CM) degrees of freedom are typically supposed both decoupled from R as well as possibly subject to the different kinds of the environment (e.g. to the harmonic bath in quantum Brownian motion [37], and the references therein). In order to describe this, introducing the S' 's environment, $\mathcal{V}$, into the model (2) is straightforward: as long as the two environments, $\mathcal{E}$ and $\mathcal{V}$, are decoupled from each other, nothing changes in our considerations, except the S system is now described by a proper master equation providing $\rho_S(t) \neq \rho_S^2(t)$. Finally, as distinguished above, the tensor-product state (5) satisfies both P - and C -criteria [26] for classicality.

Appendix D. The origin of Equation (7)

Originally, the DISD model [46] is developed for the purposes of combating decoherence in quantum computation hardware.

It's a tripartite system of interest, $S + S' + \mathcal{E}$, defined by the Hamiltonian:

$$H = H_S + H_{S'} + H_E + H_{SS'} + H_{S'E}, \quad (\text{D.1})$$

where the double subscript denotes interactions. While not assuming anything about any of the subsystems, S , S' and $\mathcal{E}$, the model assumes the interaction $H_{S'E}$ dominates the total system's dynamics and the system S' is in the initial state $|p\rangle_{S'}$ satisfying the "robustness" condition, $H_{S'E}|p\rangle_{S'}|\chi\rangle_E = |p\rangle_{S'}|\chi_p\rangle_E$; the strength of $H_{S'E}$ is denoted by C while the strength of interaction $H_{SS'}$ is denoted by c .

Then the use of the standard perturbation procedure for the (normalized) initial state $|\Psi^p\rangle_{SS'E} = \sum_k C_k |k\rangle_S \otimes |p\rangle_{S'} \otimes \sum_j \beta_j |j\rangle_E$, one obtains the *exact* total-system's state:

$$|\Psi^p(t)\rangle_{SS'E} = \left(\sum_k C_k(t) \exp(-it\lambda_{kp}^1/\hbar) |k\rangle_S \right) \otimes \exp(-it\lambda/\hbar) |p\rangle_{S'} \otimes \left(\sum_j \beta'_j(t) \exp(-it\lambda^{kpj}/\hbar) |j\rangle_E \right) + |O(\epsilon, t)\rangle_{SS'E}. \quad (\text{D.2})$$

In Eq. (D.2): $C_k(t) \equiv C_k \exp(-it_S \langle k|H_S|k\rangle_S/\hbar)$, $\lambda \equiv_{S'} \langle p|H_{S'}|p\rangle_{S'}$, $\beta'_j(t) \equiv \beta_j \exp(-it(C\kappa_{kj} + \langle j|H_E|j\rangle_E))$. κ_{pj} represents an eigenvalue of $H_{S'E}$, $\lambda_{pk}^1 \equiv_{SS'} \langle pk|H_{SS'}|pk\rangle_{SS'}$ is the first-order correction and λ^{kpj} is of the order of the second-order correction to the eigenvalues of $H_{S'E}$, while $\epsilon \sim c/C$. Due to $\lambda^{kpj} \sim c/C$, after a time interval $\tau' > C/c$, the induced correlations of S and E become non-negligible.

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