



## Some identities of the degenerate Krawtchouk polynomials

Anis Riahi<sup>a,\*</sup>, Mohamed Rhaima<sup>b</sup>, Hamza Ghoudi<sup>c</sup>, Lazhar Kthiri<sup>a</sup>

<sup>a</sup>Department of Mathematics, Nabeul Preparatory Institute for Engineering Studies, Carthage University, Tunisia

<sup>b</sup>Department of Statistics and Operations Research, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia

<sup>c</sup>Departments of mathematics, MODAL'X UMR9023, UFR SEGMI, BAT G, 200, Ave de la Republique, 92000 Nanterre, University of Paris Nanterre, France

**Abstract.** In this paper, we introduce a novel class of *degenerate Krawtchouk polynomials* associated with degenerate Pascal random variables characterized by parameters  $r > 0$  and  $q \in (0, 1)$ . We derive a variety of new identities involving these polynomials, particularly focusing on combinatorial identities linked to the  $n$ -th moments of the degenerate Pascal distribution, as well as their connections to special numbers and polynomials. Furthermore, we extend our framework to define and investigate a two-parameter generalization, referred to as the two-variable degenerate Krawtchouk polynomials, arising from degenerate Pascal random variables with parameters  $q$  and  $\alpha$ . We explore the algebraic and analytic properties of these polynomials and establish their relationships with the central moments of the associated distributions, as well as with various classes of special numbers and polynomials.

### 1. Introduction

Special polynomials and their extensions have been extensively studied by mathematicians and physicists due to their significant roles in diverse scientific disciplines, including physics, applied mathematics, engineering, and related research fields. These polynomials arise in areas such as differential equations, number theory, functional analysis, quantum mechanics, and mathematical physics, often providing powerful analytical tools for solving complex problems. Among these, degenerate versions of special polynomials have emerged as an active area of research, yielding numerous arithmetic and combinatorial results. Notably, symmetric identities for degenerate polynomials have been widely investigated.

The study of degenerate polynomials was pioneered by Carlitz [4], who introduced degenerate Stirling, Bernoulli, and Euler polynomials by replacing the exponential function  $e^z$  with  $(1 + \lambda z)^{1/\lambda}$  in their generating functions. This approach recovers the classical polynomials in the limit as  $\lambda \rightarrow 0$ . In recent years, research has expanded to explore various degenerate forms of special polynomials and numbers, including degenerate Stirling numbers (first and second kind), degenerate Bernstein polynomials, degenerate Bell numbers and polynomials, degenerate poly-Genocchi polynomials, degenerate Euler and Bernoulli polynomials,

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\* Corresponding author: Anis Riahi

Email addresses: [ariahi@yahoo.fr](mailto:ariahi@yahoo.fr) (Anis Riahi), [mrhaima.c@ksu.edu.sa](mailto:mrhaima.c@ksu.edu.sa) (Mohamed Rhaima), [h.ghoudi@parisnanterre.fr](mailto:h.ghoudi@parisnanterre.fr) (Hamza Ghoudi), [kthirilazher@hotmail.com](mailto:kthirilazher@hotmail.com) (Lazhar Kthiri)

ORCID iDs: <https://orcid.org/0000-0001-6233-7387> (Anis Riahi), <https://orcid.org/0000-0002-2400-9275> (Mohamed Rhaima), <https://orcid.org/0009-0000-3004-3296> (Hamza Ghoudi), (Lazhar Kthiri)

degenerate gamma random variables, and degenerate Laplace transforms (see [10, 12, 13, 15–17, 20–22, 25] and references therein). Building upon these developments, Haroon and Khan [9] investigated degenerate Hermite-Bernoulli polynomials using  $p$ -adic fermionic integrals, leading to new identities that establish connections with Daehee polynomials in a unified and generalized framework. Duran and Acikgoz [5, 6] introduced generalized degenerate Gould-Hopper-based Stirling polynomials of the second kind and the degenerate Poisson distribution as well as is related to degenerate Bell polynomials such that the moment generating function of the familiar Poisson distribution coincides with the generating function of the usual Bell polynomials. They provided several novel representations of these polynomials not only in terms of classical fully degenerate Bell polynomials but also via generalized degenerate Gould-Hopper-based Bernoulli, Euler, Genocchi polynomials and the connection for these polynomials related to the unsigned Stirling numbers of the first kind. In a related direction, Further contributions include the work of Araci [1] and [2], who studied a new class of generating functions for type 2 Bernoulli polynomials and a new family of Daehee polynomials called degenerate  $q$ -Daehee polynomials with weight  $\alpha$ , deriving several identities involving type 2 Euler polynomials and Stirling numbers of the second kind. Gomaa and Magar [8] explored generalized Fubini-type polynomials, utilizing generating functions to derive various combinatorial identities, relations, and probabilistic applications. Frontczak and Tomovski [7] provided probabilistic interpretations for summation formulas involving generalized Bernoulli and Euler-Genocchi polynomials of order  $(r, m)$ , enriching the analytical framework of these polynomial families. Recent work by Kim and Kim [18] introduced probabilistic variants of degenerate Stirling numbers of the second kind and degenerate Bell polynomials, grounded in random variables that satisfy certain moment conditions. Luo et al. [23] developed probabilistic extensions of degenerate Bernoulli and Euler polynomials, leading to the formulation of probabilistic degenerate  $r$ -Stirling numbers of the second kind and two-variable probabilistic degenerate Fubini polynomials. Most recently, Waseem Ahmad Khan et al. [14] investigated a probabilistic form of the degenerate Jindalrae and Jindalrae-Stirling polynomials of the second kind. They developed some new properties and formulas, including explicit expressions, symmetric identity, recurrence relations and summations formulas. These advancements continue to deepen the interplay between combinatorial structures, probabilistic methods, and degenerate special functions, revealing new pathways for analytical exploration and interdisciplinary applications.

During last years, one witnesses the growth of interest to the study of compound Poisson process and application of these to different problems of probability theory, stochastic analysis and mathematical physics. The Pascal process is an important example of compound Poisson processes. Krawtchouk polynomials, originally introduced by Mikhail Kravchuk (Krawtchouk), represent a foundational class of orthogonal polynomials with deep connections to combinatorics and probability theory. A comprehensive overview of Kravchuk's contributions and the historical development of these polynomials up to 2004 is available in [27]. These polynomials possess intrinsic mathematical richness and have found widespread applications across diverse scientific fields, including quantum probability, coding theory, electromagnetic wave propagation, and image processing (see [24] about the notion of its Darboux transformations and [28] for applications in image analysis). Motivated by the extensive literature on degenerate polynomials and their probabilistic interpretations, we investigate a new family of degenerate Krawtchouk polynomials associated with the degenerate Pascal random variables, parameterized by  $r > 0$  and  $0 < q < 1$ . In this work, we derive novel explicit expressions, establish a variety of recurrence relations, and uncover intricate identities that connect these polynomials with classical and degenerate combinatorial quantities such as the Stirling numbers of the first kind, degenerate Stirling numbers of the second kind, degenerate derangement numbers, and degenerate Frobenius-Euler polynomials.

This study not only expands the framework of degenerate orthogonal polynomials but also opens avenues for further exploration. Potential directions include applications in probability theory, the study of differential equations, and the derivation of symmetric identities. Although the translation of these theoretical results into applied domains such as physics or engineering remains a challenge, it holds promising potential. Future research will aim to bridge this gap and uncover concrete applications of degenerate Krawtchouk polynomials in computational and applied sciences.

The structure of the paper is as follows. In Section 2, we introduce a novel class of degenerate Krawtchouk polynomials defined in terms of the degenerate Pascal random variable with parameters  $r > 0$  and  $0 < q < 1$ .

We explore their probabilistic foundations by expressing these polynomials through the  $n$ -th moments of the corresponding degenerate Pascal distribution. Several identities and recurrence relations are derived, highlighting their combinatorial significance and analytical properties. In Section 3, we extend this framework by defining a two-parameter family of degenerate Krawtchouk polynomials associated with degenerate Pascal random variables characterized by parameters  $0 < q < 1$  and  $0 < \alpha < 1$ . We demonstrate that these polynomials correspond to degenerate central moments of the underlying distribution and establish new explicit formulas and relationships that enrich their theoretical structure and potential applicability.

We begin by recalling essential definitions and notations that will be used throughout this work. The symbols  $\mathbb{C}$ ,  $\mathbb{R}$ ,  $\mathbb{N}$ , and  $\mathbb{N}^*$  denote the sets of complex numbers, real numbers, natural numbers (positive integers), and non-negative integers, respectively.

A Pascal random variable (also known as a negative binomial random variable) models the number of failures before achieving  $r$  successes in a sequence of independent Bernoulli trials, where each trial results in a success with probability  $p \in (0, 1)$ . The probability mass function (pmf) of such a random variable  $X$ , taking values in  $\mathbb{N}^*$ , is given by

$$Pr([X = k]) := \mathcal{NB}_{p,r}(\{k\}) = p^r \binom{-r}{k} (-q)^k, \quad \text{where } q := 1 - p,$$

and for  $\gamma \in \mathbb{R}$ ,  $k \in \mathbb{N}^*$ , the generalized binomial coefficient is defined by

$$\binom{\gamma}{k} = \frac{\gamma(\gamma - 1) \cdots (\gamma - k + 1)}{k!}.$$

It is clear that the moment generating function of the measure  $\mathcal{NB}_{p,r}$  is given by

$$\mathbb{E}(e^{tX}) = p^r \sum_{k=0}^{\infty} \binom{-r}{k} (-qe^t)^k = \left( \frac{p}{1 - qe^t} \right)^r =: \Psi(t, q), \quad (1)$$

where  $\mathbb{E}(w)$  indicates the expectation value of the random variable  $w$ . Multiplicative renormalization framework [3] says that the multiplicative renormalization  $\Psi(t, q)$  associated with the Pascal measure  $\mathcal{NB}_{p,r}$  can be considered as a generating function of the so-called classical Krawtchouk polynomials  $\mathcal{K}_n^{(r)}(q)$ , i.e.,

$$\Psi(t, q) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathcal{K}_n^{(r)}(q). \quad (2)$$

For completeness, we recall the definition of the Stirling numbers of the second kind, denoted  $S_2(n, k)$ , which satisfy the identity

$$x^n = \sum_{k=0}^n S_2(n, k) (x)_k, \quad (n \geq 0),$$

where

$$(x)_k := \begin{cases} 1 & \text{if } k = 0 \\ x(x - 1) \cdots (x - k + 1) & \text{if } k \in \mathbb{N}^*, \end{cases}$$

and it is well known that [4, 16]

$$\frac{1}{k!} (e^t - 1)^k = \sum_{n=k}^{\infty} S_2(n, k) \frac{t^n}{n!}, \quad (k \geq 0). \quad (3)$$

Apply the binomial series expansion and using Eq. (3) to show that

$$\begin{aligned}
 \sum_{n=0}^{\infty} \mathcal{K}_n^{(r)}(q) \frac{t^n}{n!} &= p^r (1 - qe^t)^{-r} \\
 &= p^r \left(1 - q - q(e^t - 1)\right)^{-r}, \\
 &= p^r \left[\left(1 - q\right)\left(1 - \frac{q}{1 - q}(e^t - 1)\right)\right]^{-r} \\
 &= \frac{p^r}{(1 - q)^r} \left[1 - \frac{q}{1 - q}(e^t - 1)\right]^{-r} \\
 &= \sum_{k=0}^{\infty} (-1)^k \binom{-r}{k} \left(\frac{q}{p}\right)^k k! \frac{(e^t - 1)^k}{k!} \\
 &= \sum_{k=0}^{\infty} (-1)^k \binom{-r}{k} \left(\frac{q}{p}\right)^k k! \sum_{n=k}^{\infty} S_2(n, k) \frac{t^n}{n!} \\
 &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n (-1)^k \binom{-r}{k} \left(\frac{q}{p}\right)^k k! S_2(n, k)\right) \frac{t^n}{n!} \\
 &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n (-1)^k (-r)_k \left(\frac{q}{p}\right)^k S_2(n, k)\right) \frac{t^n}{n!}.
 \end{aligned}$$

Thus we conclude that classical Krawtchouk polynomials possess the following form

$$\mathcal{K}_n^{(r)}(q) = \sum_{k=0}^n (-1)^k (-r)_k \left(\frac{q}{p}\right)^k S_2(n, k). \quad (4)$$

To define degenerate extensions of classical functions and polynomials, we introduce the degenerate exponential function. For any  $\lambda \in \mathbb{R}$  and  $z \in \mathbb{C}$ , the degenerate exponential function is defined as

$$e_{\lambda}^x(z) = (1 + \lambda z)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{z^n}{n!}, \quad e_{\lambda}(z) := e_{\lambda}^1(z) = (1 + \lambda z)^{\frac{1}{\lambda}}, \quad (5)$$

where  $(x)_{n,\lambda}$  is the degenerate falling factorial defined by

$$(x)_{n,\lambda} := \begin{cases} 1, & \text{if } n = 0, \\ x(x - \lambda)(x - 2\lambda) \cdots (x - (n - 1)\lambda), & \text{if } n \in \mathbb{N}^*, \end{cases}$$

with the convention  $(0)_{n,\lambda} := 1$ . In [11], T. Kim introduced the *degenerate Stirling numbers of the second kind*, denoted by  $S_{2,\lambda}(n, k)$ , as a generalization of the classical Stirling numbers of the second kind. These numbers appear in the expansion of the degenerate falling factorial  $(x)_{n,\lambda}$  in terms of the classical falling factorials  $(x)_k$ , and are defined by the identity

$$(x)_{n,\lambda} = \sum_{k=0}^n S_{2,\lambda}(n, k) (x)_k, \quad \text{for } n \geq 0.$$

The generating function for the degenerate Stirling numbers of the second kind is given by

$$\frac{1}{k!} (e_{\lambda}(t) - 1)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n, k) \frac{t^n}{n!}, \quad (k \geq 0). \quad (6)$$

## 2. Degenerate Krawtchouk polynomials associated with degenerate Pascal random variables

**Definition 2.1.** For  $\lambda \in \mathbb{R}$ ,  $X$  is the degenerate Pascal random variable with parameters  $r > 0$  and  $0 < q < 1$  if its probability mass function is given by

$$Pr([X = k]) := \mathcal{NB}_{p,r}^{(\lambda)}(\{k\}) = p^r \frac{(-r)_{k,\lambda}}{k!} (-q)^k. \quad (7)$$

The moment generating function of the degenerate Pascal distribution  $\mathcal{NB}_{p,r}^{(\lambda)}$  is as follows:

$$\mathbb{E}[e_\lambda^X(t)] = \sum_{k=0}^{\infty} p^r \frac{(-r)_{k,\lambda}}{k!} (-qe_\lambda(t))^k = p^r (1 - \lambda q e_\lambda(t))^{-\frac{r}{\lambda}} = p^r e_\lambda^{-r}(-qe_\lambda(t)). \quad (8)$$

Accordingly, we introduce a new family of polynomials, referred to as degenerate Krawtchouk polynomials, given by

$$p^r e_\lambda^{-r}(-qe_\lambda(t)) = \sum_{n=0}^{\infty} \mathcal{K}_{n,\lambda}^{(r)}(q) \frac{t^n}{n!}. \quad (9)$$

We next aim to derive a serval of combinatorial identities involving these polynomials.

**Theorem 2.2.** Let  $X$  be a degenerate Pascal random variables with parameters  $r > 0$  and  $0 < q < 1$ , we have

$$\mathcal{K}_{n,\lambda}^{(r)}(q) = \mathbb{E}[(X)_{n,\lambda}]. \quad (10)$$

*Proof.* Firstly, we observe that

$$\begin{aligned} \mathbb{E}[(X)_{n,\lambda}] &= \mathbb{E}[X(X-\lambda) \cdots (X-(n-1)\lambda)] \\ &= p^r \sum_{k=0}^{\infty} \frac{(-r)_{k,\lambda}}{k!} (-q)^k (k)_{n,\lambda}. \end{aligned} \quad (11)$$

On the other hand, from (5) and (11), we have

$$\begin{aligned} \sum_{n=0}^{\infty} \mathcal{K}_{n,\lambda}^{(r)}(q) \frac{t^n}{n!} &= p^r \sum_{k=0}^{\infty} \frac{(-r)_{k,\lambda}}{k!} (-q)^k e_\lambda^k(t) \\ &= p^r \sum_{k=0}^{\infty} \frac{(-r)_{k,\lambda}}{k!} (-q)^k \left( \sum_{n=0}^{\infty} (k)_{n,\lambda} \frac{t^n}{n!} \right) \\ &= \sum_{n=0}^{\infty} \left( p^r \sum_{k=0}^{\infty} \frac{(-r)_{k,\lambda}}{k!} (-q)^k (k)_{n,\lambda} \right) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \mathbb{E}[(X)_{n,\lambda}] \frac{t^n}{n!}. \end{aligned} \quad (12)$$

A term-by-term comparison of equation (12) completes the proof.  $\square$

**Theorem 2.3.** For  $n \in \mathbb{N}$ , we have

$$\mathcal{K}_{n,\lambda}^{(r)}(q) = \frac{p^r}{e_\lambda^r(-q)} \sum_{k=0}^n \left( \frac{q}{1-\lambda q} \right)^k (r)_{k,-\lambda} S_{2,\lambda}(n, k). \quad (13)$$

*Proof.* From (5) and (6), and using the fact that  $(-r)_{k,\lambda} = (-1)^k(r)_{k,-\lambda}$  we have

$$\begin{aligned}
 \sum_{n=0}^{\infty} \mathcal{K}_{n,\lambda}^{(r)}(q) \frac{t^n}{n!} &= p^r e_{\lambda}^{-r}(-qe_{\lambda}(t)) = p^r (1 - \lambda q e_{\lambda}(t))^{-\frac{r}{\lambda}} \\
 &= p^r \left(1 + \lambda q' + \lambda q' (e_{\lambda}(t) - 1)\right)^{-\frac{r}{\lambda}}, \quad q' = -q \\
 &= p^r \left[ \left(1 + \lambda q'\right) \left(1 + \frac{\lambda q'}{1 + \lambda q'} (e_{\lambda}(t) - 1)\right) \right]^{-\frac{r}{\lambda}} \\
 &= \frac{p^r}{(1 + \lambda q')^{\frac{r}{\lambda}}} \left[1 + \frac{\lambda q'}{1 + \lambda q'} (e_{\lambda}(t) - 1)\right]^{-\frac{r}{\lambda}} \\
 &= \frac{p^r}{(1 + \lambda q')^{\frac{r}{\lambda}}} \sum_{k=0}^{\infty} (-r)_{k,\lambda} \left(\frac{q'}{1 + \lambda q'}\right)^k \frac{(e_{\lambda}(t) - 1)^k}{k!} \\
 &= \frac{p^r}{(1 + \lambda q')^{\frac{r}{\lambda}}} \sum_{k=0}^{\infty} (-r)_{k,\lambda} \left(\frac{q'}{1 + \lambda q'}\right)^k \sum_{n=k}^{\infty} S_{2,\lambda}(n, k) \frac{t^n}{n!} \\
 &= \frac{p^r}{(1 - \lambda q)^{\frac{r}{\lambda}}} \sum_{n=0}^{\infty} \left(\sum_{k=0}^n (-r)_{k,\lambda} \left(\frac{-q}{1 - \lambda q}\right)^k S_{2,\lambda}(n, k)\right) \frac{t^n}{n!} \\
 &= \frac{p^r}{e_{\lambda}^r(-q)} \sum_{n=0}^{\infty} \left(\sum_{k=0}^n (r)_{k,-\lambda} \left(\frac{q}{1 - \lambda q}\right)^k S_{2,\lambda}(n, k)\right) \frac{t^n}{n!}.
 \end{aligned} \tag{14}$$

A straightforward comparison of coefficients in equation (14) establishes the required identity.  $\square$

**Theorem 2.4.** For  $n \in \mathbb{N}$ , we have

$$\mathcal{K}_{n,\lambda}^{(r)}(q) = \sum_{l=0}^{\infty} \left( \sum_{k=0}^l \sum_{j=0}^{l-k} (-1)^l \binom{l}{k} r^j S_1(l-k, j) (-r)_{k,\lambda} (k)_{n,\lambda} \right) \frac{q^l}{l!}. \tag{15}$$

*Proof.* For  $n \geq 0$ , the Stirling numbers of the first kind are defined as

$$(x)_n = \sum_{k=0}^n S_1(n, k) x^k.$$

Then for  $k \geq 0$ , it is well known that (see [26])

$$\frac{1}{k!} (\log(1+t))^k = \sum_{n=k}^{\infty} S_1(n, k) \frac{t^n}{n!}. \tag{16}$$

From (16), we observe

$$\begin{aligned}
 e_{\lambda}^m(x) &= e^{\frac{m}{\lambda} \log(1+\lambda x)} = \sum_{k=0}^m \left(\frac{m}{\lambda}\right)^k \frac{(\log(1+\lambda x))^k}{k!} \\
 &= \sum_{k=0}^{\infty} m^k \lambda^{-k} \sum_{n=k}^{\infty} S_1(n, k) \frac{\lambda^n x^n}{n!} \\
 &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n m^k \lambda^{n-k} S_1(n, k)\right) \frac{x^n}{n!}.
 \end{aligned} \tag{17}$$

Then one can see that

$$\begin{aligned} p^r &= \left(1 - \lambda \frac{q}{\lambda}\right)^{\frac{r\lambda}{\lambda}} = e^{-r\lambda} \left(\frac{q}{\lambda}\right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^n (-r\lambda)^k (-\lambda)^{n-k} S_1(n, k)\right) \frac{q^n}{\lambda^n n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n (-1)^n r^k S_1(n, k)\right) \frac{q^n}{n!}. \end{aligned}$$

Hence by using Theorem 2.2 we get

$$\begin{aligned} \mathcal{K}_{n,\lambda}^{(r)}(q) &= \mathbb{E}[(X)_{n,\lambda}] \\ &= \sum_{m=0}^{\infty} \left(\sum_{j=0}^m (-1)^m r^j S_1(m, j)\right) \frac{q^m}{m!} \sum_{k=0}^{\infty} \frac{(-r)_{k,\lambda}}{k!} (-q)^k (k)_{n,\lambda} \\ &= \sum_{l=0}^{\infty} \left(\sum_{k=0}^l \sum_{j=0}^{l-k} (-1)^{l-k} r^j S_1(l-k, j)\right) \frac{(-r)_{k,\lambda}}{k!} (-q)^k (k)_{n,\lambda} \frac{q^{l-k}}{(l-k)!} \\ &= \sum_{l=0}^{\infty} \left(\sum_{k=0}^l \sum_{j=0}^{l-k} (-1)^l \binom{l}{k} r^j S_1(l-k, j) (-r)_{k,\lambda} (k)_{n,\lambda}\right) \frac{q^l}{l!}. \end{aligned}$$

Thus, the statement is established.  $\square$

**Theorem 2.5.** For  $n \in \mathbb{N}$ , we have

$$\mathcal{K}_{n,\lambda}^{(r)}(q) = \sum_{l=0}^{\infty} \sum_{n_1=0}^l \sum_{k_2=0}^{l-n_1} \sum_{k_1=0}^{n_1} \sum_{k=0}^n \binom{l}{n_1} \frac{(-1)^{l+k_1} r^{k_1+k_2} n_1^k \lambda^{n+n_1-k_1-k} q^l}{l!} S_1(l-n_1, k_2) S_1(n_1, k_1) S_1(n, k).$$

*Proof.* By using (17) we have

$$e^{-r\lambda} \left(\frac{q}{\lambda}\right) = \sum_{n_2=0}^{\infty} \left(\sum_{k_2=0}^{n_2} (-r\lambda)^{k_2} (-\lambda)^{n_2-k_2} S_1(n_2, k_2)\right) \frac{q^{n_2}}{\lambda^{n_2} n_2!}$$

and

$$e_{\lambda}^{-r}(-qe_{\lambda}(t)) = \sum_{n_1=0}^{\infty} \left(\sum_{k_1=0}^{n_1} (-r)^{k_1} \lambda^{n_1-k_1} S_1(n_1, k_1)\right) \frac{(-q)^{n_1}}{n_1!} e_{\lambda}^{n_1}(t).$$

Hence we get

$$\begin{aligned} &\sum_{n=0}^{\infty} \mathcal{K}_{n,\lambda}^{(r)}(q) \frac{t^n}{n!} \\ &= e^{-r\lambda} \left(\frac{q}{\lambda}\right) e_{\lambda}^{-r}(-qe_{\lambda}(t)) \\ &= \sum_{l=0}^{\infty} \sum_{n_1=0}^l \sum_{k_2=0}^{l-n_1} (-r\lambda)^{k_2} (-\lambda)^{l-n_1-k_2} S_1(l-n_1, k_2) \frac{q^{l-n_1}}{\lambda^{l-n_1} (l-n_1)!} \\ &\quad \sum_{k_1=0}^{n_1} (-r)^{k_1} \lambda^{n_1-k_1} S_1(n_1, k_1) \frac{(-q)^{n_1}}{n_1!} \sum_{n=0}^{\infty} \left(\sum_{k=0}^n n_1^k \lambda^{n-k} S_1(n, k)\right) \frac{t^n}{n!} \\ &= \sum_{l=0}^{\infty} \left(\sum_{n_1=0}^l \sum_{k_2=0}^{l-n_1} \sum_{k_1=0}^{n_1} \binom{l}{n_1} (-1)^{l+k_1} r^{k_1+k_2} \lambda^{n_1-k_1} S_1(l-n_1, k_2) S_{1,\lambda}(n_1, k_1)\right) \frac{q^l}{l!} \sum_{n=0}^{\infty} \left(\sum_{k=0}^n n_1^k \lambda^{n-k} S_1(n, k)\right) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left[\sum_{l=0}^{\infty} \sum_{n_1=0}^l \sum_{k_2=0}^{l-n_1} \sum_{k_1=0}^{n_1} \sum_{k=0}^n \binom{l}{n_1} \frac{(-1)^{l+k_1} r^{k_1+k_2} n_1^k \lambda^{n+n_1-k_1-k} q^l}{l!} S_1(l-n_1, k_2) S_1(n_1, k_1) S_1(n, k)\right] \frac{t^n}{n!}. \end{aligned}$$

The result follows immediately from a direct comparison of coefficients on both sides in the above equation and the proof is completed.  $\square$

From [15], Kim et al. naturally considered the degenerate derangement polynomials and degenerate Frobenius-Euler polynomials, respectively, which are given by

$$\frac{1}{1-t} e_{\lambda}^{-1}(t) e_{\lambda}^x(t) = \sum_{n=0}^{\infty} d_{n,\lambda}(x) \frac{t^n}{n!} = \frac{1}{1-t} e_{\lambda}^{x-1}(t) \quad (18)$$

and

$$\frac{1-s}{e_{\lambda}(t)-s} e_{\lambda}^x(t) = \sum_{n=0}^{\infty} h_{n,\lambda}(x|s) \frac{t^n}{n!}. \quad (19)$$

When  $x = 0$ ,  $h_{n,\lambda}^*(u) := h_{n,\lambda}(0|u)$  are called the degenerate Frobenius-Euler numbers and we obtain

$$\frac{1-s}{e_{\lambda}(t)-s} = \sum_{n=0}^{\infty} h_{n,\lambda}^*(s) \frac{t^n}{n!}. \quad (20)$$

**Theorem 2.6.** For  $n \in \mathbb{N}$ , we have

$$\frac{1}{1+q} \sum_{l=0}^n \binom{n}{l} h_{l,\lambda}^*(-q^{-1}) \mathcal{K}_{n-l,\lambda}^{(r)}(q) = p^r \lambda^n \sum_{k=0}^n \sum_{l=0}^{\infty} \frac{(-q)^l}{l!} d_{l,\lambda}(1-r) l^k \lambda^{-k} S_1(n, k). \quad (21)$$

*Proof.* By using (17) and (18) we have

$$\frac{1}{1+qe_{\lambda}(t)} e_{\lambda}^{-r}(-qe_{\lambda}(t)) = \sum_{n=0}^{\infty} d_{n,\lambda}(1-r) \frac{(-qe_{\lambda}(t))^n}{n!}.$$

Thus we obtain

$$\begin{aligned} \frac{1}{1+qe_{\lambda}(t)} \sum_{n=0}^{\infty} \mathcal{K}_{n,\lambda}^{(r)}(q) \frac{t^n}{n!} &= \frac{1}{1+qe_{\lambda}(t)} p^r e_{\lambda}^{-r}(-qe_{\lambda}(t)) \\ &= \sum_{l=0}^{\infty} p^r d_{l,\lambda}(1-r) \frac{(-qe_{\lambda}(t))^l}{l!} \\ &= \sum_{l=0}^{\infty} p^r d_{l,\lambda}(1-r) \frac{(-q)^l}{l!} \sum_{n=0}^{\infty} \left( \sum_{k=0}^{\infty} l^k \lambda^{n-k} S_1(n, k) \right) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left( \sum_{k=0}^n \sum_{l=0}^{\infty} p^r d_{l,\lambda}(1-r) \frac{(-q)^l}{l!} l^k \lambda^{n-k} S_1(n, k) \right) \frac{t^n}{n!}. \end{aligned} \quad (22)$$

On the other hand by using (20) we get

$$\begin{aligned} \frac{1}{1+qe_{\lambda}(t)} \sum_{n=0}^{\infty} \mathcal{K}_{n,\lambda}^{(r)}(q) \frac{t^n}{n!} &= \frac{q^{-1}}{e_{\lambda}(t) + q^{-1}} \sum_{n=0}^{\infty} \mathcal{K}_{n,\lambda}^{(r)}(q) \frac{t^n}{n!} \\ &= \frac{q^{-1}}{1+q^{-1}} \frac{1-(-q^{-1})}{e_{\lambda}(t) - (-q^{-1})} \sum_{n=0}^{\infty} \mathcal{K}_{n,\lambda}^{(r)}(q) \frac{t^n}{n!} \\ &= \frac{1}{1+q} \sum_{l=0}^{\infty} h_{l,\lambda}^*(-q^{-1}) \frac{t^l}{l!} \sum_{n=0}^{\infty} \mathcal{K}_{n,\lambda}^{(r)}(q) \frac{t^n}{n!} \\ &= \frac{1}{1+q} \sum_{n=0}^{\infty} \left( \sum_{l=0}^n \binom{n}{l} h_{l,\lambda}^*(-q^{-1}) \mathcal{K}_{n-l,\lambda}^{(r)}(q) \right) \frac{t^n}{n!}. \end{aligned} \quad (23)$$

Thus, from (22) and (23) we have

$$\frac{1}{1+q} \sum_{l=0}^n \binom{n}{l} h_{l,\lambda}^*(-q^{-1}) \mathcal{K}_{n-l,\lambda}^{(r)}(q) = p^r \lambda^n \sum_{k=0}^n \sum_{l=0}^{\infty} \frac{(-q)^l}{l!} d_{l,\lambda} (1-r) l^k \lambda^{-k} S_1(n, k).$$

□

**Theorem 2.7.** For  $n \in \mathbb{N}$ , we have

$$\frac{d}{dq} \mathcal{K}_{n,\lambda}^{(r)}(q) = \frac{r}{1-\lambda q} \sum_{l=0}^n \binom{n}{l} h_{l,\lambda} (1|\lambda^{-1}q^{-1}) \mathcal{K}_{n-l,\lambda}^{(r)}(q) - \frac{r}{p} \mathcal{K}_{n,\lambda}^{(r)}(q). \quad (24)$$

*Proof.* First we note that

$$\frac{d}{dt} e_{\lambda}^x(t) = x e_{\lambda}^{x-\lambda}(t). \quad (25)$$

By using (19) and (25), we get

$$\begin{aligned} \frac{d}{dq} \left( \sum_{n=0}^{\infty} \mathcal{K}_{n,\lambda}^{(r)}(q) \frac{t^n}{n!} \right) &= \frac{d}{dq} \left( (1-q)^r e_{\lambda}^{-r}(-qe_{\lambda}(t)) \right) \\ &= -r(1-q)^{r-1} e_{\lambda}^{-r}(-qe_{\lambda}(t)) + r(1-q)^r e_{\lambda}(t) e_{\lambda}^{-r-\lambda}(-qe_{\lambda}(t)) \\ &= \left[ r e_{\lambda}(t) e_{\lambda}^{-\lambda}(-qe_{\lambda}(t)) - \frac{r}{1-q} \right] (1-q)^r e_{\lambda}^{-r}(-qe_{\lambda}(t)) \\ &= \left[ r \frac{e_{\lambda}(t)}{1-\lambda q e_{\lambda}(t)} - \frac{r}{1-q} \right] \sum_{m=0}^{\infty} \mathcal{K}_{m,\lambda}^{(r)}(q) \frac{t^m}{m!} \\ &= r \left[ \frac{(-\lambda q)^{-1}}{1-(\lambda q)^{-1}} \frac{1-(\lambda q)^{-1}}{e_{\lambda}(t) - (\lambda q)^{-1}} e_{\lambda}(t) - \frac{1}{p} \right] \sum_{m=0}^{\infty} \mathcal{K}_{m,\lambda}^{(r)}(q) \frac{t^m}{m!} \\ &= r \left[ \frac{1}{1-\lambda q} \sum_{l=0}^{\infty} h_{l,\lambda} (1|\lambda^{-1}q^{-1}) \frac{t^l}{l!} - \frac{1}{p} \right] \sum_{m=0}^{\infty} \mathcal{K}_{m,\lambda}^{(r)}(q) \frac{t^m}{m!} \\ &= \frac{r}{1-\lambda q} \sum_{n=0}^{\infty} \sum_{l=0}^n \binom{n}{l} h_{l,\lambda} (1|\lambda^{-1}q^{-1}) \mathcal{K}_{n-l,\lambda}^{(r)}(q) \frac{t^n}{n!} - \frac{r}{p} \sum_{n=0}^{\infty} \mathcal{K}_{n,\lambda}^{(r)}(q) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left[ \frac{r}{1-\lambda q} \sum_{l=0}^n \binom{n}{l} h_{l,\lambda} (1|\lambda^{-1}q^{-1}) \mathcal{K}_{n-l,\lambda}^{(r)}(q) - \frac{r}{p} \mathcal{K}_{n,\lambda}^{(r)}(q) \right] \frac{t^n}{n!}. \end{aligned} \quad (26)$$

Thus, by equating the corresponding coefficients in (26), we obtain the desired identity. □

**Theorem 2.8.** For  $n \in \mathbb{N}$ , we have

$$\mathcal{K}_{n+1,\lambda}^{(r)}(q) = p^r \sum_{m=1}^{\infty} \frac{(r)_{m,-\lambda}}{(m-1)!} q^m (m-\lambda)_{n,\lambda}. \quad (27)$$

*Proof.* By using (5) and (25), we get

$$\begin{aligned}
 \frac{d}{dt} \left( \sum_{n=0}^{\infty} \mathcal{K}_{n,\lambda}^{(r)}(q) \frac{t^n}{n!} \right) &= p^r \frac{d}{dt} \left( e_{\lambda}^{-r}(-qe_{\lambda}(t)) \right) \\
 &= p^r \frac{d}{dt} \left( 1 - \lambda q e_{\lambda}(t) \right)^{-\frac{r}{\lambda}} \\
 &= p^r \frac{d}{dt} \left[ \sum_{m=0}^{\infty} \binom{-r/\lambda}{m} (-1)^m (\lambda q)^m e_{\lambda}^m(t) \right] \\
 &= p^r \sum_{m=0}^{\infty} (-1)^m \frac{(-r)_{m,\lambda}}{m!} q^m m e_{\lambda}^{m-1}(t) \\
 &= p^r \sum_{m=1}^{\infty} (-1)^m \frac{(-r)_{m,\lambda}}{(m-1)!} q^m \sum_{n=0}^{\infty} (m-1)_{n,\lambda} \frac{t^n}{n!} \\
 &= p^r \sum_{n=0}^{\infty} \left( \sum_{m=1}^{\infty} \frac{(r)_{m-1,\lambda}}{(m-1)!} q^m (m-1)_{n,\lambda} \right) \frac{t^n}{n!}.
 \end{aligned} \tag{28}$$

On the other hand, we have

$$\frac{d}{dt} \left( \sum_{n=0}^{\infty} \mathcal{K}_{n,\lambda}^{(r)}(q) \frac{t^n}{n!} \right) = \sum_{n=1}^{\infty} \mathcal{K}_{n,\lambda}^{(r)}(q) \frac{t^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \mathcal{K}_{n+1,\lambda}^{(r)}(q) \frac{t^n}{n!}. \tag{29}$$

Therefore, by comparing the coefficients of (28) and (29) we obtain the desired result.  $\square$

### 3. The two variables Degenerate Krawtchouk polynomials

In this section, we introduce a new class of two-variable degenerate Krawtchouk polynomials associated with degenerate Pascal random variables characterized by the parameters  $0 < q < 1$  and  $0 < \alpha < 1$ . We proceed to derive some explicit expressions and identities for these polynomials, highlighting their structural properties and potential connections to known combinatorial constructs.

**Definition 3.1.** The two-variable degenerate Krawtchouk polynomials are defined via the generating function:

$$\sum_{n=0}^{\infty} \mathcal{K}_{n,\lambda}^{(r)}(q, \alpha) \frac{t^n}{n!} = p^r e_{\lambda}^{-r}(-qe_{\lambda}(t)) e_{\lambda}^{\alpha-q}(t). \tag{30}$$

**Theorem 3.2.** Let  $X$  be a degenerate Pascal random variables with two parameters  $q$  and  $\alpha$ . For  $n \in \mathbb{N}$ , we have

$$\mathcal{K}_{n,\lambda}^{(r)}(q, \alpha) = \mathbb{E}[(X - q + \alpha)_{n,\lambda}]. \tag{31}$$

*Proof.* By using (7), we have

$$\begin{aligned}
 \mathbb{E}[e_{\lambda}^{X-q+\alpha}(t)] &= \sum_{k=0}^{\infty} e_{\lambda}^{k-q+\alpha}(t) \mathcal{NB}_{p,r}^{(\lambda)}(\{k\}) \\
 &= e_{\lambda}^{\alpha-q}(t) \sum_{k=0}^{\infty} e_{\lambda}^k(t) p^r \frac{(-r)_{k,\lambda}}{k!} (-q)^k \\
 &= p^r e_{\lambda}^{\alpha-q}(t) \sum_{k=0}^{\infty} \frac{(-r)_{k,\lambda}}{k!} (-qe_{\lambda}(t))^k \\
 &= p^r e_{\lambda}^{\alpha-q}(t) e_{\lambda}^{-r}(-qe_{\lambda}(t)) \\
 &= \sum_{n=0}^{\infty} \mathcal{K}_{n,\lambda}^{(r)}(q, \alpha) \frac{t^n}{n!}.
 \end{aligned} \tag{32}$$

On the other hand, we observe that

$$\mathbb{E}\left[e_\lambda^{X-q+\alpha}(t)\right] = \mathbb{E}\left[\sum_{n=0}^{\infty} (X-q+\alpha)_{n,\lambda} \frac{t^n}{n!}\right] = \sum_{n=0}^{\infty} \mathbb{E}\left[(X-q+\alpha)_{n,\lambda}\right] \frac{t^n}{n!}. \quad (33)$$

Therefore, by comparing the coefficients of (32) and (33) we obtain the desired result.  $\square$

**Theorem 3.3.** For  $n \in \mathbb{N}$ , we have

$$\mathcal{K}_{n,\lambda}^{(r)}(q, \alpha) = \sum_{l=0}^n \binom{n}{l} \mathcal{K}_{l,\lambda}^{(r)}(q) (\alpha - q)_{n-l,\lambda}. \quad (34)$$

*Proof.* From (5) and (9), we have

$$\begin{aligned} \sum_{n=0}^{\infty} \mathcal{K}_{n,\lambda}^{(r)}(q, \alpha) \frac{t^n}{n!} &= p^r e_\lambda^{-r}(-qe_\lambda(t)) e_\lambda^{\alpha-q}(t) \\ &= \sum_{l=0}^{\infty} \mathcal{K}_{l,\lambda}^{(r)}(q) \frac{t^l}{l!} \sum_{m=0}^{\infty} (\alpha - q)_{m,\lambda} \frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty} \left( \sum_{l=0}^n \binom{n}{l} \mathcal{K}_{l,\lambda}^{(r)}(q) (\alpha - q)_{n-l,\lambda} \right) \frac{t^n}{n!}. \end{aligned} \quad (35)$$

Therefore, by comparing the coefficients on both sides of (35), we obtain the desired result.  $\square$

The identity below plays a crucial role in the proof of the next theorem:

$$(x + y)_{n,\lambda} = \sum_{k=0}^n \binom{n}{k} (x)_{k,\lambda} (y)_{n-k,\lambda}, \quad n \geq 0.$$

For more details see [19].

**Theorem 3.4.** For  $n \in \mathbb{N}$ , we have

$$\mathcal{K}_{n,\lambda}^{(r)}(q, \alpha) = \frac{p^r}{e_\lambda^r(-q)} \sum_{l=0}^n \sum_{k=0}^l \binom{n}{l} (\alpha - q)_{n-l,\lambda} \left(\frac{q}{1-\lambda q}\right)^k (r)_{k,-\lambda} S_{2,\lambda}(l, k). \quad (36)$$

*Proof.* From Theorem 2.2, Theorem 2.3 and Theorem 3.2, we have

$$\begin{aligned} \mathcal{K}_{n,\lambda}^{(r)}(q, \alpha) &= \mathbb{E}\left[(X - q + \alpha)_{n,\lambda}\right] \\ &= \sum_{l=0}^n \binom{n}{l} (\alpha - q)_{n-l,\lambda} \mathbb{E}[(X)_{l,\lambda}] \\ &= \sum_{l=0}^n \binom{n}{l} (\alpha - q)_{n-l,\lambda} \mathcal{K}_{l,\lambda}^{(r)}(q) \\ &= \sum_{l=0}^n \binom{n}{l} (\alpha - q)_{n-l,\lambda} \frac{p^r}{e_\lambda^r(-q)} \sum_{k=0}^l \left(\frac{q}{1-\lambda q}\right)^k (r)_{k,-\lambda} S_{2,\lambda}(l, k) \\ &= \frac{p^r}{e_\lambda^r(-q)} \sum_{l=0}^n \sum_{k=0}^l \binom{n}{l} (\alpha - q)_{n-l,\lambda} \left(\frac{q}{1-\lambda q}\right)^k (r)_{k,-\lambda} S_{2,\lambda}(l, k). \end{aligned}$$

Thus, we get the desired result.  $\square$

**Corollary 3.5.** Let  $X$  be a degenerate Pascal random variables with parameters  $r > 0$  and  $0 < q < 1$ . For  $n \geq 0$ , we get the Pascal degenerate central moments of  $X$  as follows:

$$\mathbb{E}[(X - q)_{n,\lambda}] = \frac{p^r}{e_\lambda^r(-q)} \sum_{l=0}^n \sum_{k=0}^l \binom{n}{l} \left(\frac{q}{1-\lambda q}\right)^k (-q)_{n-l,\lambda}(r)_{k,-\lambda} S_{2,\lambda}(l, k).$$

**Theorem 3.6.** For  $n \in \mathbb{N}$ , we have

$$\mathcal{K}_{n,\lambda}^{(r)}(q, \alpha) = \sum_{l=0}^n \sum_{k=0}^l \binom{n}{l} (\alpha - q)^k \lambda^{l-k} S_1(l, k) \mathcal{K}_{n-l,\lambda}^{(r)}(q). \quad (37)$$

*Proof.* By using (9) and (17), we have

$$\begin{aligned} \sum_{n=0}^{\infty} \mathcal{K}_{n,\lambda}^{(r)}(q, \alpha) \frac{t^n}{n!} &= p^r e_\lambda^{-r}(-q e_\lambda(t)) e_\lambda^{\alpha-q}(t) \\ &= \sum_{j=0}^{\infty} \mathcal{K}_{j,\lambda}^{(r)}(q) \frac{t^j}{j!} \sum_{l=0}^{\infty} \left( \sum_{k=0}^l (\alpha - q)^k \lambda^{l-k} S_1(l, k) \right) \frac{t^l}{l!} \\ &= \sum_{n=0}^{\infty} \left( \sum_{l=0}^n \sum_{k=0}^l \binom{n}{l} (\alpha - q)^k \lambda^{l-k} S_1(l, k) \mathcal{K}_{n-l,\lambda}^{(r)}(q) \right) \frac{t^n}{n!}. \end{aligned} \quad (38)$$

Consequently, identification of the coefficients on both sides of equation (38) completes the proof.  $\square$

**Theorem 3.7.** For  $n \in \mathbb{N}$ , we have

$$\frac{1}{1+q} \sum_{l=0}^n \binom{n}{l} h_{l,\lambda}^*(-q^{-1}) \mathcal{K}_{n-l,\lambda}^{(r)}(q, \alpha) = p^r \sum_{k=0}^n \sum_{l=0}^{\infty} \frac{(-q)^l}{l!} (l + \alpha - q)^k \lambda^{n-k} d_{l,\lambda}(1-r) S_1(n, k). \quad (39)$$

*Proof.* By using (9), (17) and (18), we have

$$\begin{aligned} \frac{1}{1+q e_\lambda(t)} \sum_{n=0}^{\infty} \mathcal{K}_{n,\lambda}^{(r)}(q, \alpha) \frac{t^n}{n!} &= \frac{1}{1+q e_\lambda(t)} p^r e_\lambda^{-r}(-q e_\lambda(t)) e_\lambda^{\alpha-q}(t) \\ &= p^r \sum_{l=0}^{\infty} d_{l,\lambda}(1-r) \frac{(-q e_\lambda(t))^l}{l!} e_\lambda^{\alpha-q}(t) \\ &= p^r \sum_{l=0}^{\infty} d_{l,\lambda}(1-r) \frac{(-q)^l}{l!} e_\lambda^{l+\alpha-q}(t) \\ &= p^r \sum_{l=0}^{\infty} d_{l,\lambda}(1-r) \frac{(-q)^l}{l!} \sum_{n=0}^{\infty} \left( \sum_{k=0}^n (l + \alpha - q)^k \lambda^{n-k} S_1(n, k) \right) \frac{t^n}{n!} \\ &= p^r \sum_{n=0}^{\infty} \left( \sum_{k=0}^n \sum_{l=0}^{\infty} \frac{(-q)^l}{l!} (l + \alpha - q)^k \lambda^{n-k} d_{l,\lambda}(1-r) S_1(n, k) \right) \frac{t^n}{n!}. \end{aligned} \quad (40)$$

On the other hand, by the same way of (23), we get

$$\frac{1}{1+q e_\lambda(t)} \sum_{n=0}^{\infty} \mathcal{K}_{n,\lambda}^{(r)}(q, \alpha) \frac{t^n}{n!} = \frac{1}{1+q} \sum_{n=0}^{\infty} \sum_{l=0}^n \binom{n}{l} h_{l,\lambda}^*(-q^{-1}) \mathcal{K}_{n-l,\lambda}^{(r)}(q, \alpha) \frac{t^n}{n!}. \quad (41)$$

Thus, by comparing the coefficients of (40) and (41), we get the desired result.  $\square$

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