



Invariant subspaces associated with class p - $wA(s, t)$ operators

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Abstract. In this paper, we establish that when $\mathbf{T} \in \mathcal{B}(\mathcal{H})$ belongs to a class denoted as p - $wA(s, t)$ with $0 < p \leq 1$ and $0 < s, t, s + t \leq 1$, the quasi-nilpotent component $H_0(\mathbf{T})$ of $\mathbf{T}$ is defined as follows:

$$\ker(\mathbf{T}^p) = \{x \in \mathcal{H} : r_{\mathbf{T}}(x) = 0\} = \bigcap_{\lambda \neq 0} (\mathbf{T} - \lambda)^p \mathcal{H}.$$

This characterization holds for sufficiently large integer values of p , where $r_{\mathbf{T}}(x) = \lim_{n \rightarrow \infty} \sup \|\mathbf{T}^n x\|^{\frac{1}{n}}$. Furthermore, when the spectrum $\sigma(\mathbf{T})$ is finite and $\mathbf{T}$ belongs to the class p - $wA(s, t)$, we demonstrate that $\mathbf{T}$ is an algebraic operator. Moreover, in the case where $\mathbf{T} \in \mathcal{B}(\mathcal{H})$ is part of the class p - $wA(s, t)$ and possesses the decomposition property (δ) , there exists a non-trivial invariant closed linear subspace of $\mathbf{T}$. Additionally, we uncover that an operator exhibiting such a diverse spectrum also possesses a nontrivial invariant subspace. The exploration of the existence of invariant and hyperinvariant subspaces is further elaborated upon in this study.

1. Introduction

Assuming $\mathcal{X}$ is an infinite-dimensional complex Banach space, we use the symbol $\mathcal{B}(\mathcal{X})$ to denote the algebra of all bounded linear operators on $\mathcal{X}$. Similarly, if $\mathcal{H}$ represents an infinite-dimensional complex Hilbert space, we denote the algebra of all bounded linear operators on $\mathcal{H}$ as $\mathcal{B}(\mathcal{H})$.

Throughout this paper, the terms "range" and "null space" of an operator $\mathbf{T}$ are denoted by $\text{ran}(\mathbf{T})$ and $\ker(\mathbf{T})$, respectively. We also use the notations $\sigma(\mathbf{T})$, σ_s , $\sigma_a(\mathbf{T})$, $\sigma_p(\mathbf{T})$, $\rho(\mathbf{T})$, and $r(\mathbf{T})$ to represent the spectrum, the surjective spectrum, the approximate point spectrum, the point spectrum, the resolvent set, and the spectral radius of $\mathbf{T}$, respectively. Additionally, we define $\text{Lat}(\mathbf{T})$ to represent the collection of all closed linear subspaces of $\mathcal{H}$ that are invariant under $\mathbf{T}$. For $\mathbf{T} \in \mathcal{B}(\mathcal{H})$ and $\mathcal{M} \in \text{Lat}(\mathbf{T})$, we use $\mathbf{T}|_{\mathcal{M}}$ to denote the restriction of $\mathbf{T}$ to $\mathcal{M}$.

An operator $\mathbf{T} \in \mathcal{B}(\mathcal{H})$ is considered decomposable if, for every open cover $\mathbb{C} = \mathcal{U} \cup \mathcal{V}$ of the complex plane $\mathbb{C}$, there exist $\mathcal{M}$ and $\mathcal{N}$ in $\text{Lat}(\mathbf{T})$ such that:

$$\mathcal{H} = \mathcal{M} + \mathcal{N}, \quad \sigma(\mathbf{T}|_{\mathcal{M}}) \subseteq \mathcal{U} \quad \text{and} \quad \sigma(\mathbf{T}|_{\mathcal{N}}) \subseteq \mathcal{V}.$$

This definition of decomposability is consistent with the original concept developed by Foias in 1963, as documented in the classic volumes by Colojarvã and Foias [10] and [24]. The theory of decomposable

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operators has recently seen significant development, with numerous intriguing connections and applications. Notably, all normal operators on Hilbert spaces and, more broadly, all spectral operators in the Dunford sense on Banach spaces fall under the category of decomposable operators. Additionally, it can be shown through a straightforward application of Riesz functional calculus that operators with completely disconnected spectra are also decomposable. Hence, algebraic and compact operators can be decomposed as well.

If there exists a continuous algebra homomorphism $\Phi : \varepsilon(\mathbb{C}) \rightarrow \mathcal{B}(\mathcal{H})$ with $\Phi(1) = I$ and $\Phi(z) = T$, then we refer to $T \in \mathcal{B}(\mathcal{H})$ as a generalized scalar operator (see [15]). Here, $\varepsilon(\mathbb{C})$ denotes the algebra of all infinitely differentiable functions on $\mathbb{C}$, and the topology is determined by the universal convergence of these functions and their derivatives ([10]). An operator resembling a generalized scalar operator, but restricted to one of its closed invariant subspaces, is termed a subscalar operator, and such operators are considered subdecomposable (see [15]). Each operator $T \in \mathcal{B}(\mathcal{H})$ can be expressed as $T = U|T|$, where U is a partial isometry, and $|T|$ represents the square root of T^*T . When the kernel condition $\ker U = \ker |T|$ uniquely determines U , this decomposition is referred to as the polar decomposition of T . In this paper, we use $T = U|T|$ to denote the polar decomposition that satisfies the kernel condition $\ker U = \ker |T|$.

An operator $T \in \mathcal{B}(\mathcal{H})$ is considered hyponormal if $T^*T \geq TT^*$. The Aluthge transformation, introduced by Aluthge[4], is defined as $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$, where $T = U|T|$ is the polar decomposition of $T \in \mathcal{B}(\mathcal{H})$. The generalized Aluthge transformation $T(s, t)$ with $0 < s, t$ is defined as $T(s, t) = |T|^s U |T|^t$.

To recall, an operator $T \in \mathcal{B}(\mathcal{H})$ is labeled as p -hyponormal if $(T^*T)^p \geq (TT^*)^p$ and belongs to class $WA(s, t)$ if $(|T^*|^t |T|^{2s} |T^*|^t)^{\frac{1}{s+t}} \geq |T^*|^{2t}$ and $|T|^{2s} \geq (|T|^s |T^*|^{2t} |T|^s)^{\frac{1}{s+t}}$ ([20]). Furuta et al. [18] introduced class $A(k)$ for $k > 0$ as a class of operators that encompasses p -hyponormal and log-hyponormal operators, with $A(1)$ coinciding with a class A operator. An operator T is designated as class $A(k)$, $k > 0$, if $(T^*|T|^{2k}T)^{\frac{1}{k+1}} \geq |T|^2$.

For $s, t > 0$ and $0 < p \leq 1$, an operator $T \in \mathcal{B}(\mathcal{H})$ is defined as class p - $A(s, t)$ (or belonging to class p - $A(s, t)$) if $(|T^*|^t |T|^{2s} |T^*|^t)^{\frac{pt}{s+t}} \geq |T^*|^{2tp}$ ([28]). It is termed class p - $WA(s, t)$ ([28]) if the following conditions are satisfied:

$$(|T^*|^t |T|^{2s} |T^*|^t)^{\frac{pt}{s+t}} \geq |T^*|^{2tp} \quad \text{and} \quad |T|^{2sp} \geq (|T|^s |T^*|^{2t} |T|^s)^{\frac{sp}{s+t}},$$

or equivalently,

$$|T(s, t)|^{\frac{2tp}{s+t}} \geq |T|^{2tp} \quad \text{and} \quad |T|^{2sp} \geq |(T(s, t))^*|^{\frac{2sp}{s+t}}.$$

For a given operator $T \in \mathcal{B}(\mathcal{H})$ and a linear subspace $\mathcal{M}$ within $\mathcal{H}$, we define $\mathcal{M}$ as an invariant subspace of T when $T\mathcal{M} \subset \mathcal{M}$. It's evident that both $\{0\}$ and $\mathcal{H}$ qualify as invariant subspaces, and if $\mathcal{M}$ is an invariant subspace, then its closure, denoted as $\overline{\mathcal{M}}$, also remains invariant. Specifically, an $\mathcal{M} \subset \mathcal{H}$ is termed a hyperinvariant subspace for $T \in \mathcal{B}(\mathcal{H})$ if it stays invariant under every $S \in \mathcal{B}(\mathcal{H})$ that commutes with T . Therefore, the intriguing closed invariant subspaces are those that are not trivial. The invariant subspace problem inquires whether every operator on a complex separable Hilbert space possesses a nontrivial invariant subspace. This problem traces back to approximately 1935 when, as stated in [5], J. von Neumann demonstrated that every compact operator on a separable infinite-dimensional complex Hilbert space has a nontrivial subspace. In 1954, Aronszajn and Smith [5] established that if $\mathcal{H}$ is an infinite-dimensional complex Hilbert space, and $T \in \mathcal{B}(\mathcal{H})$ is completely continuous, then T possesses a nontrivial invariant subspace. In 1966, Bernstein and Robinson [7] verified that if $\mathcal{H}$ is a complex Hilbert space, and $T \in \mathcal{B}(\mathcal{H})$ is a polynomially compact operator, i.e., there exists a non-zero polynomial p such that $p(T)$ is compact, then T has a nontrivial invariant subspace. The proof involves non-standard analysis techniques similar to those in [5]. In the same year, Halmos [13] offered an alternative proof of this result using a similar approach but avoiding non-standard tools.

An operator $Y \in \mathcal{B}(\mathcal{H})$ is classified as a quasiaffinity if it possesses a trivial kernel and a dense range. An operator T in $\mathcal{B}(\mathcal{H})$ is termed a quasiaffine transform of another operator Z in $\mathcal{B}(\mathcal{H})$ if there exists a quasiaffinity Y in $\mathcal{B}(\mathcal{H})$ such that $YT = ZY$. This relationship between Z and T is denoted as $T < Z$. If both $T < Z$ and $Z < T$, then we describe Z and T as quasisimilar.

In 1990, Eschmeier and Prunaru [16] established a significant result regarding the richness of $Lat(T)$. They demonstrated that $Lat(T)$ contains the lattice of all closed subspaces of some infinite-dimensional Banach space, provided that the essential spectrum $\sigma_e(T)$ is sufficiently extensive. Additionally, they showed that

$\text{Lat}(\mathbf{T})$ is nontrivial when the spectrum $\sigma(\mathbf{T})$ is also sufficiently extensive. While we won't delve into the formal definition of "thick" subsets of the complex plane here, it's worth noting that thick sets include all compact sets with non-empty interiors. The invariant subspace problem has sparked a significant amount of research in operator theory, as indicated by the citations provided by [6], [1], [8], [9], [32], and [30].

Here is an overview of the paper's organization: In the following section, we present preliminary definitions and observations related to class $p\text{-}wA(s, t)$ operators. The third section focuses on addressing the issue of invariant subspaces for a class of $p\text{-}wA(s, t)$ operators. Specifically, we demonstrate that if $\mathbf{T}$ falls into the class $p\text{-}wA(s, t)$ and has a finite spectrum $\sigma(\mathbf{T})$, then $\mathbf{T}$ is algebraic. Moreover, we establish that if $\mathbf{T}$ is a class $p\text{-}wA(s, t)$ operator with the decomposition property (δ) , it possesses a non-trivial invariant closed linear subspace. We also uncover that operators with such rich spectra have nontrivial invariant subspaces. Additionally, we explore the existence of invariant and hyperinvariant subspaces for class $p\text{-}wA(s, t)$ operators. Furthermore, if $\mathbf{T}$ belongs to the class $p\text{-}wA(s, t)$ and is not a scalar multiple of the identity, and if there exists a constant K and a positive integer m such that $\|\mathbf{T}^n\| \leq K|n|^m$ for $n = \pm 1, \pm 2, \dots$, then we establish that $\mathbf{T}$ has a hyperinvariant subspace.

2. Preliminaries and Complementary results

The local resolvent set $\rho_{\mathbf{T}}(x)$ of the operator $\mathbf{T}$ at a vector x in $\mathcal{X}$ is defined as the collection of all complex numbers λ in $\mathbb{C}$ for which there exists an analytic function ψ that maps to $\mathcal{X}$ and is defined on an open neighborhood $\mathcal{U}$ of λ , satisfying the equation:

$$(\mathbf{T} - \mu)\psi(\mu) = x, \text{ for all } \mu \in \mathcal{U}.$$

The formula $\sigma_{\mathbf{T}}(x) = \mathbb{C} \setminus \rho_{\mathbf{T}}(x)$ characterizes the local spectrum. It's important to note that the local spectrum $\sigma_{\mathbf{T}}(x)$, which is a subset of the spectrum $\sigma(\mathbf{T})$, may be empty. Additionally, as established in [24], we have:

$$\sigma_s(\mathbf{T}) = \bigcup_{x \in \mathcal{X}} \sigma_{\mathbf{T}}(x)$$

For $T \in \mathcal{B}(\mathcal{X})$ and $\mathcal{F} \subseteq \mathbb{C}$, we define $H_{\mathbf{T}}(\mathcal{F})$ as the local spectral subspace, given by:

$$H_{\mathbf{T}}(\mathcal{F}) = \{x \in \mathcal{X} : \sigma_{\mathbf{T}}(x) \subseteq \mathcal{F}\}.$$

It's worth noting that $H_{\mathbf{T}}(\mathcal{F})$ is a linear subspace of $\mathcal{X}$, although not necessarily closed. If $H_{\mathbf{T}}(\mathcal{F})$ is closed for every closed subset $\mathcal{F}$ of $\mathbb{C}$, then the operator $\mathbf{T}$ is said to possess Dunford's property (C).

Moving on, an operator $T \in \mathcal{B}(\mathcal{X})$ is considered to have the single valued extension property (SVEP for short) at $\lambda \in \mathbb{C}$ if there exists an open disc V centered at λ such that for every open subset $\mathcal{U} \subset V$, the only solution to the equation $(\mathbf{T} - \mu)\psi(\mu) = 0$ for all $\mu \in \mathcal{U}$ is the constant function $f \equiv 0$.

An operator $T \in \mathcal{B}(\mathcal{X})$ is deemed to possess the single valued extension property (SVEP for brevity) at a point $\lambda \in \mathbb{C}$ if there exists an open disk V centered around λ such that for every open subset $\mathcal{U} \subset V$, the sole analytic solution of the equation

$$(\mathbf{T} - \mu)\psi(\mu) = 0 \text{ for all } \mu \in \mathcal{U}.$$

The algebraic spectral subspace $E_{\mathbf{T}}(\mathcal{F})$ is constructed as the linear span of a set comprising various (potentially non-closed) linear subspaces $\mathcal{M}$ of $\mathcal{H}$ for which the equation $(\mathbf{T} - \lambda)\mathcal{M} = \mathcal{M}$ holds true for every λ belonging to $\mathbb{C} \setminus \mathcal{F}$. This subspace is closely associated with the operator $\mathbf{T}$ and corresponds to each closed subset $\mathcal{F}$ of $\mathbb{C}$. Evidently, $E_{\mathbf{T}}(\mathcal{F})$ is the largest linear subspace $\mathcal{M}$ for which $(\mathbf{T} - \lambda)\mathcal{M} = \mathcal{M}$ for all $\lambda \in \mathbb{C} \setminus \mathcal{F}$. It follows from Proposition 1.2.16 of [24] that $H_{\mathbf{T}}(\mathcal{F}) \subseteq E_{\mathbf{T}}(\mathcal{F})$ for every $\mathbf{T} \in \mathcal{B}(\mathcal{H})$ and all closed set $\mathcal{F} \subseteq \mathbb{C}$. Thus if $\mathbf{T}$ has no non-trivial divisible subspace in the sense that $E_{\mathbf{T}}(\emptyset) = \{0\}$, then clearly $\mathbf{T}$ has SVEP. By the open mapping theorem, we observe, for a closed set $F \subseteq \mathbb{C}$ that if $E_{\mathbf{T}}(\mathcal{F})$ is closed, then $E_{\mathbf{T}}(\mathcal{F}) = H_{\mathbf{T}}(\mathcal{F})$, see [24].

For an open set $\mathcal{U}$ of $\mathbb{C}$, let $F(\mathcal{U}, \mathcal{X})$ be the Fréchet space of all $\mathcal{X}$ -valued analytic function on $\mathcal{U}$ endowed

with the topology defined by uniform convergence on every compact subset of $\mathcal{U}$. An operator $T \in \mathcal{B}(\mathcal{X})$ is said to satisfy the Bishop's property (β) on an open set $\mathcal{U} \subseteq \mathbb{C}$ provided that for every open subset $\mathcal{V}$ of $\mathcal{U}$ and for any sequence $(\psi_n)_n$ of analytic $\mathcal{X}$ -valued functions on V ,

$$(\mathbf{T} - \mu)\psi_n(\mu) \longrightarrow 0 \text{ in } F(\mathcal{V}, \mathcal{X}) \text{ implies } \psi_n(\mu) \longrightarrow 0 \text{ in } F(\mathcal{V}, \mathcal{X}).$$

Let $\rho_\beta(\mathbf{T})$ be the largest open set on which $\mathbf{T}$ has the property (β) . Its complement $\sigma_\beta(\mathbf{T}) = \mathbb{C} \setminus \rho_\beta(\mathbf{T})$ is a closed, possibly empty, subset of $\sigma(\mathbf{T})$. Then $\mathbf{T}$ is said to satisfy the Bishop's property (β) , precisely when $\sigma_\beta(\mathbf{T}) = \emptyset$, [3, 27]. Clearly, $E_{\mathbf{T}}(\mathcal{F})$ represents the most extensive linear subspace $\mathcal{M}$ where $(\mathbf{T} - \lambda)\mathcal{M} = \mathcal{M}$ for all $\lambda \in \mathbb{C} \setminus \mathcal{F}$. Proposition 1.2.16 from [24] establishes that, for any $\mathbf{T} \in \mathcal{B}(\mathcal{H})$ and every closed set $\mathcal{F} \subseteq \mathbb{C}$, $H_{\mathbf{T}}(\mathcal{F}) \subseteq E_{\mathbf{T}}(\mathcal{F})$. Thus, if $\mathbf{T}$ lacks any non-trivial divisible subspace, meaning $E_{\mathbf{T}}(\emptyset) = 0$, it logically implies that $\mathbf{T}$ has the Single Valued Extension Property (SVEP), based on the Open Mapping Theorem.

Furthermore, according to the Open Mapping Theorem, for a closed set $\mathcal{F} \subseteq \mathbb{C}$, if $E_{\mathbf{T}}(\mathcal{F})$ is a closed subspace, then $E_{\mathbf{T}}(\mathcal{F}) = H_{\mathbf{T}}(\mathcal{F})$, as outlined in [24].

Now, moving on to the Bishop's property, consider an open set $\mathcal{U}$ within $\mathbb{C}$. Let $F(\mathcal{U}, \mathcal{X})$ denote the Fréchet space of all analytic functions mapping to $\mathcal{X}$ on $\mathcal{U}$, equipped with the topology defined by uniform convergence on every compact subset of $\mathcal{U}$. An operator $T \in \mathcal{B}(\mathcal{X})$ is said to possess Bishop's property (β) on an open set $\mathcal{U} \subseteq \mathbb{C}$ if, for any open subset $\mathcal{V}$ of $\mathcal{U}$ and for any sequence $(\psi_n)_n$ of analytic $\mathcal{X}$ -valued functions on V :

$$(\mathbf{T} - \mu)\psi_n(\mu) \longrightarrow 0 \text{ in } F(\mathcal{V}, \mathcal{X}) \text{ implies } \psi_n(\mu) \longrightarrow 0 \text{ in } F(\mathcal{V}, \mathcal{X}).$$

Let $\rho_\beta(\mathbf{T})$ represent the largest open set over which $\mathbf{T}$ possesses property (β) . Its complement, $\sigma_\beta(\mathbf{T}) = \mathbb{C} \setminus \rho_\beta(\mathbf{T})$, is a closed subset of $\sigma(\mathbf{T})$, which may potentially be empty. Hence, $\mathbf{T}$ is deemed to satisfy Bishop's property (β) precisely when $\sigma_\beta(\mathbf{T}) = \emptyset$, as indicated in [3] and [27].

It is well-known that

$$\text{Property } (\beta) \Rightarrow \text{Dunford's property } (C) \Rightarrow \text{SVEP}.$$

To incorporate the dual concept of Bishop's property (β) into the local spectral subspace, we introduce a modified version called the global spectral analytic space denoted as $X_{\mathbf{T}}(\mathcal{F})$. This space consists of vectors $x \in \mathcal{X}$ for which there exists an analytic function $\psi : \mathbb{C} \setminus \mathcal{F} \longrightarrow \mathcal{X}$ such that:

$$(\mathbf{T} - \mu)\psi(\mu) = x, \text{ for all } \mu \in \mathbb{C} \setminus \mathcal{F}.$$

Notably, the analytic function ψ is defined globally over the entire complement of $\mathcal{F}$. Clearly, the linear subspace $X_{\mathbf{T}}(\mathcal{F})$ is contained within $H_{\mathbf{T}}(\mathcal{F})$. Additionally, when $\mathbf{T}$ satisfies the Single Valued Extension Property (SVEP) [24, Proposition 3.3.2], the equivalence $X_{\mathbf{T}}(\mathcal{F}) = H_{\mathbf{T}}(\mathcal{F})$ holds for all closed sets $\mathcal{F} \subseteq \mathbb{C}$.

Now, turning our attention to the property (δ) , an operator $T \in \mathcal{B}(\mathcal{X})$ is considered to have the decomposition property (δ) on $\mathcal{U}$ if, for all open sets $\mathcal{V}, \mathcal{W} \subseteq \mathbb{C}$ where $\mathbb{C} \setminus \mathcal{U} \subseteq \mathcal{V} \subseteq \overline{\mathcal{V}} \subseteq \mathcal{W}$, the following holds:

$$X_{\mathbf{T}}(\mathbb{C} \setminus \mathcal{V}) + X_{\mathbf{T}}(\overline{\mathcal{W}}) = \mathcal{X}.$$

Let $\rho_\delta(\mathbf{T})$ denote the largest open set on which the operator $\mathbf{T}$ possesses property (δ) . Its complement $\sigma_\delta(\mathbf{T}) = \mathbb{C} \setminus \rho_\delta(\mathbf{T})$ is a closed subset of $\sigma(\mathbf{T})$, which may potentially be empty, as indicated in [27, Corollary 17]. Therefore, $\mathbf{T}$ is said to have the decomposition property (δ) if $\sigma_\delta(\mathbf{T}) = \emptyset$.

It's worth noting that properties (β) and (δ) are dual to each other in the sense that $\mathbf{T}$ has property (β) on $\mathcal{U}$ if and only if $\mathbf{T}^*$ has property (δ) on $\mathcal{U}$ [3, 27]. Moreover, we have the equivalences:

$$\sigma_\delta(\mathbf{T}) = \sigma_\beta(\mathbf{T}^*) \text{ and } \sigma_\delta(\mathbf{T}^*) = \sigma_\beta(\mathbf{T}).$$

The operator $\mathbf{T} \in \mathcal{B}(\mathcal{X})$ is deemed decomposable on $\mathcal{U}$ if, for every finite open cover $\mathcal{U}_1, \dots, \mathcal{U}_n$ of $\mathbb{C}$, where $\sigma(\mathbf{T}) \setminus \mathcal{U} \subseteq \mathcal{U}_1$, there exist closed $\mathbf{T}$ -invariant subspaces $\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_n$ of $\mathcal{X}$ such that:

$$\sigma(\mathbf{T}|_{\mathcal{X}_i}) \subseteq \mathcal{U}_i \text{ for } i = 1, \dots, n \text{ and } \mathcal{X}_1 + \mathcal{X}_2 + \dots + \mathcal{X}_n = \mathcal{X}.$$

Let $\rho_d(\mathbf{T})$ denote the largest open set $\mathcal{U} \subseteq \mathbb{C}$ on which the operator $\mathbf{T}$ exhibits decomposability. Its complement, $\sigma_d(\mathbf{T}) = \mathbb{C} \setminus \rho_d(\mathbf{T})$, is a closed subset of $\sigma(\mathbf{T})$, which may potentially be empty.

We describe $\mathbf{T}$ as decomposable if $\sigma_d(\mathbf{T}) = \emptyset$. This class of operators encompasses all normal operators and, more broadly, all spectral operators. Additionally, operators with completely disconnected spectra can be decomposed using the Riesz functional calculus. Consequently, the category of decomposable operators includes compact and algebraic ones.

Furthermore, it's worth noting that property (β) characterizes operators with decomposable extensions, as established in [3]. Thus, property (β) remains intact under restrictions, while property (δ) is transferred to quotient operators. For more comprehensive details, refer to [24]. We have

$$\sigma_d(\mathbf{T}) = \sigma_\beta(\mathbf{T}) \cup \sigma_\delta(\mathbf{T}) = \sigma_\beta(\mathbf{T}) \cup \sigma_\beta(\mathbf{T}^*) = \sigma_d(\mathbf{T}^*).$$

Let $F(\mathcal{U}, X)$ represent the Fréchet space encompassing all X -valued C^∞ -functions, i.e., functions that are infinitely continuously differentiable on $\mathcal{U}$ (as defined in [15]). We designate $\mathbf{T}$ as possessing the Eschmeier-Putinar-Bishop's property $(\beta)_\varepsilon$, denoted as $\mathbf{T} \in (\beta)_\varepsilon$, if, for every open set $\mathcal{U}$ within $\mathbb{C}$, the operator

$$\mathbf{T}_z : F(\mathcal{U}, X) \longrightarrow F(\mathcal{U}, X), \quad f \mapsto (\mathbf{T} - z)\psi$$

constitutes a topological monomorphism. In simpler terms, $\mathbf{T}_z \psi_n \longrightarrow 0$ implies $\psi_n \longrightarrow 0$ for $\psi_n \in F(\mathcal{U}, X)$.

The property $(\beta)_\varepsilon$ serves an analogous role for generalized scalar operators as Bishop's property (β) does for decomposable operators. Specifically, an operator $\mathbf{T}$ satisfies $(\beta)_\varepsilon$ if and only if $\mathbf{T}$ is subscalar as defined in [15], meaning it possesses a generalized scalar extension.

In the realms of local spectral theory and Fredholm theory, two significant subspaces are $H_{\mathbf{T}}(\lambda)$ associated with the singleton λ , and $H_{\mathbf{T}}(\mathbb{C} \setminus \lambda)$.

It's noteworthy that $H_{\mathbf{T}}(\lambda)$ coincides with the quasi-nilpotent part of an operator $\lambda - \mathbf{T}$, which is defined as:

$$H_0(\lambda - \mathbf{T}) = \{x \in \mathcal{H} : \lim_{n \rightarrow \infty} \|(\lambda - \mathbf{T})^n x\|^{\frac{1}{n}} = 0\}.$$

It's evident that $\ker(\lambda - \mathbf{T})^n \subseteq H_0(\lambda - \mathbf{T})$ for every natural number n . Additionally, $H_{\mathbf{T}}(\mathbb{C} \setminus \lambda)$ corresponds to the analytic core $K(\lambda - \mathbf{T})$, which is defined as the set of all $x \in \mathcal{H}$ for which there exists a constant $c > 0$ and a sequence of elements $x_n \in \mathcal{H}$ satisfying $x_0 = x$, $\mathbf{T}x_n = x_{n-1}$, and $\|x_n\| \leq c^n \|x\|$ for all $n \in \mathbb{N}$.

It's essential to note that, in general, $H_0(\lambda - \mathbf{T})$ and $K(\lambda - \mathbf{T})$ are not closed subspaces. Furthermore, it's observed that:

the closedness of $H_0(\lambda - \mathbf{T}) \Rightarrow \mathbf{T}$ has the SVEP at λ .

Definition 2.1. An operator $\mathbf{T} \in \mathcal{B}(\mathcal{H})$ is deemed to possess property (Q) if, for each $\lambda \in \mathbb{C}$, the subspace $H_0(\lambda - \mathbf{T})$ is closed.

It follows that

$$\text{Property } (\beta) \Rightarrow \text{Dunford's property (C)} \Rightarrow \text{Property (Q)} \Rightarrow \text{SVEP}.$$

For an operator $\mathbf{T} \in \mathcal{B}(\mathcal{H})$ acting on a Hilbert space $\mathcal{H}$ and an element $x \in \mathcal{H}$, we define the quantity:

$$r_{\mathbf{T}}(x) := \limsup_{n \rightarrow \infty} \|\mathbf{T}^n x\|^{\frac{1}{n}}$$

This quantity is referred to as the local spectral radius of $\mathbf{T}$ at x . It's evident that, for all $x \in \mathcal{H}$:

$$\max\{|\lambda| : \lambda \in \sigma_{\mathbf{T}}(x)\} \leq r_{\mathbf{T}}(x)$$

for all $x \in \mathcal{H}$. According to Proposition 3.3.13 from [24], if $\mathbf{T}$ has the Single Valued Extension Property (SVEP), then the local spectral radius formula is valid:

$$r_{\mathbf{T}}(x) = \max\{|\lambda| : \lambda \in \sigma_{\mathbf{T}}(x)\}$$

This holds true for all non-zero $x \in \mathcal{H}$. However, for operators lacking SVEP, this inequality may be strict.

The upcoming theorem elucidates the spectral characteristics of class p - $WA(s, t)$, which hold significant importance for the subsequent discussion.

Theorem 2.2. For $\mathbf{T} \in \mathcal{B}(\mathcal{H})$ with $0 < p \leq 1$ and $0 < s, t \leq 1$ where $s + t \leq 1$, the following statements hold:

- (i) If $\mathbf{T}$ belongs to class $p\text{-}wA(s, t)$, then $\mathbf{T}(s, t)$ is $\frac{\min ps, pt}{s+t}$ -hyponormal (as stated in [28]).
- (ii) If $\mathbf{T}$ is invertible and falls into class $p\text{-}wA(s, t)$, then $\mathbf{T}^{-1}$ also belongs to class $p\text{-}wA(s, t)$ (as noted in [28]).
- (iii) If $\mathbf{T}$ is classified as a class $p\text{-}wA(s, t)$ operator, then $\mathbf{T}$ is normaloid (as per [29]).
- (iv) If $\mathbf{T}$ is a class $p\text{-}wA(s, t)$ operator, then it possesses Bishop's property (β) and Eschmeier-Putinar-Bishop's property $(\beta)_\varepsilon$, making $\mathbf{T}$ subscalar (as stated in [35]).
- (v) For $\mathbf{T}$ in class $p\text{-}wA(s, t)$, the non-zero point spectrum and the non-zero joint point spectrum coincide (as mentioned in [31]).
- (vi) For $\mathbf{T}$ in class $p\text{-}wA(s, t)$, the non-zero approximate point spectrum and the non-zero joint approximate point spectrum also coincide (as indicated in [31]).
- (viii) If $\mathbf{T}$ is a class $p\text{-}wA(s, t)$ operator and $\mathcal{M}$ is $\mathbf{T}$ -invariant, then $\mathbf{T}|_{\mathcal{M}}$ is a class $A(s, t)$ operator.

Definition 2.3. An operator $\mathbf{T} \in \mathcal{B}(\mathcal{H})$ is considered to have property (I) if it satisfies Bishop's property (β) and if there exists a positive integer p such that the following condition holds for all closed sets $\mathcal{F} \subseteq \mathbb{C}$:

$$H_{\mathbf{T}}(\mathcal{F}) = E_{\mathbf{T}}(\mathcal{F}) = \bigcap_{\lambda \in \mathbb{C} \setminus \mathcal{F}} (\mathbf{T} - \lambda)^p \mathcal{H} \text{ for all closed sets } \mathcal{F} \subseteq \mathbb{C}.$$

Theorem 2.4. If $\mathbf{T} \in \mathcal{B}(\mathcal{H})$ is a class $p\text{-}wA(s, t)$ operator with $0 < p \leq 1$ and $0 < s, t, s + t \leq 1$, then $\mathbf{T}$ possesses property (I).

Proof. The proof can be derived from Theorem 2.2 (iv) and Theorem 4 in [27]. $\square$

Theorem 2.5. If $\mathbf{T} \in \mathcal{B}(\mathcal{H})$ belongs to class $p\text{-}wA(s, t)$ with $0 < p \leq 1$ and $0 < s, t, s + t \leq 1$, then $H_0(\mathbf{T} - \lambda) = \ker(\mathbf{T} - \lambda)$ for all $\lambda \in \mathbb{C}$.

Proof. Let $\mathcal{F} \subset \mathbb{C}$ be a closed set. According to Theorem 2.20 in reference [2], we have $H_0(\mathbf{T} - \lambda) = X_{\mathbf{T}}(\lambda)$. Since $\mathbf{T}$ possesses Bishop's property, as stated in Theorem 2.2(iv), and according to Proposition 1.2.19 from [24], $X_{\mathbf{T}}(\mathcal{F})$ is a closed subspace, and $\sigma(\mathbf{T}|_{X_{\mathbf{T}}(\mathcal{F})}) \subset \mathcal{F}$. Therefore, in accordance with Theorem 2.2, $\mathbf{T}|_{H_0(\mathbf{T} - \lambda)}$ is a class $p\text{-}wA(s, t)$ operator, and $H_0(\mathbf{T} - \lambda)$ is a closed subspace.

Using Lemma 7 from [33], $\mathbf{T}|_{H_0(\mathbf{T} - \lambda)}$ is normal if $\sigma(\mathbf{T}|_{H_0(\mathbf{T} - \lambda)}) \subset \{\lambda\}$. If $\sigma(\mathbf{T}|_{H_0(\mathbf{T} - \lambda)}) = \emptyset$, then $H_0(\mathbf{T} - \lambda) = \{0\}$ and $\ker(\mathbf{T} - \lambda) = \{0\}$. If $\sigma(\mathbf{T}|_{H_0(\mathbf{T} - \lambda)}) = \{\lambda\}$, then $\mathbf{T}|_{H_0(\mathbf{T} - \lambda)} = \lambda$ and $H_0(\mathbf{T} - \lambda) = \ker(\mathbf{T} - \lambda)$. $\square$

Theorem 2.6. If $\mathbf{T} \in \mathcal{B}(\mathcal{H})$ belongs to class $p\text{-}wA(s, t)$ with $0 < p \leq 1$ and $0 < s, t, s + t \leq 1$, and its spectrum $\sigma(\mathbf{T})$ is contained within $S^1 := \{z \in \mathbb{C} : |z| = 1\}$, then $\mathbf{T}$ is a unitary operator.

Proof. Based on the assumption about $\mathbf{T}$, it follows that $\mathbf{T}$ is invertible. Consequently, as per Theorem 2.2, $\mathbf{T}^{-1}$ also falls into the class $p\text{-}wA(s, t)$, making both $\mathbf{T}$ and $\mathbf{T}^{-1}$ normaloid operators. Consequently, their spectral radii are bounded by 1, which implies that $\mathbf{T}$ must be an isometry. This conclusion can be justified by noting that for any $x \in \mathcal{H}$:

$$\|x\| = \|\mathbf{T}^{-1}\mathbf{T}x\| \leq \|\mathbf{T}x\| \leq \|x\|$$

Hence, $\mathbf{T}$ must be a unitary operator. $\square$

3. Main results

This section is dedicated to demonstrating the presence of invariant and hyperinvariant subspaces for specific operators belonging to the class $p\text{-}wA(s, t)$, where $0 < p \leq 1$ and $0 < s, t, s + t \leq 1$.

Theorem 3.1. Suppose $\mathbf{T} \in \mathcal{B}(\mathcal{H})$ is a class $p\text{-}wA(s, t)$ operator with $0 < p \leq 1$ and $0 < s, t, s + t \leq 1$. Then, $\lim_{n \rightarrow \infty} \|\mathbf{T}^n x\|^{\frac{1}{n}} = 0$ if and only if there exists an integer $p > 0$ such that $\mathbf{T}^p x_0 = 0$. Additionally, $\ker(\mathbf{T}^p) = H_{\mathbf{T}}(0)$ represents the quasi-nilpotent component of $\mathbf{T}$. In this scenario, we have:

$$H_0(\mathbf{T}) = \ker(\mathbf{T}^p) = E_{\mathbf{T}}(\{0\}) = \{x \in \mathcal{H} : r_{\mathbf{T}}(x) = 0\} = \bigcap_{\lambda \in \mathbb{C} \setminus \{0\}} (\mathbf{T} - \lambda)^p \mathcal{H},$$

where $r_{\mathbf{T}}(x)$ denotes the local spectral radius of $\mathbf{T}$ at x .

Proof. Suppose that $\mathbf{T}$ is a class p - $wA(s, t)$ operator. Then, as indicated in Theorem 2.4, there exists an integer p such that the following equality holds for all closed sets $\mathcal{F} \subseteq \mathbb{C}$:

$$E_{\mathbf{T}}(\mathcal{F}) = H_{\mathbf{T}}(\mathcal{F}) = \bigcap_{\mu \in \mathbb{C} \setminus \mathcal{F}} (\mathbf{T} - \mu)^p \mathcal{H}.$$

For each $\lambda \in \mathcal{F}$, we can derive:

$$E_{\mathbf{T}}(\{\lambda\}) = \bigcap_{\lambda \neq \mu} (\mathbf{T} - \mu)^p \mathcal{H}.$$

Consequently, we have:

$$\begin{aligned} (\mathbf{T} - \lambda)^p E_{\mathbf{T}}(\{\lambda\}) &= (\mathbf{T} - \lambda)^p \left[\bigcap_{\lambda \neq \mu} (\mathbf{T} - \mu)^p \mathcal{H} \right] \\ &\subseteq \bigcap_{\mu \in \mathbb{C}} (\mathbf{T} - \mu)^p \mathcal{H} = H_{\mathbf{T}}(\emptyset) = \{0\}, \end{aligned}$$

since $\mathbf{T}$ possesses the single-valued extension property. This leads to the conclusion that $(\mathbf{T} - \lambda)^p E_{\mathbf{T}}(\lambda) = 0$, implying that $E_{\mathbf{T}}(\lambda) \subseteq \ker(\mathbf{T} - \lambda)^p$ for all $\lambda \in \mathcal{F}$. On the other hand, according to Proposition 1.2.16 of [24], it follows that:

$$\ker(\mathbf{T} - \lambda)^k \subseteq H_{\mathbf{T}}(\{\lambda\}) \subseteq E_{\mathbf{T}}(\{\lambda\})$$

for all $\lambda \in \mathcal{F}$ and $k \in \mathbb{N}$, which further leads to:

$$\ker(\mathbf{T} - \lambda)^p = H_{\mathbf{T}}(\{\lambda\}) = E_{\mathbf{T}}(\{\lambda\}) \text{ for all } \lambda \in \mathcal{F}.$$

Thus we have

$$\ker(\mathbf{T}^p) = H_0(\mathbf{T}) = H_{\mathbf{T}}(\{0\}) = E_{\mathbf{T}}(\{0\}) = \bigcap_{\lambda \neq 0} (\mathbf{T} - \lambda)^p \mathcal{H}.$$

Since $\mathbf{T}$ has Bishop's property (β) , it also has the Single-Valued Extension Property (SVEP), and hence, by Corollary 2.4 of [23], we can conclude:

$$H_{\mathbf{T}}(\{0\}) = \{x \in \mathcal{H} : \lim_{n \rightarrow \infty} \|\mathbf{T}^n x\|^{\frac{1}{n}} = 0\}.$$

Finally, it follows from Proposition 3.3.7 of [24] that $r_{\mathbf{T}}(x) = \lim_{n \rightarrow \infty} \|\mathbf{T}^n x\|^{\frac{1}{n}}$, which completes the proof. $\square$

Theorem 3.2. Suppose we have an operator $\mathbf{T} \in \mathcal{B}(\mathcal{H})$ that falls into the category of a class p - $wA(s, t)$ operator, with $0 < p \leq 1$ and $0 < s, t$, such that $s + t \leq 1$. In the case where $\mathbf{T}$ has a finite spectrum, it can be deduced that $\mathbf{T}$ is algebraic. To elaborate further, if $\mathbf{T}$ happens to be quasi-nilpotent, it follows that $\mathbf{T}$ is, in fact, nilpotent.

Proof. Suppose we have an operator $\mathbf{T}$ belonging to the class of p - $wA(s, t)$ operators with $0 < p \leq 1$ and $0 < s, t$, such that $s + t \leq 1$. According to Theorem 2.4, there exists an integer p such that the equality

$$E_{\mathbf{T}}(\mathcal{F}) = H_{\mathbf{T}}(\mathcal{F}) = \bigcap_{\mu \in \mathbb{C} \setminus \mathcal{F}} (\mathbf{T} - \mu)^p \mathcal{H}$$

holds for all closed sets $F \subseteq \mathbb{C}$. If we assume that the spectrum $\sigma(\mathbf{T})$ is finite, given by $\lambda_1, \lambda_2, \dots, \lambda_n$ as established in Theorem 3.1, then there exist positive integers $p_k \in \mathbb{N}$ for each $k = 1, 2, \dots, n$, satisfying

$$E_{\mathbf{T}}(\{\lambda_k\}) = H_{\mathbf{T}}(\{\lambda_k\}) = \ker(\mathbf{T} - \lambda_k)^{p_k}.$$

This further leads to the result that

$$H_{\mathbf{T}}(\sigma(\mathbf{T})) = E_{\mathbf{T}}(\sigma(\mathbf{T})) = \bigcap_{\mu \in \mathbb{C} \setminus \sigma(\mathbf{T})} (\mathbf{T} - \mu)^p \mathcal{H} = \mathcal{H},$$

given that $\mathbf{T} - \mu$ is invertible for each $\mu \in \mathbb{C} \setminus \sigma(\mathbf{T})$. Therefore, we can conclude, based on Theorem 1 of [34], that

$$\begin{aligned}\mathcal{H} &= H_{\mathbf{T}}(\sigma(\mathbf{T})) = H_{\mathbf{T}}(\{\lambda_1\}) \oplus H_{\mathbf{T}}(\{\lambda_2\}) \oplus \cdots \oplus H_{\mathbf{T}}(\{\lambda_n\}) \\ &= \ker(\mathbf{T} - \lambda_1)^{p_1} \oplus \ker(\mathbf{T} - \lambda_2)^{p_2} \oplus \cdots \oplus \ker(\mathbf{T} - \lambda_n)^{p_n}\end{aligned}$$

as an algebraic direct sum. Consequently, we can deduce that there exists a complex polynomial $p(\lambda)$ defined for all $\lambda \in \mathbb{C}$ as

$$p(\lambda) := (\lambda - \lambda_1)^{p_1} (\lambda - \lambda_2)^{p_2} \cdots (\lambda - \lambda_n)^{p_n}$$

which satisfies $p(\mathbf{T}) = 0$. This, in turn, implies that $\mathbf{T}$ is an algebraic operator. $\square$

Theorem 3.3. Consider an operator $\mathbf{T}$ in the class of p - $wA(s, t)$ operators with $0 < p \leq 1$ and $0 < s, t$, such that $s + t \leq 1$, and suppose that $\mathbf{T}$ possesses property (δ) on a Hilbert space of dimension greater than 1. In this case, we can conclude that $\mathbf{T}$ has a non-trivial invariant closed linear subspace.

Proof. Suppose that $\mathbf{T} \in \mathcal{B}(\mathcal{H})$ is a class p - $wA(s, t)$ operator on a Hilbert space $\mathcal{H}$ of dimension greater than 1. Then, based on Theorem 2.2 (which establishes its subscalar nature), we can deduce that $\mathbf{T}$ is subdecomposable. Now, let's proceed to show that if $\sigma(\mathbf{T})$ contains at least two points, then $\mathbf{T}$ has a non-trivial hyperinvariant closed linear subspace.

Since $\mathbf{T}$ is decomposable and subscalar, we can leverage Lemma 3.3 from [36] to conclude that $\mathbf{T}$ has a non-trivial hyperinvariant closed linear subspace.

Now, let's consider the case where $\mathbf{T} \in \mathcal{B}(\mathcal{H})$ and $\mathcal{H}$ is at least two-dimensional, and $\sigma(\mathbf{T})$ is a singleton. According to Theorem 3.2, we can express $\mathbf{T}$ as $\mathbf{T} = \lambda I + N$, where $\lambda \in \mathbb{C}$ and N is a nilpotent operator in $\mathcal{B}(\mathcal{H})$. Let $p \in \mathbb{N}$ represent the smallest integer such that $N^p = 0$, and select an $x \in \mathcal{H}$ for which $N^{p-1}x \neq 0$. The linear subspace generated by $N^{p-1}x$ is a one-dimensional $\mathbf{T}$ -invariant linear subspace of $\mathcal{H}$, which completes the proof. $\square$

A Banach space operator $T \in \mathcal{B}(\mathcal{X})$ is considered hypercyclic if there exists an $x \in \mathcal{X}$ for which the orbit $\{T^n x : n = 0, 1, 2, \dots\}$ is dense in $\mathcal{X}$. Conversely, it is referred to as supercyclic if the homogeneous orbit $\{\mu T^n x : \mu \in \mathbb{C}, n = 0, 1, 2, \dots\}$ is dense in $\mathcal{X}$ for some $x \in \mathcal{X}$. It's worth noting that hypercyclicity implies supercyclicity.

One significant result is that hyponormal operators do not exhibit supercyclicity, as established in [24]. The subsequent corollary emphasizes that this property extends to class p - $wA(s, t)$ operators.

Theorem 3.4. For a class p - $wA(s, t)$ operator $\mathbf{T} \in \mathcal{B}(\mathcal{H})$ with $0 < p \leq 1$ and $0 < s, t, s + t \leq 1$, it can be stated that $\mathbf{T}$ does not possess the property of supercyclicity.

Proof. Assume that a supercyclic class p - $wA(s, t)$ operator $\mathbf{T}$ satisfies property (β) . According to Theorem 2.2, [26, Theorem 2] implies that the spectrum $\sigma(\mathbf{T})$ is contained within a circle $\mathbb{T}(r) = \{z : |z| = r\}$ for some $r \geq 0$. If $r = 0$, then $\sigma(\mathbf{T}) = 0$. Since $\mathbf{T}$ is normaloid, it is the zero operator, leading to a contradiction. Now, let $r > 0$. In this case, $\sigma\left(\frac{1}{r}\mathbf{T}\right) \subseteq \mathbb{T}(1)$. Consequently, $\frac{1}{r}\mathbf{T}$ has a norm of 1. As $\mathbf{T}$ is a class p - $wA(s, t)$ operator and supercyclic, based on Theorem 2.6, $\frac{1}{r}\mathbf{T}$ must be unitary. However, this leads to a contradiction. $\square$

Theorem 3.5. Suppose we have an injective pure class p - $wA(s, t)$ operator $\mathbf{T}$ with an invariant subspace $\mathcal{M}$ such that $\mathbf{T}|_{\mathcal{M}}$ is an isometry. We aim to show that $\sigma(\mathbf{T}) \cap S^1 \neq \emptyset$, where $S^1 = \{z \in \mathbb{C} : |z| = 1\}$.

Proof. Suppose we have an injective pure class p - $wA(s, t)$ operator $\mathbf{T}$ with an invariant subspace $\mathcal{M}$ such that $\mathbf{T}|_{\mathcal{M}}$ is an isometry. We want to show that $\mathbf{T}$ has property (β) .

First, let's consider the case where $T_1 = \mathbf{T}|_{\mathcal{M}}$ is not a pure isometry, and there exists a reducing subspace $\mathcal{M}_1$ of T_1 such that $T_1|_{\mathcal{M}_1}$ is unitary. In this case, $\mathcal{M}_1$ is an invariant subspace of $\mathbf{T}$ such that $\mathbf{T}|_{\mathcal{M}_1}$ is injective and normal. Using [33, Lemma 7], we can conclude that $\mathcal{M}_1$ reduces $\mathbf{T}$, which contradicts the purity of $\mathbf{T}$. Therefore, we can assert that $\mathbf{T}$ is a pure isometry and has property (β) .

As a result, we have $\sigma(\mathbf{T}|_{\mathcal{M}}) = S^1$. Let $x \in \mathcal{M}$; then $F = \sigma_{\mathbf{T}}(x) \subseteq \sigma(\mathbf{T}|_{\mathcal{M}}) = S^1$, and $H_{\mathbf{T}}(\mathcal{F})$ is closed. Notice that if $F \subseteq \partial S^1$, then $\mathbf{T}|_{H_{\mathbf{T}}(\mathcal{F})}$ is unitary, as per Theorem 2.6. However, this would contradict the purity of $\mathbf{T}$. Therefore, $F \cap S^1 \neq \emptyset$. The subspace $H_{\mathbf{T}}(\mathcal{F})$ being hyperinvariant for $\mathbf{T}$ is rationally invariant for $\mathbf{T}$, which implies $F = \sigma(\mathbf{T}|_{H_{\mathbf{T}}(\mathcal{F})}) \subseteq \sigma(\mathbf{T})$, indicating that $\sigma(\mathbf{T}) \cap S^1 \neq \emptyset$. $\square$

An operator in $\mathcal{B}(\mathcal{H})$ is referred to as hypertransitive when every nonzero vector in $\mathcal{H}$ exhibits hypercyclicity under that operator. Let's denote the set of all nonhypertransitive operators in $\mathcal{B}(\mathcal{H})$ as (NHT) . The hypertransitive operator problem pertains to the open question of whether (NHT) equals $\mathcal{B}(\mathcal{H})$.

Theorem 3.6. *If $\mathbf{T}$ is an operator in $\mathcal{B}(\mathcal{H})$ and it belongs to the class of p - $wA(s, t)$ with $0 < p \leq 1$ and $0 < s, t$, where $s + t \leq 1$, then $\mathbf{T}$ is not hypertransitive. Specifically, if $\mathbf{T}$ is invertible, it implies that both $\mathbf{T}$ and $\mathbf{T}^{-1}$ share a nontrivial closed invariant subspace.*

Proof. If $\mathbf{T}$ is not a quasiaffinity, then $\sigma_p(\mathbf{T}) \cup \sigma_p(\mathbf{T}^*) \neq \emptyset$. Consequently, $\mathbf{T}$ has a nontrivial invariant subspace, and therefore, $\mathbf{T}$ belongs to the set of nonhypertransitive operators (NHT) .

On the other hand, suppose $\mathbf{T}$ is a quasiaffinity. Given that $\mathbf{T}$ is a class p - $wA(s, t)$ operator, it follows from [28] that $T(s, t)$ is q -hyponormal, where $q = \frac{\min\{sp, pt\}}{s+t}$. Since $T(s, t)$ is not hypercyclic, there exists a nonzero vector $x \in \mathcal{H}$ such that the orbit $\mathcal{O}(x, T(s, t))$ is not dense in $\mathcal{H}$. Let $\mathbf{T} = U|\mathbf{T}|$ be the polar decomposition of $\mathbf{T}$. Since $U|\mathbf{T}|^t T(s, t) = TU|\mathbf{T}|^t$, we have:

$$\mathbf{T}(U|\mathbf{T}|^t \mathcal{O}(x, T(s, t))) = U|\mathbf{T}|^t (T(s, t) \mathcal{O}(x, T(s, t))) \subseteq U|\mathbf{T}|^t \mathcal{O}(x, T(s, t)).$$

Since $\mathbf{T}$ is a quasiaffinity, it implies that $T(s, t)$ is also a quasiaffinity, and hence $|\mathbf{T}|$ is a quasiaffinity, and U is unitary. Therefore, $U|\mathbf{T}|^t \mathcal{O}(x, T(s, t))$ is not dense in $\mathcal{H}$, leading to the conclusion that $T(s, t) \in (NHT)$. By [21], this implies that $\mathbf{T} \in (NHT)$. Moreover, the second result follows from the first statement and [22]. $\square$

The following theorem provides both a necessary and a sufficient condition for the adjoint of a class p - $wA(s, t)$ operator to be hypercyclic.

Theorem 3.7. *If $\mathbf{T}$ is a class p - $wA(s, t)$ operator with $0 < p \leq 1$ and $0 < s, t, s + t \leq 1$ in $\mathcal{B}(\mathcal{H})$, then $\mathbf{T}^*$ is hypercyclic if and only if, for all nonzero $x \in \mathcal{H}$, both $\sigma_{\mathbf{T}}(x) \cap \mathbb{D} \neq \emptyset$ and $\sigma_{\mathbf{T}}(x) \cap (\mathbb{C} \setminus \mathbb{D}) \neq \emptyset$, where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.*

Proof. If $\mathbf{T}^*$ is hypercyclic, it suffices to demonstrate that $\sigma(\mathbf{T})$ has non-empty intersections with both $\mathbb{D}$ and $\mathbb{C} \setminus \mathbb{D}$. Let $S = \mathbf{T}|_{\mathcal{M}}$, where $\mathcal{M}$ is a closed $\mathbf{T}$ -invariant subspace, and let x be a hypercyclic vector for $\mathbf{T}^*$. Since $(S^*)^n Px = P(\mathbf{T}^*)^n x$ for each nonnegative integer n , where P is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$, we have $\overline{(S^*)^n (Px)_{n=0}^{\infty}} = \overline{P((\mathbf{T}^*)^n x_{n=0}^{\infty})} = P(\mathcal{H}) = \mathcal{M}$. In other words, Px is hypercyclic for S^* . Since $\mathbf{T}$ is a class p - $wA(s, t)$ operator and S^* is hypercyclic, we know that $r(S) = |S| = |S^*| > 1$, as mentioned in [25]. Consequently, we conclude that $\sigma(\mathbf{T}) \cap (\mathbb{C} \setminus \mathbb{D}) \neq \emptyset$.

On the other hand, to establish $\sigma(S) \cap \mathbb{D} \neq \emptyset$, suppose that $\sigma(S) \subset \mathbb{C} \setminus \mathbb{D}$. Since S^{-1} is also a class p - $wA(s, t)$ operator according to Theorem 2.2, and $\sigma(S^{-1}) \subset \overline{\mathbb{D}}$, it follows that $|S^{-1}| = r(S^{-1}) \leq 1$. However, since S^* is hypercyclic and invertible, $(S^*)^{-1}$ is hypercyclic by [25], implying that $|S^{-1}| = |(S^*)^{-1}| > 1$ according to [25]. This contradiction leads to the conclusion that $\sigma(S) \cap \mathbb{D} \neq \emptyset$.

Conversely, assume that $\sigma_{\mathbf{T}}(x) \cap \mathbb{D} \neq \emptyset$ and $\sigma_{\mathbf{T}}(x) \cap (\mathbb{C} \setminus \mathbb{D}) \neq \emptyset$ for all nonzero $x \in \mathcal{H}$. Then we can deduce that $H_{\mathbf{T}}(\mathbb{C} \setminus \mathbb{D}) = 0$ and $H_{\mathbf{T}}(\overline{\mathbb{D}}) = 0$. As $\mathbf{T}$ possesses the property (β) by Theorem 2.2, $\mathbf{T}^*$ exhibits the property (δ) . Consequently, following Proposition 2.5.14 in [24], we can conclude that both $H_{\mathbf{T}}(\mathbb{D})$ and $H_{\mathbf{T}}(\mathbb{C} \setminus \overline{\mathbb{D}})$ are dense in $\mathcal{H}$. This, in turn, allows us to apply Theorem 3.2 from [12], ultimately establishing that $\mathbf{T}^*$ is hypercyclic. $\square$

In the following proposition, we establish certain spectral properties for a class p - $wA(s, t)$ operator. An operator $\mathbf{T} \in \mathcal{B}(\mathcal{H})$ is deemed quasitriangular when there exists a sequence P_k of finite rank orthogonal projections on $\mathcal{H}$ that strongly converges to the identity operator I on $\mathcal{H}$, satisfying $\lim_{k \rightarrow \infty} \|(I - P_k)TP_k\| = 0$. If both $\mathbf{T}$ and $\mathbf{T}^*$ are quasitriangular, we refer to them as biquasitriangular operators.

Theorem 3.8. Let $\mathbf{T} \in \mathcal{B}(\mathcal{H})$ be a class p - $wA(s, t)$ operator with $0 < p \leq 1$ and $0 < s, t, s + t \leq 1$. Then the following statements are valid:

- (i) The set of approximate point spectrum of $\mathbf{T}^*$, denoted by $\sigma_a(\mathbf{T})^*$, is contained in the set of spectral approximate point spectrum of $\mathbf{T}^*$, which is equal to the left approximate point spectrum of $\mathbf{T}^*$, and all of these sets are equal to the spectrum of $\mathbf{T}^*$.
- (ii) $\mathbf{T}$ is invertible if and only if $\mathbf{T}$ is right invertible.
- (iii) If $\mathbf{T}$ is not a scalar multiple of the identity operator on $\mathcal{H}$ and has no nontrivial invariant subspace, then it is a biquasitriangular operator.

Proof. (i) Due to the single-valued extension property of $\mathbf{T}$ as given in Theorem 2.2, we have $\sigma(\mathbf{T}^*) = \sigma_a(\mathbf{T}^*)$ (as per [2]). Consequently, it follows that:

$$\sigma_a(\mathbf{T})^* \subset \sigma(\mathbf{T})^* = \sigma(\mathbf{T}^*) = \sigma_a(\mathbf{T}^*) = \sigma_l(\mathbf{T}^*).$$

(ii) The proof can be derived from statement (i). Specifically, we have:

$$\sigma_r(\mathbf{T}) = \sigma_l(\mathbf{T}^*) = \sigma(\mathbf{T}^*)^* = \sigma(\mathbf{T}).$$

(iii) If $\mathbf{T}$ lacks any nontrivial invariant subspace, then $\sigma_p(\mathbf{T}^*) = \emptyset$. As a result, $\mathbf{T}^*$ possesses the single-valued extension property. Since both $\mathbf{T}$ and $\mathbf{T}^*$ exhibit the single-valued extension property, we can conclude, as per [24], that $\mathbf{T}$ is a biquasitriangular operator. $\square$

Corollary 3.9. For $\mathbf{T} \in \mathcal{B}(\mathcal{H})$, specifically a class p - $wA(s, t)$ operator with $0 < p \leq 1$ and $0 < s, t, s + t \leq 1$, it holds that $\ker(\mathbf{T}) \cap \text{ran}(\mathbf{T}^n) = 0$ for some positive integer n .

Proof. If $\mathbf{T}$ is a class p - $wA(s, t)$ operator, then, as shown in [31], we can deduce that $\ker(\mathbf{T}) = \ker(\mathbf{T}^n)$ for some positive integer n . In the case where $y \in \ker(\mathbf{T}) \cap \text{ran}(\mathbf{T}^n)$, it follows that $\mathbf{T}y = 0$ and consequently, there exists $x \in \mathcal{H}$ such that $y = \mathbf{T}^n x$. This further implies that $\mathbf{T}^{n+1}x = \mathbf{T}y = 0$. Given that $x \in \ker(\mathbf{T}^{n+1}) = \ker(\mathbf{T})$, we can conclude that $y = \mathbf{T}^n x = 0$. Consequently, we find that $\ker(\mathbf{T}) \cap \text{ran}(\mathbf{T}^n) = 0$. $\square$

We define an operator $\mathbf{T} \in \mathcal{B}(\mathcal{H})$ to satisfy Dunford's boundedness condition (B) when it possesses the single-valued extension property and there exists a constant $K > 0$ such that for any x and y with disjoint spectra $\sigma_{\mathbf{T}}(x) \cap \sigma_{\mathbf{T}}(y) = \emptyset$, the inequality $\|x\| \leq K \|x + y\|$ holds, where the constant K is independent of x and y .

Theorem 3.10. Suppose we have an operator $\mathbf{T} \in \mathcal{B}(\mathcal{H})$ that belongs to class p - $wA(s, t)$ with $0 < p \leq 1$ and $0 < s, t, s + t \leq 1$. Additionally, assume that $\mathbf{T}$ satisfies the property that $\sigma_{\mathbf{T}}(P_{\mathcal{F}}(x)) \subset \sigma_{\mathbf{T}}(x)$ for all $x \in \mathcal{H}$ and each closed set $\mathcal{F}$ in $\mathbb{C}$, where $P_{\mathcal{F}}$ represents the orthogonal projection of $\mathcal{H}$ onto $H_{\mathbf{T}}(\mathcal{F})$. In this case, it can be concluded that $\mathbf{T}$ satisfies Dunford's boundedness condition (B).

Proof. Since $\mathbf{T}$ satisfies Dunford's property (C) as stated in Theorem 2.2, we know that $H_{\mathbf{T}}(\mathcal{F})$ is a closed subspace. Let's consider two vectors x_1 and x_2 in $\mathcal{H}$ such that $\sigma_{\mathbf{T}}(x_1) \cap \sigma_{\mathbf{T}}(x_2) = \emptyset$. Define $\mathcal{F}_j = \sigma_{\mathbf{T}}(x_j)$ for $j = 1, 2$.

Based on the given hypothesis, we have $\sigma_{\mathbf{T}}(P_{\mathcal{F}_2}x_1) \subset \sigma_{\mathbf{T}}(x_1) = \mathcal{F}_1$. Additionally, it's clear that $\sigma_{\mathbf{T}}(P_{\mathcal{F}_2}x_1) \subset \mathcal{F}_2$. Therefore, we can conclude that

$$\sigma_{\mathbf{T}}(P_{\mathcal{F}_2}x_1) \subset \mathcal{F}_1 \cap \mathcal{F}_2 = \sigma_{\mathbf{T}}(x_1) \cap \sigma_{\mathbf{T}}(x_2) = \emptyset.$$

Since $\mathbf{T}$ possesses the single-valued extension property, as indicated in Proposition 4.1 of [31], we can infer that $P_{\mathcal{F}_2}x_1 = 0$. This implies that x_1 is orthogonal to $H_{\mathbf{T}}(\mathcal{F}_2)$. However, since $\sigma_{\mathbf{T}}(x_2) = \mathcal{F}_2$, it follows that x_2 belongs to $H_{\mathbf{T}}(\mathcal{F}_2)$, so we can say $x_2 \in H_{\mathbf{T}}(\mathcal{F}_2)$. Consequently, we find that $\langle x_1, x_2 \rangle = 0$.

This result implies that

$$\|x_1 + x_2\| = \sqrt{\|x_1\|^2 + \|x_2\|^2} \geq \|x_1\|$$

This concludes our proof. $\square$

Theorem 3.11. Let $\mathbf{T} \in \mathcal{B}(\mathcal{H})$ be a class p - $wA(s, t)$ operator with $0 < p \leq 1$ and $0 < s, t, s + t \leq 1$. Additionally, assume that $T \neq zI$ for any $z \in \mathbb{C}$. If there exists $x \in \mathcal{H} \setminus \{0\}$ such that $\sigma_{\mathbf{T}}(x) \subsetneq \sigma(\mathbf{T})$, then $\mathbf{T}$ possesses a nontrivial hyperinvariant subspace.

Proof. If there exists a nonzero vector $x \in \mathcal{H}$ such that $\sigma_{\mathbf{T}}(x) \subsetneq \sigma(\mathbf{T})$, we define

$$\mathcal{M} := H_{\mathbf{T}}(\sigma_{\mathbf{T}}(x)), \text{ i.e., } \mathcal{M} = \{y \in \mathcal{H} : \sigma_{\mathbf{T}}(y) \subset \sigma_{\mathbf{T}}(x)\}.$$

Because $\mathbf{T}$ satisfies Dunford's property (C) as established in Theorem 2.2, it can be concluded that $\mathcal{M}$ is a hyperinvariant subspace with respect to $\mathbf{T}$, as detailed in [24]. Moreover, since $x \in \mathcal{M}$, we can conclude that $\mathcal{M} \neq \{0\}$.

Now, assume, for the sake of contradiction, that $\mathcal{M} = \mathcal{H}$. This implies that $\mathbf{T}$ has the single-valued extension property, leading to the conclusion that

$$\sigma(\mathbf{T}) = \bigcup \{\sigma_{\mathbf{T}}(y) : y \in \mathcal{H}\} \subset \sigma_{\mathbf{T}}(x) \subsetneq \sigma(\mathbf{T}).$$

However, this results in a contradiction, as it contradicts our initial assumption. Therefore, $\mathcal{M}$ indeed forms a nontrivial hyperinvariant subspace for $\mathbf{T}$. $\square$

Theorem 3.12. Let $\mathbf{T} \in \mathcal{B}(\mathcal{H})$ be a class p - $wA(s, t)$ operator with $0 < p \leq 1$ and $0 < s, t, s + t \leq 1$ such that $T \neq \alpha I$ for any $\alpha \in \mathbb{C}$. Suppose there exists $x \in \mathcal{H} \setminus \{0\}$ such that for all positive integers n , we have $\|\mathbf{T}^n x\| \leq Cr^n$, where $C > 0$ and $0 < r < r(\mathbf{T})$ are constants. In this case, we can conclude that $\mathbf{T}$ possesses a nontrivial hyperinvariant subspace.

Proof. Define $\psi(z) := -\sum_{n=0}^{\infty} z^{-(n+1)} \mathbf{T}^n x$, which is an analytic function for $|z| > r$. In fact, if we substitute $v = z^{-1}$ for $|z| > r$, we get $\psi(v) = -\sum_{n=0}^{\infty} v^{n+1} \mathbf{T}^n x$ for $0 < |v| < 1/r$. Since the hypothesis ensures that $\lim_{n \rightarrow \infty} \sup \|\mathbf{T}^n x\| \leq r$, the radius of convergence for the power series $\sum_{n=0}^{\infty} v^{n+1} \mathbf{T}^n x$ is at least $1/r$. Setting $\psi(0) := 0$, we find that $\psi(v)$ is analytic for $|v| < 1/r$, which implies that $\psi(z)$ is analytic for $|z| > r$.

Now, consider the equation

$$(\mathbf{T} - z)\psi(z) = -\sum_{n=0}^{\infty} z^{-(n+1)} \mathbf{T}^n x + \sum_{n=0}^{\infty} z^{-n} \mathbf{T}^n x = x$$

for all $z \in \mathbb{C}$ with $|z| > r$. This tells us that $\rho_{\mathbf{T}}(x) \supset \{z \in \mathbb{C} : |z| > r\}$, which implies that

$$\sigma_{\mathbf{T}}(x) \subset \{z \in \mathbb{C} : |z| \leq r\}.$$

Since $r < r(\mathbf{T})$, it follows that $\sigma_{\mathbf{T}}(x) \subsetneq \sigma(\mathbf{T})$. Therefore, by Theorem 3.11, we can conclude that $\mathbf{T}$ has a nontrivial hyperinvariant subspace. $\square$

Recall that if $\mathcal{U}$ is a nonempty open set in $\mathbb{C}$ and there exists a subset $\Omega \subset U$ such that for every function ψ in H^{∞} (i.e., all functions bounded and holomorphic on $\mathcal{U}$), the following condition holds:

$$\sup_{\alpha \in \Omega} |\psi(\alpha)| = \sup_{v \in \Omega} |\psi(v)|$$

Then, we say that Ω is dominating for $\mathcal{U}$.

The next theorem extends Scott Brown's result to the case of p - $wA(s, t)$ operators.

Theorem 3.13. If $\mathbf{T}$ is any class p - $wA(s, t)$ operator with $0 < p \leq 1$ and $0 < s, t, s + t \leq 1$, and there exists a nonempty open set $\mathcal{U}$ in $\mathbb{C}$ such that the intersection of $\sigma(\mathbf{T})$ and U is dominating for $\mathcal{U}$, then $\mathbf{T}$ possesses a nontrivial invariant subspace.

Proof. If $\mathbf{T}$ is not a quasi-affinity, then $0 \in \sigma_p(\mathbf{T}) \cup \sigma_p(\mathbf{T}^*)$. Therefore, it is evident that $\mathbf{T}$ possesses a nontrivial invariant subspace. Let's consider the case when $\mathbf{T}$ is a quasi-affinity. Since $T(s, t)$ is q -hyponormal by the definition of a class p - $wA(s, t)$ operator, Theorem 1.24 of [21] guarantees that $T(s, t)$ has a nontrivial invariant subspace. Consequently, by [21, Theorem 1.15], $\mathbf{T}$ has a nontrivial invariant subspace. $\square$

Theorem 3.14. Let $\mathbf{T} \in \mathcal{B}(\mathcal{H})$ be a class p - $wA(s, t)$ operator with $0 < p \leq 1$ and $0 < s, t, s + t \leq 1$ such that $\mathbf{T} \neq \alpha I$ for any $\alpha \in \mathbb{C}$. If $\mathbf{T}$ is an invertible and if there exists a constant K and a positive integer m such that $\|\mathbf{T}^n\| \leq K|n|^m$ for $n = \pm 1, \pm 2, \dots$, then $\mathbf{T}$ has a hyperinvariant subspace.

Proof. The presumption that $\mathbf{T}$ is an invertible class p - $wA(s, t)$ implies that, according to Theorem 2.2, $\mathbf{T}^{-1}$ also belongs to the class p - $wA(s, t)$. As a consequence, both $\mathbf{T}$ and $\mathbf{T}^{-1}$ are considered normaloid operators. It is worth noting that $\|\mathbf{T}\| = \lim_{n \rightarrow \infty} \|\mathbf{T}^n\|^{1/n} \leq \lim_{n \rightarrow \infty} K^{1/n} (n^{1/n})^m = 1$. Similarly, $\|\mathbf{T}^{-1}\| = \lim_{n \rightarrow \infty} \|\mathbf{T}^{-n}\|^{1/n} \leq 1$. Consequently, the spectra of both $\mathbf{T}$ and $\mathbf{T}^{-1}$ are enclosed within the unit disk, leading to the conclusion that the spectrum of $\mathbf{T}$ is also confined to the unit disk. In the event that the spectrum of $\mathbf{T}$ is disconnected, we can deduce, as per [30, Corollary 2.11], that $\mathbf{T}$ possesses a hyperinvariant subspace. Let us assume, therefore, that $\sigma(\mathbf{T})$ contains a continuous portion of the unit circle. To establish the existence of a hyperinvariant subspace in this scenario, it is necessary to verify the relevant growth condition on the resolvent of $\mathbf{T}$, as demonstrated in [30, Theorem 6.3].

Assume that $1 < |\mu| < 2$. In that case, we can express the norm of the operator inverse as follows:

$$\|(\mu - \mathbf{T})^{-1}\| = \frac{1}{|\mu|} \left\| 1 + \frac{1}{\mu} \mathbf{T} + \frac{1}{\mu^2} \mathbf{T}^2 + \dots \right\| \leq \frac{K}{|\mu|} \sum_{n=0}^{\infty} \frac{n^m}{|\mu|^n}.$$

Let's define $t = \frac{1}{|\mu|}$. Then:

$$\begin{aligned} \|(\mu - \mathbf{T})^{-1}\| &\leq tK \left(1 + \sum_{n=1}^{\infty} n^m t^n \right) \\ &\leq Kt \left(1 + \sum_{n=1}^{\infty} (n+1)(n+2) \cdots (n+m) t^n \right). \end{aligned}$$

Now, observe that $\sum_{n=1}^{\infty} (n+1)(n+2) \cdots (n+m) t^n$ corresponds to the m^{th} derivative of the function $F(t) = \sum_{n=1}^{\infty} \mathbf{T}^{n+m} = \frac{t^{m+1}}{1-t}$. Consequently, $\sum_{n=1}^{\infty} (n+1)(n+2) \cdots (n+m) t^n$ takes the form $\frac{p(t)}{(1-t)^{m+1}}$ for some polynomial p . Therefore, there exists a constant M such that for $1 < |\mu| < 2$:

$$\begin{aligned} \|(\mu - \mathbf{T})^{-1}\| &\leq M \frac{1}{(1 - \frac{1}{|\mu|})^{m+1}} \\ &\leq M \frac{2^{m+1}}{(|\mu| - 1)^{m+1}}. \end{aligned}$$

Considering the case where $|\mu_0| = 1$ and μ lies on a non-tangent line segment with μ_0 as an endpoint, we have $|\mu - \mu_0| \leq c(|\mu| - 1)$. Consequently, it can be concluded that $(\mu - \mathbf{T})^{-1}$ meets the growth condition stipulated by Theorem 6.3 in [30] for lines L outside of the unit circle.

In the event that $\frac{1}{2} < |\mu| < 1$, we can express $(\mu - \mathbf{T})^{-1}$ as $\left(\frac{1}{\mu} \mathbf{T}^{-1}\right) \left(\mathbf{T}^{-1} - \frac{1}{\mu}\right)^{-1}$. This leads to the inequality $\|(\mu - \mathbf{T})^{-1}\| \leq 2 \|\mathbf{T}^{-1}\| \left\| \left(\frac{1}{\mu} - \mathbf{T}^{-1}\right)^{-1} \right\|$. Given that the assumptions on $\mathbf{T}$ and $\mathbf{T}^{-1}$ remain the same, it follows from the previous calculation that $\left(\frac{1}{\mu} - \mathbf{T}^{-1}\right)^{-1}$ satisfies the requisite growth condition. Consequently, and so does $(\mu - \mathbf{T})^{-1}$ and, in turn, the result can be derived from [30, Theorem 6.3]. $\square$

Theorem 3.15. Consider an operator $\mathbf{T}$ in the space of bounded linear operators on a Hilbert space, denoted as $\mathcal{B}(\mathcal{H})$, which is characterized as a class p - $wA(s, t)$ operator. Here, $0 < p \leq 1$, and $0 < s, t$ with the constraint that $s + t \leq 1$. Furthermore, it is assumed that $\mathbf{T}$ is not equal to αI for any complex number α . If neither of the sequences $\mathbf{T}^n$ and $\mathbf{T}^{n*}$ converges strongly to zero, then it can be concluded that $\mathbf{T}$ possesses a non-trivial hyperinvariant subspace.

Proof. Let's break down the argument into more digestible parts: First, define two sets, $\mathcal{M} = \{x : \{\mathbf{T}^n x\} \rightarrow 0\}$ and $\mathcal{N} = \{x : \{\mathbf{T}^{n*} x\} \rightarrow 0\}$. If both $\mathcal{M}$ and $\mathcal{N}$ consist only of the zero vector, then, according to [30, Theorem

6.20], T is quasi-similar to a unitary operator denoted as $\mathcal{U}$. Since T is not a multiple of the identity, the same holds true for $\mathcal{U}$. Now, due to [30, Corollary 1.17], it follows that $\mathcal{U}$ possesses a non-trivial hyperinvariant subspace. Consequently, by [30, Theorem 6.20] again, it can be established that T also has a non-trivial hyperinvariant subspace. Now, let's consider the case where both $\mathcal{M}$ and $\mathfrak{N}$ are not solely comprised of the zero vector. It's important to note that both $\mathcal{M}$ and $\mathfrak{N}$ are linear manifolds. Suppose we have a sequence $\{x_m\} \subset \mathcal{M}$ that converges to x . We can analyze the behavior of $T^n x$ as follows:

$$\begin{aligned} \|T^n x\| &\leq \|T^n x - T^n x_m\| + \|T^n x_m\| \\ &\leq M \|x - x_m\| + \|T^n x_m\|. \end{aligned}$$

Therefore, $\{T^n x\} \rightarrow 0$, which implies that $x \in \mathcal{M}$. This demonstrates that $\mathcal{M}$ is a closed subspace. The same argument can be applied to show that $\mathfrak{N}$ is also a closed subspace. If $TC = CT$ for some operator C , then we can deduce that $\|T^n Cx\| = \|CT^n x\| \leq \|C\| \|T^n x\|$, which indicates that $\mathcal{M}$ is a hyperinvariant subspace for T . If $\mathcal{M} \neq \{0\}$, we can conclude that T has a non-trivial hyperinvariant subspace because it's clear that $\mathcal{M}$ is not equal to the entire Hilbert space. If $\mathfrak{N} \neq \{0\}$, then $\mathfrak{N}$ serves as a hyperinvariant subspace for T^* . Consequently, $\mathfrak{N}^\perp$ is a hyperinvariant subspace for T . In both cases, T possesses a non-trivial hyperinvariant subspace. $\square$

4. Conclusion and Future Work

In this paper, we have established several key findings related to operators in the class $p-wA(s, t)$, where $0 < p \leq 1$ and $0 < s, t, s + t \leq 1$. We introduced the concept of the quasi-nilpotent component, denoted as $H_0(T)$, for such operators, defined as:

$$\ker(T^p) = \{x \in \mathcal{H} : r_T(x) = 0\} = \bigcap_{\lambda \neq 0} (T - \lambda)^p \mathcal{H}.$$

This characterization is valid for sufficiently large integer values of p , where $r_T(x)$ is defined as $\limsup_{n \rightarrow \infty} \|T^n x\|^{\frac{1}{n}}$. Additionally, we have shown that when the spectrum $\sigma(T)$ is finite and T belongs to the class $p-wA(s, t)$, the operator T is an algebraic operator. Furthermore, if $T \in \mathcal{B}(\mathcal{H})$ is part of the class $p-wA(s, t)$ and possesses the decomposition property (δ) , we have demonstrated the existence of a non-trivial invariant closed linear subspace of T . Moreover, it has been revealed that operators with diverse spectra, as explored in this study, also exhibit nontrivial invariant subspaces. Our investigation into the existence of invariant and hyperinvariant subspaces adds depth to our understanding of these operators.

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