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# Two dimensional pseudo-Chebyshev wavelets and their application in the theory of approximation of functions belonging to Hölders class

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**Abstract.** For the first time in 2022, the authors introduced the notion of *pseudo-Chebyshev wavelets* in the context of one dimension. Continuing the study in advance sense, in this article, two dimensional pseudo Chebyshev wavelets are introduced. Two dimensional pseudo Chebyshev wavelet expansion of a function of two variable is defined and verified. This research paper introduces a novel algorithm based on the two dimensional pseudo Chebyshev wavelet method to address computation problems in approximation theory. The methods are illustrated by an example and compared with prominent Chebyshev wavelet methods to demonstrate the validity and applicability of the results. The error analysis and convergence analysis of a functions in the Hölder classes have been studied by this wavelets. More over the error of approximation of functions of Holder's class have been estimated by an orthogonal projection operators of its two dimensional pseudo Chebyshev wavelets. The results of this paper are the significant developments in wavelet analysis.

## 1. Introduction

Wavelets which were relatively recently, at the start of the 1980s, have considerable interest from the mathematical community and research of many diverse areas of science and technology. A consequence of this interest is the appearance of several researchers like Daubechies [10], Chui [8], Morlet et al. [32, 33], Meyer [34], Strang [44], Natanson [36], Chui [9], Daubechies and Lagarias [11], Walter [47, 48], Islam et al.[12], Mohammadi[35], Venkatesh [46], Keshavarz et al. [13], Kumar[14], Lal et al [15, 18–22, 24, 25], Bastin [1], Biazar et al. [2], Babolian and Fattahzadeh [3, 5] in wavelet analysis as well as the different area of Mathematics and Mathematical sciences. While with the company of Fourier analysis and harmonic theory, wavelets are growing under the influence of approximation theory and fractals. Working in this direction, researchers like Strang [42], Lal et al. [26–30],and Rehman & Siddiqi [37] and many more developed the applications of wavelets in diverse area of sciences and technology.

Wavelets are only natural to look for complete orthonormal bases for the Hilbert space $L^2(\mathbb{R})$ having qualities that reflect the applications of translations and dilations. In view of these observations, orthogonal functions play an important role for a constriction of new wavelets. The approach in using wavelets is

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to transform the underlying problems into simpler approximating truncated orthogonal functions. Their are several orthogonal set of functions in $L^2(\mathbb{R})$. Among them, one of the orthogonal sets of functions is the Chebyshev polynomials. The Chebyshev polynomials $P_\vartheta(\omega)$, where $\omega \in \Omega = (0,1]$ & $\vartheta$ is a non negative integers, are more applicable for the computational problems, see [4, 6, 31, 39–41]. The pseudo Chebyshev functions of fractional degree is introduced by Ricci [38] and some of its important properties like orthogonality and more many studied by Cesarano and Ricci [7]. Lal et al [23] introduced the pseudo Chebyshev wavelet for the first time in June 2022. These wavelets possess numerous applications in Mathematics and Mathematical Sciences, particularly in the realm of fractals due to their inherent characteristics.

The fractals are everywhere continuous but nowhere differentiable functions (*see*[23]) . The fractional Brownian motion, complex Bernoulli spiral, Brownian trajectories, typical Feynmann path, and turbulent fluid motion are related to irregular fractals. The irregular fractals are specified at every point by a local Hölder's condition between any finite interval. This fact is to motivate the inspiration for considering the approximation of functions belonging to Hölders class via the two-dimensional *EPCW*. But till now no work seems to have been done to obtain the error of a function $f$ belonging to generalized Hölder's classes $H^{(\alpha,\beta)}_{(\Omega^2,\phi)}(\mathbb{R})$ where $0 < \alpha, \beta \leq 1$, using its two dimensional *EPCW* expansion.

In precise, the theorem's objectives of the research paper are as follows:

(i) To introduce the notion of algorithms to estimate the approximation of Hölders class $f \in H^{(\alpha,\beta)}_{(\Omega^2,\phi)}(\mathbb{R})$ of function by two-dimensional *EPCW*.
(ii) To discuss the convergence and error analysis of the functions $f$.
(iii) To estimate the order of error function .
(iv) To discuss the uniform convergence of a functions of the Hölders class using the two-dimensional *EPCW* series expansions.

This research paper is organised as follows: In section 2, "Definitions and Preliminaries " Functions of Hölders class in two variables and examples, Two dimensional *EPCW* and its series expansions of functions, Two Lemmas required for the proof of main results are described. In section 3, "Results: Convergence analysis" we discussed the uniform convergence of the functions of generalized Hölders class by the two dimensional extended pseudo Chebyshev wavelet series expansions. Three important corollaries related to convergence analysis are introduced by the main results. In section 4 "Results: Error analysis" we discussed the orthogonal projection operator, Function approximation and error analysis of the functions of Hölders class by the orthogonal projection operators with two dimensional *EPCW* expansions. Some of the most important corollaries related to error analysis are presented by the main theorems. In section 5, "Effectiveness of the this method" the approximated solutions and its errors have been compared with the solutions obtained by prominent *CW* method. Conclusion and impact of this research paper are provided in section 6 .

## 2. Definitions and Preliminaries

### 2.1. *Function of generalized Hölders class in two variable*

A two variable real valued function $\xi : \Omega^2 \to \mathbb{R}$, is said to be function of generalized Hölder's class

$$
\begin{aligned}
&i.e.\ \xi \in H^{(\alpha,\beta)}_{(\Omega^2,\phi)}(\mathbb{R}) \quad \Leftrightarrow \quad \text{if there exists a number } \kappa > 0 \text{ such that} \\
&|\xi(\omega+\nu,\varpi+\upsilon) - \xi(\omega,\varpi)| = \kappa(|\nu|^\alpha + |\upsilon|^\beta)\phi(\nu,\upsilon) = O(|\nu|^\alpha + |\upsilon|^\beta)\phi(\nu,\upsilon), \\
&\text{where, } \phi \text{ is uniformaly bounded and, } \phi(\nu,\upsilon) \to 0 \text{ as } \|(\nu,\upsilon)\| \to 0, [45].
\end{aligned}
$$

The function $\xi \in H^{(\alpha,\beta)}_{\Omega^2}(\mathbb{R})$ if and only if $|\xi(\omega+\nu,\varpi+\upsilon) - \xi(\omega,\varpi)| = \kappa(|\nu|^\alpha + |\upsilon|^\beta)$, where, $0 < \alpha, \beta \leq 1$.

**Remark 2.1.** *(i) If* $\phi = c$, *then* $H^{(\alpha,\beta)}_{\Omega^2}(\mathbb{R}) = H^{(\alpha,\beta)}_{(\Omega^2,\phi)}(\mathbb{R})$

*(ii) If $\xi$ be a function $\xi : \Omega^2 \to \mathbb{R}$ defined by*

$$\xi(\omega, \varpi) \quad = \quad |\omega|^{\alpha} + |\varpi|^{\beta} \quad \forall\, (\omega, \varpi)\, \in \Omega^2,\ then \xi \in\ H^{(\alpha,\beta)}_{\Omega^2}(\mathbb{R})\,.$$

### 2.2. *Two dimensional extended pseudo Chebyshev wavelets* (*EPCW*)

The concept of *PCW* was primary considered by Shyam Lal, Susheel Kumar, and collaborators [23] in June 2022. Further their two-dimensional extension broadened their applicability to multivariate problems [16]. The two dimensional extended pseudo-Chebyshev wavelets $\Psi^{\lambda}_{(\eta,\vartheta;\eta',\vartheta')}$ are defined by

$$\Psi^{\lambda}_{(\eta,\vartheta;\eta',\vartheta')}(\omega,\varpi) \quad := \quad \Psi^{(\lambda,(\varrho,\varrho'))}_{(\eta,\vartheta;\eta',\vartheta')}(\omega,\varpi) = \Psi^{(\lambda,\varrho)}_{(\eta,\vartheta)}(\omega) \times \Psi^{(\lambda,\varrho')}_{(\eta',\vartheta')}(\varpi),$$

$$\text{where}\quad \Psi^{(\lambda,\varrho)}_{(\eta,\vartheta)}(\omega) := \begin{cases} \sqrt{\frac{2}{\pi}}\lambda^{\varrho/2}P_{(\vartheta+1/2)}(\lambda^{\varrho}\omega - 2\eta + 1); & \text{for } \frac{\eta-1}{\lambda^{\varrho-1}} \le \ \omega\ \le \frac{\eta}{\lambda^{\varrho-1}}, \\ 0; & \text{otherwise, see[17]}, \end{cases}$$

where $\vartheta$, $\vartheta'$ are non negative integers and $\eta = 1, 2, 3, \cdots, \lambda^{\varrho-1} << \infty$, $\eta' = 1, 2, 3, \cdots, \lambda^{\varrho'-1} << \infty$ & $\lambda\,(\ge 2)\,, \varrho, \varrho'$ are a positive integers.

$$P_{(\vartheta+1/2)}(\omega) = \cos\left((\vartheta + 1/2)\,(arc\cos\omega)\right)\quad \vartheta = 0, 1, 2, \cdots,\ \text{and recurrence relations are given by,}$$

$$P_{\vartheta''+1/2}(\omega) = 2\omega P_{(\vartheta''-1/2)}(\omega) - P_{(\vartheta''-3/2)}(\omega),\ \text{with } P_{\pm 1/2}(\omega) = \sqrt{\frac{1+\omega}{2}},\ \ \vartheta'' \in \mathbb{N}.$$

It is noteworthy that the set of extended pseudo-Chebyshev wavelets (*EPCW*), $\left\{\Psi_{(\eta,\vartheta;\eta',\vartheta')}\right\}$, forms a semi bi-orthonormal subset of $L^2_{\Omega^2}(\mathbb{R})$ with respect to the extended weight function (*EWF*)

$$w^{(\lambda,(\varrho,\varrho'))}_{(\eta,\eta')} = w^{(\lambda,\varrho)}_{\eta} \times w^{(\lambda,\varrho')}_{\eta'},\ \text{where } w^{(\lambda,\varrho)}_{\eta} = w\left(\lambda^{\varrho}\tau - 2\eta + 1\right)\ \text{and } w(\tau) = \frac{1}{\sqrt{1-\tau^2}}.$$

### 2.3. *Wavelet Series*

A function $\xi \in L^2_{\Omega}(\mathbb{R})$ is expanded by extended pseudo-Chebyshev wavelet series as [17]:

$$\xi \quad \sim \quad \sum_{\eta=1}^{\infty}\sum_{\vartheta=0}^{\infty}\alpha_{(\eta,\vartheta)}\psi^{\lambda}_{(\eta,\vartheta)} = \sum_{\eta=1}^{\infty}\sum_{\vartheta=0}^{\infty}\left\langle \xi, \psi^{\lambda}_{(\eta,\vartheta)}\right\rangle_{w^{\varrho}_{\eta}}\psi^{\lambda}_{(\eta,\vartheta)}\ \text{where } \alpha_{(\eta,\vartheta)} = \int \xi\,(\omega)\,\psi^{\lambda}_{(\eta,\vartheta)}\,(\omega)\,w^{\varrho}_{\eta}(\omega)d\omega.$$

If $\xi \in L^2_{\Omega^2}(\mathbb{R})$ then its two dimensional *EPCW* expansion is given by

$$\xi \quad \sim \quad \sum_{\eta=1}^{\infty}\sum_{\vartheta=0}^{\infty}\sum_{\eta'=1}^{\infty}\sum_{\vartheta'=0}^{\infty}\alpha_{(\eta,\vartheta;\eta',\vartheta')}\Psi^{\lambda}_{(\eta,\vartheta;\eta',\vartheta)}\ \text{ where } \alpha_{(\eta,\vartheta;\eta',\vartheta')} = \int_{\Omega^2}\xi\psi^{\lambda}_{(\eta,\vartheta;\eta',\vartheta')}w^{(\varrho,\varrho')}_{(\eta,\eta')}d\mu. \tag{1}$$

### 2.4. *Lemmas*

The following Lemmas are required hereafter.

**Lemma 2.2.** *Let $\aleph$ be a non negative positive integer and a function $\xi : [\aleph, \infty) \to \mathbb{R}$ be a real valued monotonic decreasing function. Then*

$$\int_{\aleph}^{\infty}\xi(v)dv \le \sum_{\aleph}^{\infty}\xi(v) \le \xi(\aleph) + \int_{\aleph}^{\infty}\xi(v)dv,\ (see[16])\,.$$

**Lemma 2.3.** *Let $\xi : (X \times X') \to \mathbb{R}$ be a bounded measurable function on the non negative finite measurable spaces $(X, \mathfrak{I}, \mu)$ & $(X', \mathfrak{I}', \mu')$ and $\Omega'$ & $\Omega''$ is a subset of $X$ & $X'$ respectively. Then there exist $\kappa_0 > 0$ such that*

$$\begin{aligned} |\xi(\upsilon_0, \bar{\upsilon}_0)| &\leq \kappa_0 \mu(X \times X')\mu(\Omega' \times \Omega'') \text{ a.e., where } (\upsilon_0, \bar{\upsilon}_0) \in \Omega' \times \Omega'' \text{ In particular,} \\ &\text{if} \quad X = X' = \Omega \text{ and } (\Omega' \times \Omega'') = \left[\frac{\eta - 1}{\lambda^{\varrho-1}}, \frac{\eta}{\lambda^{\varrho-1}}\right] \times \left[\frac{\eta' - 1}{\lambda^{\varrho'-1}}, \frac{\eta'}{\lambda^{\varrho'-1}}\right], \text{where} \eta = 1, 2, 3, \cdots, \lambda^{\varrho-1} << \infty, \\ &\quad \eta' = 1, 2, 3, \cdots, \lambda^{\varrho'-1} << \infty, \lambda \geq 2. \text{ Then, } \xi\left(\frac{2\eta - 1}{\lambda^{\varrho}}, \frac{2\eta' - 1}{\lambda^{\varrho'}}\right) \leq \frac{4\kappa_0}{\lambda^{(\varrho+\varrho')}}, \quad (\text{see}[16]). \end{aligned}$$

## 3. Main results

### 3.1. Convergence Analysis

In this section we state and prove a theorem ascertaining that the two dimensional EPCW expansions for the classes of $\mathrm{H}^{(\alpha,\beta)}_{(\Omega^2,\phi)}(\mathbb{R})$, functions, converges uniformly.

**Theorem 3.1.** *If $f : \Omega^2 \to \mathbb{R}$ be a real valued function of two variable such that $f \in H^{(\alpha,\beta)}_{(\Omega^2,\phi)}(\mathbb{R})$, then the two dimensional EPCW series expansions of the function $f$,*

$$\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n'=1}^{\infty}\sum_{m'=0}^{\infty} \alpha_{(n,m;n',m')}\Psi^{\lambda}_{(n,m;n',m')} = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n'=1}^{\infty}\sum_{m'=0}^{\infty} \left\langle f, \Psi^{\lambda}_{(n,m;n',m')}\right\rangle_{w_{(k,n;k',n')}} \Psi^{\lambda}_{(n,m;n',m')}, \text{ is uniformly converges to } f.$$

**Proof of Theorem 3.1.** Consider the sequence of partial sums of the above two-dimensional *EPCW* series

$$S^f_{(\aleph,\wp;\aleph',\wp')} = \sum_{\eta=1}^{\aleph}\sum_{\vartheta=0}^{\wp-1}\sum_{\eta'=1}^{\aleph'}\sum_{\vartheta'=0}^{\wp'-1} \alpha_{(\eta,\vartheta;\eta',\vartheta')}\Psi^{(\varrho,\varrho'),\lambda}_{(\eta,\vartheta;\eta',\vartheta')} \quad \text{where} \quad \Psi^{(\varrho,\varrho'),\lambda}_{(\eta,\vartheta;\eta',\vartheta')}(\omega,\varpi) = \psi^{\varrho,\lambda}_{\eta,\vartheta}(\omega)\psi^{\varrho',\lambda}_{\eta',\vartheta'}(\varpi),$$

further suppose that $f \in H^{(\alpha,\beta)}_{(\Omega^2,\phi)}(\mathbb{R})$, and by the semi by-orthonormality of the $\left\{\Psi^{\lambda}_{(\eta,\vartheta;\eta',\vartheta')}\right\}$ in the disjoint intervals $\left[\frac{\eta-1}{\lambda^{\varrho-1}}, \frac{\eta}{\lambda^{\varrho-1}}\right] \times \left[\frac{\eta'-1}{\lambda^{\varrho'-1}}, \frac{\eta'}{\lambda^{\varrho'-1}}\right]$ and take $\aleph = \lambda^{\varrho-1}, \aleph' = \lambda^{\varrho'-1}$ $\lambda(\geq 2)$, $\varrho, \varrho' \in \mathbb{N}$, we have

$$\| f - S^f_{(2^{\varrho},\wp;\lambda^{\varrho'},\wp')} \|_2^2 = \sum_{\eta=1}^{\lambda^{\varrho-1}}\sum_{\vartheta=\wp}^{\infty}\sum_{\eta'=1}^{\lambda^{\varrho'-1}}\sum_{\vartheta=\wp'}^{\infty} \left|\alpha_{(\eta,\vartheta;\eta',\vartheta')}\right|^2 = \sum_{\vartheta=\wp}^{\infty}\sum_{\vartheta=\wp'}^{\infty}\sum_{\eta=1}^{\lambda^{\varrho-1}}\sum_{\eta'=1}^{\lambda^{\varrho'-1}} \left|\alpha_{(\eta,\vartheta;\eta',\vartheta')}\right|^2$$

Since

$$\begin{aligned} \alpha_{(\eta,\vartheta;\eta',\vartheta')} &= \int_{\mathbb{R}}\int_{\mathbb{R}} f(\omega,\varpi)\,\Psi^{\lambda}_{(\eta,\vartheta;\eta',\vartheta')}(\omega,\varpi)w^{(\varrho,\varrho'),\lambda}_{(\eta,\eta')}(\omega,\varpi)d\omega d\varpi = \int_{\mathbb{R}}\int_{\mathbb{R}} f(\omega,\varpi) - f\left(\frac{2\eta-1}{\lambda^{\varrho}}, \frac{2\eta'-1}{\lambda^{\varrho'}}\right) \\ &\times \Psi^{\lambda}_{(\eta,\vartheta;\eta',\vartheta')}(\omega,\varpi)w^{(\varrho,\varrho'),\lambda}_{(\eta,\eta')}(\omega,\varpi)d\omega d\varpi + f\left(\frac{2\eta-1}{\lambda^{\varrho}}, \frac{2\eta'-1}{\lambda^{\varrho'}}\right)\int_{\mathbb{R}}\int_{\mathbb{R}} \Psi^{\lambda}_{(\eta,\vartheta;\eta',\vartheta')}(\omega,\varpi)w^{(\varrho,\varrho'),\lambda}_{(\eta,\eta')}(\omega,\varpi)d\omega d\varpi. \end{aligned}$$

Now, $f \in \mathrm{H}^{(\alpha,\beta)}_{(\Omega^2,\phi)}(\mathbb{R})$, $\sup(\lambda^{\varrho}\omega - 2\eta + 1) = 1\ \forall\ \nu = \left(x - \frac{2\eta-1}{\lambda^{\varrho}}\right), \omega \in \left(\frac{\eta-1}{\lambda^{\varrho-1}}, \frac{\eta}{\lambda^{\varrho-1}}\right]$ & $\sup\left(\lambda^{\varrho'}\varpi - 2\eta' + 1\right) = 1, \forall\ \upsilon = \left(y - \frac{2\eta'-1}{\lambda^{\varrho'}}\right), \varpi \in \left(\frac{\eta'-1}{\lambda^{\varrho'-1}}, \frac{\eta'}{\lambda^{\varrho'-1}}\right]$ and using Lemma 2.3, we have,

$$\left|\alpha_{(\eta,\vartheta;\eta',\vartheta')}\right| \leq \left(\sup\phi(\nu,\upsilon)\,\kappa\left(\frac{1}{\lambda^{\varrho\alpha}} + \frac{1}{\lambda^{\varrho'\beta}}\right) + \frac{4\kappa_0}{\lambda^{\varrho}\lambda^{\varrho'}}\right)\int_{\frac{\eta-1}{\lambda^{\varrho-1}}}^{\frac{\eta}{\lambda^{\varrho-1}}}\int_{\frac{\eta'-1}{\lambda^{\varrho'-1}}}^{\frac{\eta'}{\lambda^{\varrho'-1}}} \left|\Psi^{\lambda}_{(\eta,\vartheta;\eta',\vartheta')}(\omega,\varpi)w^{(\varrho,\varrho'),\lambda}_{(\eta,\eta')}(\omega,\varpi)\right| d\omega d\varpi.$$

If $\varrho \neq \varrho'$ or $\eta \neq \eta'$ then $\Psi_{(\eta,\vartheta;\eta',\vartheta')} = 0$

$$\begin{aligned}
\left|\alpha_{(\eta,\vartheta;\eta,\vartheta')}\right| &\leq \left(\sup\phi(\nu,\upsilon)\,\kappa\left(\frac{1}{\lambda^{\varrho\alpha}}+\frac{1}{\lambda^{\varrho\beta}}\right)+\frac{4\kappa_0}{\lambda^{\varrho}\lambda^{\varrho}}\right)\int_{\frac{\eta-1}{\lambda^{\varrho-1}}}^{\frac{\eta}{\lambda^{\varrho-1}}}\int_{\frac{\eta-1}{\lambda^{\varrho-1}}}^{\frac{\eta}{\lambda^{\varrho-1}}}\left|\Psi^{\lambda}_{(\eta,\vartheta;\eta,\vartheta')}(\omega,\varpi)w^{(\varrho,\varrho),\lambda}_{(\eta,\eta)}(\omega,\varpi)\right|d\omega d\varpi,\\
&\leq \max\left\{\sup\phi(\nu,\upsilon)\,\kappa,2\kappa_0\right\}\left(\frac{1}{\lambda^{\varrho\alpha}}+\frac{1}{\lambda^{\varrho\beta}}+\frac{2}{\lambda^{\varrho\alpha}\lambda^{\varrho\beta}}\right)\int_{\frac{\eta-1}{\lambda^{\varrho-1}}}^{\frac{\eta}{\lambda^{\varrho-1}}}\int_{\frac{\eta-1}{\lambda^{\varrho-1}}}^{\frac{\eta}{\lambda^{\varrho-1}}}\left|\Psi^{\lambda}_{(\eta,\vartheta;\eta,\vartheta')}(\omega,\varpi)w^{(\varrho,\varrho),\lambda}_{(\eta,\eta)}(\omega,\varpi)\right|d\omega d\varpi,\\
&\leq 2\max\left\{\sup\phi(\nu,\upsilon)\,\kappa,2\kappa_0\right\}\left(\frac{1}{\lambda^{\varrho\alpha}}+\frac{1}{\lambda^{\varrho\beta}}\right)\int_{\frac{\eta-1}{\lambda^{\varrho-1}}}^{\frac{\eta}{\lambda^{\varrho-1}}}\int_{\frac{\eta-1}{\lambda^{\varrho-1}}}^{\frac{\eta}{\lambda^{\varrho-1}}}\left|\Psi^{\lambda}_{(\eta,\vartheta;\eta,\vartheta')}(\omega,\varpi)w^{(\varrho,\varrho),\lambda}_{(\eta,\eta)}(\omega,\varpi)\right|d\omega d\varpi.
\end{aligned}$$

Next,

$$\begin{aligned}
\int_{\frac{\eta-1}{\lambda^{\varrho-1}}}^{\frac{n}{\lambda^{\varrho-1}}}\Psi^{\lambda}_{(\eta,\sigma)}(\omega)w^{\varrho}_{\eta}(\omega)d\omega &= \sqrt{\frac{\lambda^{\varrho+1}}{\pi}}\int_{\frac{\eta-1}{\lambda^{\varrho-1}}}^{\frac{n}{\lambda^{\varrho-1}}}P_{(\sigma+1/2)}(\lambda^{\varrho}\omega-2\eta+1)w(\lambda^{\varrho}\omega-2\eta+1)d\omega = \frac{1}{\lambda^{\varrho}}\sqrt{\frac{\lambda^{\varrho+1}}{\pi}}\int_0^{\pi}P_{(\sigma+1/2)}(cos\theta)d\theta\\
&= \frac{(-1)^{\sigma}}{\lambda^{\varrho}}\sqrt{\frac{\lambda^{\varrho+1}}{\pi}}\frac{1}{(\sigma+1/2)}\text{ where }\sigma\text{ is non negative integers.}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left|\alpha_{(\eta,\vartheta;\eta,\vartheta')}\right| &\leq 2\max\left\{\sup\phi(\nu,\upsilon)\,\kappa,2\kappa_0\right\}\left(\frac{1}{\lambda^{\varrho\alpha}}+\frac{1}{\lambda^{\varrho\beta}}\right)\int_{\frac{\eta-1}{\lambda^{\varrho-1}}}^{\frac{\eta}{\lambda^{\varrho-1}}}\left|\Psi^{\lambda}_{(\eta,\vartheta)}(\omega)w^{\varrho,\lambda}_{\eta}(\omega)\right|d\omega\int_{\frac{\eta-1}{\lambda^{\vartheta-1}}}^{\frac{\eta}{\lambda^{\vartheta-1}}}\left|\Psi^{\lambda}_{(\eta,\vartheta')}(\varpi)w^{\varrho,\lambda}_{\eta}(\varpi)\right|d\varpi\\
&\leq 2\max\left\{\sup\phi(\nu,\upsilon)\,\kappa,2\kappa_0\right\}\left(\frac{1}{\lambda^{\varrho\alpha}}+\frac{1}{\lambda^{\varrho\beta}}\right)\frac{1}{\lambda^{\varrho}}\sqrt{\frac{\lambda^{\varrho+1}}{\pi}}\frac{1}{(\vartheta+1/2)}\frac{1}{\lambda^{\varrho}}\sqrt{\frac{\lambda^{\varrho+1}}{\pi}}\frac{1}{(\vartheta'+1/2)}\\
&= \frac{4}{\pi}\max\left\{\sup\phi(\nu,\upsilon)\,\kappa,2\kappa_0\right\}\frac{1}{\lambda^{\varrho}}\left(\frac{1}{\lambda^{\varrho\alpha}}+\frac{1}{\lambda^{\varrho\beta}}\right)\frac{1}{(\vartheta+1/2)(\vartheta'+1/2)}\\
&= \frac{4}{\pi}\max\left\{\sup\phi(\nu,\upsilon)\,\kappa,2\kappa_0\right\}\left(\frac{1}{\lambda^{\varrho(\alpha+1)}}+\frac{1}{\lambda^{\varrho(\beta+1)}}\right)\frac{1}{(\vartheta+1/2)(\vartheta'+1/2)}\\
&= \delta\left(\frac{1}{\lambda^{\varrho(\alpha+1)}}+\frac{1}{\lambda^{\varrho(\beta+1)}}\right)\frac{1}{(\vartheta+1/2)(\vartheta'+1/2)}\text{ where }\delta=\frac{4}{\pi}\max\left\{\sup\phi(\nu,\upsilon)\,\kappa,2\kappa_0\right\}
\end{aligned}$$

$$\begin{aligned}
\| f-S^{f}_{(\lambda^{\varrho},\wp;\lambda^{\varrho'},\wp')}\|_2^2 &= \sum_{\eta=1}^{\lambda^{\varrho-1}}\sum_{\vartheta=\wp}^{\infty}\sum_{\eta'=1}^{\lambda^{\varrho'-1}}\sum_{\vartheta=\wp'}^{\infty}\left|\alpha_{(\eta,\vartheta;\eta',\vartheta')}\right|^2=\sum_{\vartheta=\wp}^{\infty}\sum_{\vartheta'=\wp'}^{\infty}\sum_{\eta=1}^{\lambda^{\varrho-1}}\sum_{\eta=1}^{\lambda^{\varrho-1}}\left|\alpha_{(\eta,\vartheta;\eta,\vartheta')}\right|^2\\
&\leq \frac{\lambda^{\varrho-1}\lambda^{\varrho-1}}{4\pi^2}\sum_{\vartheta=\wp}^{\infty}\sum_{\vartheta'=\wp'}^{\infty}\left(\frac{\delta\pi}{4}\right)^2\left(\frac{1}{\lambda^{\varrho(\alpha+1)}}+\frac{1}{\lambda^{\varrho(\beta+1)}}\right)^2\left(\frac{1}{(\vartheta+1/2)(\vartheta'+1/2)}\right)^2\\
&= \delta^2\left(\frac{1}{\lambda^{\varrho\alpha+3}}+\frac{1}{\lambda^{\varrho\beta+3}}\right)^2\sum_{\vartheta=\wp}^{\infty}\sum_{\vartheta'=\wp'}^{\infty}\frac{1}{(\vartheta+1/2)^2(\vartheta'+1/2)^2}.
\end{aligned}$$

Thus,

$$\| f - S^{f}_{(\lambda^{\varrho},\wp;\lambda^{\varrho'},\wp')} \|_2^2 \quad \leq \quad \delta^2 \left( \frac{1}{\lambda^{\varrho\alpha+3}} + \frac{1}{\lambda^{\varrho\beta+3}} \right)^2 S_{(\wp,\wp')}.$$

Where

$$\begin{aligned} S_{(\wp,\wp')} &= \sum_{\vartheta=\wp}^{\infty} \sum_{\vartheta'=\wp'}^{\infty} \frac{1}{(\vartheta+1/2)^2(\vartheta'+1/2)^2} = \sum_{\vartheta=\wp}^{\infty} \frac{1}{(\vartheta+1/2)^2} \sum_{\vartheta'=\wp'}^{\infty} \frac{1}{(\vartheta'+1/2)^2} \\ &\leq \left( \frac{1}{(\wp+1/2)^2} + \int_{\wp}^{\infty} \frac{d\vartheta}{(\vartheta+1/2)^2} \right) \left( \frac{1}{(\wp'+1/2)^2} + \int_{\wp'}^{\infty} \frac{d\vartheta'}{(\vartheta'+1/2)^2} \right) \text{ by Lemma 2.2,} \\ &= \left( \frac{1}{(\wp+1/2)^2} + \frac{1}{(\wp+1/2)} \right) \left( \frac{1}{(\wp'+1/2)^2} + \frac{1}{(\wp'+1/2)} \right). \end{aligned}$$

Since

$$\begin{aligned} 0 &\leq \| f - S^{f}_{(\lambda^{\varrho},\wp;\lambda^{\varrho'},\wp')} \|_2 \leq \max\left\{ \sup \phi(\nu, \upsilon)\, \kappa, 2\kappa_0 \right\} \left( \frac{1}{\lambda^{\varrho\alpha+3}} + \frac{1}{\lambda^{\varrho'\beta+3}} \right) \frac{2}{\pi\sqrt{(\wp+1/2)(\wp'+1/2)}} \\ &= \frac{2\max\left\{ \sup \phi(\nu, \upsilon)\, \kappa, 2\kappa_0 \right\}}{\pi\sqrt{(\wp+1/2)(\wp'+1/2)}} \left( \frac{1}{\lambda^{\varrho\alpha+2}} + \frac{1}{\lambda^{\varrho'\beta+2}} \right) = \frac{2\delta}{\sqrt{(\wp+1/2)(\wp'+1/2)}} \left( \frac{1}{\lambda^{\varrho\alpha+2}} + \frac{1}{\lambda^{\varrho'\beta+2}} \right) \\ &\to 0 \text{ as } \varrho, \varrho'; \wp, \wp' \to \infty. \end{aligned}$$

Therefore, the *EPCW* series of the function $f$ is uniformly convergent to $f$. Thus, Theorem 3.1 is completely established.

**Theorem 3.2.** *If* $f : \Omega^2 \to \mathbb{R}$ *be a real valued function of two variable such that* $f \in H^{(\alpha,\beta)}_{\Omega^2}(\mathbb{R})$, *then the EPCW series expansion* $\sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \sum_{n'=1}^{\infty} \sum_{m'=0}^{\infty} \left\langle f, \Psi^{\lambda}_{(n,m;n',m')} \right\rangle_{w_{(k,n;k',n')}} \Psi^{\lambda}_{(n,m;n',m')}$, *of the function* $f$, *is uniformly converges to* $f$.

**Proof of Theorem 3.2** The proof of above theorem can be given by exactly the same line of proof of Theorem 3.1 using the fact that $\phi = c$.

### *3.2. Corollaries*

In this section, three new corollaries related to Theorems 3.2, have been established in the following forms:

**Corollary 3.3.** *If* $f \in H^{(\alpha,\beta)}_{\Omega^2}(\mathbb{R})$ *and it can be expanded as an infinite series of the EPCW for* $\vartheta = 0$ *and* $\vartheta' = 0$ *is given by*

$$\sum_{\eta=1}^{\infty} \sum_{\eta'=1}^{\infty} \alpha_{(\eta;\eta')} \Psi^{\lambda}_{(\eta;\eta')} = \sum_{\eta=1}^{\infty} \sum_{\eta'=1}^{\infty} \left\langle f, \Psi^{\lambda}_{(\eta;\eta')} \right\rangle^{(\varrho,\varrho')}_{w_{(\eta;\eta'),\lambda}} \Psi^{\lambda}_{(\eta;\eta')}, \text{ then the series converges uniformly to } f.$$

**Corollary 3.4.** *If* $f \in H^{(\alpha,\beta)}_{\Omega^2}(\mathbb{R})$ *and it can be expanded as an infinite series of the EPCW for* $\varrho = \varrho' = 1$ *is given by*

$$\sum_{\vartheta=0}^{\infty} \sum_{\vartheta'=0}^{\infty} \left\langle f, \Psi^{\lambda}_{(\vartheta;\vartheta')} \right\rangle_{w^{(1,1),\lambda}_{(1;1)}} \Psi^{\lambda}_{(\vartheta;\vartheta')}, \text{ then the series converges uniformly to } f.$$

**Corollary 3.5.** *If $f$ is single variable real valued function in the class $H^{\alpha}_{\Omega}(\mathbb{R})$ and it can be expanded as an infinite series of the EPCW for $\varrho = \eta = 1$ is given by*

$$\sum_{\vartheta=0}^{\infty} \alpha_{\vartheta}\Psi_{\vartheta} = \sum_{\vartheta=0}^{\infty} \left\langle f, \psi^{\lambda}_{\vartheta}\right\rangle_{w^{1}_{1}} \psi^{\lambda}_{\vartheta}, \text{ where } \alpha_{\vartheta} = \int_{\mathbb{R}} f(\omega)\bar{\psi}^{\lambda}_{\vartheta}(\omega) w(2\omega-1)d\omega, \text{ then the series converges uniformly to } f.$$

## 4. Error analysis

### 4.1. Orthogonal Projection Operator

An orthogonal projection operator is a surjective map $\Phi_{\eta} : L^{2}_{\Omega} \to V_{\eta}$ defined by (*see*[43])

$$\Phi_{\eta}(f) = \sum_{m=0}^{\infty} \left\langle f, \Psi^{\lambda}_{(\eta,\vartheta)}\right\rangle_{w^{\varrho}_{\eta}} \Psi^{\lambda}_{(\eta,\vartheta)}, \text{ fixed } \eta = 1,2,3,\cdots \lambda^{\varrho-1} << \infty, \ \varrho \in \mathbb{N}.$$

The two dimensional orthogonal projection operator $\Phi_{(\eta,\eta')} : L^{2}_{\Omega^2} \to V_{(\eta,\eta')}$ is given by

$$\begin{aligned}\Phi_{(\eta,\eta')}(f) &= \sum_{\vartheta'=0}^{\infty}\sum_{\vartheta=0}^{\infty} \alpha_{(\vartheta;\vartheta')}\Psi^{\lambda}_{(\vartheta;\vartheta')}, \text{ fixed } \eta = 1,2,3,\cdots \lambda^{\varrho-1} << \infty, \quad \eta' = 1,2,3,\cdots \lambda^{\varrho'-1} << \infty, \quad \varrho,\varrho' \in \mathbb{N}, \\ &= \sum_{\vartheta=0}^{\infty}\sum_{\vartheta'=0}^{\infty} \left\langle f, \Psi^{\lambda}_{(\vartheta;\vartheta')}\right\rangle_{w^{(\varrho,\varrho'),\lambda}_{(\eta;\eta')}} \Psi^{\lambda}_{(\vartheta;\vartheta')}(\omega,\varpi), \text{ where } \alpha_{(\vartheta,\vartheta')} = \int_{\mathbb{R}}\int_{\mathbb{R}} f(\omega,\varpi)\Psi^{\lambda}_{\vartheta;\vartheta'} w^{(\varrho,\varrho'),\lambda}_{\eta;\eta'}(\omega,\varpi)d\omega d\varpi.\end{aligned}$$

### 4.2. Function Approximation

A two dimensional real valued function $f$ defined on $\Omega^2$ may be expanded in terms of the two dimensional *EPCW* series (1). If an infinite series (1) is approximated by an orthogonal projection operators $\Phi_{(\eta,\eta')}$, then

$$f \approx f_0 = \sum_{\eta=1}^{2^{\varrho-1}}\sum_{\vartheta=0}^{\wp-1}\sum_{\eta'=1}^{2^{\varrho'-1}}\sum_{\vartheta'=0}^{\wp'-1} \alpha_{\eta,\vartheta;\eta',\vartheta'}\Psi_{\eta,\vartheta;\eta',\vartheta'} = \langle \Upsilon, \Psi\rangle = \Upsilon^{\tau}\Psi \text{ where } \Upsilon^{\tau} \text{ indicates transpose of a matrix } \Upsilon,$$

where $\Upsilon$ and $\Psi$ are $2^{\varrho-1}\wp 2^{\varrho'-1}\wp' \times 1$ matrices and $\langle \Upsilon, \Psi\rangle$ is an inner product of column vectors $\Upsilon$ and $\Psi$ (*see*[23]).

### 4.3. Error of Wavelet Approximation

The error $\zeta_{(\aleph,\wp)}$ of wavelet approximation of a function $f$ by the orthogonal projection operators $\Phi_{(\aleph,\wp)}$ is defined by (*see*[49])

$$\zeta_{(\aleph,\wp)}(f) = \inf_{\Phi_{(\aleph,\wp)}(f)} \|\Phi_{(\aleph,\wp)}(f_0) - f\|_2 \ \text{ where } \aleph = \lambda^{\varrho-1} \text{ and } \varrho \in \mathbb{N}.$$

If error $\zeta_{(\aleph,\wp)}(f) \to 0$ as $\aleph \to \infty$ or $\wp \to \infty$ then $\Phi_{(\aleph,\wp)}(f_0)$ is called the best wavelet approximation of a function $f \in L^{2}_{\Omega}(\mathbb{R})$ (*see*[49]).

The error $\zeta_{(\aleph,\wp;\aleph',\wp')}(f)$ of two dimensional EPCW approximation of a function $f \in L^{2}_{\Omega^2}(\mathbb{R})$ by the operators $\Phi_{(\aleph,\wp;\aleph',\wp')}$ is given by

$$\zeta_{(\aleph,\wp;\aleph',\wp')}(f) = \inf_{\Phi_{(\aleph,\wp;\aleph',\wp')}(f)} \|\Phi_{(\aleph,\wp;\aleph',\wp')}(f_0) - f\|_2.$$

If error $\zeta_{(\aleph,\wp;\aleph',\wp')}(f) \to 0$ as $\aleph,\aleph' \to \infty$ or $\wp,\wp' \to \infty$ then $\Phi_{(\aleph,\wp;\aleph',\wp')}(f_0)$ is called the best wavelet approximation of a function $f \in L^{2}_{\Omega^2}(\mathbb{R})$.

*4.4. Main Results*

In this section, we develop a new theorems related to approximation error of function using orthogonal projection operators, ascertaining that the two dimensional *EPCW* expansion of a function $f$, in the Hölder's class $\mathrm{H}^{(\alpha,\beta)}_{(\Omega^2,\phi)}(\mathbb{R})$.

**Theorem 4.1.** *Let $f : \Omega^2 \to \mathbb{R}$ be a function belongs to class $H^{(\alpha,\beta)}_{(\Omega^2,\phi)}(\mathbb{R})$, and two dimensional EPCW expansion is given by*

$$\sum_{\eta=1}^{\infty}\sum_{\vartheta=0}^{\infty}\sum_{\eta'=1}^{\infty}\sum_{\vartheta'=0}^{\infty}\alpha_{(\eta,\vartheta;\eta',\vartheta')}\Psi^{\lambda}_{(\eta,\vartheta;\eta',\vartheta')}.$$ *Then the error function $\zeta_{(\aleph,\wp;\aleph',\wp')}(f)$ of $f$ converges uniformly to zero $\hat{\theta}$.*

**Proof:** Following the proof of theorem 3.1 we have

$$\begin{aligned} 0 \leq \| \zeta_{(\aleph,\wp;\aleph',\wp')}(f) \|_2^2 &\leq \frac{\delta^2}{16}\left(\frac{1}{\aleph^{\alpha}}+\frac{1}{\aleph'^{\beta}}\right)^2 \sum_{\vartheta=\wp}^{\infty}\sum_{\vartheta'=\wp'}^{\infty}\frac{1}{(\vartheta+1/2)^2(\vartheta'+1/2)^2} \leq \frac{\aleph_0^2}{(\wp+1/2)(\wp'+1/2)} \text{ by Lemma 2.2} \\ &\to 0 \text{ as } \wp \text{ or } \wp' \to \infty. \end{aligned}$$

Therefore, error function $\zeta_{(\aleph,\wp;\aleph',\wp')}(f)$ uniformly converges to zero function $\hat{\theta}$. Thus, Theorem 4.1 is completely established.

**Theorem 4.2.** *If $f : \Omega^2 \to \mathbb{R}$ be a function belongs to class $H^{(\alpha,\beta)}_{\Omega^2}(\mathbb{R})$, and two dimensional EPCW expansion is given by $\sum_{\eta=1}^{\infty}\sum_{\vartheta=0}^{\infty}\sum_{\eta'=1}^{\infty}\sum_{\vartheta'=0}^{\infty}\alpha_{(\eta,\vartheta;\eta',\vartheta')}\Psi^{\lambda}_{(\eta,\vartheta;\eta',\vartheta')}$, then the error function $\zeta_{(\aleph,\wp;\aleph',\wp')}(f)$ of $f$ converges uniformly to $\hat{\theta}$.*

**Proof of Theorem 4.2** The proof of above theorem can be given by exactly the same line of proof of Theorem 4.1 using the fact that $\phi = c$.

*4.5. Corollaries*

In this section, three new corollaries related to Theorem 4.1, has been established in the following forms:

**Corollary 4.3.** *If $f \in H^{(\alpha,\beta)}_{\Omega^2}(\mathbb{R})$, then the two dimensional EPCW error function $\zeta_{(\aleph;\aleph')}(f)$ of the target functions $f$ by the orthogonal projection operators $\Phi_{(\aleph;\aleph')}(f)$ are satisfy*

$$\left|\zeta_{(\aleph,\aleph')}(f)\right| = O\left(\frac{1}{\aleph^{\alpha}}+\frac{1}{\aleph'^{\beta}}\right).$$

**Corollary 4.4.** *If $f \in H^{(\alpha,\beta)}_{\Omega^2}(\mathbb{R})$, then the two dimensional EPCW error function $\zeta_{(\wp;\wp')}(f)$ of a $\xi$ by the operators $\Phi_{(\wp,\wp')}(f)$ are satisfy*

$$\left|\Phi^{\xi}_{(\wp,\wp')}\right| = O\left(\frac{1}{\lambda^{\alpha+1}}+\frac{1}{\lambda^{\beta+1}}\right)\left(\frac{1}{\sqrt{(\wp+1/2)(\wp'+1/2)}}\right) \textit{for } 0<\alpha,\beta\leq 1.$$

**Corollary 4.5.** *If $f$ is a single real valued function in the class $H^{\alpha}_{\Omega}(\mathbb{R})$, then the EPCW error function $\zeta_{\wp}(f)$ of a function $f$ by $\Phi_{\wp}(f)$ are satisfy*

$$\left|\zeta_{\wp}(f)\right| = O\left(\frac{1}{\lambda^{\alpha}(\wp+1/2)}\right).$$

## 5. Effectiveness of the *EPCW*/*PCW* approximation method

In this section, we calculate the approximation of a function and it effectiveness show by an example, more over this results are compared with *PCW* and *CW* approximation methods.

*5.1. Illustrative Examples*

**Example 5.1.** *Consider the single variable real valued function $\xi$ on $\Omega$,*

$$\xi(\omega) = \begin{cases} \omega^{1/2} - 2\omega^{3/2} - 3\omega^{5/2} + 4\omega^{7/2} & \textit{for} \quad \omega \in \Omega, \\ 0 \quad \textit{otherwise.} \end{cases}$$

*Next, we calculate the approximated valued for the function $\xi$, by the PCW approximation method.*

*In the Corollary 3.5, if $\lambda = 2$ and*

$$\xi_0^{\wp}(\omega) \quad = \quad \sum_{\vartheta=0}^{\wp-1} \alpha_\vartheta \Psi_\vartheta(\omega) \;\&\; \alpha_\vartheta = \langle \xi, \Psi_\vartheta \rangle_{w_1^1}.$$

*Next, we evaluate $\xi_0^1(\omega)$, $\xi_0^2(\omega)$, $\xi_0^3(x)$, $\xi_0^4(x)$, $\zeta_1^{\xi}(\omega)$, $\zeta_2^{\xi}(\omega)$, $\zeta_3^{\xi}(\omega)$, $\zeta_4^{\xi}(\omega)$ and $\xi_0^{\wp}(\omega)$ & $\zeta_{\wp}^{\xi}(\omega)$. If,*

$$\Upsilon^{\wp} = \left(\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_{\wp-1}\right)^{\tau} \quad \& \quad \Psi^{\wp} = \left(\Psi_0, \Psi_1, \Psi_2, \cdots, \Psi_{\wp-1}\right)^{\tau}$$

$$\xi_0 \quad = \quad \sum_{\vartheta=0}^{\infty} \alpha_\vartheta \Psi_\wp = \lim_{\wp\to\infty} \sum_{\vartheta=0}^{\wp-1} \alpha_\vartheta \Psi_\vartheta, = \lim_{\wp\to\infty} \langle \Upsilon^{\wp}, \Psi^{\wp} \rangle = \lim_{\wp\to\infty} \left((\Upsilon^{\wp})^{\tau}\, \Psi^{\wp}\right) = \lim_{\wp\to\infty} \xi_0^{\wp}, \textit{ where } \alpha_\vartheta = \int_0^1 \xi(\omega)\Psi_\vartheta(\omega) w(2\omega-1)d\omega,$$

*and $\xi_0(\omega)$ is called approximated valued function of the targeted problem $\xi(\omega)$ using PCW series expansion method. Now, we calculate $\alpha_\vartheta$ for $\vartheta \geq 0$,* $\alpha_0 = \int_0^1 \xi(\omega)\Psi_0(\omega)w_1^1(\omega)d\omega \approx -0.1662$, $\alpha_1 = \int_0^1 \xi(\omega)\Psi_1(\omega)w_1^1(\omega)d\omega \approx -0.1108$, $\alpha_2 = \int_0^1 \xi(\omega)\Psi_2(\omega)w_1^1(\omega)d\omega \approx 0.2216$, $\alpha_3 = \int_0^1 \xi(\omega)\Psi_3(\omega)w_1^1(\omega)d\omega \approx 0.0554$, $\alpha_4 = \int_0^1 \xi(\omega)\Psi_4(\omega)w_1^1(\omega)d\omega = 0$, $\alpha_\vartheta = 0$, *for* $\vartheta \geq 4$, *i.e.* $\Upsilon^{\wp} = (-0.1662, -0.1108, 0.2216, 0.0554, 0, \cdots, 0)^{\tau}$.

*Since the approximated and its error of function $\xi$ of an order $\wp$ using PCW series expansion method are given by*

$$\xi_0^{\wp}(\omega) = \sum_{\vartheta=0}^{\wp-1} \alpha_\vartheta \Psi_\vartheta(\omega) = (\Upsilon^{\wp})^{\tau}\, \Psi^{\wp}(\omega) \quad \textit{and} \quad \zeta_{\wp}^{\xi}(\omega) = \sum_{\vartheta=\wp}^{\infty} \alpha_\vartheta \Psi_\vartheta \textit{ respectively.}$$

*Therefore*

$$\begin{aligned} \xi_0^{\wp}(\omega) &= -0.1662\, \Psi_0(\omega) - 0.1108\, \Psi_1(\omega) + 0.2216\, \Psi_2(\omega) + 0.0554\, \Psi_3(\omega) + 0\Psi_4(\omega) + 0\Psi_5(\omega) + \cdots + 0\Psi_{(\wp-1)}(\omega), \\ &= \sum_{\vartheta=0}^{(4-1)} \alpha_\vartheta \Psi_\vartheta(\omega), = \xi_0^4(\omega), \textit{ and approximation is given by} \end{aligned}$$

$$\xi_0 = \lim_{\wp\to\infty} \sum_{\vartheta=0}^{\wp-1} \alpha_\theta \Psi_\vartheta = \lim_{\wp\to\infty} \left( \sum_{\vartheta=0}^{4-1} \alpha_\theta \Psi_\vartheta + \sum_{\vartheta=4}^{\wp-1} \alpha_\theta \Psi_\vartheta \right) = \lim_{\wp\to\infty} \left(\xi_0^4 + \hat{\theta}\right) = \xi_0^4,$$

*moreover the error of targeted function is zero* $\hat{\theta} = \zeta_{\wp}^{\xi}(\omega) = \sum_{\vartheta=4}^{\infty} \alpha_\vartheta \Psi_\vartheta(\omega)$ *of the order four. The energy of signal $\xi$ and approximated signal $\xi_0^4$ are given by*

$$\|\, \xi \,\|_2^2 \quad = \quad \langle \xi, \xi \rangle_{w_1^1} = \int_0^1 (\xi(\omega))^2\, \omega\, (2\omega - 1)\, d\omega \simeq \lim_{\wp\to\infty} \sum_{\vartheta=0}^{\wp-1} |\alpha_\vartheta|^2 = \|\, \xi_0^4 \,\|_2^2 = \|\, \xi_0 \,\|_2^2\,.$$

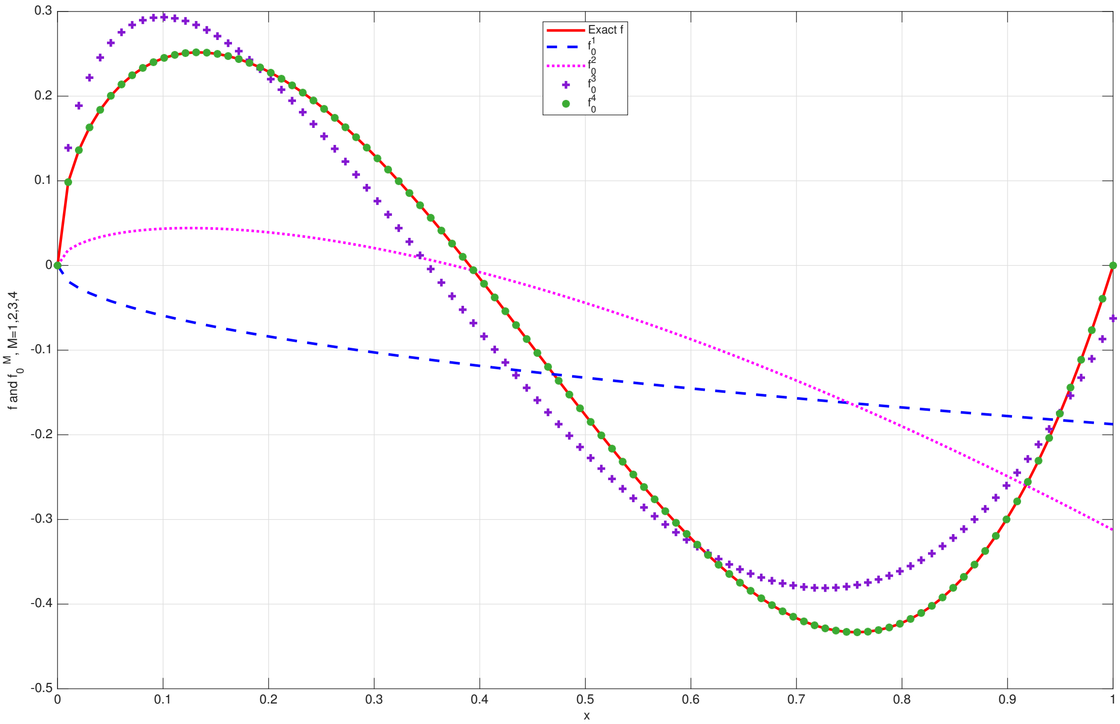


Figure 1: Graph of $\xi$ and truncated $\xi_0^{\wp}$ for $\wp = 1, 2, 3, 4$.

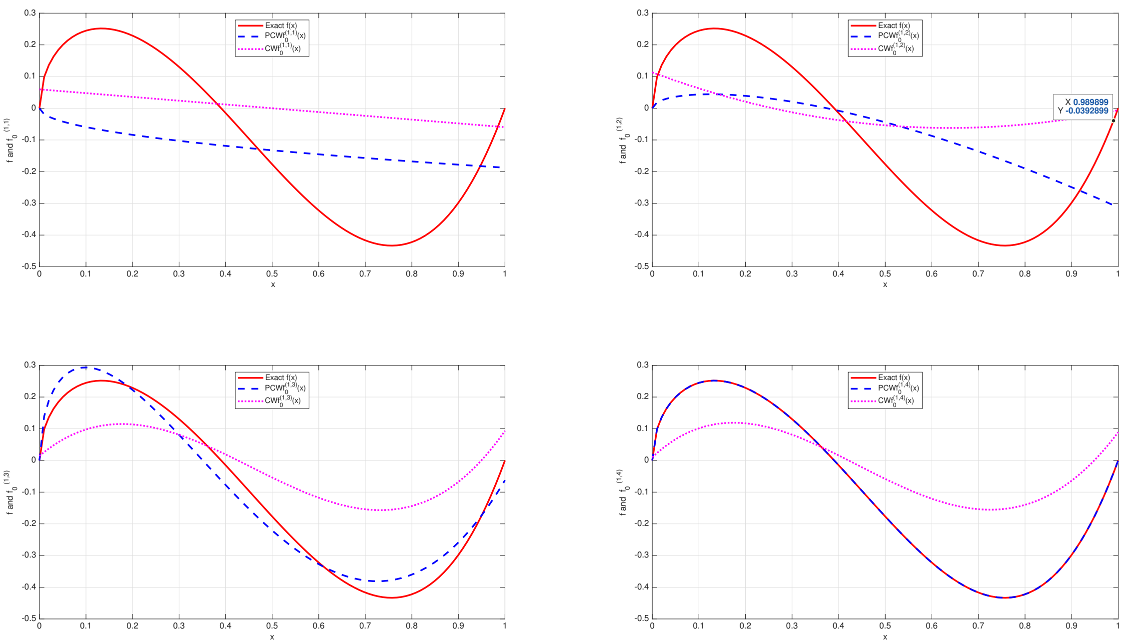


Figure 2: Graph of exact $\xi$ and $\xi_0^{\wp}$ by PCW & CW methods, for $\wp = 1, 2, 3, 4$.

| $t$ | 0.0 0 | 0.25 | 0.50 | 0.75 | 1.00 |
|---|---|---|---|---|---|
| $\xi$ | 0 | 0.1875 | -0.1767766953 | -0.4330127019 | 0 |
| PCW$\xi_0^0$ | 0 | -0.09375 | -0.1325825215 | -0.1623797632 | -0.1875 |
| CW$\xi_0^0$ | 0.05961994694 | 0.02980997347 | 0 | -0.02980997347 | -0.05961994694 |
| PCW$\xi_0^1$ | 0 | 0.03125 | -0.04419417382 | -0.1623797632 | -0.3125 |
| CW$\xi_0^1$ | 0.1134524414 | 0.002893726238 | -0.05383249446 | -0.0567262207 | -0.005787452476 |
| PCW$\xi_0^2$ | 0 | 0.15625 | -0.2209708691 | -0.3788861142 | -0.0625 |
| CW$\xi_0^2$ | 0.01425310335 | 0.1020930643 | -0.05383249446 | -0.1559255587 | 0.09341188557 |
| PCW$\xi_0^3$ | 0 | 0.1875 | -0.1767766953 | -0.4330127019 | 0 |
| CW$\xi_0^3$ | 0.009970812508 | 0.1042342097 | -0.0581147853 | -0.1537844133 | 0.08912959473 |
| PCW$\xi_0^4$ | 0 | 0.1875 | -0.1767766953 | -0.4330127019 | 0 |
| CW$\xi_0^4$ | 0.003403968134 | 0.1009507875 | -0.0581147853 | -0.1505009911 | 0.0956964391 |

Table 1: Comparison between $\xi, \xi_0^{\wp}$ for $\wp = 0, 1, 2, 3, 4$.

| $t$ | 0.0 0 | 0.25 | 0.50 | 0.75 | 1.00 |
|---|---|---|---|---|---|
| PCW$\zeta^1$ | 0 | 0.28125 | 0.04419417382 | 0.2706329387 | 0.1875 |
| CW$\zeta^1$ | 0.05961994694 | 0.1576900265 | 0.1767766953 | 0.4032027284 | 0.05961994694 |
| PCW$\zeta^2$ | 0 | 0.15625 | 0.1325825215 | 0.2706329387 | 0.3125 |
| CW$\zeta^2$ | 0.1134524414 | 0.1846062738 | 0.1229442008 | 0.3762864812 | 0.005787452476 |
| PCW$\zeta^3$ | 0 | 0.03125 | 0.04419417382 | 0.05412658774 | 0.0625 |
| CW$\zeta^3$ | 0.01425310335 | 0.08540693572 | 0.1229442008 | 0.2770871431 | 0.09341188557 |
| PCW$\zeta^4$ | 0 | 0 | 0 | 0 | 0 |
| CW$\zeta^4$ | 0.009970812508 | 0.0832657903 | 0.11866191 | 0.2792282886 | 0.08912959473 |
| PCW$\zeta^5$ | 0 | 0 | 0 | 0 | 0 |
| CW$\zeta^5$ | 0.003403968134 | 0.08654921248 | 0.11866191 | 0.2825117108 | 0.0956964391 |

Table 2: $\zeta^{\wp}$ using PCW & CW methods for $\wp = 1, 2, 3, 4, 5$.

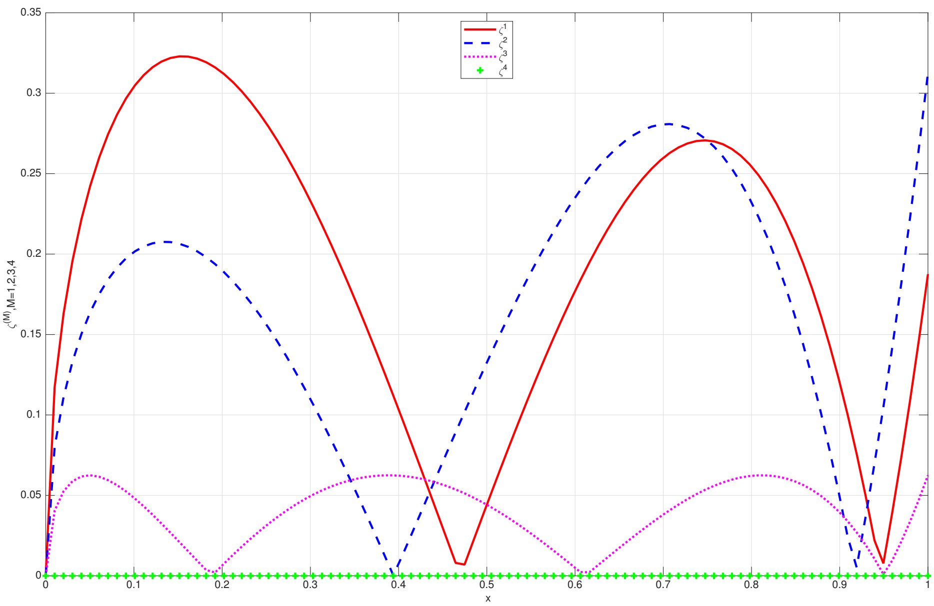


Figure 3: Graph of errors $\zeta_1^{\xi}, \zeta_2^{\xi}, \xi_3^{\xi}, \xi_4^{\zeta}$ for different order $\wp = 1, 2, 3, 4$.

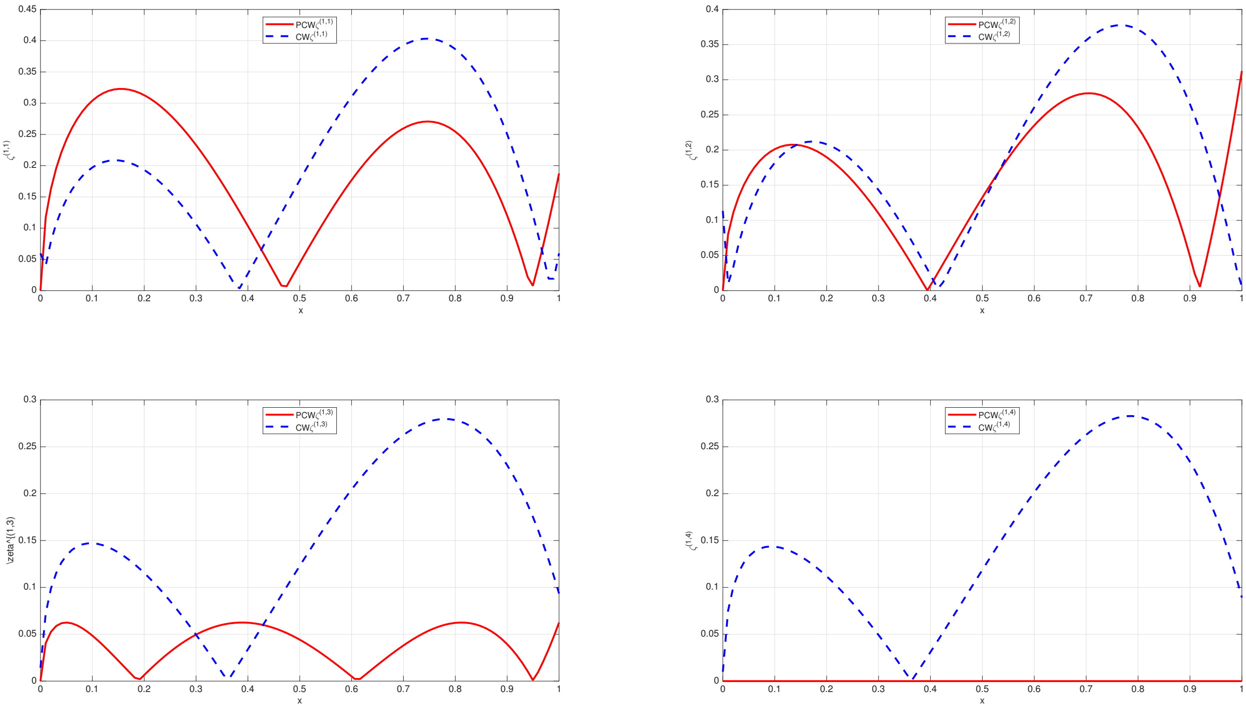

Figure 4: Graph of errors $\zeta_1^\xi, \zeta_2^\xi, \zeta_3^\xi, \zeta_4^\xi,$ by PCW & CW methods.

**Example 5.2.** *In this example, we consider a function defined over two variables, $f$.*

$$f(\omega, u) = \begin{cases} \omega^{1/2}u^{3/2} - 2\omega^{3/2}u^{1/2} - 3\omega^{5/2}u^{7/2} + 4\omega^{7/2}u^{5/2} & \textit{for} \quad (\omega, u) \in \Omega^2, \\ 0 \quad \textit{otherwise.} \end{cases}$$

*In the Corollary 3.4, if $\lambda = 2$, and, if we consider the Pseudo-Chebyshev series expansions of the function, for*

$$\varrho = \varrho' = 1 \Rightarrow \eta = \eta' = 1,$$

*then,*

$$\begin{aligned} f \quad &\sim \quad \sum_{\vartheta=0}^{\infty}\sum_{\vartheta'=0}^{\infty} \alpha_{(\vartheta;\vartheta')}\psi_\vartheta\psi_{\vartheta'} \\ &= \quad -0.3206\psi_0(\omega)\psi_0(u) + 0.2500\psi_0(\omega)\psi_1(u) - 0.0537\psi_0(\omega)\psi_2(u) - 0.0230\psi_0(\omega)\psi_3(u) \\ &- \quad 0.1511\psi_1(\omega)\psi_0(u) + 0.0805\psi_1(\omega)\psi_1(u) - 0.0161\psi_1(\omega)\psi_2(u) + 0.0192\psi_1(\omega)\psi_3(u) \\ &+ \quad 0.5100\psi_2(\omega)\psi_0(u) + 0.2684\psi_2(\omega)\psi_1(u) + 0.0644\psi_2(\omega)\psi_2(u) + 0.0038\psi_2(\omega)\psi_3(u) \\ &+ \quad 0.0537\psi_3(\omega)\psi_0(u) + 0.0268\psi_3(\omega)\psi_1(u) + 0.0054\psi_3(\omega)\psi_2(u) + 0\psi_3(\omega)\psi_3(u) + \cdots \\ &= \quad f_0, \end{aligned}$$

$$\textit{where} \quad \alpha_{(\vartheta;\vartheta')} = \int_{\Omega^2} f\psi_\vartheta w_1^1 \psi_{\vartheta'} w_1^1 d\mu = \int_{\omega=0}^{1}\int_{u=0}^{1} f(\omega, u)\psi(\omega)w(2\omega - 1)\psi(u)w(2u - 1)d\omega du.$$

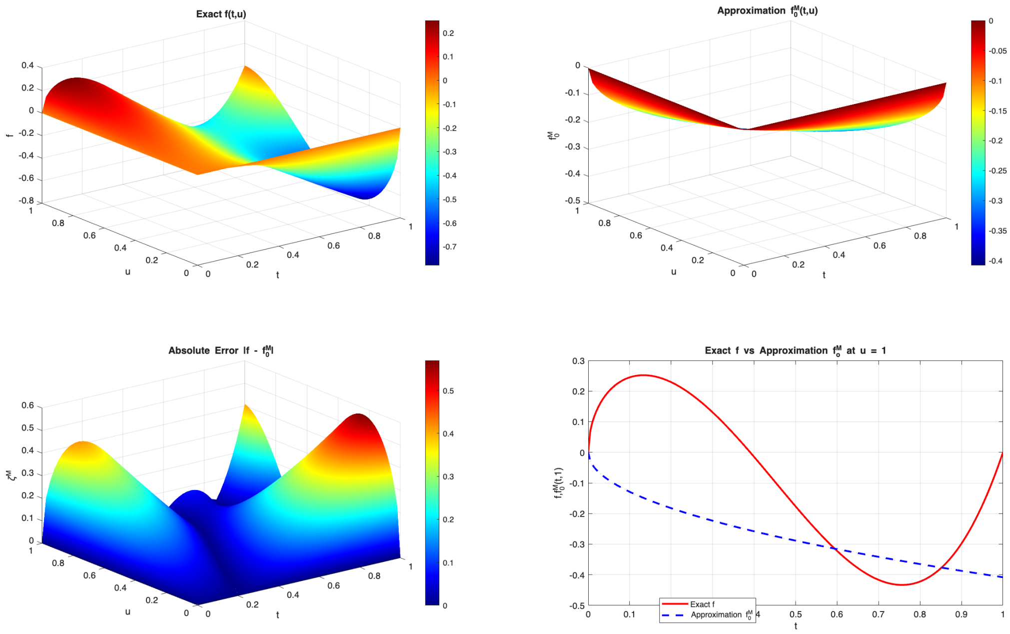

Figure 5: Graph of $f, f_0, |f - f_0|$ in 3D and $f, f_0$ in 2D for $M = 1$

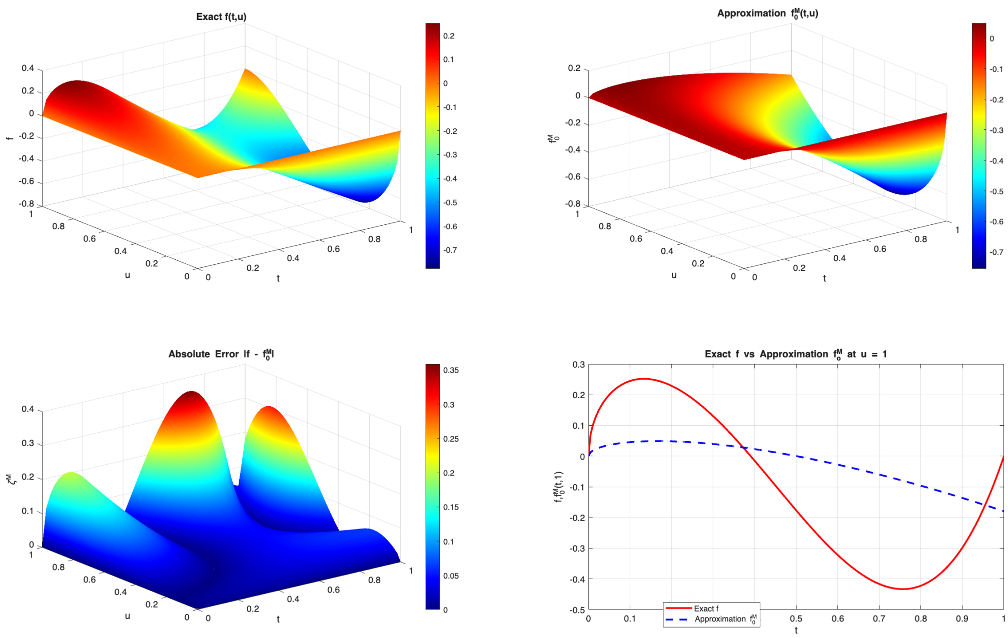

Figure 6: Graph of $f, f_0, |f - f_0|$ in 3D and $f, f_0$ in 2D for $M = 2$

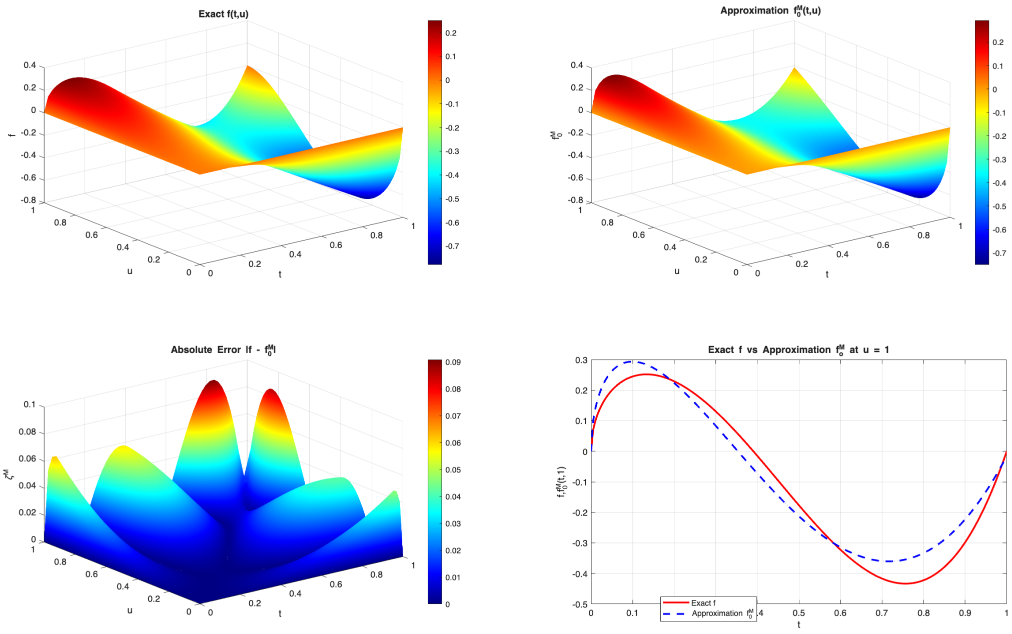


Figure 7: Graph of $f, f_0, |f - f_0|$ in 3D and $f, f_0$ in 2D for $M = 3$

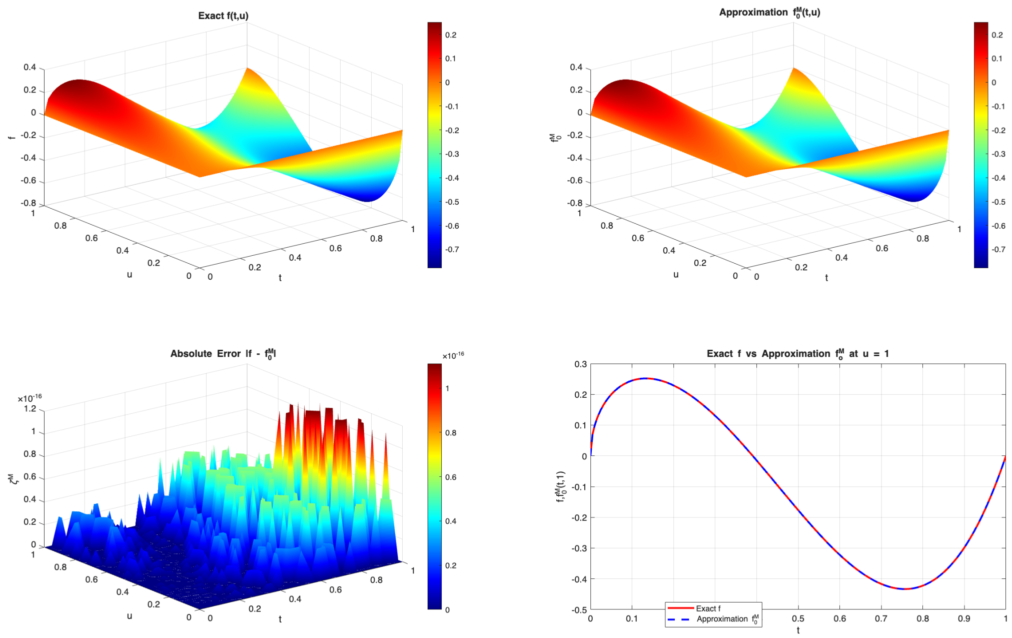


Figure 8: Graph of $f, f_0, |f - f_0|$ in 3D and $f, f_0$ in 2D for $M = 4$

## 6. Conclusions

(i) Since $\zeta_{(\aleph,\wp;\aleph',\wp')}(f) \to 0$ as $\aleph, \aleph' \to \infty$ or $\wp, \wp' \to \infty$ in above results. Therefore the EPCW approximations determined in this results are best possible in the wavelet analysis (*see*[49]).

(ii) Some most important corollaries 3.3,3.4,3.5 and 4.3,4.4,4.5 have been derived from our main Theorems 3.2 and 4.2 respectively.

(iii) Independent proofs of these corollaries 3.3,3.4,3.5, 4.3,4.4 and 4.5 can be developed for specific contributions of these estimates in wavelet analysis.

(iv) Figures (1, 2, 3&4) and Tables (1&2) are shows that the $PCW$ method is more effective rather than $CW$ method in the case of fractional degree.

(v) Figures (1, 2, 3 &4) are shows that how to error functions are rapidly converges to zero functions by the $PCW$ method rather than $CW$ method in this case.

(vi) The outcomes obtained for the two-dimensional case are effectively demonstrated through Example 5.2. In this example, by fixing the variable $u = 1$, the problem formulation aligns closely with that of Example 5.1. This equivalence becomes apparent when analyzing the respective solutions. Specifically, Figures 5, 6, 7, and 8 visually highlight the structural similarities and reinforce the consistency of the proposed approach across both one- and two-dimensional settings. Such correspondence not only validates the generality of the method but also showcases its adaptability in higher-dimensional approximation scenarios.

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**Conflict of interest** The authors declare that there is no conflict of interest.

**Data availability** My manuscript has no associated data.

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