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On the solutions of third-order fuzzy problems with symmetric triangular fuzzy numbers

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Abstract. This paper is on third-order fuzzy problems with symmetric triangular fuzzy numbers. We study solutions of the problems via fuzzy Laplace transform method. We give comparison results of the problems. Numerical examples are solved. Graphics of the solutions are drawn for each alpha level sets. Conclusions are given.

1. Introduction

Dynamical systems under uncertainty are modeled by means of fuzzy differential equations. The topic of fuzzy differential equations has been rapidly growing in recent years. Firstly, Zadeh introduced fuzzy number and fuzzy arithmetic [36]. Chang and Zadeh first introduced the concept of fuzzy derivative [11], followed by Dubois and Prade [12]. Dubois and Prade used the extension principle. Puri and Ralescu generalized the Hukuhara derivative of a set-valued function [29]. Hukuhara derivative has a drawback: It becomes fuzzier as time goes. So, the fuzzy solution behaves quite differently from the crisp solution. Hüllermeier interpreted the fuzzy differential equation as a family of differential inclusions [19]. Bede and Gal introduced the strongly generalized differentiability [6]. The strongly generalized differentiability was studied [7, 8, 10, 14, 26]. Another approach to solve fuzzy differential equations is Zadeh's extension principle [3, 27].

Abbasbandy and Allahviranloo, Abbasbandy et al. and Allahviranloo et al. introduced the numerical methods for solving fuzzy differential equations [1, 2, 4]. Numerical methods were studied in many papers [15, 16, 20, 21, 24, 33, 34].

Fuzzy differential equations and corresponding fuzzy initial and boundary value problems are solved by the fuzzy Laplace transform method. The fuzzy Laplace transform method is practically the most important operational method. Problems are solved directly. So, fuzzy Laplace transform method was used in many papers [9, 13, 17, 18, 31, 32, 35].

After a preliminary section, we study the solutions of third-order fuzzy problems with positive symmetric triangular fuzzy numbers using the fuzzy Laplace transform method. Also, in section 4, we present numerical examples to illustrate our problems. At the end of the paper, we give some conclusions.

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2. Preliminaries

Definition 2.1 ([7]). *A fuzzy number is a function $\hat{u}:\mathbb{R} \to [0,1]$ verifying the following four assumptions:*
(a) $\hat{u}$ is normal,
(b) $\hat{u}$ is convex fuzzy set,
(c) $\hat{u}$ is upper semi-continuous on $\mathbb{R}$,
(d) $cl\{x \in \mathbb{R} \mid \hat{u}(x) > 0\}$ is compact, where cl denotes the closure of a subset.
Let $\mathbb{R}_F$ be the set of all fuzzy number on $\mathbb{R}$.

Definition 2.2 ([26]). *Let $\hat{u} \in \mathbb{R}_F$. The α-level set of $\hat{u}$ is*

$$[\hat{u}]^\alpha = \begin{cases} \{x \in \mathbb{R} \mid \hat{u}(x) \geq \alpha\}, & \text{if } 0 < \alpha \leq 1 \\ cl\{x \in \mathbb{R} \mid \hat{u}(x) > 0\}, & \text{if } \alpha = 0 \end{cases}.$$

Definition 2.3 ([5]). *A fuzzy number $\hat{u}$ in parametric form is a pair $\left[\underline{\hat{u}}_\alpha, \overline{\hat{u}}_\alpha\right]$ of functions $\underline{\hat{u}}_\alpha, \overline{\hat{u}}_\alpha$, $0 \leq \alpha \leq 1$, which satisfty the following requirements:*
1. $\underline{\hat{u}}_\alpha$ is bounded non-decreasing left-continuous in $(0,1]$, right-continuous at $\alpha = 0$.
2. $\overline{\hat{u}}_\alpha$ is bounded non-increasing left-continuous in $(0,1]$, right-continuous at $\alpha = 0$.
3. $\underline{\hat{u}}_\alpha \leq \overline{\hat{u}}_\alpha, 0 \leq \alpha \leq 1$.

Definition 2.4 ([26]). *For $\hat{u}_1$, $\hat{u}_2 \in \mathbb{R}_F$ and $\lambda \in \mathbb{R}$,*

$$[\hat{u}_1 + \hat{u}_2]^\alpha = [\hat{u}_1]^\alpha + [\hat{u}_2]^\alpha,$$

$$[\lambda \hat{u}_1]^\alpha = \lambda [\hat{u}_1]^\alpha.$$

Definition 2.5 ([23]). *If $\hat{A}$ is a symmetric triangular number with support $\left[\underline{a}, \overline{a}\right]$, the α-level set of $\hat{A}$ is*

$$\left[\hat{A}\right]^\alpha = \left[\underline{a} + \left(\frac{\overline{a} - \underline{a}}{2}\right)\alpha, \overline{a} - \left(\frac{\overline{a} - \underline{a}}{2}\right)\alpha\right].$$

Definition 2.6 ([22]). *Let $\hat{u}_1$, $\hat{u}_2 \in \mathbb{R}_F$. If there exists $\hat{u}_3 \in \mathbb{R}_F$ such that $\hat{u}_1 = \hat{u}_2 + \hat{u}_3$ then $\hat{u}_3$ is called the Hukuhara difference of $\hat{u}_1$ and $\hat{u}_2$, which we denote by $\hat{u}_1 \ominus \hat{u}_2$.*

Definition 2.7 ([28]). *Let $\hat{f} : (a,b) \longrightarrow \mathbb{R}_F$ and $t_0 \in (a,b)$. If there exists $\hat{f}'(t_0) \in \mathbb{R}_F$ such that for all $h > 0$ sufficiently small, exists $\hat{f}(t_0+h) \ominus \hat{f}(t_0)$, $\hat{f}(t_0) \ominus \hat{f}(t_0-h)$ and the limits hold*

$$\lim_{h \longrightarrow 0^+} \frac{\hat{f}(t_0+h) \ominus \hat{f}(t_0)}{h} = \lim_{h \longrightarrow 0^+} \frac{\hat{f}(t_0) \ominus \hat{f}(t_0-h)}{h} = \hat{f}'(t_0),$$

$\hat{f}$ is Hukuhara differentiable at t_0.

Definition 2.8 ([28]). *Let $\hat{f} : (a,b) \longrightarrow \mathbb{R}_F$ and $t_0 \in (a,b)$.*

1) $\hat{f}$ is said to be (i)-differentiable at t_0, if there exists $\hat{f}'(t_0) \in \mathbb{R}_F$ such that for all $h > 0$ sufficiently small, exists $\hat{f}(t_0+h) \ominus \hat{f}(t_0)$, $\hat{f}(t_0) \ominus \hat{f}(t_0-h)$ and the limits hold

$$\lim_{h \longrightarrow 0^+} \frac{\hat{f}(t_0+h) \ominus \hat{f}(t_0)}{h} = \lim_{h \longrightarrow 0^+} \frac{\hat{f}(t_0) \ominus \hat{f}(t_0-h)}{h} = \hat{f}'(t_0).$$

2) $\hat{f}$ is said to be (ii)-differentiable at t_0, if there exists $\hat{f}'(t_0) \in \mathbb{R}_F$ such that for all $h > 0$ sufficiently small, exists $\hat{f}(t_0) \ominus \hat{f}(t_0+h)$, $\hat{f}(t_0-h) \ominus \hat{f}(t_0)$ and the limits hold

$$\lim_{h \longrightarrow 0^+} \frac{\hat{f}(t_0) \ominus \hat{f}(t_0+h)}{-h} = \lim_{h \longrightarrow 0^+} \frac{\hat{f}(t_0-h) \ominus \hat{f}(t_0)}{-h} = \hat{f}'(t_0).$$

Theorem 2.9 ([10]). *Let* $\hat{f}:(a,b)\longrightarrow \mathbb{R}_F$ *and* $\left[\hat{f}(t)\right]^{\alpha}=\left[\underline{\hat{f}}_{\alpha}(t),\overline{\hat{f}}_{\alpha}(t)\right]$, *for all* $\alpha\in[0,1]$.

1) If $\hat{f}$ *is* (i) *differentiable, then* $\underline{\hat{f}}_{\alpha}$ *and* $\overline{\hat{f}}_{\alpha}$ *differentiable functions and* $\left[\hat{f}'(t)\right]^{\alpha}=\left[\underline{\hat{f}}'_{\alpha}(t),\overline{\hat{f}}'_{\alpha}(t)\right]$.

2) If $\hat{f}$ *is* (ii)*-differentiable, then* $\underline{\hat{f}}_{\alpha}$ *and* $\overline{\hat{f}}_{\alpha}$ *differentiable functions and* $\left[\hat{f}'(t)\right]^{\alpha}=\left[\overline{\hat{f}}'_{\alpha}(t),\underline{\hat{f}}'_{\alpha}(t)\right]$.

Definition 2.10 ([30]). *The fuzzy Laplace transform of fuzzy function* $\hat{f}$ *is*

$$\hat{F}(s)=L\left(\hat{f}(t)\right)=\int_0^{\infty}e^{-st}\hat{f}(t)\,dt=\left[\lim_{\rho\longrightarrow\infty}\int_0^{\rho}e^{-st}\underline{\hat{f}}(t)\,dt\,,\ \lim_{\rho\longrightarrow\infty}\int_0^{\rho}e^{-st}\overline{\hat{f}}(t)\,dt\right],$$

$$\hat{F}(s,\alpha)=L\left(\left[\hat{f}(t)\right]^{\alpha}\right)=\left[L\left(\underline{\hat{f}}_{\alpha}(t)\right),L\left(\overline{\hat{f}}_{\alpha}(t)\right)\right],$$

$$L\left(\underline{\hat{f}}_{\alpha}(t)\right)=\int_0^{\infty}e^{-st}\underline{\hat{f}}_{\alpha}(t)\,dt=\lim_{\rho\longrightarrow\infty}\int_0^{\rho}e^{-st}\underline{\hat{f}}_{\alpha}(t)\,dt\,,$$

$$L\left(\overline{\hat{f}}_{\alpha}(t)\right)=\int_0^{\infty}e^{-st}\overline{\hat{f}}_{\alpha}(t)\,dt=\lim_{\rho\longrightarrow\infty}\int_0^{\rho}e^{-st}\overline{\hat{f}}_{\alpha}(t)\,dt.$$

Theorem 2.11 ([5]). *Let* $\hat{f}'(t)$ *be an integrable fuzzy function and* $\hat{f}(t)$ *is primitive of* $\hat{f}'(t)$ *on* $(0,\infty]$. *Then,*

$$L\left(\hat{f}'(t)\right)=sL\left(\hat{f}(t)\right)\ominus\hat{f}(0)$$

where $\hat{f}$ *is* (i) *differentiable or*

$$L\left(\hat{f}'(t)\right)=\left(-\hat{f}(0)\right)\ominus\left(-sL\left(\hat{f}(t)\right)\right)$$

where f *is* (ii) *differentiable.*

Theorem 2.12 ([25]). *Suppose that* $\hat{f}(t)$, $\hat{f}'(t)$ *and* $\hat{f}''(t)$ *are continuous fuzzy-valued functions on* $(0,\infty]$ *and of exponential order and* $\hat{f}'''(t)$ *is piecewise continuous fuzzy-valued function on* $(0,\infty]$ *with*

$$\hat{f}(t)=\left[\underline{\hat{f}}(t,\alpha),\overline{\hat{f}}(t,\alpha)\right],$$

then

$$L\left(\hat{f}'''(t)\right)=s^3L\left(\hat{f}(t)\right)\ominus s^2\hat{f}(0)\ominus s\hat{f}'(0)\ominus\hat{f}''(0).$$

Theorem 2.13 ([5]). *Let* $\hat{f}_1(t)$ *and* $\hat{f}_2(t)$ *be continuous fuzzy-valued functions suppose that* c_1, c_2 *are constant, then*

$$L\left(c_1\hat{f}_1(t)+c_2\hat{f}_2(t)\right)=c_1L\left(\hat{f}_1(t)\right)+c_2L\left(\hat{f}_2(t)\right).$$

3. Main results

We investigate the solutions of the following problems

i)

$$\hat{y}'''=[\widehat{\tau}]^{\alpha}\hat{y}',\ \ t>0 \tag{1}$$

$$\begin{cases} \hat{y}(0) = [\widehat{\mu}]^{\alpha} \\ \hat{y}'(0) = [\widehat{\nu}]^{\alpha} \\ \hat{y}''(0) = \left[\widehat{\xi}\right]^{\alpha} \end{cases}, \tag{2}$$

and

ii)

$$\hat{y}''' = -[\widehat{\tau}]^{\alpha} \hat{y}', \quad t > 0 \tag{3}$$

$$\begin{cases} \hat{y}(0) = [\widehat{\mu}]^{\alpha} \\ \hat{y}'(0) = [\widehat{\nu}]^{\alpha} \\ \hat{y}''(0) = \left[\widehat{\xi}\right]^{\alpha} \end{cases}, \tag{4}$$

via fuzzy Laplace transform method, where

$$[\widehat{\tau}]^{\alpha} = \left[\underline{\widehat{\tau}}_{\alpha}, \overline{\widehat{\tau}}_{\alpha}\right], [\widehat{\mu}]^{\alpha} = \left[\underline{\widehat{\mu}}_{\alpha}, \overline{\widehat{\mu}}_{\alpha}\right], \; [\widehat{\nu}]^{\alpha} = \left[\underline{\widehat{\nu}}_{\alpha}, \overline{\widehat{\nu}}_{\alpha}\right], \left[\widehat{\xi}\right]^{\alpha} = \left[\underline{\widehat{\xi}}_{\alpha}, \overline{\widehat{\xi}}_{\alpha}\right]$$

are positive symmetric triangular fuzzy numbers, $\hat{y}(t)$ is a positive fuzzy function and the Laplace transform of $\hat{y}(t)$ is $\hat{L}(\hat{y}(t)) = \hat{Y}(s)$.

i) From the fuzzy differential equation (1), we obtain the equation

$$s^3 \hat{Y}(s) \ominus s^2 \hat{y}(0) \ominus s\hat{y}'(0) \ominus \hat{y}''(0) = [\widehat{\tau}]^{\alpha} \left(s\hat{Y}(s) \ominus \hat{y}(0)\right)$$

via fuzzy Laplace transform. From this, using the fuzzy arithmetic, the equations

$$\underline{\hat{Y}}_{\alpha}(s)\left(s^3 - \underline{\widehat{\tau}}_{\alpha} s\right) = \underline{\hat{y}}_{\alpha}(0)\left(s^2 - \underline{\widehat{\tau}}_{\alpha}\right) + s\underline{\hat{y}}'_{\alpha}(0) + \underline{\hat{y}}''_{\alpha}(0),$$

$$\overline{\hat{Y}}_{\alpha}(s)\left(s^3 - \overline{\widehat{\tau}}_{\alpha} s\right) = \overline{\hat{y}}_{\alpha}(0)\left(s^2 - \overline{\widehat{\tau}}_{\alpha}\right) + s\overline{\hat{y}}'_{\alpha}(0) + \overline{\hat{y}}''_{\alpha}(0)$$

are obtained. Using initial conditions Eq. (2), yields $\underline{\hat{Y}}_{\alpha}(s)$, $\overline{\hat{Y}}_{\alpha}(s)$ as

$$\underline{\hat{Y}}_{\alpha}(s) = \frac{\underline{\widehat{\mu}}_{\alpha}}{s} + \frac{\underline{\widehat{\nu}}_{\alpha}}{s^2 - \underline{\widehat{\tau}}_{\alpha}} + \frac{\underline{\widehat{\xi}}_{\alpha}}{s\left(s^2 - \underline{\widehat{\tau}}_{\alpha}\right)},$$

$$\overline{\hat{Y}}_{\alpha}(s) = \frac{\overline{\widehat{\mu}}_{\alpha}}{s} + \frac{\overline{\widehat{\nu}}_{\alpha}}{s^2 - \overline{\widehat{\tau}}_{\alpha}} + \frac{\overline{\widehat{\xi}}_{\alpha}}{s\left(s^2 - \overline{\widehat{\tau}}_{\alpha}\right)}.$$

Taking inverse Laplace transform, the solution of the problem (1)-(2) are obtained as

$$\underline{\hat{y}}_{\alpha}(t) = \underline{\widehat{\mu}}_{\alpha} + \frac{\underline{\widehat{\nu}}_{\alpha}}{\sqrt{\underline{\widehat{\tau}}_{\alpha}}} sinh\left(\sqrt{\underline{\widehat{\tau}}_{\alpha}} t\right) + \frac{\underline{\widehat{\xi}}_{\alpha}}{\underline{\widehat{\tau}}_{\alpha}}\left(cosh\left(\sqrt{\underline{\widehat{\tau}}_{\alpha}} t\right) - 1\right),$$

$$\overline{\hat{y}}_{\alpha}(t) = \overline{\widehat{\mu}}_{\alpha} + \frac{\overline{\widehat{\nu}}_{\alpha}}{\sqrt{\overline{\widehat{\tau}}_{\alpha}}} sinh\left(\sqrt{\overline{\widehat{\tau}}_{\alpha}} t\right) + \frac{\overline{\widehat{\xi}}_{\alpha}}{\overline{\widehat{\tau}}_{\alpha}}\left(cosh\left(\sqrt{\overline{\widehat{\tau}}_{\alpha}} t\right) - 1\right),$$

$$[\hat{y}(t)]^{\alpha} = \left[\underline{\hat{y}}_{\alpha}(t), \overline{\hat{y}}_{\alpha}(t)\right].$$

ii) If we apply the fuzzy Laplace transform method to equation (3), we have

$$s^3\hat{Y}(s) \ominus s^2\hat{y}(0) \ominus s\hat{y}'(0) \ominus \hat{y}''(0) = -[\widehat{\tau}]^\alpha \left(s\hat{Y}(s) \ominus \hat{y}(0)\right).$$

From this, the equations

$$s^3\underline{\hat{Y}}_\alpha(s) + \overline{\widehat{\tau}}_\alpha s\overline{\hat{Y}}_\alpha(s) = s^2\underline{\hat{y}}_\alpha(0) + s\underline{\hat{y}}'_\alpha(0) + \underline{\hat{y}}''_\alpha(0) + \overline{\widehat{\tau}}_\alpha\overline{\hat{y}}_\alpha(0),$$

$$s^3\overline{\hat{Y}}_\alpha(s) + \underline{\widehat{\tau}}_\alpha s\underline{\hat{Y}}_\alpha(s) = s^2\overline{\hat{y}}_\alpha(0) + s\overline{\hat{y}}'_\alpha(0) + \overline{\hat{y}}''_\alpha(0) + \underline{\widehat{\tau}}_\alpha\underline{\hat{y}}_\alpha(0)$$

are obtained. Using the conditions (4), we obtain the following equations

$$s^2\underline{\hat{Y}}_\alpha(s) + \overline{\widehat{\tau}}_\alpha\overline{\hat{Y}}_\alpha(s) = s\underline{\widehat{\mu}}_\alpha + \underline{\widehat{\nu}}_\alpha + \frac{\left(\underline{\widehat{\xi}}_\alpha + \overline{\widehat{\tau}}_\alpha\overline{\widehat{\mu}}_\alpha\right)}{s}, \tag{5}$$

$$s^2\overline{\hat{Y}}_\alpha(s) + \underline{\widehat{\tau}}_\alpha\underline{\hat{Y}}_\alpha(s) = s\overline{\widehat{\mu}}_\alpha + \overline{\widehat{\nu}}_\alpha + \frac{\left(\overline{\widehat{\xi}}_\alpha + \underline{\widehat{\tau}}_\alpha\underline{\widehat{\mu}}_\alpha\right)}{s}. \tag{6}$$

From this, making the necessarry operations, $\underline{\hat{Y}}_\alpha(s)$ is obtained as

$$\underline{\hat{Y}}_\alpha(s) = \frac{s^3\underline{\widehat{\mu}}_\alpha}{s^4 - \underline{\widehat{\tau}}_\alpha\overline{\widehat{\tau}}_\alpha} + \frac{s^2\underline{\widehat{\nu}}_\alpha}{s^4 - \underline{\widehat{\tau}}_\alpha\overline{\widehat{\tau}}_\alpha} + \frac{s\underline{\widehat{\xi}}_\alpha}{s^4 - \underline{\widehat{\tau}}_\alpha\overline{\widehat{\tau}}_\alpha} - \frac{\overline{\widehat{\tau}}_\alpha\overline{\widehat{\nu}}_\alpha}{s^4 - \underline{\widehat{\tau}}_\alpha\overline{\widehat{\tau}}_\alpha} - \frac{\overline{\widehat{\tau}}_\alpha\left(\overline{\widehat{\xi}}_\alpha - \underline{\widehat{\tau}}_\alpha\underline{\widehat{\mu}}_\alpha\right)}{s\left(s^4 - \underline{\widehat{\tau}}_\alpha\overline{\widehat{\tau}}_\alpha\right)}.$$

By the inverse Laplace transform, we get the lower solution of the problem

$$\underline{\hat{y}}_\alpha(t) = \left(\underline{\widehat{\mu}}_\alpha - \frac{\overline{\widehat{\xi}}_\alpha}{2\underline{\widehat{\tau}}_\alpha}\right)\left(cosh\left(\sqrt[4]{\underline{\widehat{\tau}}_\alpha\overline{\widehat{\tau}}_\alpha}t\right) + cos\left(\sqrt[4]{\underline{\widehat{\tau}}_\alpha\overline{\widehat{\tau}}_\alpha}t\right)\right) + \frac{\underline{\widehat{\nu}}_\alpha}{2\sqrt[4]{\underline{\widehat{\tau}}_\alpha\overline{\widehat{\tau}}_\alpha}}\left(sinh\left(\sqrt[4]{\underline{\widehat{\tau}}_\alpha\overline{\widehat{\tau}}_\alpha}t\right) + sin\left(\sqrt[4]{\underline{\widehat{\tau}}_\alpha\overline{\widehat{\tau}}_\alpha}t\right)\right)$$

$$+\frac{\underline{\widehat{\xi}}_\alpha}{2\sqrt{\underline{\widehat{\tau}}_\alpha\overline{\widehat{\tau}}_\alpha}}\left(cosh\left(\sqrt[4]{\underline{\widehat{\tau}}_\alpha\overline{\widehat{\tau}}_\alpha}t\right) - cos\left(\sqrt[4]{\underline{\widehat{\tau}}_\alpha\overline{\widehat{\tau}}_\alpha}t\right)\right) - \frac{\sqrt[4]{\overline{\widehat{\tau}}_\alpha\overline{\widehat{\nu}}_\alpha}}{2\sqrt[4]{\left(\underline{\widehat{\tau}}_\alpha\right)^3}}\left(sinh\left(\sqrt[4]{\underline{\widehat{\tau}}_\alpha\overline{\widehat{\tau}}_\alpha}t\right) - sin\left(\sqrt[4]{\underline{\widehat{\tau}}_\alpha\overline{\widehat{\tau}}_\alpha}t\right)\right) + \frac{\overline{\widehat{\xi}}_\alpha}{\underline{\widehat{\tau}}_\alpha} - \underline{\widehat{\mu}}_\alpha.$$

Similarly, we obtain the upper solution $\overline{\hat{y}}_\alpha(t)$ of the problem as

$$\overline{\hat{y}}_\alpha(t) = \left(\overline{\widehat{\mu}}_\alpha - \frac{\underline{\widehat{\xi}}_\alpha}{2\overline{\widehat{\tau}}_\alpha}\right)\left(cosh\left(\sqrt[4]{\underline{\widehat{\tau}}_\alpha\overline{\widehat{\tau}}_\alpha}t\right) + cos\left(\sqrt[4]{\underline{\widehat{\tau}}_\alpha\overline{\widehat{\tau}}_\alpha}t\right)\right) + \frac{\overline{\widehat{\nu}}_\alpha}{2\sqrt[4]{\underline{\widehat{\tau}}_\alpha\overline{\widehat{\tau}}_\alpha}}\left(sinh\left(\sqrt[4]{\underline{\widehat{\tau}}_\alpha\overline{\widehat{\tau}}_\alpha}t\right) + sin\left(\sqrt[4]{\underline{\widehat{\tau}}_\alpha\overline{\widehat{\tau}}_\alpha}t\right)\right)$$

$$+\frac{\overline{\widehat{\xi}}_\alpha}{2\sqrt{\underline{\widehat{\tau}}_\alpha\overline{\widehat{\tau}}_\alpha}}\left(cosh\left(\sqrt[4]{\underline{\widehat{\tau}}_\alpha\overline{\widehat{\tau}}_\alpha}t\right) - cos\left(\sqrt[4]{\underline{\widehat{\tau}}_\alpha\overline{\widehat{\tau}}_\alpha}t\right)\right) - \frac{\sqrt[4]{\underline{\widehat{\tau}}_\alpha\underline{\widehat{\nu}}_\alpha}}{2\sqrt[4]{\left(\widehat{\tau}_\alpha\right)^3}}\left(sinh\left(\sqrt[4]{\underline{\widehat{\tau}}_\alpha\overline{\widehat{\tau}}_\alpha}t\right) - sin\left(\sqrt[4]{\underline{\widehat{\tau}}_\alpha\overline{\widehat{\tau}}_\alpha}t\right)\right) + \frac{\underline{\widehat{\xi}}_\alpha}{\overline{\widehat{\tau}}_\alpha} - \overline{\widehat{\mu}}_\alpha.$$

4. Numerical examples

Example 4.1. *We consider the problem with fuzzy initial conditions*

$$\hat{y}^{'''} = \left[\hat{1}\right]^{\alpha} \hat{y}^{'}, \tag{7}$$

$$\hat{y}(0) = \left[\hat{1}\right]^{\alpha},\ \hat{y}^{'}(0) = \left[\hat{2}\right]^{\alpha},\ \hat{y}^{''}(0) = \left[\hat{3}\right]^{\alpha}, \tag{8}$$

where

$$\left[\hat{1}\right]^{\alpha} = [\alpha, 2-\alpha], \left[\hat{2}\right]^{\alpha} = [1+\alpha, 3-\alpha], \left[\hat{3}\right]^{\alpha} = [2+\alpha, 4-\alpha].$$

Solving (7)-(8), *we get*

$$\underline{\hat{y}}_{\alpha}(t) = \alpha + \left(1 + \frac{1}{\alpha}\right) sinh\left(\sqrt{\alpha} t\right) + \left(1 + \frac{2}{\alpha}\right)\left(cosh\left(\sqrt{\alpha} t\right) - 1\right), \tag{9}$$

$$\overline{\hat{y}}_{\alpha}(t) = 2 - \alpha + \left(1 + \frac{1}{2-\alpha}\right) sinh\left(\sqrt{(2-\alpha)} t\right) + \left(1 + \frac{2}{2-\alpha}\right)\left(cosh\left(\sqrt{(2-\alpha)} t\right) - 1\right) \tag{10}$$

$$\left[\hat{y}(t)\right]^{\alpha} = \left[\underline{\hat{y}}_{\alpha}(t), \overline{\hat{y}}_{\alpha}(t)\right]. \tag{11}$$

Example 4.2. *We consider the following fuzzy initial value problem:*

$$\hat{y}^{'''} = -\left[\hat{1}\right]^{\alpha} \hat{y}^{'}, \tag{12}$$

$$\hat{y}(0) = \left[\hat{1}\right]^{\alpha},\ \hat{y}^{'}(0) = \left[\hat{2}\right]^{\alpha},\ \hat{y}^{''}(0) = \left[\hat{3}\right]^{\alpha}. \tag{13}$$

Solving the problem (12)-(13), *the solution is obtained as*

$$\begin{aligned}
\underline{\hat{y}}_{\alpha}(t) &= \left(\alpha + \frac{1}{2} - \frac{2}{\alpha}\right)\left(cosh\left(\sqrt[4]{\alpha(2-\alpha)} t\right) + cos\left(\sqrt[4]{\alpha(2-\alpha)} t\right)\right) \\
&+ \left(\frac{1+\alpha}{2\sqrt[4]{\alpha(2-\alpha)}}\right)\left(sinh\left(\sqrt[4]{\alpha(2-\alpha)} t\right) + sin\left(\sqrt[4]{\alpha(2-\alpha)} t\right)\right) \\
&+ \left(\frac{2+\alpha}{2\sqrt{\alpha(2-\alpha)}}\right)\left(cosh\left(\sqrt[4]{\alpha(2-\alpha)} t\right) - cos\left(\sqrt[4]{\alpha(2-\alpha)} t\right)\right) \\
&- \left(\frac{\sqrt[4]{(2-\alpha)}(3-\alpha)}{2\sqrt[4]{\alpha^{3}}}\right)\left(sinh\left(\sqrt[4]{\alpha(2-\alpha)} t\right) - sin\left(\sqrt[4]{\alpha(2-\alpha)} t\right)\right) + \frac{4}{\alpha} - 1 - \alpha,
\end{aligned} \tag{14}$$

$$\begin{aligned}
\overline{\hat{y}}_{\alpha}(t) &= \left(2 - \alpha - \frac{1}{2}\left(\frac{2+\alpha}{2-\alpha}\right)\right)\left(cosh\left(\sqrt[4]{\alpha(2-\alpha)} t\right) + cos\left(\sqrt[4]{\alpha(2-\alpha)} t\right)\right) \\
&+ \left(\frac{3-\alpha}{2\sqrt[4]{\alpha(2-\alpha)}}\right)\left(sinh\left(\sqrt[4]{\alpha(2-\alpha)} t\right) + sin\left(\sqrt[4]{\alpha(2-\alpha)} t\right)\right)
\end{aligned}$$

$$+\left(\frac{4-\alpha}{2\sqrt{\alpha(2-\alpha)}}\right)\left(cosh\left(\sqrt[4]{\alpha(2-\alpha)}t\right)-cos\left(\sqrt[4]{\alpha(2-\alpha)}t\right)\right)$$

$$-\left(\frac{\sqrt[4]{\alpha}(1+\alpha)}{2\sqrt[4]{(2-\alpha)^{3}}}\right)\left(sinh\left(\sqrt[4]{\alpha(2-\alpha)}t\right)-sin\left(\sqrt[4]{\alpha(2-\alpha)}t\right)\right)+\frac{2+\alpha}{2-\alpha}+\alpha-2, \tag{15}$$

$$[\hat{y}(t)]^{\alpha}=\left[\underline{\hat{y}}_{\alpha}(t),\overline{\hat{y}}_{\alpha}(t)\right]. \tag{16}$$

The found solutions Eq. (9)-(11) *and Eq.* (14)-(16) *must be valid fuzzy functions. For this, the graphs of the solutions are analyzed:*

According to Definition 2.3 , and Fig. 1-6, Eq. (9)-(11) *and Eq.* (14)-(16) *are valid fuzzy functions. However, fuzzy solution* (14)-(16) *approaches faster to the crisp solution.*

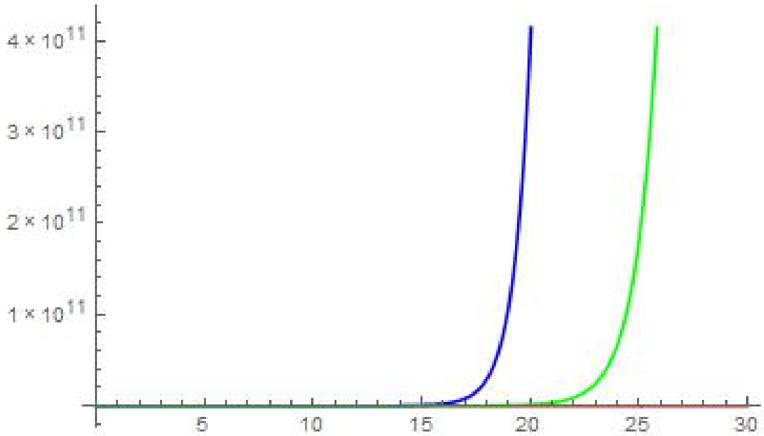

Figure 1. Graphic of the solution (9)-(11) for $\alpha = 0.3$

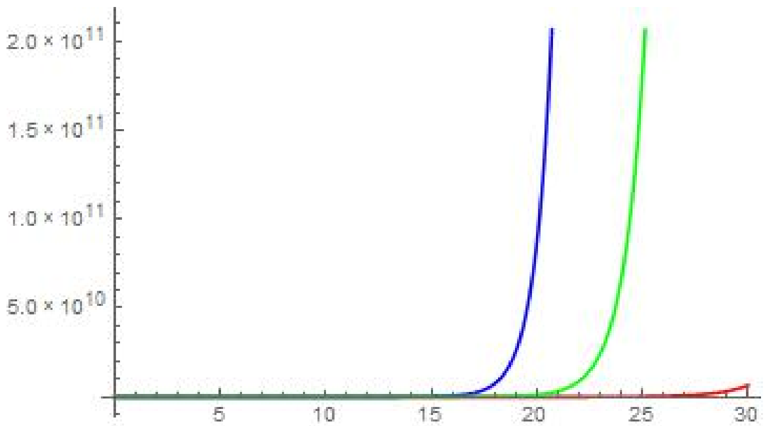

Figure 2. Graphic of the solution (9)-(11) for $\alpha = 0.5$

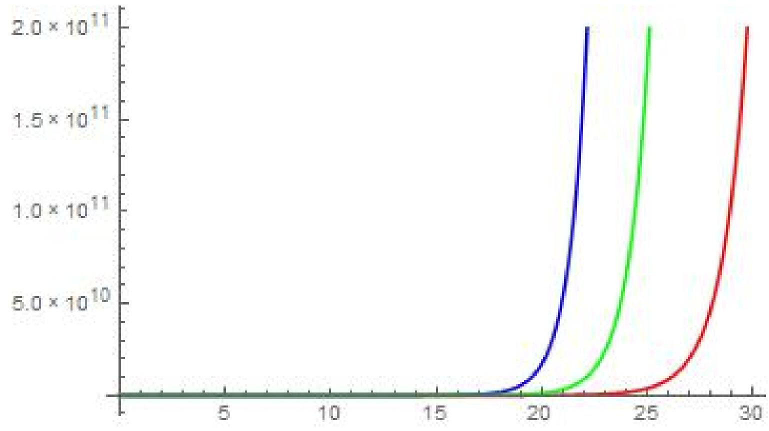

Figure 3. Graphic of the solution (9)-(11) for $\alpha = 0.7$

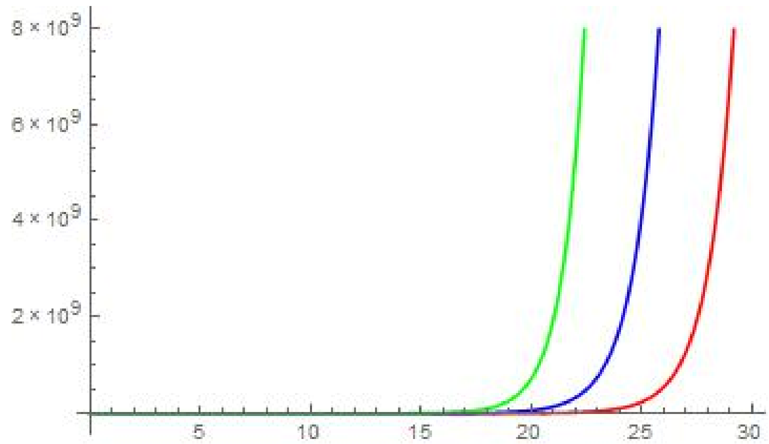

Figure 4. Graphic of the solution (14)-(16) for $\alpha = 0.3$

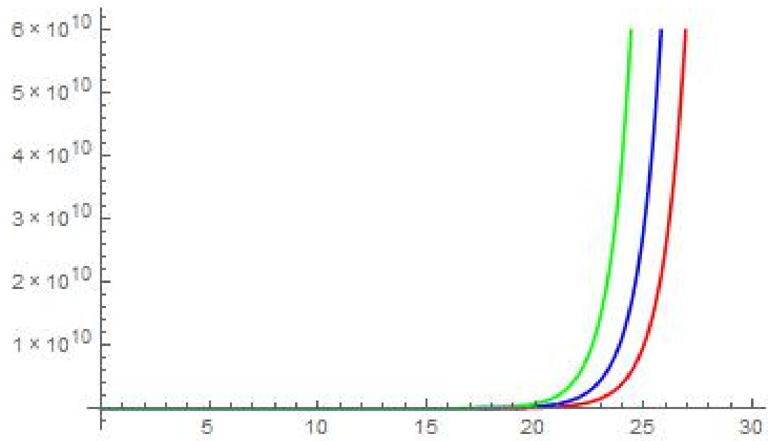

Figure 5. Graphic of the solution (14)-(16) for $\alpha = 0.5$

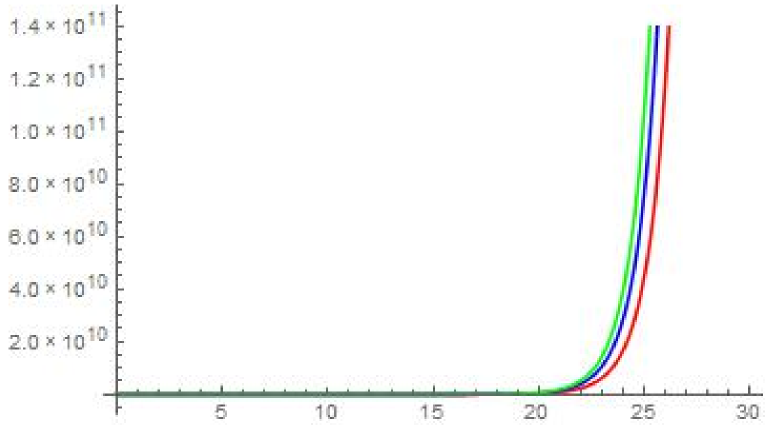

Figure 6. Graphic of the solution (14)-(16) for $\alpha = 0.7$

$$red \to \underline{\hat{y}}_{\alpha}(x)\,, blue \to \overline{\hat{y}}_{\alpha}(x)\,, green \to \underline{\hat{y}}_{1}(x) = \overline{\hat{y}}_{1}(x)$$

5. Conclusions

In this paper, third-order fuzzy problems with symmetric triangular fuzzy numbers are investigated. Solutions are found using the fuzzy Laplace transform method. Numerical examples are solved. Graphics of solutions are drawn.

In this study, it is seen that the solutions are valid fuzzy functions, but one of the fuzzy solutions approaches faster to the crisp solution.

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