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Exploration of k-hypergeometric polynomials and their mathematical implications

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Abstract. In current study, we focus on a mathematical concept called the k-hypergeometric polynomials. These polynomials are constructed using a mathematical tool called the Pochhammer k-symbol, as introduced by [R. Diaz, E. Pariguan, On hypergeometric functions and pochhammer k-symbol, Divulg. Mat. 15(2007), 179-192]. We develop several theorems related to these k-hypergeometric polynomials. Using these theorems, we derive two important functions: a multilinear generating function and a multilateral generating function for k-hypergeometric polynomials. These functions play a crucial role in our analysis. Furthermore, extend our research to explore the concept of the k-fractional secondary driver. This extension is based on the properties of k-hypergeometric polynomials and another mathematical entity known as the beta k-function. To make these connections, we utilize the Riemann-Liouville k-fractional process, as described by [G. Rahman, S. Nisar Mubeen, K. Sooppy, On generalized k-fractional derivative operator, AIMS Math. 5(2020), 1936-1945]. This has allowed us to establish some novel results, which are analogous to well-known mathematical transformations like the Mellin transformation. Additionally, we explore the relationships between our findings and other mathematical functions, such as hypergeometric and Appell' k-functions. In the last section of our paper, we delve into the relationship between k-hypergeometric polynomials and two specific mathematical functions: We also provide an integral representation of k-hypergeometric polynomials. Overall, our research paper contributes to the understanding of k-hypergeometric polynomials and their connections to various mathematical functions and transformations.

1. Introduction

In the realm of applied sciences, special functions serve as pivotal mathematical tools, often defined through improper integrals or infinite series. Over the last five decades, a plethora of research has emerged that expands upon these well-established functions, including but not limited to the gamma and beta functions as well as the Gauss hypergeometric function. These extensions not only enrich theoretical discourse but also enhance practical applications across various scientific domains. Recent studies have highlighted significant advancements in the k-generalization of these special polynomials. Works by authors such [7, 8]

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have provided rigorous frameworks for understanding these generalizations, while researchers like [3] and [22] have contributed further insights into their applications. This growing body of literature underscores a collective endeavor to refine our comprehension of special functions and their utility. Importantly, this ongoing exploration is motivated by both historical study and contemporary needs within applied sciences. The extension and generalization of familiar special functions allow for more versatile solutions to complex problems encountered in fields such as physics, engineering, and quantitative finance. As we continue to delve into this rich area of mathematics drawing inspiration from works such as those cited there lies a promising future for innovative approaches that could revolutionize our understanding and application of these essential mathematical constructs. In conclusion, the advancement in special functions through recent research signifies not merely an academic pursuit but a vital step toward enhancing practical methodologies within applied sciences. The collaborations among various scholars ultimately pave the way for new discoveries that resonate beyond pure mathematics into real-world implications.

The hypergeometric polynomials $S_n^{(\alpha,\beta)}(x)$ are defined by [5]

$$S_n^{(\alpha,\beta)}(x) = \binom{\alpha+n-1}{n} {}_2F_1\left[-n,\beta;\alpha;x\right], \ (x,\alpha,\beta \in \mathbb{C}).$$

Where ${}_2F_1$ indicates Gauss's hypergeometric series whose natural generalization of an arbitrary number of p numerator and q denominator parameters ($p,q \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$) is called and showed by the generalized hypergeometric series ${}_pF_q$ defined by

$$\begin{aligned} {}_pF_q\left[\begin{array}{cc} \alpha_1,\cdots,\alpha_p; & \\ \beta_1,\cdots,\beta_q; & x \end{array}\right] &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots \left(\alpha_p\right)_n}{(\beta_1)_n \cdots \left(\beta_q\right)_n} \frac{x^n}{n!} \\ &= {}_pF_q\left[\alpha_1,\cdots,\alpha_p;\ \beta_1,\cdots,\beta_q;x\right]. \end{aligned}$$

Here $(\lambda)_v$ denotes the Pochhammer symbol described (in terms of gamma function) by

$$\begin{aligned} (\lambda)_v &= \frac{\Gamma(\lambda+v)}{\Gamma(\lambda)} \ \left(\lambda \in \mathbb{C}\diagdown\mathbb{Z}_0^-\right) \\ &= \begin{cases} 1, & \text{if } v=0\,;\ \lambda \in \mathbb{C}\diagdown\{0\} \\ \lambda(\lambda+1)\cdots(\lambda+n-1), & \text{if } v=n\in\mathbb{N}\,;\ \lambda\in\mathbb{C} \end{cases} \end{aligned}$$

and $\mathbb{Z}_0^-$ indicates the set of nonpositive integers and $\Gamma(\lambda)$ familiar Gamma function. These polynomials have the following linear generating function [21]:

$$\begin{aligned} \sum_{n=0}^{\infty} S_n^{(\alpha,\beta)}(x)t^n &= (1-t)^{-\alpha}\,(1+\frac{xt}{1-t})^{-\beta} \\ &= (1-t)^{-\alpha}\,(1+\frac{xt}{1-t})^{-\beta} \\ &= (1-t)^{\beta-\alpha}\,[1-(1-x)\,t]^{-\beta},\ \left(|t|<\min,\left\{1,|1-x|^{-1}\right\}\right). \end{aligned}$$

In mathematical discourse, the pursuit of completeness necessitates a meticulous examination of foundational concepts. This essay delineates a structured approach to the preliminary section of our investigation by segmenting it into three distinct subsections. Such an organization is warranted by the extensive array of theorems and definitions that underpin our subsequent analyses. The clarity and precision afforded by this method enhance comprehension and facilitate a robust framework for future explorations. The first subsection introduces essential definitions related to k-gamma, k-beta, and their k-analogues within hypergeometric functions. These concepts are pivotal as they form the core building blocks for our forthcoming inquiries. By elucidating these terms, we lay the groundwork necessary for understanding their interrelations and applications in more complex scenarios. Moving forward, we delve into k-generalized

F_1 functions, which represent classical Appell functions in a generalized context. This subsection not only broadens the scope of classical analysis but also highlights how these generalizations can be employed to derive new results within mathematical frameworks that involve multiple variables. Finally, we conclude this preliminary section with an exploration of fractional calculus specifically focusing on Riemann–Liouville fractional derivatives and their k-generalizations. We will present several fundamental results that are essential for our investigations in later sections. The inclusion of these vital theorems emphasizes their significance in analytical procedures involving both traditional calculus and its fractional counterparts. By establishing these conventions early on, we aim to ensure clarity in our mathematical and analyses. In summary, organizing our preliminary section into three distinct subsections not only enhances the accessibility of critical definitions and properties but also prepares the reader for deeper engagement with the results that follow. This systematic approach underscores the importance of solid foundation in mathematical investigation. In the realm of special functions, the k-gamma and k-beta functions play a crucial role in various mathematical analyses. Defined as extensions of their traditional counterparts, these functions were explored by Diaz et al. [7] and Mubeen et al. [11], who provided significant relations and identities that enhance their applicability in diverse fields. The definitions of these functions are foundational to understanding their behavior and interrelations. The study progresses with the introduction of the k-hypergeometric function, a generalization that permits comprehensive integral representations and various pertinent formulas [10, 11]. This function serves as a bridge between discrete mathematics and continuous analysis, facilitating advancements across multiple disciplines. In 2015, researchers introduced the k-generalization of F_1 Appell′ functions, further expanding our analytical toolkit. Utilizing fundamental relations associated with the Pochhammer k-symbol allowed for the derivation of contiguous function relations alongside integral representations relevant to these Appell functions [8, 12]. These advancements underscore the versatility of special functions in addressing complex mathematical problems. The exploration into fractional derivatives has fostered numerous extensions; among them is Riemann-Liouville's k-fractional derivative operator. This operator was meticulously studied by authors such as [3, 14] and [16]. To facilitate our understanding moving forward into more intricate discussions related to fractional calculus applications involving special functions defined above, we will remind readers about both Riemann-Liouville fractional derivatives along with their respective generalizations while presenting key theoretical frameworks applicable throughout subsequent sections. Through this synthesis of knowledge encompassing definitions from k-function theory to contemporary applications within fractional calculus methodologies outlined herein-one can appreciate not merely individual components but rather an interconnected landscape contributing towards advancing mathematics holistically.

Definition 1.1. *[7] For $x \in \mathbb{C}$ and $k \in \mathbb{R}^+$, the integral representation of k-gamma function Γ_k is defined by*

$$\Gamma_k(x) = \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}} dt \tag{1}$$

where $Re(x) > 0$.

Definition 1.2. *[7] For $x, y \in \mathbb{C}$ and $k \in \mathbb{R}^+$, the k-beta function B_k is defined by*

$$B_k(x, y) = \frac{1}{k} \int_0^1 t^{\frac{x}{k}-1} (1-t)^{\frac{y}{k}-1} dt$$

where $Re(x) > 0$ and $Re(y) > 0$.

Proposition 1.3. *[7] Let $k \in \mathbb{R}^+$.The k-gamma function Γ_k and the k-beta function B_k fulfill the following properties,*

$$\Gamma_k(x+k) = x\Gamma_k(x),$$

$$\Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma_k\left(\frac{x}{k}\right),$$

$$B_k(x,y) = \frac{1}{k}\,B_k\left(\frac{x}{k},\frac{y}{k}\right),$$

$$B_k(x,y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}. \tag{2}$$

Definition 1.4. *Let $x \in \mathbb{C}$, $k \in \mathbb{R}^+$ and $n \in \mathbb{N}$. Then, the Pochhammer k-symbol is defined in [7] by*

$$(x)_{n,k} = x(x+k)(x+2k)\ldots(x+(n-1)k). \tag{3}$$

In particular, we denote $(x)_{0,k} := 1$. If $k = 1$ in the expression (3), the classical Pochammer symbol $(x)_n$ is obtained.

Definition 1.5. *[13] For $n \in \mathbb{N}$, $k \in \mathbb{R}^+$, we define (n,k) factorial as*

$$(n,k)! = k^n n!. \tag{4}$$

Using the above results, we see that

$$(0,k)! = 1.$$

By Pochhammer k-symbol and relation (4), we get

$$(1)_{n,k} = \left(\frac{1}{k}+n-1,k\right)!.$$

Proposition 1.6. *[7] If $\alpha \in \mathbb{C}$ and $m,n \in \mathbb{N}$ then for $k \in \mathbb{R}^+$, we have*

$$(\alpha)_{n,k} = \frac{\Gamma_k(\alpha+nk)}{\Gamma_k(\alpha)}, \tag{5}$$

$$(\alpha)_{m+n,k} = (\alpha)_{m,k}(\alpha+mk)_{n,k} \tag{6}$$

where $(\alpha)_{n,k}$ denote the Pochhammer k-symbol, respectively.

Lemma 1.7. *[7] For any $\alpha \in \mathbb{C}$ and $k \in \mathbb{R}^+$, the following identity holds*

$$\sum_{n=0}^{\infty}(\alpha)_{n,k}\frac{x^n}{n!} = (1-kx)^{-\frac{\alpha}{k}} \tag{7}$$

where $|x| < \frac{1}{k}$.

Definition 1.8. *Providing that $x \in \mathbb{C}$, $k \in \mathbb{R}^+$ and $0 < Re(\beta) < Re(\gamma)$, then the the k-hypergeometric function is defined in [11] as*

$$\begin{aligned} {}_2F_{1,k}\left[\begin{array}{cc} \alpha, & \beta \\ & \gamma \end{array};x\right] &= {}_2F_1\left[\begin{array}{c} (\alpha,k) \quad (\beta,k) \\ (\gamma,k) \end{array};x\right] \qquad (8) \\ &= \sum_{m=0}^{\infty}\frac{(\alpha)_{m,k}(\beta)_{m,k}}{(\gamma)_{m,k}}\frac{x^m}{m!}. \end{aligned}$$

Theorem 1.9. *Providing that $x \in \mathbb{C}$, $k \in \mathbb{R}^+$ and $0 < Re(\beta) < Re(\gamma)$, then the integral representation of the k-hypergeometric function is given in [6, 11] as*

$$ {}_2F_{1,k}\left[\begin{array}{cc} \alpha, & \beta \\ & \gamma \end{array} ;x\right] = \frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)} \int_0^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{\gamma-\beta}{k}-1}(1-kxt)^{-\frac{\alpha}{k}}\,dt. $$

Theorem 1.10. *[10] Assume that $k \in \mathbb{R}^+$ and $Re(\gamma-\beta) > 0$, then*

$$ {}_2F_1\left[\begin{array}{cc} (\alpha,1) & (\beta,k) \\ & (\gamma,k) \end{array} ;1\right] = \sum_{m=0}^{\infty} \frac{(\alpha)_m(\beta)_{m,k}}{(\gamma)_{m,k}}\frac{1}{m!} := \frac{\Gamma_k(\gamma)\Gamma_k(\gamma-\beta-k\alpha)}{\Gamma_k(\gamma-\beta)\Gamma_k(\gamma-k\alpha)}. \tag{9} $$

For the special case $\alpha = -n$ in (9)

$$ \sum_{m=0}^{\infty} \frac{(-n)_{m,1}(\beta)_{m,k}}{(\gamma)_{m,k}}\frac{1}{m!} = \frac{(\gamma-\beta)_{n,k}}{(\gamma)_{n,k}}. \tag{10} $$

Definition 1.11. *[12] Let $k \in \mathbb{R}^+$, $x, y \in \mathbb{C}$, $\alpha, \beta, \beta', \gamma \in \mathbb{C}$ and $n \in \mathbb{N}$. Then, the Appell' k-function $F_{1,k}$ with the parameters $\alpha, \beta, \beta', \gamma$ is given by*

$$ F_{1,k}\left(\alpha,\beta,\beta';\gamma;x,y\right) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n,k}(\beta)_{m,k}(\beta')_{n,k}}{(\gamma)_{m+n,k}}\frac{x^m}{m!}\frac{y^n}{n!} \tag{11} $$

where $\gamma \neq 0, -1, -2...$ and $|x| < \frac{1}{k}$, $|y| < \frac{1}{k}$.

By taking $k = 1$ in expression (11), the following Appell' function is obtained [21]:

$$ F_1\left(\alpha,\beta,\beta';\gamma;x,y\right) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}. $$

Theorem 1.12. *[12] Assume that $k \in \mathbb{R}^+$, $x, y \in \mathbb{R}$, $0 < Re(\alpha) < Re(\gamma)$, then the integral representation of the k-hypergeometric function is as follows;*

$$ F_{1,k}\left(\alpha,\beta,\beta';\gamma;x,y\right) = \frac{\Gamma_k(\gamma)}{k\Gamma_k(\alpha)\Gamma_k(\gamma-\alpha)} \int_0^1 t^{\frac{\alpha}{k}-1}(1-t)^{\frac{\gamma-\alpha}{k}-1}(1-kxt)^{-\frac{\beta}{k}}(1-kyt)^{-\frac{\beta'}{k}}\,dt. $$

Definition 1.13. *[14] The k-analogue of Riemann-Liouville fractional derivative of order μ is defined by*

$$ {}_kD_z^{\mu}\{f(z)\} = \frac{1}{k\Gamma_k(-\mu)} \int_0^z f(t)(z-t)^{-\frac{\mu}{k}-1}\,dt \tag{12} $$

where $Re(\mu) < 0$ and $k \in \mathbb{R}^+$. Particularly, for the case $m-1 < Re(\mu) < m$, where $m = 1, 2, ...$ (12) is written by

$$ {}_kD_z^{\mu}\{f(z)\} = \frac{d^m}{dz^m}\,{}_kD_z^{\mu-mk}\{f(z)\} = \frac{d^m}{dz^m}\left\{\frac{1}{k\Gamma_k(-\mu+mk)} \int_0^z f(t)(z-t)^{-\frac{\mu}{k}+m-1}\,dt\right\}. \tag{13} $$

Taking $k = 1$ in (13) yields the well-known Riemann-Liouville fractional derivative of order μ [21] .

The main aim of this paper is to introduce the generating function of k-hypergeometric polynomials and to obtain some of its properties. The sum expression and some new generating functions for k-hypergeometric polynomials will be given and their special cases will be studied. Also, integral representations and Mellin transform will be given. Finally, conclusions will be given.

2. The Sum Expression and Generating Function Relations for k-Hypergeometric Polynomials

In the realm of mathematical analysis, k-hypergeometric polynomials emerge as a significant extension of traditional hypergeometric functions. These polynomials not only encapsulate a broader class of series but also facilitate numerous applications across combinatorial theory and mathematical physics. This essay outlines the fundamental sum expressions and generating function relations pertinent to k-hypergeometric polynomials, drawing on methodologies established in several key studies [17, 19] and [20]. The sum expression for k-hypergeometric polynomials is characterized by its intricate dependence on parameters that define both the polynomial's structure and its convergence properties. Specifically, these sums are articulated through a finite series that incorporates multiple variables each representing distinct dimensions within the hypergeometric framework. The systematic arrangement of these parameters allows for an efficient evaluation of their coefficients, which relates closely to combinatorial identities. Moreover, generating functions play an instrumental role in understanding the behavior of k-hypergeometric polynomials. As elucidated in referenced articles, these functions serve as a powerful tool to encode sequences derived from polynomial evaluations into a singular analytic expression. By doing so, they enable mathematicians to derive recurrence relations and transformation formulas that are essential for deeper insights into polynomial behavior under various conditions. Furthermore, recent contributions have expanded upon traditional methods by introducing novel approaches to manipulating generating functions associated with k-hypergeometric forms. The interplay between sum expressions and their corresponding generating functions underscores their duality where one can often be expressed or transformed into the other through intricate transformations or limiting processes. In conclusion, the exploration of sum expressions and generating function relations surrounding k-hypergeometric polynomials reveals profound connections within mathematical theory. As highlighted by foundational works [17, 19] and [20], these relationships not only enhance our comprehension of hypergeometric structures but also pave pathways toward future research endeavors aimed at uncovering new properties inherent in this expansive classof polynomials. For further reading, you may want to refer to the following sources that delve deeper into the study of k-hypergeometric polynomials and their generating function relations:

Definition 2.1. *The generating function of k-hypergeometric polynomials is defined by the following relation:*

$$\sum_{n=0}^{\infty} S_{n,k}^{\left(\frac{\alpha}{k},\frac{\beta}{k}\right)}(x)\, t^n = (1-kt)^{-\frac{\alpha}{k}} \left(1+\frac{kxt}{1-kt}\right)^{-\frac{\beta}{k}}, \tag{14}$$

where $k \in \mathbb{R}^+$, $|t| < \min\left\{\frac{1}{k}, \left|\frac{1}{k(1-x)}\right|\right\}$.

Some values of the k-hypergeometric polynomials $S_{n,k}^{\left(\frac{\alpha}{k},\frac{\beta}{k}\right)}(x)$ can be given as follows:

$$\begin{aligned}
S_{0,k}^{\left(\frac{\alpha}{k},\frac{\beta}{k}\right)}(x) &= 1,\\
S_{1,k}^{\left(\frac{\alpha}{k},\frac{\beta}{k}\right)}(x) &= \alpha-\beta x,\\
S_{2,k}^{\left(\frac{\alpha}{k},\frac{\beta}{k}\right)}(x) &= \frac{\alpha(\alpha+k)}{2!}\left(1-\frac{2\beta}{\alpha}x+\frac{\beta(\beta+k)}{\alpha(\alpha+k)}x^2\right),\\
S_{3,k}^{\left(\frac{\alpha}{k},\frac{\beta}{k}\right)}(x) &= \frac{\alpha(\alpha+k)(\alpha+2k)}{3!}\left(1-\frac{3\beta}{\alpha}x+\frac{3\beta(\beta+k)}{\alpha(\alpha+k)}x^2-\frac{\beta(\beta+k)(\beta+2k)}{\alpha(\alpha+k)(\alpha+2k)}x^3\right),\\
S_{4,k}^{\left(\frac{\alpha}{k},\frac{\beta}{k}\right)}(x) &= \frac{\alpha(\alpha+k)(\alpha+2k)(\alpha+3k)}{4!}\left[1-\frac{4\beta}{\alpha}x+\frac{6\beta(\beta+k)}{\alpha(\alpha+k)}x^2\right.\\
&\quad \left.-\frac{4\beta(\beta+k)(\beta+2k)}{\alpha(\alpha+k)(\alpha+2k)}x^3+\frac{\beta(\beta+k)(\beta+2k)(\beta+3k)}{\alpha(\alpha+k)(\alpha+2k)(\alpha+3k)}x^4\right].
\end{aligned}$$

It is observed that the special case $k = 1$ of equation (14) immediately reduces to the hypergeometric polynomials in [5].

Theorem 2.2. *The explicit form of k-hypergeometric polynomials is as follows:*

$$S_{n,k}^{\left(\frac{\alpha}{k},\frac{\beta}{k}\right)}(x) = k^n \frac{(\alpha)_{n,k}}{(n,k)!}\, {}_2F_1\left[\begin{array}{cc} (-n,1)\,, & (\beta,k) \\ & (\alpha,k) \end{array} ;x\right] \tag{15}$$

where $x \in \mathbb{C}$, $k \in \mathbb{R}^+$ and $Re(\alpha) > Re(\beta) > 0$.

Proof. We first recall the following double series equality [21]:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n} A(m,n) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A(m,n+m). \tag{16}$$

By using (4), (7) and equality (15) also applying (16) recursively we observe that

$$\begin{aligned}
\sum_{n=0}^{\infty} S_{n,k}^{\left(\frac{\alpha}{k},\frac{\beta}{k}\right)}(x)\, t^n &= (1-kt)^{-\frac{\alpha}{k}}(1+\frac{kxt}{1-kt})^{-\frac{\beta}{k}} \\
&= (1-kt)^{-\frac{\alpha}{k}} \sum_{m=0}^{\infty} (\beta)_{m,k} \frac{(-xt)^m}{(1-kt)^m\, m!} \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(\beta)_{m,k}\,(\alpha)_{m+n,k}\,(-1)^m\, x^m t^{n+m}}{(\alpha)_{m,k}\, n!m!} \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{(\beta)_{m,k}\,(\alpha)_{n,k}\,(-1)^m\, x^m t^n}{(\alpha)_{m,k}\,(n-m)!m!} \\
&= \sum_{n=0}^{\infty}\left(k^n \frac{(\alpha)_{n,k}}{(n,k)!} \sum_{m=0}^{\infty} \frac{(-n)_{m,1}\,(\beta)_{m,k}}{(\alpha)_{m,k}} \frac{x^m}{m!}\right) t^n.
\end{aligned}$$

By comparing the coefficients of t^n on both sides of the equation, the desired result can be obtained. □

Corollary 2.3. *Theorem 2.2 can alternatively be proved using relations (12) and (13). However, due to the extensive steps and technical details required by this approach, the present proof method, which is more concise and efficient, is used in this study.*

Theorem 2.4. *The k-hypergeometric polynomials satisfy the following addition formula:*

$$S_{n,k}^{\left(\frac{\alpha_1+\alpha_2}{k},\frac{\beta_1+\beta_2}{k}\right)}(x) = \sum_{m=0}^{n} S_{n-m,k}^{\left(\frac{\alpha_1}{k},\frac{\beta_1}{k}\right)}(x) S_{m,k}^{\left(\frac{\alpha_2}{k},\frac{\beta_2}{k}\right)}(x) \tag{17}$$

where $x \in \mathbb{C}$, $k \in \mathbb{R}^+$.

Proof. Replacing α by $\alpha_1 + \alpha_2$ and β by $\beta_1 + \beta_2$ in (14), we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} S_{n,k}^{\left(\frac{\alpha_1+\alpha_2}{k},\frac{\beta_1+\beta_2}{k}\right)}(x)t^n &= (1-kt)^{-\frac{\alpha_1+\alpha_2}{k}}(1-\frac{kxt}{1-kt})^{-\frac{\beta_1+\beta_2}{k}} \\
&= (1-kt)^{-\frac{\alpha_1}{k}}(1-\frac{kxt}{1-kt})^{-\frac{\beta_1}{k}}(1-kt)^{-\frac{\alpha_2}{k}}(1-\frac{kxt}{1-kt})^{-\frac{\beta_2}{k}}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} S_{n,k}^{\left(\frac{\alpha_1}{k},\frac{\beta_1}{k}\right)}(x)\, t^n \sum_{m=0}^{\infty} S_{m,k}^{\left(\frac{\alpha_2}{k},\frac{\beta_2}{k}\right)}(x)\, t^m \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} S_{n,k}^{\left(\frac{\alpha_1}{k},\frac{\beta_1}{k}\right)}(x)\, S_{m,k}^{\left(\frac{\alpha_2}{k},\frac{\beta_2}{k}\right)}(x)\, t^{n+m} \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{n} S_{n-m,k}^{\left(\frac{\alpha_1}{k},\frac{\beta_1}{k}\right)}(x)\, S_{m,k}^{\left(\frac{\alpha_2}{k},\frac{\beta_2}{k}\right)}(x)\, t^{n}.
\end{aligned}
$$

By comparing the coefficients of t^n on both sides of the equation, the desired result can be obtained. □

Theorem 2.5. *The following equation represents the generating function for the k-hypergeometric polynomials:*

$$
\sum_{n=0}^{\infty} \binom{n+m}{m} S_{n+m,k}^{\left(\frac{\alpha}{k},\frac{\beta}{k}\right)}(x)t^n = (1-kt)^{\frac{\beta-\alpha}{k}-m}(1-k(1-x)t)^{-\frac{\beta}{k}} S_{m,k}^{\left(\frac{\alpha}{k},\frac{\beta}{k}\right)}\left(\frac{x}{1-k(1-x)t}\right), \tag{18}
$$

where $|t| < \min\left\{\frac{1}{k}, \left|\frac{1}{k-kx}\right|\right\}$ *and* $k \in \mathbb{R}^+$.

Proof. Replacing t by $u+t$ in (14), we obtain

$$
\sum_{n=0}^{\infty} S_{n,k}^{\left(\frac{\alpha}{k},\frac{\beta}{k}\right)}(x)(t+u)^n = (1-ku-kt)^{\frac{\beta-\alpha}{k}}(1-k(1-x)t-k(1-x)u)^{-\frac{\beta}{k}}
$$

$$
\sum_{n=0}^{\infty} S_{n,k}^{\left(\frac{\alpha}{k},\frac{\beta}{k}\right)}(x)(t+u)^n = (1-kt)^{\frac{\beta-\alpha}{k}}\left(1-k\frac{u}{1-kt}\right)^{\frac{\beta-\alpha}{k}}(1-k(1-x)t)^{-\frac{\beta}{k}}\left(1-\frac{k(1-x)}{1+\frac{kxt}{1-kt}}\frac{u}{1-kt}\right)^{-\frac{\beta}{k}}
$$

$$
\sum_{n=0}^{\infty} S_{n,k}^{\left(\frac{\alpha}{k},\frac{\beta}{k}\right)}(x)(t+u)^n = (1-kt)^{\frac{\beta-\alpha}{k}}(1-k(1-x)t)^{-\frac{\beta}{k}}\left(1-k\left(\frac{u}{1-kt}\right)\right)^{\frac{\beta-\alpha}{k}}\left(1-k\left(1-\frac{x}{1-k(1-x)t}\right)\frac{u}{1-kt}\right)^{-\frac{\beta}{k}}
$$

$$
\sum_{n=0}^{\infty}\sum_{m=0}^{n} \binom{n}{m} S_{n,k}^{\left(\frac{\alpha}{k},\frac{\beta}{k}\right)}(x)\, t^{n-m}u^m = (1-kt)^{\frac{\beta-\alpha}{k}}(1-k(1-x)t)^{-\frac{\beta}{k}}\sum_{m=0}^{\infty} S_{m,k}^{\left(\frac{\alpha}{k},\frac{\beta}{k}\right)}\left(\frac{x}{1-k(1-x)t}\right)\left(\frac{u}{1-kt}\right)^m
$$

$$
\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \binom{n+m}{m} S_{n+m,k}^{\left(\frac{\alpha}{k},\frac{\beta}{k}\right)}(x)\, t^n u^m = \sum_{m=0}^{\infty}(1-kt)^{\frac{\beta-\alpha}{k}-m}(1-k(1-x)t)^{-\frac{\beta}{k}} S_{m,k}^{\left(\frac{\alpha}{k},\frac{\beta}{k}\right)}\left(\frac{x}{1-k(1-x)t}\right)u^m.
$$

From the coeffcients of u^m on the both sides of the last equality, one can get the wanted result. □

Remark 2.6. *If we take $k=1$ in Theorem 2.5, we obtain the generating function for generalized hypergeometric polynomials $S_{n+m}^{(\alpha,\beta)}(x)$ in [5].*

3. Generating Functions

This section focuses on deriving several families of multilinear and multilateral generating functions for the k-hypergeometric polynomials. These polynomials, characterized by (15), (17) and (18), are explicitly expressed in (15). The derivation employs a similar approach to that used in $S_{n+m}^{(\alpha,\beta)}(x)$, [1, 4, 9] and [18]. To begin, we state the following theorem.

Theorem 3.1. *Corresponding to an identically non-vanishing function $\Omega_{\mu}(y_1, ..., y_r)$ of r complex variables $y_1, ..., y_r$ ($r \in \mathbb{N}$) and of complex order μ, ψ let*

$$\Lambda_{\mu,\psi}[y_1, ..., y_r; \zeta] := \sum_{i=0}^{\infty} a_i \Omega_{\mu+\psi i}(y_1, ..., y_r)\zeta^i, \quad (a_i \neq 0)$$

and

$$\Theta_{n,p;k}^{\mu,\psi}(x; y_1, ..., y_r; \zeta) := \sum_{j=0}^{[n/p]} a_j S_{n-pj,k}^{\left(\frac{\alpha}{k},\frac{\beta}{k}\right)}(x)\, \Omega_{\mu+\psi j}(y_1, ..., y_r)\zeta^j$$

where the notation $[n/p]$ means the greatest integer less than or equal n/p. Then, for $p \in \mathbb{N}$, $k \in \mathbb{R}^+$, we have

$$\begin{aligned} &\sum_{n=0}^{\infty} \Theta_{n,p;k}^{\mu,\psi}(x; y_1, ..., y_r; \frac{\eta}{t^p})t^n \\ =\ & (1-kt)^{-\frac{\alpha}{k}} \left(1 + \frac{kxt}{1-kt}\right)^{-\frac{\beta}{k}} \Lambda_{\mu,\psi}[y_1, ..., y_r; \eta] \end{aligned} \tag{19}$$

provided that each member of (19) exists.

Proof. For convenience, let S denote the first member of the assertion (19) of Theorem 3.1. Then,

$$S = \sum_{n=0}^{\infty} \sum_{j=0}^{[n/p]} a_j S_{n-pj,k}^{\left(\frac{\alpha}{k},\frac{\beta}{k}\right)}(x)\, \Omega_{\mu+\psi j}(y_1, ..., y_r)\eta^j t^{n-pj}.$$

Replacing n by $n + pj$, we may write that

$$\begin{aligned} S &= \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} a_j S_{n,k}^{\left(\frac{\alpha}{k},\frac{\beta}{k}\right)}(x)\, \Omega_{\mu+\psi j}(y_1, ..., y_r)\eta^j t^n \\ &= \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} a_j S_{n,k}^{\left(\frac{\alpha}{k},\frac{\beta}{k}\right)}(x)\, \Omega_{\mu+\psi j}(y_1, ..., y_r)\eta^j t^n \\ &= \sum_{n=0}^{\infty} S_{n;k}^{\left(\frac{\alpha}{k},\frac{\beta}{k}\right)}(x)\, t^n \sum_{j=0}^{\infty} a_j \Omega_{\mu+\psi j}(y_1, ..., y_r)\eta^j \\ &= (1-kt)^{-\frac{\alpha}{k}} \left(1 + \frac{kxt}{1-kt}\right)^{-\frac{\beta}{k}} \Lambda_{\mu,\psi}[y_1, ..., y_r; \eta] \end{aligned}$$

which completes the proof. □

Theorem 3.2. *Corresponding to an identically non-vanishing function $\Omega_j(y_1,...,y_r)$ of r complex variables $(y_1,...,y_r)$ $(r \in \mathbb{N})$ and of complex order $\mu, \psi, \alpha_1, \alpha_2, \beta_1, \beta_2$ let*

$$\Lambda^{n,p}_{\mu,\psi,\alpha_{1;k},\alpha_{2;k},\beta_{1;k},\beta_{2;k}}[x;y_1,...,y_r;z] := \sum_{j=0}^{[n/p]} a_j S_{n-pj,k}^{\left(\frac{\alpha_1+\beta_1}{k},\frac{\alpha_2+\beta_2}{k}\right)}(x)\Omega_{\mu+\psi j}(y_1,...,y_r)z^j$$

where $a_j \neq 0$, $n,p \in \mathbb{N}$. Then, for $p \in \mathbb{N}$, $k \in \mathbb{R}^+$, we have

$$\sum_{j=0}^{n}\sum_{l=0}^{[j/p]} a_l S_{n-j,k}^{\left(\frac{\alpha_1}{k},\frac{\beta_1}{k}\right)}(x)\, S_{j-pl,k}^{\left(\frac{\alpha_2}{k},\frac{\beta_2}{k}\right)}(x)\,\Omega_{\mu+\psi l}(y_1,...,y_r)z^l = \Lambda^{n,p}_{\mu,\psi,\alpha_{1;k},\alpha_{2;k},\beta_{1;k},\beta_{2;k}}[x;y_1,...,y_r;z] \tag{20}$$

provided that each member of (20) exists.

Proof. For convenince, let T denote the first member of the assertion (20) of Theorem 3.2. Then, upon subsituting for the polynomials $S_{n,k}^{\left(\frac{\alpha}{k},\frac{\beta}{k}\right)}(x)$ by (17) into the left-hand side of (20), we get

$$\begin{aligned}
T &= \sum_{j=0}^{n}\sum_{l=0}^{[j/p]} a_l S_{n-j,k}^{\left(\frac{\alpha_1}{k},\frac{\beta_1}{k}\right)}(x)\, S_{j-pl,k}^{\left(\frac{\alpha_2}{k},\frac{\beta_2}{k}\right)}(x)\,\Omega_{\mu+\psi l}(y_1,...,y_r)z^l \\
&= \sum_{l=0}^{[n/p]}\sum_{j=0}^{n-pl} a_l S_{n-j-pl,k}^{\left(\frac{\alpha_1}{k},\frac{\beta_1}{k}\right)}(x)\, S_{j,k}^{\left(\frac{\alpha_2}{k},\frac{\beta_2}{k}\right)}(x)\,\Omega_{\mu+\psi l}(y_1,...,y_r)z^l \\
&= \sum_{l=0}^{[n/p]} a_l \left(\sum_{j=0}^{n-pl} S_{n-j-pl,k}^{\left(\frac{\alpha_1}{k},\frac{\beta_1}{k}\right)}(x)\, S_{j,k}^{\left(\frac{\alpha_2}{k},\frac{\beta_2}{k}\right)}(x)\right)\Omega_{\mu+\psi l}(y_1,...,y_r)z^l \\
&= \sum_{l=0}^{[n/p]} a_l S_{n-pl,k}^{\left(\frac{\alpha_1+\beta_1}{k},\frac{\alpha_2+\beta_2}{k}\right)}(x)\,\Omega_{\mu+\psi l}(y_1,...,y_r)z^l \\
&= \Lambda^{n,p}_{\mu,\psi,\alpha_{1;k},\alpha_{2;k},\beta_{1;k},\beta_{2;k}}[x;y_1,...,y_r;z].
\end{aligned}$$

□

Theorem 3.3. *Corresponding to an identically non-vanishing function $\Omega_j(y_1,...,y_r)$ of r complex variables $(y_1,...,y_r)$ $(r \in \mathbb{N})$ and $\mu \in \mathbb{C}$, $k \in \mathbb{R}^+$ let*

$$\Lambda_{\mu,p,q;k}[x;y_1,...,y_r;z] := \sum_{i=0}^{\infty} a_i S_{m+iq,k}^{\left(\frac{\alpha}{k},\frac{\beta}{k}\right)}(x)\,\Omega_{\mu+pi}(y_1,...,y_r)z^i$$

where $a_i \neq 0$ and

$$\Theta_{\mu,p,q}(y_1,...,y_r;z) := \sum_{j=0}^{[i/q]} \binom{m+i}{i-qj} a_j \Omega_{\mu+pj}(y_1,...,y_r)z^j.$$

Then, for $p,q \in \mathbb{N}$; we have

$$\begin{aligned}
&\sum_{i=0}^{\infty} S_{m+i,k}^{\left(\frac{\alpha}{k},\frac{\beta}{k}\right)}(x)\,\Theta_{\mu,p,q}(y_1,...,y_r;z)t^i \\
&= (1-kt)^{\frac{\beta-\alpha}{k}-m}(1-k(1-x)t)^{-\frac{\beta}{k}}\Lambda_{\mu,p,q;k}\left[\frac{x}{1-k(1-x)t};y_1,...,y_r;z\left(\frac{t}{1-kt}\right)^q\right].
\end{aligned} \tag{21}$$

provided that each member of (21) exists.

Proof. For convenience, let H denote the first member of the assertion (21) of Theorem 3.3. Then,

$$\sum_{i=0}^{\infty} S_{m+i,k}^{\left(\frac{\alpha}{k},\frac{\beta}{k}\right)}(x) \sum_{j=0}^{[i/q]} \binom{m+i}{i-qj} a_j \Omega_{\mu+pj}(y_1,...,y_r) z^j t^i.$$

Replacing i by $i+qj$ and then using (18), we may write that

$$\begin{aligned}
H &= \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} a_j \binom{m+i+qj}{i} S_{m+i+qj,k}^{\left(\frac{\alpha}{k},\frac{\beta}{k}\right)}(x)\, \Omega_{\mu+pj}(y_1,...,y_r) z^j t^{i+qj} \\
&= \sum_{j=0}^{\infty} a_j \left(\sum_{i=0}^{\infty} \binom{m+i+qj}{i} S_{m+i+qj,k}^{\left(\frac{\alpha}{k},\frac{\beta}{k}\right)}(x)\, t^i \right) \Omega_{\mu+pj}(y_1,...,y_r)\, (zt^q)^j \\
&= \sum_{j=0}^{\infty} a_j \Omega_{\mu+pj}(y_1,...,y_r)\, (zt^q)^j \\
&\quad \times \left[(1-kt)^{\frac{\beta-\alpha}{k}-m-qj} (1-k(1-x)t)^{-\frac{\beta}{k}} S_{m+qj,k}^{\left(\frac{\alpha}{k},\frac{\beta}{k}\right)} \left(\frac{x}{1-k(1-x)t} \right) \right] \\
&= (1-kt)^{\frac{\beta-\alpha}{k}-m} (1-k(1-x)t)^{-\frac{\beta}{k}} \Lambda_{\mu,p,q;k} \left[\frac{x}{1-k(1-x)t}; y_1,...,y_r; z\left(\frac{t}{1-kt}\right)^q \right],
\end{aligned}$$

which completes the proof. □

4. Exploring the Implications of Multivariable Functions in Terms of Simpler Functions

In the realm of mathematical analysis, the study of multivariable functions is a critical area that fosters deeper insights into complex systems. The function denoted as $\Omega_{\mu+\psi j}(y_1,...,y_r)$ $j \in \mathbb{N}_0$, $r \in \mathbb{N}$, where j belongs to the set of non-negative integers $\mathbb{N}_0$ and r represents a natural number in $\mathbb{N}$, serves as an illustrative example. Expressing such multivariable functions in terms of simpler one-variable or multivariable functions not only enhances our understanding but also opens avenues for practical applications derived from established theoretical frameworks. The primary benefit of rephrasing $\Omega_{\mu+\psi j}(y_1,...,y_r)$ involves simplifying the intricate relationships among its variables. By breaking down this multivariable function into more manageable components such as linear or polynomial expressions we can leverage existing mathematical theorems more effectively. For instance, if we can represent $\Omega_{\mu+\psi j}(y_1,...,y_r)$ through a series expansion or via product forms involving simpler functions like exponential or logarithmic functions, we may apply well-known results such as Taylor's theorem or properties related to convergence and continuity. Furthermore, this simplification facilitates computational efficiency and analytical tractability.

We first set $r=1$ and

$$\Omega_{\mu+\psi j}(y_1) = L_{\alpha,\beta,m,\mu+\psi j}(y_1)$$

in Theorem 3.1, where the modified Laguerre polynomials, denoted by $L_{\alpha,\beta,m,n}(x)$ [17], generated by

$$(1-\beta t)^{-m} \exp\left(\frac{\alpha x t}{\beta t - 1}\right) = \sum_{n=0}^{\infty} L_{\alpha,\beta,m,n}(x) t^n, \quad \left(\left|\beta t\right| < 1\right). \tag{22}$$

Thus, we obtain the following result, which supply a class of two-sided generating functions for the univariate extension of modified Lagurre polynomials $L_{\alpha,\beta,m,\mu+\psi j}(y_1)$ and the k-hypergeometric polynomials $S_{n,k}^{\left(\frac{\alpha}{k},\frac{\beta}{k}\right)}(x)$.

Example 4.1. *If*

$$\Lambda_{\mu,\psi}[y_1;\zeta] := \sum_{j=0}^{\infty} a_j\, L_{\alpha,\beta,m,\mu+\psi j}(y_1)\zeta^j \quad (a_j \neq 0, \mu, \psi \in \mathbb{C})$$

then, we have

$$\sum_{n=0}^{\infty}\sum_{j=0}^{[n/p]} a_j S_{n-pj,k}^{\left(\frac{\alpha}{k},\frac{\beta}{k}\right)}(x)\, L_{\alpha,\beta,m,\mu+\psi j}(y_1)\frac{\zeta^j}{t^{pj}}t^n = (1-kt)^{-\frac{\alpha}{k}}\left(1+\frac{kxt}{1-kt}\right)^{-\frac{\beta}{k}}\Lambda_{\mu,\psi}[y_1;\zeta].$$

Using the generating relation (22) for the univariate polynomials $L_{\alpha,\beta,m,\mu+\psi j}(y_1)$ and getting $a_j = 1$, $\mu = 0$, $\psi = 1$, we find that

$$\sum_{n=0}^{\infty}\sum_{j=0}^{[n/p]} S_{n-pj,k}^{\left(\frac{\alpha}{k},\frac{\beta}{k}\right)}(x)\, L_{\alpha,\beta,m,j}(y_1)\zeta^j t^{n-pj} = (1-kt)^{-\frac{\alpha}{k}}\left(1+\frac{kxt}{1-kt}\right)^{-\frac{\beta}{k}}(1-\beta\zeta)^{-m}\exp\left(\frac{\alpha y_1\zeta}{\beta\zeta-1}\right).$$

If we set

$$r = 1,\ y_1 = x \ \text{ and } \Omega_{\mu+\psi j}(x) = S_{\mu+\psi j,k}^{\left(\frac{\alpha_3}{k},\frac{\beta_3}{k}\right)}(x)$$

in Theorem 3.2, we have the following bilinear generating functions the k-hypergeometric polynomials.

Example 4.2. *If*

$$\Lambda^{n,p}_{\mu,\psi,\alpha_{1;k},\alpha_{2;k},\beta_{1;k},\beta_{2;k}}[x;x;z] := \sum_{j=0}^{[n/p]} a_j S_{n-pj,k}^{\left(\frac{\alpha_1+\beta_1}{k},\frac{\alpha_2+\beta_2}{k}\right)}(x)\, S_{\mu+\psi j,k}^{\left(\frac{\alpha_3}{k},\frac{\beta_3}{k}\right)}(x)\, z^j \quad \left(a_j \neq 0,\ \ \mu,\psi \in \mathbb{C},\ k \in \mathbb{R}^+\right),$$

then, we have

$$\sum_{j=0}^{n}\sum_{l=0}^{[j/p]} a_l S_{n-j,k}^{\left(\frac{\alpha_1}{k},\frac{\beta_1}{k}\right)}(x)\, S_{j-pl,k}^{\left(\frac{\alpha_2}{k},\frac{\beta_2}{k}\right)}(x)\, S_{\mu+\psi l,k}^{\left(\frac{\alpha_3}{k},\frac{\beta_3}{k}\right)}(x)\, z^l = \Lambda^{n,p}_{\mu,\psi,\alpha_{1;k},\alpha_{2;k},\beta_{1;k},\beta_{2;k}}[x;x;z].$$

Using (17) and $a_l = 1$, $\mu = 0$, $\psi = 1$, $p = 1$, $z^l = 1$

$$\begin{aligned}
\sum_{j=0}^{n}\sum_{l=0}^{j} S_{n-j,k}^{\left(\frac{\alpha_1}{k},\frac{\beta_1}{k}\right)}(x)\, S_{j-l,k}^{\left(\frac{\alpha_2}{k},\frac{\beta_2}{k}\right)}(x)\, S_{l,k}^{\left(\frac{\alpha_3}{k},\frac{\beta_3}{k}\right)}(x) &= \sum_{j=0}^{n} S_{n-j,k}^{\left(\frac{\alpha_1}{k},\frac{\beta_1}{k}\right)}(x)\sum_{l=0}^{j} S_{j-l,k}^{\left(\frac{\alpha_2}{k},\frac{\beta_2}{k}\right)}(x)\, S_{l,k}^{\left(\frac{\alpha_3}{k},\frac{\beta_3}{k}\right)}(x) \\
&= \sum_{j=0}^{n} S_{n-j,k}^{\left(\frac{\alpha_1}{k},\frac{\beta_1}{k}\right)}(x)\, S_{j,k}^{\left(\frac{\alpha_2+\alpha_3}{k},\frac{\beta_2+\beta_3}{k}\right)}(x) \\
&= S_{n,k}^{\left(\frac{\alpha_1+\alpha_2+\alpha_3}{k},\frac{\beta_1+\beta_2+\beta_3}{k}\right)}(x).
\end{aligned}$$

If we set

$$r = 1,\ \Omega_{\mu+\psi j}(y_1) = P_{\mu+\psi j}^{(\alpha,\beta)}(y_1)$$

in Theorem 3.3, where the classical Jacobi polynomials $P_n^{(\alpha,\beta)}(y)$ is generating by [15],

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)\, t^n = \frac{2^{\alpha+\beta}}{\rho}(1-t+\rho)^{-\alpha}(1-t+\rho)^{-\beta}\left\{\rho = \left(1-2xt+t^2\right)^{\frac{1}{2}},\ |\,\rho\,| < 1\right\}$$

we obtain a family of the bilateral generating functions for the classical Jacobi polynomials and the k-hypergeometric polynomials as follow:

Example 4.3. *If*

$$\begin{aligned}\Lambda_{\mu,p,q;k}[x;y_1;z] \quad : \quad &= \sum_{i=0}^{\infty} a_i S_{m+iq,k}^{\left(\frac{\alpha}{k},\frac{\beta}{k}\right)}(x)\, P_{\mu+pi}^{(\alpha,\beta)}(y_1)\, z^i \\ &(a_i \neq 0,\ m \in \mathbb{N}_0,\ \ \mu,\psi \in \mathbb{C})\end{aligned}$$

and

$$\Theta_{\mu,p,q}(y_1;z) := \sum_{j=0}^{[i/q]} \binom{m+i}{i-qj} a_j P_{\mu+pj}^{(\alpha,\beta)}(y_1)\, z^j$$

where $p,q \in \mathbb{N}$, *then we have*

$$\sum_{i=0}^{\infty} S_{m+i,k}^{\left(\frac{\alpha}{k},\frac{\beta}{k}\right)}(x)\Theta_{\mu,p,q}(y_1;z)\, t^i \quad = \quad (1-kt)^{\frac{\beta-\alpha}{k}-m}(1-k(1-x)t)^{-\frac{\beta}{k}}\Lambda_{\mu,p,q;k}\left[\frac{x}{1-k(1-x)t};y_1;z\left(\frac{t}{1-kt}\right)^q\right].$$

Thus, for any appropriate choice of the coefficients a_j $(j \in \mathbb{N}_0)$, if the multivariable functions $\Omega_{\mu+\psi j}(y_1,...,y_r)$, $r \in \mathbb{N}$, are expressed as a convenient product of several simpler functions, the results of Theorem 3.1, Theorem 3.2 and Theorem 3.3 can be utilized to derive various families of multilinear and multilateral generating functions for the k-hypergeometric polynomials explicitly defined in (14).

5. Theorems and Proofs on k-Hypergeometric Polynomials and the Riemann-Liouville k-Fractional Derivative

In this section, we will show some theorems and proofs about k-hypergeometric polynomials. Moreover we recall the following definition of fractional derivatives from and give a new extension called Rimann-Liouville k-fractional derivative.

Theorem 5.1. *Another expression of k-hypergeometric polynomials is as follows:*

$$S_{n,k}^{\left(\frac{\alpha}{k},\frac{\beta}{k}\right)}(x) = k^n \frac{(\alpha)_{n,k}}{(n,k)!}(1-kx)^{-\frac{\beta}{k}}\ {}_2F_1\left[\begin{matrix}(\alpha+n,1), & (\beta,k) \\ & (\alpha,k)\end{matrix};-\frac{x}{1-kx}\right],$$

where ${}_2F_1$ *is the k-hypergeometric function given in (8).*

Proof. If we apply expressions (7) and (8) to the right side of the theorem, we get

$$\begin{aligned}& k^n \frac{(\alpha)_{n,k}}{(n,k)!}(1-kx)^{-\frac{\beta}{k}}\ {}_2F_1\left[\begin{matrix}(\alpha+n,1), & (\beta,k) \\ & (\alpha,k)\end{matrix};-\frac{x}{1-kx}\right] \\ = \quad & k^n \frac{(\alpha)_{n,k}}{(n,k)!}\sum_{m=0}^{\infty}\frac{(-1)^m(\alpha+n)_{m,1}(\beta)_{m,k}x^m}{(\alpha)_{m,k}\,m!}\sum_{h=0}^{\infty}\frac{(\beta+km)_{h,k}x^h}{h!} \\ = \quad & k^n \frac{(\alpha)_{n,k}}{(n,k)!}\sum_{m=0}^{\infty}\sum_{h=0}^{\infty}\frac{(-1)^m(\alpha+n)_{m,1}(\beta)_{m,k}(\beta+km)_{h,k}x^{m+h}}{(\alpha)_{m,k}\,h!m!}.\end{aligned} \tag{23}$$

If we apply the expression (6) to the expression (23)

$$\begin{aligned}& k^n \frac{(\alpha)_{n,k}}{(n,k)!}\ (1-kx)^{-\frac{\beta}{k}}\ {}_2F_1\left[\begin{matrix}(\alpha+n,1), & (\beta,k) \\ & (\alpha,k)\end{matrix};-\frac{x}{1-kx}\right] \\ = \quad & k^n \frac{(\alpha)_{n,k}}{(n,k)!}\sum_{m=0}^{\infty}\sum_{h=0}^{\infty}\frac{(-1)^m(\alpha+n)_{m,1}(\beta)_{m+h,k}x^{m+h}}{(\alpha)_{m,k}\,m!h!}.\end{aligned} \tag{24}$$

Replacing h by $h-m$ in (24), we obtain

$$\begin{aligned}
& k^n \frac{(\alpha)_{n,k}}{(n,k)!} \; (1-kx)^{-\frac{\beta}{k}} \; {}_2F_1\left[\begin{array}{cc} (\alpha+n,1), & (\beta,k) \\ & (\alpha,k) \end{array} ; -\frac{x}{1-kx} \right] \\
= \; & k^n \frac{(\alpha)_{n,k}}{(n,k)!} \sum_{h=0}^{\infty} \sum_{m=0}^{h} \frac{(-1)^m (\alpha+n)_{m,1} (\beta)_{h,k} x^h}{(\alpha)_{m,k} \, m!(h-m)!} \\
= \; & k^n \frac{(\alpha)_{n,k}}{(n,k)!} \sum_{h=0}^{\infty} \left(\sum_{m=0}^{\infty} \frac{(\alpha+n)_{m,1} (-h)_{m,1}}{(\alpha)_{m,k} \, m!} (1)^m \right) \frac{(\beta)_{h,k} x^h}{h!} \\
= \; & k^n \frac{(\alpha)_{n,k}}{(n,k)!} \sum_{h=0}^{\infty} {}_2F_1\left[\begin{array}{cc} (-h,1) & (\alpha+n,1) \\ & (\alpha,k) \end{array} ; 1 \right] \frac{(\beta)_{h,k} x^h}{h!}.
\end{aligned} \tag{25}$$

Considering (10) in the expression (25), we attain

$$\begin{aligned}
& k^n \frac{(\alpha)_{n,k}}{(n,k)!} \; (1-kx)^{-\frac{\beta}{k}} \; {}_2F_1\left[\begin{array}{cc} (\alpha+n,1), & (\beta,k) \\ & (\alpha,k) \end{array} ; -\frac{x}{1-kx} \right] \\
= \; & k^n \frac{(\alpha)_{n,k}}{(n,k)!} \sum_{h=0}^{\infty} \frac{(-n)_{h,1} (\beta)_{h,k}}{(\alpha)_{h,k}} \frac{x^h}{h!} \\
= \; & k^n \frac{(\alpha)_{n,k}}{(n,k)!} \, {}_2F_1\left[\begin{array}{cc} (-n,1), & (\beta,k) \\ & (\alpha,k) \end{array} ; x \right] \\
= \; & S_{n,k}^{\left(\frac{\alpha}{k},\frac{\beta}{k}\right)}(x).
\end{aligned}$$

□

Corollary 5.2. *Theorem 5.1 demonstrates that the explicit form of the k-hypergeometric polynomials can also be obtained by an alternative approach.*

Theorem 5.3. *The following Mellin transform formula holds true:*

$$\begin{aligned}
& M\left[e^{-\omega} \, {}_kD_z^{\mu} \left\{ z^{\frac{\eta}{k}} \sum_{n=0}^{\infty} S_{n,k}^{\left(\frac{\alpha}{k},\frac{\beta}{k}\right)}(x) \, z^n \right\} ; \phi \right] \\
= \; & \frac{\Gamma\left(\phi\right) \Gamma_k(\eta+k)}{\Gamma_k(\eta-\mu+k)} z^{\frac{\eta-\mu}{k}} F_{1,k}(\eta+k, \beta, \alpha+mk; \eta-\mu+k; z, -xz)
\end{aligned}$$

where $F_{1,k}$ is the Appell' k- function given in (11), $Re(-\mu)<0$, $Re(\eta)>0$, $Re(\alpha)>0$, $Re\left(\phi\right)>0$, $max\{|z|, |-xz|\} < \frac{1}{k}$, $k \in \mathbb{R}^+$.

Proof. Applying the Mellin transform on definition (13), we have

$$\begin{aligned}
& M\left[e^{-\omega} \, {}_kD_z^{\mu} \left\{ z^{\frac{\eta}{k}} \sum_{n=0}^{\infty} S_{n,k}^{\left(\frac{\alpha}{k},\frac{\beta}{k}\right)}(x) \, z^n \right\} ; \; \phi \right] \\
= \; & M\left[e^{-\omega} \, {}_kD_z^{\mu} \left\{ z^{\frac{\eta}{k}} \sum_{n=0}^{\infty} (\alpha)_{n,k} \frac{z^n}{n!} \sum_{m=0}^{\infty} (\beta)_{m,k} \frac{(-xz)^m}{(1-kz)^m \, m!} \right\} ; \; \phi \right] \\
= \; & M\left[e^{-\omega} \, {}_kD_z^{\mu} \left\{ z^{\frac{\eta}{k}} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\beta)_{m,k} (\alpha)_{m+n,k} (-x)^m z^{n+m}}{(\alpha)_{m,k} \, n!m!} \right\} ; \; \phi \right]
\end{aligned}$$

$$= M\left[e^{-\omega} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(\alpha)_{m+n,k}(\beta)_{m,k}(-x)^m}{(\alpha)_{m,k}\, n!m!}\, {}_kD_z^{\mu}\left\{z^{\frac{\eta+mk+nk}{k}}\right\};\ \phi\right]$$

$$= \int_0^{\infty} e^{-\omega}\omega^{\phi-1}\left(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(\alpha)_{m+n,k}(\beta)_{m,k}(-x)^m}{(\alpha)_{m,k}\, n!m!}\, {}_kD_z^{\mu}\left\{z^{\frac{\eta+mk+nk}{k}}\right\}\right)d\omega$$

$$= \int_0^{\infty} e^{-\omega}\omega^{\phi-1}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(\alpha)_{m+n,k}(\beta)_{m,k}(-x)^m}{(\alpha)_{m,k}\, n!m!}\frac{1}{k\Gamma_k(-\mu)}\int_0^{z} t^{\frac{\eta+nk+mk+k}{k}-1}(z-t)^{-\frac{\mu}{k}-1}\,dtd\omega \tag{26}$$

interchanging the order of integrations (26) in above equation, we get

$$M\left[e^{-\omega}\, {}_kD_z^{\mu}\left\{z^{\frac{\eta}{k}}\sum_{n=0}^{\infty} S_{n,k}^{\left(\frac{\alpha}{k},\frac{\beta}{k}\right)}(x)\, z^n\right\};\ \phi\right]$$

$$= \int_0^{\infty} e^{-\omega}\omega^{\phi-1}\left(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(\alpha)_{m+n,k}(\beta)_{m,k}(-x)^m}{(\alpha)_{m,k}\, n!m!}\frac{1}{k\Gamma_k(-\mu)}\int_0^{1}(uz)^{\frac{\eta+nk+mk+k}{k}-1}(z-uz)^{-\frac{\mu}{k}-1}zdu\right)d\omega$$

$$= \int_0^{\infty} e^{-\omega}\omega^{\phi-1}\left(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(\alpha)_{m+n,k}(\beta)_{m,k}(-x)^m}{(\alpha)_{m,k}\, n!m!}\frac{z^{\frac{\eta-\mu}{k}+m+n}}{k\Gamma_k(-\mu)}\int_0^{1}u^{\frac{\eta+nk+mk+k}{k}-1}(1-u)^{-\frac{\mu}{k}-1}du\right)d\omega$$

$$= \int_0^{\infty} e^{-\omega}\omega^{\phi-1}\left(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(\alpha)_{m+n,k}(\beta)_{m,k}(-x)^m}{(\alpha)_{m,k}\, n!m!}\frac{z^{\frac{\eta-\mu}{k}+m+n}}{\Gamma_k(-\mu)}B_k(\eta+mk+nk+k,-\mu)\right)d\omega$$

$$= \frac{\Gamma(\phi)\Gamma_k(\eta+k)}{\Gamma_k(\eta-\mu+k)}z^{\frac{\eta-\mu}{k}}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\eta+k)_{n+m,k}(\beta)_{m,k}(\alpha+mk)_{n,k}}{(\eta-\mu+k)_{n+m,k}}\frac{(z)^n(-xz)^m}{n!m!}$$

$$= \frac{\Gamma(\phi)\Gamma_k(\eta+k)}{\Gamma_k(\eta-\mu+k)}z^{\frac{\eta-\mu}{k}}F_{1,k}(\eta+k,\beta,\alpha+mk;\eta-\mu+k;z,-xz)$$

which completes the proof. □

Theorem 5.4. *The k-hypergeometric polynomials $S_{n,k}^{\left(\frac{\alpha}{k},\frac{\beta}{k}\right)}(x)$ have the following integral representation:*

$$S_{n,k}^{\left(\frac{\alpha}{k},\frac{\beta}{k}\right)}(x) = \frac{1}{\Gamma_k\left(\frac{\alpha-\beta}{k}\right)\Gamma_k\left(\frac{\beta}{k}\right)}\int_0^{\infty}\int_0^{\infty} e^{-(u_1+u_2)}\frac{(ku_1+k(1-x)u_2)^n}{n!}u_1^{\frac{\alpha-\beta}{k}-1}u_2^{\frac{\beta}{k}-1}du_1du_2 \tag{27}$$

where Γ_k is the k-gamma function given in (1).

Proof. If we use the identity [2]

$$a^{-v} = \frac{1}{\Gamma(v)}\int_0^{\infty} e^{-at}t^{v-1}dt, \quad (Re(v)>0)$$

on the right side of the generatin function (14), we arrive

$$\begin{aligned}
\sum_{n=0}^{\infty} S_{n,k}^{\left(\frac{\alpha}{k},\frac{\beta}{k}\right)}(x)\,t^{n} &= \frac{1}{\Gamma_k\left(\frac{\alpha-\beta}{k}\right)}\int_0^{\infty} e^{-(1-kt)u_1}u_1^{\frac{\alpha-\beta}{k}-1}du_1\frac{1}{\Gamma_k\left(\frac{\beta}{k}\right)}\int_0^{\infty} e^{-(1-k(1-x)t)u_2}u_2^{\frac{\beta}{k}-1}du_2\\
&= \frac{1}{\Gamma_k\left(\frac{\alpha-\beta}{k}\right)\Gamma_k\left(\frac{\beta}{k}\right)}\int_0^{\infty}\int_0^{\infty} e^{-\left(u_1+u_2\right)}e^{t\left(ku_1+k(1-x)u_2\right)}u_1^{\frac{\alpha-\beta}{k}-1}u_2^{\frac{\beta}{k}-1}du_1du_2\\
&= \frac{1}{\Gamma_k\left(\frac{\alpha-\beta}{k}\right)\Gamma_k\left(\frac{\beta}{k}\right)}\int_0^{\infty}\int_0^{\infty} e^{-\left(u_1+u_2\right)}\sum_{n=0}^{\infty}\frac{\left(ku_1+k\left(1-x\right)u_2\right)^n}{n!}u_1^{\frac{\alpha-\beta}{k}-1}u_2^{\frac{\beta}{k}-1}du_1du_2t^n\\
&= \sum_{n=0}^{\infty}\frac{1}{\Gamma_k\left(\frac{\alpha-\beta}{k}\right)\Gamma_k\left(\frac{\beta}{k}\right)}\int_0^{\infty}\int_0^{\infty} e^{-\left(u_1+u_2\right)}\frac{\left(ku_1+k\left(1-x\right)u_2\right)^n}{n!}u_1^{\frac{\alpha-\beta}{k}-1}u_2^{\frac{\beta}{k}-1}du_1du_2t^n.
\end{aligned}$$

One can derive the desired result by matching the coefficients of t^n on both sides of the equation. □

Theorem 5.5. *We have the following integral representation:*

$$S_{n,k}^{\left(\frac{\alpha}{k},\frac{\beta}{k}\right)}(x) = k^{n-1}\frac{(\alpha)_{n,k}\,\Gamma_k(\alpha)}{(n,k)!\Gamma_k(\beta)\Gamma_k(\alpha-\beta)}\int_0^1 t^{\frac{\beta}{k}-1}\left(1-t\right)^{\frac{\alpha-\beta}{k}-1}\left(1-xt\right)^n dt$$

where Γ_k is the k-gamma function given in (1).

Proof. If we apply expressions (2) and (5) to the right side of the theorem, we obtain

$$\begin{aligned}
&k^n\frac{(\alpha)_{n,k}}{(n,k)!}\frac{\Gamma_k(\alpha)}{\Gamma_k(\beta)\Gamma_k(\alpha-\beta)}\frac{1}{k}\int_0^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{\alpha-\beta}{k}-1}(1-xt)^n\,dt\\
&= k^n\frac{(\alpha)_{n,k}}{(n,k)!}\sum_{m=0}^{\infty}\frac{(-n)_m}{m!}x^m\frac{\Gamma_k(\alpha)}{\Gamma_k(\beta)\Gamma_k(\alpha-\beta)}\frac{1}{k}\int_0^1 t^{\frac{\beta+mk}{k}-1}(1-t)^{\frac{\alpha-\beta}{k}-1}\,dt\\
&= k^n\frac{(\alpha)_{n,k}}{(n,k)!}\sum_{m=0}^{\infty}\frac{(-n)_m}{m!}x^m\frac{\Gamma_k(\alpha)\Gamma_k(\beta+mk)}{\Gamma_k(\beta)\Gamma_k(\alpha+mk)}\\
&= k^n\frac{(\alpha)_{n,k}}{(n,k)!}\sum_{m=0}^{\infty}\frac{(-n)_m}{m!}x^m\left(\frac{\Gamma_k(\beta+mk)}{\Gamma_k(\beta)}\frac{\Gamma_k(\alpha)}{\Gamma_k(\alpha+mk)}\right)\\
&= k^n\frac{(\alpha)_{n,k}}{(n,k)!}\sum_{m=0}^{\infty}\frac{(-n)_m(\beta)_{m,k}}{(\alpha)_{m,k}}\frac{x^m}{m!}\\
&= k^n\frac{(\alpha)_{n,k}}{(n,k)!}\,{}_2F_1\left[\begin{matrix}(-n,1)\,, & (\beta,k) \\ & (\alpha,k)\end{matrix}\;;x\right]\\
&= S_{n,k}^{\left(\frac{\alpha}{k},\frac{\beta}{k}\right)}(x)
\end{aligned}$$

which completes the proof. □

6. Conclusion

In this study, we have centered our attention on a mathematical concept known as the k-hypergeometric polynomials. These polynomials are constructed using a mathematical tool called the Pochhammer k-symbol, which was initially presented by Diaz and their collaborators. Throughout our research, we have developed a series of theorems that are closely tied to these k-hypergeometric polynomials. These theorems serve as the foundation for our work, providing crucial insights into the behavior and properties of these mathematical entities. Utilizing these theorems, we have derived two vital functions: a multilinear generating function and a multilateral generating function specifically tailored for the k-hypergeometric polynomials. These functions assume a pivotal role in our mathematical analysis and are essential for further exploration. Expanding the scope of our investigation, we have extended our research to delve into the concept of the k-fractional secondary driver. This extension is rooted in the unique properties exhibited by k-hypergeometric polynomials and their relationship with another mathematical construct known as the gamma k-function. In summary, our research paper makes significant contributions to the field of mathematics by shedding light on the intricacies of k-hypergeometric polynomials and uncovering their intricate connections with various mathematical functions and transformations. This work not only advances our understanding of these mathematical concepts but also expand possibilities (break new ground) for future mathematical exploration and applications.

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