Filomat 39:26 (2025), 9347–9362
https://doi.org/10.2298/FIL2526347B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A new generalization of Laguerre-based Appell polynomials with two parameters

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Abstract. In this paper, we define a new generalization of Laguerre-based Appell polynomials with two parameters. We obtain a recurrence relation, a lowering operator, a integro-partial raising operator, a integro-partial differential equation for this new polynomial family. We introduce subpolynomials of these polynomials, namely Laguerre-based Hermite-Frobenius Euler polynomials, Laguerre-based Miller-Lee polynomials and generalizations of Laguerre-based Hermite polynomials and obtain various properties of them. We also show 3D graphs of these subfamilies and graphs of the distribution of their real roots.

1. Introduction

The theory of special functions plays a crucial role in both theoretical and applied mathematics. Within this framework, polynomial sequences stand out due to their rich structure and widespread applications. Among them, Appell polynomials have attracted significant attention thanks to their differential properties and generating functions. These polynomials $A_m(x)$ are defined by the differential relation [12, 18, 31, 35]

$$\frac{d}{dx}A_m(x) = mA_{m-1}(x), \quad A_0(x) \neq 0, \tag{1}$$

and have the generating function given by

$$a(t)e^{xt} = \sum_{m=0}^{\infty} A_m(x)\frac{t^m}{m!} \tag{2}$$

where

$$a(t) = \sum_{m=0}^{\infty} A_m \frac{t^m}{m!}, \quad A_0 \neq 0. \tag{3}$$

2020 *Mathematics Subject Classification*. Primary 33C65; Secondary 11B83, 11B68.
Keywords. Laguerre based-Appell polynomials, differential equations, determinant form.
Received: 21 January 2025; Revised: 21 July 2025; Accepted: 12 August 2025
Communicated by Miodrag Spalević
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In recent years, researchers have focused on extending the classical Appell structure by incorporating various special function bases. In this context, numerous Appell-type polynomial families associated with different bases have been proposed and studied. Notable examples include the Hermite-based Appell polynomials, Gould–Hopper Appell polynomials, Hann Appell polynomials, other related generalizations [3–5, 11, 13–16, 22, 25, 26, 29, 32]. These families allow the construction of richer algebraic frameworks beyond the classical Appell polynomials and facilitate the exploration of novel analytical properties. Such constructions are often referred to as hybrid polynomial families, as they combine classical and special function structures. Hybrid Appell polynomials have demonstrated effectiveness in solving differential equations, modeling physical phenomena, deriving operational identities, and establishing combinatorial identities. Their flexibility and structural diversity make them valuable tools in both pure and applied mathematical research. Among the various hybrid constructions, one particularly notable class arises from incorporating Laguerre polynomials into the Appell framework. The properties of various polynomial families in the literature continue to be investigated through different methods. One of this method, the factorization method, is widely used in the literature to derive differential equations associated with special functions and polynomial families [1, 5, 11, 14–16, 24].

As a classical family of polynomials, Laguerre polynomials possess rich analytical properties such as recurrence relations, well-defined differential equations. These features make them highly suitable for building structured generalizations of existing polynomial families. Two-variable Laguerre polynomials $L_m := L_m(x, y)$ are defined by Dattoli and Torre [7] as follows

$$L_m = m! \sum_{k=0}^{m} \frac{(-1)^k x^k y^{m-k}}{(m-k)!\,(k!)^2}. \tag{4}$$

The generating function of this polynomial is as follows [8]

$$e^{yt} C_0(xt) = \sum_{m=0}^{\infty} L_m \frac{t^m}{m!} \tag{5}$$

where the 0-th order Tricomi function is denoted by $C_0(x)$. The definition of Tricomi functions of order m-th is given in [8]

$$C_m(x) = \sum_{s=0}^{\infty} \frac{(-1)^s x^s}{s!\,(m+s)!}, \quad m \in \mathbb{N}_0. \tag{6}$$

The classical Laguerre polynomials, along with their numerous generalizations, have been the focus of extensive research due to their analytical richness and applicability [2, 6, 7, 9, 10, 17, 19, 20, 23, 27, 28, 30, 33, 34, 36]. One of them, Laguerre-based Appell polynomials ${}_LA_m(x, y)$ are defined in [27] with the following generating function

$$a(t)\, e^{yt} C_0(xt) = \sum_{m=0}^{\infty} {}_LA_m(x, y) \frac{t^m}{m!}. \tag{7}$$

Motivated by these developments, the present study focuses on the class of Laguerre-based Appell polynomials with two parameters, which has received increasing attention in the literature. The generalization of the Laguerre-based Appell polynomials with two parameters framework provides a unifying structure that enables the systematic construction of various rich subfamilies. Thanks to the flexibility of this approach, numerous well-known and novel polynomial families can be obtained as special or extended cases. In particular, the generalized forms of Laguerre-based Hermite-Frobenius–Euler, Miller–Lee and Hermite polynomials emerge naturally within this framework. This not only broadens the scope of the theory but also simplifies the derivation of their fundamental properties such as generating functions, recurrence relations, and differential equations. The paper proceeds as follows: In section 2, a new generalization of Laguerre-based Appell polynomials with two parameters is introduced and its various properties such as a recurrence relation, a lowering operator, a integro-partial raising operator, a integro-partial differential equation are shown. In section 3, the subfamilies of these polynomials are given and their corresponding properties are derived. In addition, 3D graphical plots of these subfamilies of polynomials and the graph of the distribution of their real roots are shown.

2. The generalization of Laguerre-based Appell polynomials with two parameters

In this section, a new generalization of Laguerre-based Appell polynomials with two parameters will be defined and the recurrence relation, shift operators that decrease and increase the degree of the polynomial and the integro-partial differential equation will be obtained for this polynomial family.

Definition 2.1. *The generalization of Laguerre-based Appell polynomials with two parameters* ${}_L\mathcal{A}_m^{(a,b)} := {}_L\mathcal{A}_m^{(a,b)}(x,y,z)$ *is defined with the help of the following explicit representation:*

$$ {}_L\mathcal{A}_m^{(a,b)} = \sum_{j=0}^{m}\sum_{l=0}^{j}\sum_{r=0}^{l}\binom{m}{j}\binom{j}{l}\binom{l}{r}\alpha_{m-j}\beta_{j-l}(y)\frac{(-1)^r(ax)^{l-r}(bz)^r}{r!} \tag{8} $$

where

$$ A(t)=\sum_{k=0}^{\infty}\alpha_k\frac{t^k}{k!},\quad \phi(y,t)=\sum_{k=0}^{\infty}\beta_k(y)\frac{t^k}{k!}\quad \text{and}\quad a,b\in\mathbb{R}\setminus\{0\}. \tag{9} $$

Theorem 2.2. $\left\{{}_L\mathcal{A}_m^{(a,b)}\right\}$ *is a generalization of Laguerre-based Appell polynomial sequence with two parameters if and only if it has the generating function given by*

$$ A(t)e^{axt}\phi(y,t)C_0(bzt)=\sum_{m=0}^{\infty}{}_L\mathcal{A}_m^{(a,b)}\frac{t^m}{m!}. \tag{10} $$

Proof. Let $\left\{{}_L\mathcal{A}_m^{(a,b)}\right\}$ be the generalization of Laguerre-based Appell polynomials with two parameters. Then, by equality (8), we have

$$ \sum_{m=0}^{\infty}{}_L\mathcal{A}_m^{(a,b)}\frac{t^m}{m!}=\sum_{m=0}^{\infty}\sum_{j=0}^{m}\sum_{l=0}^{j}\sum_{r=0}^{l}\binom{m}{j}\binom{j}{l}\binom{l}{r}\alpha_{m-j}\beta_{j-l}(y)\frac{(-1)^r(ax)^{l-r}(bz)^r}{r!}\frac{t^m}{m!}. $$

Using the Cauchy product and the relevant arrangements, we obtain the generating function given in (10). On the other hand, let $\left\{{}_L\mathcal{A}_m^{(a,b)}\right\}$ have the generating function given in eq. (10). Then, when we apply the Cauchy product using the series representations of e^{axt} and $C_0(bzt)$, and eq. (9), we obtain eq. (8). This leads us to the generalization of Laguerre-based Appell polynomials with two parameters $\left\{{}_L\mathcal{A}_m^{(a,b)}\right\}$. Thus the proof is completed. □

Remark 2.3. *It should be noted that we inspired from the two versions of the Hermite polynomials (physicist's, probabilistic's) and define the polynomial family in* (8) *in terms of the two parameters a and b. This inspiration can be easily observed from the equivalent definition (generating function) in* (10)*. In the following sections, this point of view can be examined from the generating functions given in the equations* (38) *and* (46).

Remark 2.4. *When* $x=0$, $z\to -x$ *and* $b=1$ *are taken in eq.* (10)*, it provides the following generating function*

$$ A(t)e_1(xt)\phi(y,t)=\sum_{m=0}^{\infty}{}_l\mathcal{A}_m(x,y)\frac{t^m}{m!} \tag{11} $$

which is given in [33].

Remark 2.5. *When in eq.* (10) *is written with $B(t)$ instead of $\phi(y,t)$ and $a = b = 1$ is taken, we have the twice-iterated Laguerre-based Appell polynomials in [17].*

Remark 2.6. *When we take $q \to 1^-$ in the family of the polynomials in [2], we obtain the case of eq.* (10) *where $a = b = 1$.*

Remark 2.7. *When we take $m = 1$, $-r_3 \to bz$ and $r_1 \to ax$ in the family of the polynomials in [36], we arrive at the generating function given in eq.* (10).

Theorem 2.8. *The following statements are provided for the sequence of the generalization of Laguerre-based Appell polynomials with two parameters:*
(i) *The generalization of Laguerre-based Appell polynomials with two parameters $\left\{{}_L\mathcal{A}_m^{(a,b)}\right\}$ has the following determinant representation:*

$$
{}_L\mathcal{A}_m^{(a,b)} = \frac{(-1)^m}{(\nu_0)^{m+1}} \begin{vmatrix} \Theta_0^{(a,b)} & \Theta_1^{(a,b)} & \cdots & \Theta_{m-1}^{(a,b)} & \Theta_m^{(a,b)} \\ \nu_0 & \nu_1 & \cdots & \nu_{m-1} & \nu_m \\ 0 & \nu_0 & \cdots & \binom{m-1}{1}\nu_{m-2} & \binom{m}{1}\nu_{m-1} \\ 0 & 0 & \cdots & \binom{m-1}{2}\nu_{m-3} & \binom{m}{2}\nu_{m-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \nu_0 & \binom{m}{m-1}\nu_1 \end{vmatrix} \tag{12}
$$

where $\sum_{m=0}^{\infty} \Theta_m^{(a,b)} \frac{t^m}{m!} = e^{axt}\phi(y,t)\, C_0(bzt)$, $\frac{1}{A(t)} = \sum_{k=0}^{\infty} \nu_k \frac{t^k}{k!}$ *and* $\Theta_m^{(a,b)} := \Theta_m^{(a,b)}(x,y,z)$.
(ii) *The generalization of Laguerre-based Appell polynomials with two parameters $\left\{{}_L\mathcal{A}_m^{(a,b)}\right\}$ satisfies the following derivative relations:*

$$
D_x \left\{{}_L\mathcal{A}_m^{(a,b)}\right\} = ma\,{}_L\mathcal{A}_{m-1}^{(a,b)}, \tag{13}
$$

$$
-D_z z D_z \left\{{}_L\mathcal{A}_m^{(a,b)}\right\} = mb\,{}_L\mathcal{A}_{m-1}^{(a,b)}. \tag{14}
$$

Proof. To show that (i), using the series representation of $\frac{1}{A(t)}$ as follows:

$$
[A(t)]^{-1} = \sum_{k=0}^{\infty} \nu_k \frac{t^k}{k!},
$$

using the generating function (10), we get

$$
e^{axt}\phi(y,t)\, C_0(bzt) = \left(\sum_{k=0}^{\infty} \nu_k \frac{t^k}{k!}\right)\left(\sum_{m=0}^{\infty} {}_L\mathcal{A}_m^{(a,b)} \frac{t^m}{m!}\right).
$$

Hence

$$
\sum_{m=0}^{\infty} \Theta_m^{(a,b)} \frac{t^m}{m!} = \left(\sum_{k=0}^{\infty} \nu_k \frac{t^k}{k!}\right)\left(\sum_{m=0}^{\infty} {}_L\mathcal{A}_m^{(a,b)} \frac{t^m}{m!}\right).
$$

Applying the Cauchy product and by comparing the coefficients of $\frac{t^m}{m!}$ from the polynomial equation, we obtain

$$
\Theta_m^{(a,b)} = \sum_{k=0}^{m} \binom{m}{k} \nu_k\, {}_L\mathcal{A}_{m-k}^{(a,b)}, \quad m \in \mathbb{N}_0.
$$

From here we obtain the system of equations. Then, using Cramer's rule followed by elementary row operations, we obtain the determinantal form stated in (12).

To show that (*ii*) firstly, when we apply the D_x operator to the generating function (10) of ${}_L\mathcal{A}_m^{(a,b)}$ and basic algebraic operations, we obtain the derivative relation given in eq. (13).
Secondly, when we apply the $-D_z z D_z$ operator to the generating funciton (10) of ${}_L\mathcal{A}_m^{(a,b)}$ and basic algebraic operations, we obtain the derivative relation given in eq. (14). Therefore, the proof is completed. □

Theorem 2.9. *The generalization of Laguerre-based Appell polynomials with two parameters has the following recurrence relation*

$$ {}_L\mathcal{A}_{m+1}^{(a,b)} = \sum_{k=0}^{m} \binom{m}{k} \gamma_k \, {}_L\mathcal{A}_{m-k}^{(a,b)} + ax \, {}_L\mathcal{A}_m^{(a,b)} + \sum_{k=0}^{m} \binom{m}{k} \rho_k(y) \, {}_L\mathcal{A}_{m-k}^{(a,b)} - bD_z^{-1} \, {}_L\mathcal{A}_m^{(a,b)} \tag{15} $$

where

$$ \frac{A'(t)}{A(t)} = \sum_{k=0}^{\infty} \gamma_k \frac{t^k}{k!}, \quad \frac{\phi_t(y,t)}{\phi(y,t)} = \sum_{k=0}^{\infty} \rho_k(y) \frac{t^k}{k!}, \quad \phi_t(y,t) = \frac{\partial}{\partial t}\phi(y,t) \text{ and } D_z^{-1} \text{ is the inverse of } D_z. \tag{16} $$

Proof. Taking the derivative with respect to t on both sides of (10), we have

$$ \begin{aligned} \sum_{m=0}^{\infty} {}_L\mathcal{A}_{m+1}^{(a,b)} \frac{t^m}{m!} = \; & \frac{A'(t)}{A(t)} A(t) e^{axt} \phi(y,t) C_0(bzt) \\ & + axA(t) e^{axt} \phi(y,t) C_0(bzt) \\ & + \frac{\phi_t(y,t)}{\phi(y,t)} A(t) e^{axt} \phi(y,t) C_0(bzt) \\ & + \left(\sum_{m=0}^{\infty} \frac{(-1)^{m+1}(m+1)(bz)^{m+1} t^m}{[(m+1)!]^2} \right) A(t) e^{axt} \phi(y,t). \end{aligned} $$

Using (16), we have

$$ \begin{aligned} \sum_{m=0}^{\infty} {}_L\mathcal{A}_{m+1}^{(a,b)} \frac{t^m}{m!} = & \left(\sum_{k=0}^{\infty} \gamma_k \frac{t^k}{k!} \right) \left(\sum_{m=0}^{\infty} {}_L\mathcal{A}_m^{(a,b)} \frac{t^m}{m!} \right) + \left(\sum_{k=0}^{\infty} \rho_k(y) \frac{t^k}{k!} \right) \left(\sum_{m=0}^{\infty} {}_L\mathcal{A}_m^{(a,b)} \frac{t^m}{m!} \right) \\ & + ax \sum_{m=0}^{\infty} {}_L\mathcal{A}_m^{(a,b)} \frac{t^m}{m!} + \left(\sum_{m=0}^{\infty} \frac{(-1)^{m+1}(m+1)(bz)^{m+1} t^m}{[(m+1)!]^2} \right) A(t) e^{axt} \phi(y,t). \end{aligned} $$

Hence, when use the Cauchy product, we obtain

$$ \begin{aligned} \sum_{m=0}^{\infty} {}_L\mathcal{A}_{m+1}^{(a,b)} \frac{t^m}{m!} = & \sum_{m=0}^{\infty} \sum_{k=0}^{m} \binom{m}{k} \gamma_k \, {}_L\mathcal{A}_{m-k}^{(a,b)} \frac{t^m}{m!} + \sum_{m=0}^{\infty} \sum_{k=0}^{m} \binom{m}{k} \rho_k(y) \, {}_L\mathcal{A}_{m-k}^{(a,b)} \frac{t^m}{m!} \\ & + ax \sum_{m=0}^{\infty} {}_L\mathcal{A}_m^{(a,b)} \frac{t^m}{m!} + \left(\sum_{m=0}^{\infty} \frac{(-1)^{m+1}(m+1)(bz)^{m+1} t^m}{[(m+1)!]^2} \right) A(t) e^{axt} \phi(y,t). \end{aligned} $$

Taking the derivative of both sides of the last equation with respect to z, we get

$$ \begin{aligned} \sum_{m=0}^{\infty} D_z \left\{ {}_L\mathcal{A}_{m+1}^{(a,b)} \right\} \frac{t^m}{m!} = & \sum_{m=0}^{\infty} \sum_{k=0}^{m} \binom{m}{k} \gamma_k D_z \left\{ {}_L\mathcal{A}_{m-k}^{(a,b)} \right\} \frac{t^m}{m!} + \sum_{m=0}^{\infty} \sum_{k=0}^{m} \binom{m}{k} \rho_k(y) \, D_z \left\{ {}_L\mathcal{A}_{m-k}^{(a,b)} \right\} \frac{t^m}{m!} \\ & + ax \sum_{m=0}^{\infty} D_z \left\{ {}_L\mathcal{A}_m^{(a,b)} \right\} \frac{t^m}{m!} - b \sum_{m=0}^{\infty} {}_L\mathcal{A}_m^{(a,b)} \frac{t^m}{m!}. \end{aligned} $$

Applying D_z^{-1} to both sides oh the above equation, we get

$$\begin{aligned}\sum_{m=0}^{\infty} {}_L\mathcal{A}_{m+1}^{(a,b)} \frac{t^m}{m!} &= \sum_{m=0}^{\infty}\sum_{k=0}^{m}\binom{m}{k}\gamma_k\, {}_L\mathcal{A}_{m-k}^{(a,b)}\frac{t^m}{m!} + \sum_{m=0}^{\infty}\sum_{k=0}^{m}\binom{m}{k}\rho_k(y)\, {}_L\mathcal{A}_{m-k}^{(a,b)}\frac{t^m}{m!} \\ &\quad + ax\sum_{m=0}^{\infty} {}_L\mathcal{A}_{m}^{(a,b)}\frac{t^m}{m!} - b\sum_{m=0}^{\infty} D_z^{-1}\, {}_L\mathcal{A}_{m}^{(a,b)}\frac{t^m}{m!}. \end{aligned} \tag{17}$$

Thus, equating the coefficients of $\frac{t^m}{m!}$ on both sides of the (17) completes the proof. □

Theorem 2.10. *The generalization of Laguerre-based Appell polynomials with two parameters satisfies the lowering operator, integro-partial raising operator and integro-partial differential equation provided by*

$${}_xL_m^- := \frac{1}{ma}D_x, \tag{18}$$

$${}_xL_m^+ := \sum_{k=0}^{m}\frac{\gamma_k}{k!a^k}D_x^k + \sum_{k=0}^{m}\frac{\rho_k(y)}{k!a^k}D_x^k + ax - bD_z^{-1}, \tag{19}$$

$$\left[\sum_{k=0}^{m}\frac{\gamma_k}{k!a^k}D_x^{k+1} + \sum_{k=0}^{m}\frac{\rho_k(y)}{k!a^k}D_x^{k+1} + axD_x - bD_z^{-1}D_x - ma\right]{}_L\mathcal{A}_m^{(a,b)} = 0. \tag{20}$$

Proof. Using the eq. (13), it is clear that the lowering operator is given as

$${}_xL_m^- := \frac{1}{ma}D_x.$$

Applying the lowering operator k times to the term ${}_L\mathcal{A}_m$ we can write it as follows:

$$\begin{aligned}{}_L\mathcal{A}_{m-k}^{(a,b)} &= \left[{}_xL_{m-k+1}^-\, {}_xL_{m-k+2}^- \cdots\ {}_xL_m^-\right]{}_L\mathcal{A}_m^{(a,b)} \\ &= \left[\frac{1}{(m-k+1)a}D_x\frac{1}{(m-k+2)a}D_x\ldots\ \frac{1}{ma}D_x\right]{}_L\mathcal{A}_m^{(a,b)} \\ &= \frac{(m-k)!}{m!a^k}D_x^k\left\{{}_L\mathcal{A}_m^{(a,b)}\right\}. \end{aligned} \tag{21}$$

Upon substituting from (21) in (15), we get

$${}_L\mathcal{A}_{m+1}^{(a,b)} = \left[\sum_{k=0}^{m}\frac{\gamma_k}{k!a^k}D_x^k + \sum_{k=0}^{m}\frac{\rho_k(y)}{k!a^k}D_x^k + ax - b\,D_z^{-1}\right]{}_L\mathcal{A}_m^{(a,b)}.$$

Hence the raising operator is given by

$${}_xL_m^+ := \sum_{k=0}^{m}\frac{\gamma_k}{k!a^k}D_x^k + \sum_{k=0}^{m}\frac{\rho_k(y)}{k!a^k}D_x^k + ax - b\,D_z^{-1}.$$

Then, we apply the factorization method to obtain the differential equation of the generalization of Laguerre based Appell polynomials as follows

$${}_xL_{m+1}^-\, {}_xL_m^+\left({}_L\mathcal{A}_m^{(a,b)}\right) = {}_L\mathcal{A}_m^{(a,b)},$$

we get

$${}_L\mathcal{A}_m^{(a,b)} = \frac{1}{(m+1)a}D_x\left[\sum_{k=0}^{m}\frac{\gamma_k}{k!a^k}D_x^k + \sum_{k=0}^{m}\frac{\rho_k(y)}{k!a^k}D_x^k + ax - bD_z^{-1}\right]{}_L\mathcal{A}_m^{(a,b)}$$

and rearranging the terms we can write

$$\left[\sum_{k=0}^{m}\frac{\gamma_k}{k!a^k}D_x^{k+1}+\sum_{k=0}^{m}\frac{\rho_k\left(y\right)}{k!a^k}D_x^{k+1}+axD_x-bD_z^{-1}D_x-ma\right]{}_L\mathcal{A}_m^{(a,b)}=0.$$

So the proof is completed. □

3. Examples

In this section, the subpolynomial families of generalized Laguerre-based Appell polynomials formed by special choices of $A\left(t\right)$ and $\phi\left(y,t\right)$ functions are examined. For this newly obtained subpolynomial family, the recurrence relation, lowering operator, integro-partial raising operator, integro-partial differential equation and determinant represantation are obtained. Also 3D graphical plots of these subfamilies of polynomials and the graph of the distribution of real roots are shown.

3.1. Generalization of Laguerre-based Hermite-Frobenius Euler polynomials

When $a=b=1$ is taken in the eq. (10), the generalization of Laguerre-based Hermite-Frobenius Euler polynomials are defined as follows ${}_LE_m^F:={}_LE_m^F\left(x,y,z;\lambda\right)$,

$$\left(\frac{1-\lambda}{e^t-\lambda}\right)e^{xt+yt^2}C_0\left(zt\right)=\sum_{m=0}^{\infty}{}_LE_m^F\left(x,y,z;\lambda\right)\frac{t^m}{m!},\quad \lambda\in\mathbb{C},\ \lambda\neq1 \tag{22}$$

where

$$A\left(t\right)=\frac{1-\lambda}{e^t-\lambda}\quad\text{and}\quad\phi\left(y,t\right)=e^{yt^2}. \tag{23}$$

Corollary 3.1. *The generalization of Laguerre-based Hermite-Frobenius Euler polynomials satisfies the following recurrence relation*

$${}_LE_{m+1}^F=\frac{1}{1-\lambda}\sum_{k=0}^{m}\binom{m}{k}{}_LE_k^Fe_{m-k}^F\left(\lambda\right)+x\,{}_LE_m^F+2my\,{}_LE_{m-1}^F-D_z^{-1}\,{}_LE_m^F \tag{24}$$

where the coefficients $e_l^F\left(\lambda\right)$ are connected by the Frobenius-Euler polynomials $E_l^F\left(x;\lambda\right)$ via the following expansion [21]

$$e_l^F\left(\lambda\right):=-\sum_{i=0}^{l}\frac{1}{2^i}\binom{l}{i}E_{l-i}^F\left(\frac{1}{2},\lambda\right). \tag{25}$$

We note that $E_l^F\left(x,\lambda\right)$ have the generating function from [21]

$$\frac{1-\lambda}{e^t-\lambda}e^{xt}=\sum_{l=0}^{\infty}E_l^F\left(x,\lambda\right)\frac{t^l}{l!}. \tag{26}$$

Corollary 3.2. *The generalization of Laguerre-based Hermite-Frobenius Euler polynomials satisfies the lowering operator, integro-partial raising operator and integro-partial differential equation as follows*

$$\left({}_xL_m^F\right)^-:=\frac{1}{m}D_x, \tag{27}$$

$$\left({}_xL_m^F\right)^+:=\frac{1}{1-\lambda}\sum_{k=0}^{m}D_x^{m-k}\frac{e_{m-k}^F\left(\lambda\right)}{(m-k)!}+x+2yD_x-D_z^{-1}, \tag{28}$$

$$\left[\frac{1}{1-\lambda}\sum_{k=0}^{m}D_x^{m-k+1}\frac{e_{m-k}^F\left(\lambda\right)}{(m-k)!}+x\,D_x+2yD_x^2-D_z^{-1}D_x-m\right]{}_LE_m^F=0. \tag{29}$$

Corollary 3.3. *The generalization of Laguerre-based Hermite-Frobenius Euler polynomials satisfies the following determinant representation*

$$
{}_{L}E_{m}^{F}=(-1)^{m}\begin{vmatrix}
{}_{f}\Theta_{0} & {}_{f}\Theta_{1} & \cdots & {}_{f}\Theta_{m-1} & {}_{f}\Theta_{m}\\
1 & -\frac{1}{\lambda-1} & \cdots & -\frac{1}{\lambda-1} & -\frac{1}{\lambda-1}\\
0 & 1 & \cdots & -\frac{1}{\lambda-1}\binom{m-1}{1} & -\frac{1}{\lambda-1}\binom{m}{1}\\
0 & 0 & \cdots & -\frac{1}{\lambda-1}\binom{m-1}{2} & -\frac{1}{\lambda-1}\binom{m}{2}\\
\vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & \cdots & 1 & -\frac{1}{\lambda-1}\binom{m}{m-1}
\end{vmatrix} \tag{30}
$$

where $\sum\limits_{m=0}^{\infty} {}_{f}\Theta_{m}\frac{t^{m}}{m!}=e^{xt+yt^{2}}C_{0}\left(zt\right)$.

The 3D surface plots of the generalization of Laguerre-based Hermite-Frobenius Euler polynomials ${}_{L}E_{2}^{F}\left(x,y,z;3\right)$ (Figure 1(a)) and the graph of the distribution of real roots for the this polynomials ${}_{L}E_{2}^{F}\left(x,y,z;3\right)$ (Figure 1(b)) are shown in Figure 1.

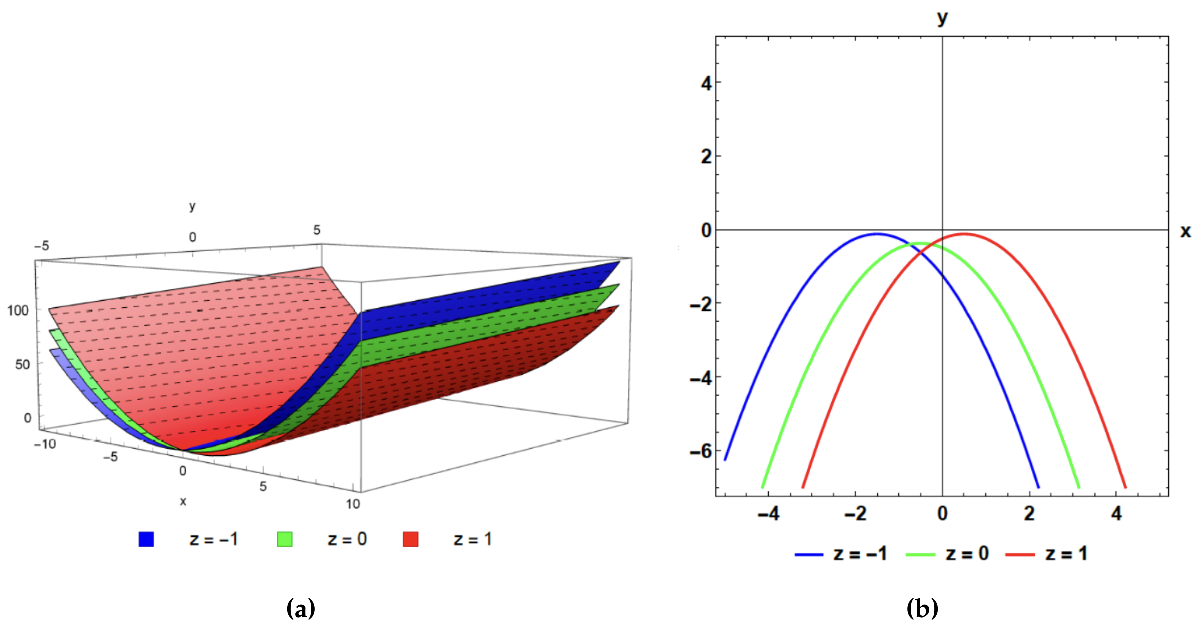
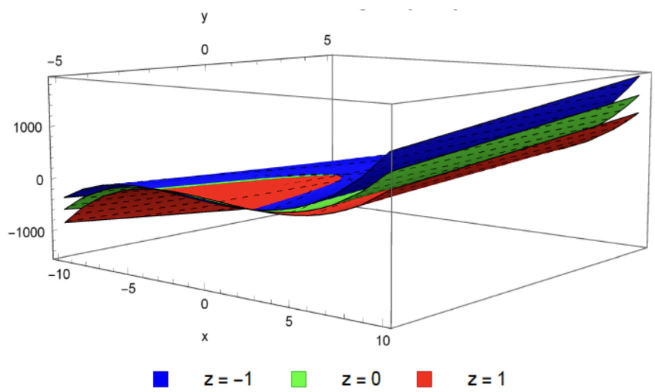
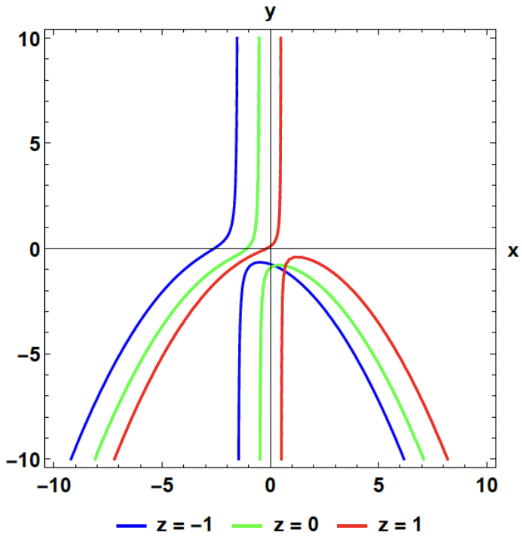


Figure 1: Figures related to ${}_{L}E_{2}^{F}\left(x,y,z;3\right)$

The 3D surface plots of the generalization of Laguerre-based Hermite-Frobenius Euler polynomials ${}_{L}E_{3}^{F}\left(x,y,z;3\right)$ (Figure 2(a)) and the graph of the distribution of real roots for the this polynomials ${}_{L}E_{3}^{F}\left(x,y,z;3\right)$ (Figure 2(b)) are shown in Figure 2.



(a)



(b)

Figure 2: Figures related to ${}_{L}E_{3}^{F}(x, y, z; 3)$

3.2. *Generalization of Laguerre-based Miller-Lee polynomials*

When $a = b = 1$ is taken in the eq. (10), the generalization of Laguerre-based Miller-Lee polynomials are defined as follows ${}_{L}\mathcal{M}_m := {}_{L}\mathcal{M}_m(x, y, z; n)$,

$$\frac{1}{(1-t)^{n+1}} e^{xt+yt^2} C_0(zt) = \sum_{m=0}^{\infty} {}_{L}\mathcal{M}_m \frac{t^m}{m!} \tag{31}$$

where

$$A(t) = \frac{1}{(1-t)^{n+1}} \quad \text{and} \quad \phi(y, t) = e^{yt^2}. \tag{32}$$

Corollary 3.4. *The generalization of Laguerre-based Miller-Lee polynomials satisfies the following recurrence relation*

$${}_{L}\mathcal{M}_{m+1} = \sum_{k=0}^{m} \binom{m}{k} k!\,(n+1)\,{}_{L}\mathcal{M}_{m-k} + x\,{}_{L}\mathcal{M}_m + 2my\,{}_{L}\mathcal{M}_{m-1} - D_z^{-1}\,{}_{L}\mathcal{M}_m. \tag{33}$$

Corollary 3.5. *The generalization of Laguerre-based Miller-Lee polynomials satisfies the lowering operator, integro-partial raising operator and integro-partial differential equation as follows*

$$\left({}_{x}L_m^{\mathcal{M}}\right)^{-} := \frac{1}{m} D_x, \tag{34}$$

$$\left({}_{x}L_m^{\mathcal{M}}\right)^{+} := \sum_{k=0}^{m} (n+1) D_x^k + x + 2yD_x - D_z^{-1}, \tag{35}$$

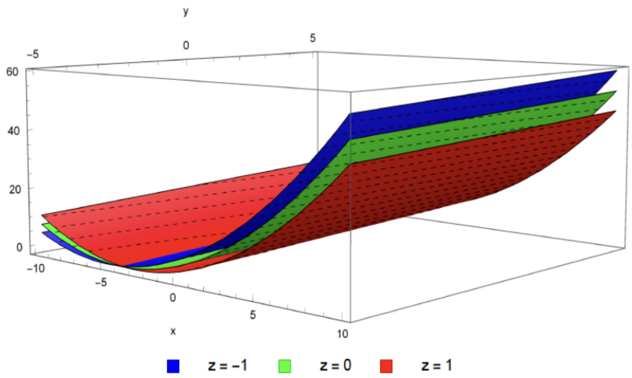
$$\left[\sum_{k=0}^{m} (n+1) D_x^{k+1} + x D_x + 2yD_x^2 - D_z^{-1} D_x - m\right] {}_{L}\mathcal{M}_m = 0. \tag{36}$$

Corollary 3.6. *The generalization of Laguerre-based Miller-Lee polynomials satisfies the following determinant representation*

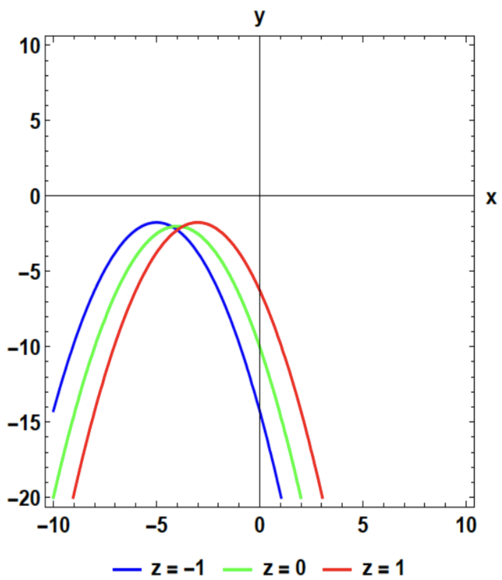
$$
{}_{L}\mathcal{M}_{m} = (-1)^{m} \begin{vmatrix} {}_{M}\Theta_{0} & {}_{M}\Theta_{1} & {}_{M}\Theta_{2} & \cdots & {}_{M}\Theta_{m} \\ 1 & -(n+1) & n(n+1) & \cdots & (-1)^{m}[n+1]_{m} \\ 0 & 1 & -\binom{2}{1}(n+1) & \cdots & \binom{m}{1}(-1)^{m-1}[n+1]_{m-1} \\ 0 & 0 & 1 & \cdots & \binom{m}{2}(-1)^{m-2}[n+1]_{m-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\binom{m}{m-1}(n+1) \end{vmatrix} \tag{37}
$$

where $[n]_{m} = n(n-1)\ldots(n-m+1)$ *with* $[n]_{0} = 1$ *and* $\sum_{m=0}^{\infty} {}_{M}\Theta_{m} \frac{t^{m}}{m!} = e^{xt+yt^{2}} C_{0}(zt)$.

The 3D surface plots of the generalization of Laguerre-based Miller-Lee polynomials ${}_{L}\mathcal{M}_{2}(x, y, z; 3)$ (Figure 3(a)) and the graph of the distribution of real roots for the this polynomials ${}_{L}\mathcal{M}_{2}(x, y, z; 3)$ (Figure 3(b)) are shown in Figure 3.



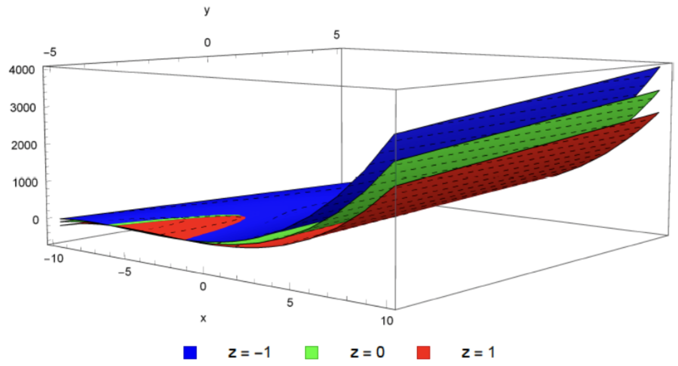
(a)



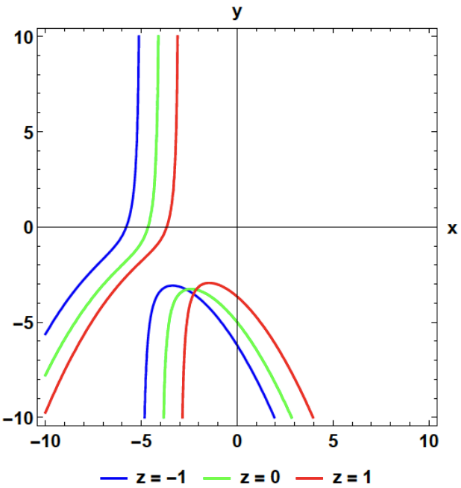
(b)

Figure 3: Figures related to ${}_{L}\mathcal{M}_{2}(x, y, z; 3)$

The 3D surface plots of the generalization of Laguerre-based Miller-Lee polynomials ${}_{L}\mathcal{M}_{3}(x, y, z; 3)$ (Figure 4(a)) and the graph of the distribution of real roots for the this polynomials ${}_{L}\mathcal{M}_{3}(x, y, z; 3)$ (Figure 4(b)) are shown in Figure 4.



(a)



(b)

Figure 4: Figures related to ${}_L\mathcal{M}_3\,(x, y, z; 3)$

3.3. Generalizations of Laguerre-based Hermite polynomials

- When $a = b = 1$ is taken in the eq. (10), the Laguerre-based probabilist's bivariate Hermite polynomials are defined as follows ${}_LHe_m := {}_LHe_m\,(x, y, z)$,

$$e^{(x+y)t-\frac{t^2}{2}}C_0\,(zt) = \sum_{m=0}^{\infty} {}_LHe_m \frac{t^m}{m!} \tag{38}$$

where

$$A\,(t) = e^{-\frac{t^2}{2}} \quad \text{and} \quad \phi\,(y, t) = e^{yt}. \tag{39}$$

Remark 3.7. *We should point out here that when we take $a\,(t) = e^{-\frac{t^2}{2}}$, $y \to x + y$ and $x \to z$ in the generating function in* (7)*, we obtain eq.* (38) *and write the following relation:*

$${}_LA_m\,(x + y, z) = {}_LHe_m\,(x, y, z)\,. \tag{40}$$

Corollary 3.8. *The Laguerre-based probabilist's bivariate Hermite polynomials satisfies the following recurrence relation*

$${}_LHe_{m+1} = -m{}_LHe_{m-1} + x\,{}_LHe_m + y{}_LHe_m - D_z^{-1}{}_LHe_m. \tag{41}$$

Corollary 3.9. *The Laguerre-based probabilist's bivariate Hermite polynomials satisfies the lowering operator, integro-partial raising operator and integro-partial differential equation as follows*

$$\left({}_xL_m^{He}\right)^- := \frac{1}{m}D_x, \tag{42}$$

$$\left({}_xL_m^{He}\right)^+ := -D_x + y + x - D_z^{-1}, \tag{43}$$

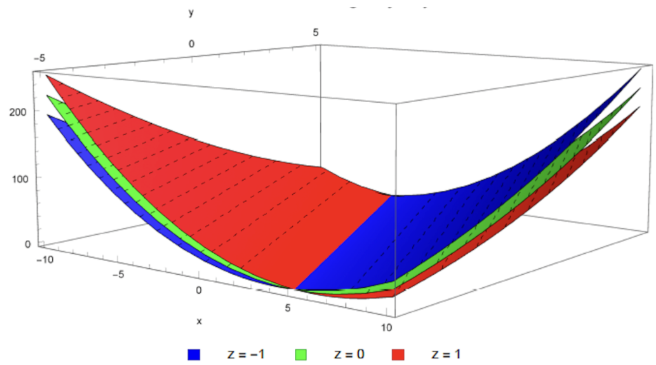
$$\left[-D_x^2 + (x + y)\,D_x - D_z^{-1}D_x - m\right]{}_LHe_m = 0. \tag{44}$$

Corollary 3.10. *The Laguerre-based probabilist's bivariate Hermite polynomials satisfies the following determinant representation*

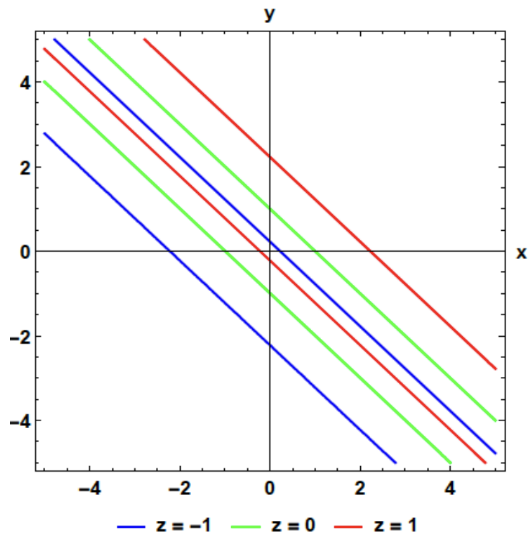
$$ {}_LHe_m = (-1)^m \begin{vmatrix} {}_h\Theta_0 & {}_h\Theta_1 & {}_h\Theta_2 & \cdots & {}_h\Theta_{m-1} & {}_h\Theta_m \\ 1 & 0 & 1 & \cdots & \gamma_{m-1} & \gamma_m \\ 0 & 1 & 0 & \cdots & \binom{m-1}{1}\gamma_{m-2} & \binom{m}{1}\gamma_{m-1} \\ 0 & 0 & 1 & \cdots & \binom{m-1}{2}\gamma_{m-3} & \binom{m}{2}\gamma_{m-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \binom{m}{m-1}\gamma_m \end{vmatrix} \tag{45} $$

where $\sum_{m=0}^{\infty} {}_h\Theta_m \frac{t^m}{m!} = e^{(x+y)t} C_0(zt)$ *and* $\gamma_1, \gamma_2, \ldots, \gamma_m$ *are the coefficients of the Maclaurin series of the function* $e^{-\frac{t^2}{2}}$.

The 3D surface plots of the Laguerre-based probabilist's bivariate Hermite polynomials ${}_LHe_2(x, y, z; 3)$ (Figure 5(a)) and the graph of the distribution of real roots for the this polynomials ${}_LHe_2(x, y, z; 3)$ (Figure 5(b)) are shown in Figure 5.



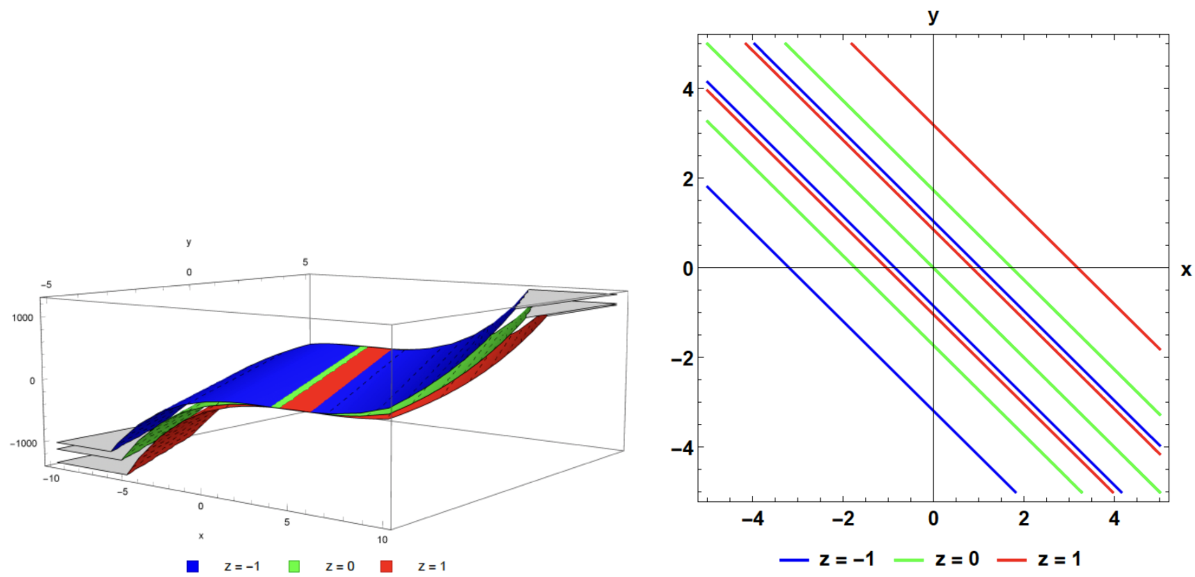
(a)



(b)

Figure 5: Figures related to ${}_LHe_2(x, y, z; 3)$

The 3D surface plots of Laguerre-based probabilist's bivariate Hermite polynomials ${}_LHe_3(x, y, z; 3)$ (Figure 6(a)) and the graph of the distribution of real roots for the this polynomials ${}_LHe_3(x, y, z; 3)$ (Figure 6(b)) are shown in Figure 6.



(a) **(b)**

Figure 6: Figures related to ${}_LHe_3\,(x, y, z; 3)$

- When $a = 2, b = 1$ is taken in the eq. (10), the Laguerre-based physicist's bivariate Hermite polynomials are defined as follows ${}_LH_m := {}_LH_m\,(x, y, z)$,

$$e^{(2x+y)t}C_0\,(zt) = \sum_{m=0}^{\infty} {}_LH_m \frac{t^m}{m!} \tag{46}$$

where

$$A\,(t) = e^{-t^2} \quad \text{and} \quad \phi\,(y, t) = e^{yt}. \tag{47}$$

Corollary 3.11. *The Laguerre-based physicist's bivariate Hermite polynomials satisfies the following recurrence relation*

$${}_LH_{m+1} = -2m_LH_{m-1} + 2x\,{}_LH_m + y_LH_m - D_z^{-1}{}_LH_m. \tag{48}$$

Corollary 3.12. *The Laguerre-based physicist's bivariate Hermite polynomials satisfies the lowering operator, integro-partial raising operator and integro-partial differential equation as follows*

$$\left({}_xL_m^H\right)^- := \frac{1}{m}D_x, \tag{49}$$

$$\left({}_xL_m^H\right)^+ := -2D_x + y + 2x - D_z^{-1}, \tag{50}$$

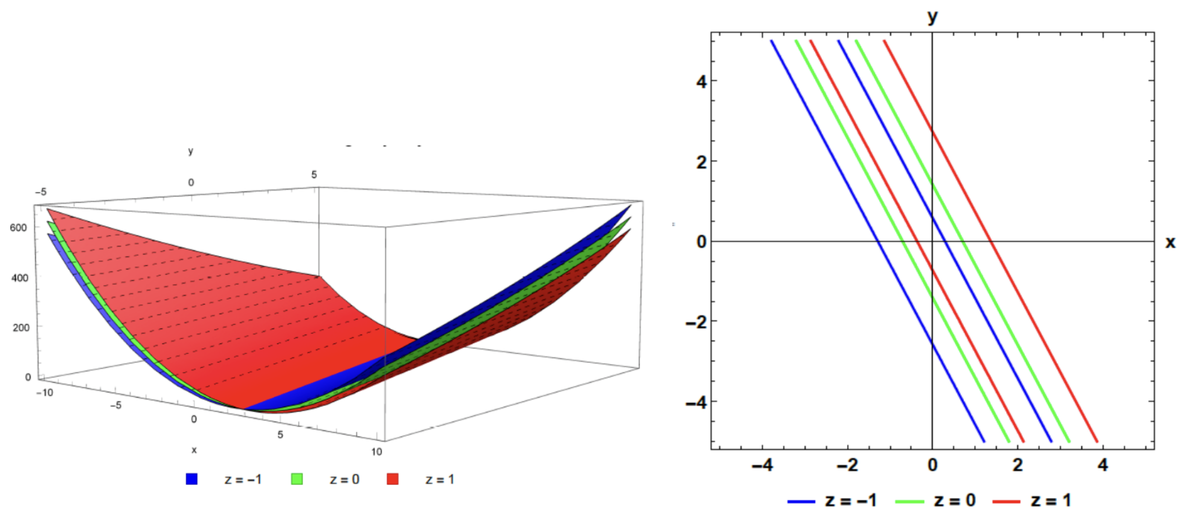
$$\left[-2D_x^2 + (2x + y)\,D_x - D_z^{-1}D_x - m\right]{}_LH_m = 0. \tag{51}$$

Corollary 3.13. *The Laguerre-based physicist's bivariate Hermite polynomials satisfies the following determinant* *representation*

$$
{}_LHe_m = (-1)^m \begin{vmatrix} {}_H\Theta_0 & {}_H\Theta_1 & {}_H\Theta_2 & \cdots & {}_H\Theta_{m-1} & {}_H\Theta_m \\ 1 & 0 & 2 & \cdots & \eta_{m-1} & \eta_m \\ 0 & 1 & 0 & \cdots & \binom{m-1}{1}\eta_{m-2} & \binom{m}{1}\eta_{m-1} \\ 0 & 0 & 1 & \cdots & \binom{m-1}{2}\eta_{m-3} & \binom{m}{2}\eta_{m-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \binom{m}{m-1}\eta_m \end{vmatrix} \tag{52}
$$

where $\sum_{m=0}^{\infty} {}_H\Theta_m \frac{t^m}{m!} = e^{(2x+y)t} C_0(zt)$ *and* $\eta_1, \eta_2, \ldots, \eta_m$ *are the coefficients of the Maclaurin series of the function* e^{-t^2}.

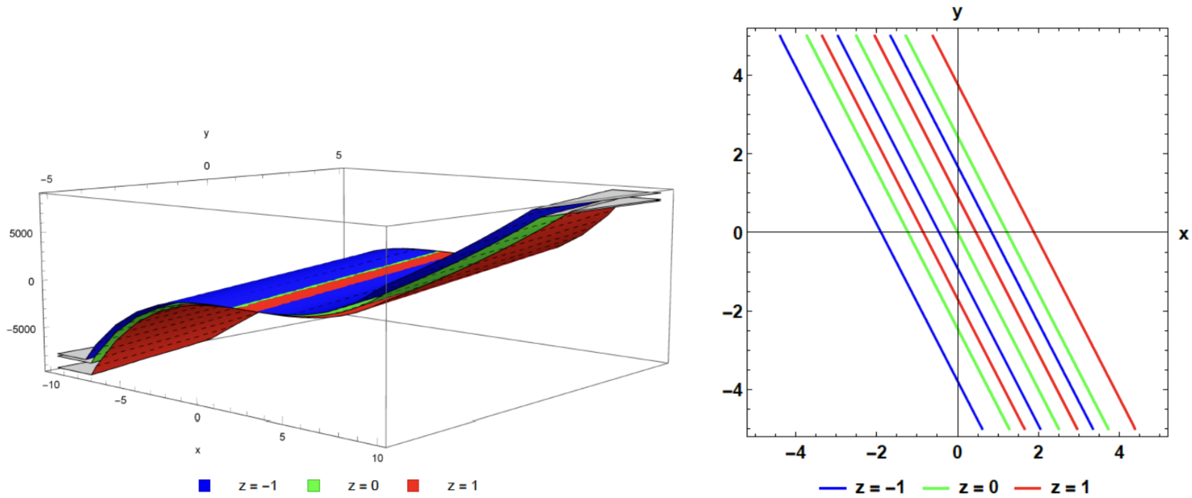
The 3D surface plots of the Laguerre-based physicist's bivariate Hermite polynomials ${}_LH_2(x, y, z; 3)$ (Figure 7(a)) and the graph of the distribution of real roots for the this polynomials ${}_LH_2(x, y, z; 3)$ (Figure 7(b)) are shown in Figure 7.



(a) **(b)**

Figure 7: Figures related to ${}_LH_2(x, y, z; 3)$

The 3D surface plots of the Laguerre-based physicit's bivariate Hermite polynomials ${}_LH_3(x, y, z; 3)$ (Figure 8(a)) and the graph of the distribution of real roots for the this polynomials ${}_LH_3(x, y, z; 3)$ (Figure 8(b)) are shown in Figure 8.



(a) **(b)**

Figure 8: Figures related to ${}_LH_3\,(x, y, z; 3)$

4. Conclusion

In this study, a new generalization of Laguerre-based Appell polynomials with two parameters has been introduced, and various properties of this structure have been examined. The recurrence relation, lowering operator, integro-partial raising operators, integro-partial differential equation and determinant representation obtained for the polynomial family provide opportunities for further theoretical and applied investigations. In addition, several special subfamilies derived from the general structure such as Laguerre-based Hermite-Frobenius-Euler and Miller–Lee polynomials have been defined, and sample graphical representations of these families have been presented. The results obtained suggest that this polynomial family can be examined from broader perspectives. In particular, by introducing new parameters, more general hybrid polynomial families may be constructed. The generating functions and determinant forms presented in this study may contribute to the development of new polynomial identities. Moreover, considering their relationship with classical q-polynomial sequences, these structures may offer potential applications in combinatorics and symbolic computation.

Acknowledgment.

We would like to thank The Scientific and Technological Research Council of Türkiye (TÜBİTAK) for the TÜBİTAK BİDEB 2211-A General Domestic Doctorate Scholarship Program that supported the first author. We would like to thank the editor and reviewers for their valuable suggestions and comments.

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