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About the uniqueness of generalized Drazin T-Riesz inverses

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Abstract. For two complex unital Banach algebras $\mathcal{A}$ and $\mathcal{B}$ and a homomorphism algebra $T : \mathcal{A} \to \mathcal{B}$ having the strong Riesz property, we propose a new formula for generalized Drazin T-Riesz inverses, and we give a new Laurent expansion for the resolvent of a given generalized Drazin T-Riesz invertible element. Most importantly, necessary and sufficent conditions to have the uniqueness of generalized Drazin T-Riesz inverse are given.

1. Introduction

Throughout this paper, $\mathcal{A}$ and $\mathcal{B}$ denote complex infinite dimensional Banach algebras with respective units $1_{\mathcal{A}}$ and $1_{\mathcal{B}}$. We denote by $\mathcal{A}^{-1}$, $QN(\mathcal{A})$, and $Idemp(\mathcal{A})$ the sets of all invertible, quasinilpotent, and idempotent elements of $\mathcal{A}$, respectively.

For $a \in \mathcal{A}$, the set of all elements commuting with a is defined as follows:

$$comm(a) = \{x \in \mathcal{A} \ : \ xa = ax\}.$$

We define the spectrum and the resolvent of an element $a \in B$ related to the closed unital subalgebra B of $\mathcal{A}$ respectively by

1. $\sigma_B(a) = \{\lambda \in \mathbb{C} \ : \ (\lambda 1_B - a) \notin B^{-1}\}$, if $B = \mathcal{A}$, then we write $\sigma(a)$ instead of $\sigma_{\mathcal{A}}(a)$.
2. $\rho_B(a) = \{\lambda \in \mathbb{C} \ : \ (\lambda 1_B - a) \in B^{-1}\} = \mathbb{C} \setminus \sigma_B(a)$, if $B = \mathcal{A}$, we simply write $\rho(a)$ rather than $\rho_{\mathcal{A}}(a)$.

In the case where $p \in Idemp(\mathcal{A})$, we have that $p\mathcal{A}p$ is a Banach algebra of unit p. We recall that for a complex subset K, the set of accumulation points of K, the set of isolated points of K and the frontier of K are respectively denoted by $acc\,K$, $iso\,K$, and ∂K. We denote the disk and the circle of the center $\mu \in \mathbb{C}$ and of the radius $r > 0$ by $D(\mu, r)$ and $C(\mu, r)$ respectively.

The concept of Drazin invertible elements was introduced by M.P. Drazin in 1958 [9] in the context of semigroups and associative rings. An element $a \in \mathcal{A}$ is said to be *Drazin invertible* if there exists $b \in comm(a)$ such that

$$bab = b, \text{ and } aba - a \text{ is nilpotent.}$$

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If such an element b exists, then it is unique and is called the Drazin inverse of a.

A non-invertible element a is Drazin invertible if and only if 0 is a pole of finite order of the resolvent map of a.

The notion of generalized Drazin invertible elements was introduced by J.J. Koliha [14]. An element $a \in \mathcal{A}$ is *generalized Drazin invertible* if there exists $b \in comm(a)$ such that

$$bab = b, \text{ and } aba - a \text{ is quasinilpotent.}$$

If b exists, then it is unique, and is called the *generalized Drazin inverse* of a, we denote it by a^{gD}. As a characterization of non-invertible generalized Drazin invertible elements, a is generalized Drazin invertible if and only if $0 \in iso\,\sigma(a)$ [14].

In what comes next, we focus our study on non-invertible elements of the given Banach algebra $\mathcal{A}$. Recall that a linear operator $T : \mathcal{A} \to \mathcal{B}$ is called a homomorphism if $T(ab) = TaTb$ for all $a, b \in \mathcal{A}$ and $T(1_{\mathcal{A}}) = 1_{\mathcal{B}}$. Let H and K be two subsets of $\mathcal{A}$, following [22], we define and denote the subset of the sum of commuting elements of H and K by

$$H +_{comm} K = \{h + k \ : \ (h, k) \in H \times K\ ,\ hk = kh\}.$$

For a given homomorphism of Banach algebras $T : \mathcal{A} \to \mathcal{B}$, we recall the following definitions (see [18, 24]):

1. $a \in \mathcal{A}$ is *T-Fredholm* if $a \in T^{-1}(\mathcal{B}^{-1})$.
2. The *T-Fredholm spectrum* is defined and denoted by
$$\sigma_{T,e}(a) = \{\lambda \in \mathbb{C} \ : \ \lambda 1_{\mathcal{B}} - T(a) \notin \mathcal{B}^{-1}\} = \sigma_{\mathcal{B}}(Ta).$$
3. The *T-Fredholm resolvent* set is defined and denoted by $\rho_{T,e}(a) = \mathbb{C} \setminus \sigma_{T,e}(a)$;
4. $a \in \mathcal{A}$ is *T-Browder* if $a \in \mathcal{A}^{-1} +_{comm} T^{-1}(0)$, the class of T-Browder elements is denoted by $\mathcal{B}_T(\mathcal{A})$.
5. The *T-Browder spectrum* is defined by $\sigma_{T,b}(a) = \{\lambda \in \mathbb{C} \ : \ \lambda 1_{\mathcal{A}} - a$ is not T-Browder$\}$, In the case where $\mathcal{A}$ is a semi-simple Banach algebra, we use the notation $\sigma_b(a)$.
6. The *T-Browder resolvent* set is defined by $\rho_{T,b}(a) = \mathbb{C} \setminus \sigma_{T,b}(a)$.
7. We say that $d \in \mathcal{A}$ is a *T-Riesz* element if $T(d) \in QN(\mathcal{B})$, i.e, $\sigma_{T,e}(a) = \{0\}$.
8. We say that the homomorphism T has the *Riesz property* if for an arbitrary $a \in \mathcal{A}$, we have $a \in T^{-1}(0) \Rightarrow acc\,\sigma(a) \subset \{0\}$.
9. We say that T possesses the *strong Riesz property* if the following inclusion holds for an arbitrary $a \in \mathcal{A}$, $\partial\sigma(a) \subset \sigma_{T,e}(a) \cup iso\,\sigma(a)$.

Obviously, T has the strong Riesz property implies that T has the Riesz property. If T has the Riesz property, we have by Corollary 3.6 [18] that $\sigma_{T,b}(a) = \sigma_{T,e}(a) \cup acc\,\sigma(a)$, hence we can define a *T-Riesz point* μ of $\sigma(a)$ as follows $\mu \in \rho_{T,e}(a) \cap iso\,\sigma(a)(= \sigma(a) \setminus \sigma_{T,b}(a))$. The set of all T-Riesz points of a is denoted by $\pi_{T,00}(a) := \sigma(a) \setminus \sigma_{T,b}(a)$.

Let $a \in \mathcal{A}$, we recall that a complex subset $\sigma \subset \sigma(a)$ is a *spectral set* of a if it is both open and closed (clopen) in $\sigma(a)$ following the induced topology on $\sigma(a)$ of $\mathbb{C}$.

Now, assume that there exists a spectral set σ of a. Then the *spectral idempotent* $p_{\sigma,a}$ of a corresponding to σ is

$$p_{\sigma,a} = \frac{1}{2\pi i}\int_{\Gamma^+} (\lambda - a)^{-1} d\lambda,$$

where Γ^+ is a positive (counter clockwise) oriented Cauchy contour surrounding σ and its exterior is containg $\sigma(a) \setminus \sigma$, when there is no confusion, we simply write Γ instead of Γ^+. Also, Γ^- is meant to be the negative (clockwise) oriented Cauchy contour Γ. We easily remark that $p_{\sigma,a}$ commutes with each element which commutes with a.

Recently, S. C. Živković-Zlatanović introduced the concept of generalized Drazin T-Riesz invertible elements in a Banach algebra $\mathcal{A}$.

Definition 1.1. *[24, Definition 4.4] Let $T : \mathcal{A} \to \mathcal{B}$ be a homomorphism of Banach algebras. An element $a \in \mathcal{A}$ is generalized Drazin T-Riesz invertible if there is $b \in comm(a)$ such that*

$$bab = b, \text{ and } a - aba \text{ is T-Riesz.}$$

In this case, $p = 1 - ab$ is called the idempotent related to the generalized Drazin T-Riesz inverse b of a.

The concept of generalized Drazin T-Riesz invertibility generalizes the one of generalized Drazin-Riesz invertibility introduced in [23] for bounded operators and in [2] for elements lying on a semisimple Banach algebra. For a semi-simple Banach algebra $\mathcal{A}$, we denote by $S_{\mathcal{A}}$ the socle of $\mathcal{A}$, in the case of a given semi-simple Banach algebra $\mathcal{A}$, the homomorphism T is the canonical projection $\pi : \mathcal{A} \to \mathcal{A}/S_{\mathcal{A}}$. In this case, generalized Drazin T-Riesz invertible elements are simply called generalized Drazin-Riesz invertible elements.

Generalized Drazin T-Riesz invertible elements are characterized as follows.

Theorem 1.2. *[24, Theorem 4.10] Let $T : \mathcal{A} \to \mathcal{B}$ be a homomorphism having the strong Riesz property, and $a \in \mathcal{A}$. The following statements are equivalent:*

(i) a is generalized Drazin T-Riesz invertible;
(ii) There exists an idempotent $p \in comm(a)$such that $a + p$ is T-Browder and ap is T-Riesz;
(iii) There exists an idempotent $p \in \mathcal{A}$ such that $a + p$ is invertible in $\mathcal{A}$ and ap is T-Riesz;
(iv) $0 \notin acc\, \sigma_{T,b}(a)$.

Definition 1.3. *[2, Definition 4.1] Let $a \in \mathcal{A}$ be a non invertible element with a spectral set σ which contain 0. The Drazin inverse of a relative to σ is defined by*

$$a^{D,\sigma} = (a - \xi p_{\sigma,a})^{-1}(1 - p_{\sigma,a}),$$

for some $\xi \in \mathbb{C}$ such that $|\xi| > 2r$ where $r = \sup_{\lambda\in\sigma}|\lambda|$.

As witnessed for the Drazin inverse and the generalized Drazin inverse, their existence implies automatically their uniqueness, this is not the case of generalized Drazin-Riesz inverses in the context of bounded operators. Indeed, if a bounded operator A is generalized Drazin-Riesz invertible such that $0 \in acc\, \sigma(A)$, then A disposes automatically of infinitely many generalized Drazin-Riesz inverses by virtue of Theorem 2.3 [1]. Therefore, finding sufficient and necessary conditions to reach the uniqueness of generalized Drazin T-Riesz inverses of an element $a \in \mathcal{A}$ in the case where $0 \in iso\, \sigma(a)$ turns out to be of crucial importance.

The purpose of this paper is to show when the uniqueness of the generalized Drazin T-Riesz inverse of a given generalized Drazin T-Riesz invertible element holds. Section 2 is devoted to generalize some results of [1] and [2] in order to give a new formula for generalized Drazin T-Riesz inverses, and to make tools which will enable us to treat the uniqueness of generalized Drazin T-Riesz inverses. Also, we give a counterexample that shows the existence of a non-spectral idempotent related to a generalized Drazin-Riesz invertible bounded operator. In Section 3, we focus our interest on the uniqueness of generalized Drazin T-Riesz inverses. Section 4 is devoted to study the uniqueness of generalized Drazin-Riesz inverses in the case of unital semisimple complex Banach algebras, this will be done using in one hand algebraic geometry notions of connectedness, on the other hand, we will use functional analysis notions of the connectedness of the character set. Examples are provided for practical understanding of these characterizations. Finally, in the last section, we apply the characterizations found in sections 3 and 4 to give more specific characterizations of the uniqueness of generalized Drazin-Riesz inverses in the case of bounded operators acting on a Banach space or a Hilbert space.

2. Preliminary results

Here and elsewhere, the homomorphism $T : \mathcal{A} \to \mathcal{B}$ has the strong Riesz property. Most of the results lying in this section are a generalization of some results of [2] that are made in the context of semi-simple Banach algebras.

Let $a \in \mathcal{A}$ be generalized Drazin T-Riesz invertible, and $0 \in acc\,\sigma(a)$. By [24, Corollary 4.13], we have $\omega(a) = \{0\} \cup \{\lambda_n \,:\, n \in \mathbb{N}\}$ is a closed set, where λ_n are non-zero T-Riesz points of a, and $\sigma = \sigma(a) \setminus \omega(a)$ is a spectral set of $\sigma(a)$.

Hence $\sigma_n = \omega(a) \setminus \{\lambda_1, ..., \lambda_n\} = \{0, \lambda_{n+1}, \lambda_{n+2}, ...\}$ and $\sigma'_n = \sigma \cup \{\lambda_1, ..., \lambda_n\}$ form two spectral sets of $\sigma(a) = \sigma_n \sqcup \sigma'_n$, where $\sqcup$ means a disjoint union.

Following the proof of [24, Theorem 2.2] and as $\mathcal{B}_T(\mathcal{A})$ forms a regularity, we have $a^{D,\sigma_n} = (a - p_{\sigma_n,a})^{-1}(1 - p_{\sigma_n,a})$ is a generalized Drazin T-Riesz inverse of a.

We mean by appropriate $n_0 \in \mathbb{N}$ in the case where $0 \in \; iso\,\sigma_{T,b}(a)$, that for a large enough n_0, σ_{n_0} will be contained on a given disk $D(0, r_{n_0})$ where $r_{n_0} < \min(\frac{1}{4}, \frac{1}{\|(a+p_{\sigma_{n_0}})^{-1}\|})$, and $D(0, r_{n_0}) \cap \sigma'_{n_0} = \emptyset$.

Theorem 2.1. *Let $0 \in iso\; \sigma_{T,b}(a)$, and let $a^{D,\sigma_{n_0}}$ be a generalized Drazin T-Riesz inverse of a with σ_{n_0} is a spectral set for some appropriate n_0. Then, for all $\lambda \in D(0, r) \setminus \sigma_{n_0}$, we have*

$$(\lambda - a)^{-1}(1 - p_{\sigma_{n_0},a}) = -\sum_{k=0}^{+\infty} \lambda^k (a^{D,\sigma_{n_0}})^{k+1}. \tag{2.1}$$

Proof. Let $p_{\sigma_{n_0},a}$ be the spectral idempotent of a relative to σ_{n_0}. Hence, by choosing $\xi = -1$ in Definition 1.3, we get

$$a^{D,\sigma_{n_0}} = (a + p_{\sigma_{n_0},a})^{-1}(1 - p_{\sigma_{n_0},a}).$$

Let $\lambda \in D(0, r) \setminus \sigma_{n_0}$, then $(\lambda - a)$ is invertible in $\mathcal{A}$. Also, it is easy to conclude that $(\lambda - 1) \in \rho(ap_{\sigma_{n_0},a})$ since $|\lambda| < \frac{1}{4}$ and $\sigma(ap_{\sigma_{n_0},a}) \subsetneq D(0, \frac{1}{4})$. Thus $(\lambda - 1) - ap_{\sigma_{n_0},a}$ is invertible in $\mathcal{A}$. From

$$\begin{aligned} \lambda - (a + p_{\sigma_{n_0},a}) &= ((\lambda - 1) - a)p_{\sigma_{n_0},a} + (\lambda - a)(1 - p_{\sigma_{n_0},a}) \\ &= ((\lambda - 1) - ap_{\sigma_{n_0},a})p_{\sigma_{n_0},a} + (\lambda - a)(1 - p_{\sigma_{n_0},a}), \qquad (\star) \end{aligned}$$

and according to [8, Lemma 2.1], it follows that $\lambda - (a + p_{\sigma_{n_0},a})$ is invertible in $\mathcal{A}$.
We have for all $\lambda \in D(0, r) \setminus \sigma_{n_0}$

$$\begin{aligned} &(1 - \lambda(a + p_{\sigma_{n_0},a})^{-1}) \sum_{k=0}^{q} \lambda^k (a + p_{\sigma_{n_0},a})^{-k} \\ &\qquad = \sum_{k=0}^{q} \lambda^k (a + p_{\sigma_{n_0},a})^{-k} - \sum_{k=0}^{q} \lambda^{k+1} (a + p_{\sigma_{n_0},a})^{-k-1} \\ &\qquad = 1 - \lambda^{q+1}(a + p_{\sigma_{n_0},a})^{-q-1}. \end{aligned}$$

As $\|\lambda(a + p_{\sigma_{n_0},a})^{-1}\| < 1$, it follows that

$$\|\lambda^q (a + p_{\sigma_{n_0},a})^{-q}\| \underset{q \longrightarrow \infty}{\longrightarrow} 0.$$

From (⋆), we obtain for all $\lambda \in D(0,r) \setminus \sigma_{n_0}$,

$$\begin{aligned}(\lambda - a)^{-1}(1 - p_{\sigma_{n_0},a}) &= (\lambda - (a + p_{\sigma_{n_0},a}))^{-1}(1 - p_{\sigma_{n_0},a}) \\ &= -\sum_{k=0}^{+\infty} \lambda^k (a + p_{\sigma_{n_0},a})^{-k}(a + p_{\sigma_{n_0},a})^{-1}(1 - p_{\sigma_{n_0},a}) \\ &= -\sum_{k=0}^{+\infty} \lambda^k ((a + p_{\sigma_{n_0},a})^{-1}(1 - p_{\sigma_{n_0},a}))^{k+1} \\ &= -\sum_{k=0}^{+\infty} \lambda^k (a^{D,\sigma_{n_0}})^{k+1}.\end{aligned}$$

□

Corollary 2.2. *Let $0 \in iso\, \sigma_{T,b}(a)$ and let $a^{D,\sigma_{n_0}}$ be a generalized Drazin T-Riesz inverse of some appropriate n_0, with its corresponding spectral idempotent $p_{\sigma_{n_0},a}$ of a, with $0 < r_{n_0} < \frac{1}{4}$. Then*

$$a^{D,\sigma_{n_0}} = \lim_{\substack{\lambda \longrightarrow 0 \\ \lambda \in D(0,r_{n_0}) \setminus \sigma_{n_0}}} (a - \lambda)^{-1}(1 - p_{\sigma_{n_0},a}).$$

Proof. As $\lambda \longrightarrow 0$, we have $|\lambda| < \frac{1}{\|(a+p_{\sigma_{n_0}})^{-1}\|}$, hence, Multiplying by $(1 - p_{\sigma_{n_0},a})$ both sides of equality

$$(\lambda - a)^{-1} = (\lambda - ap_{\sigma_{n_0},a})^{-1} p_{\sigma_{n_0},a} - \sum_{k=0}^{+\infty} \lambda^k (a^{D,\sigma_{n_0}})^{k+1},$$

we obtain

$$(\lambda - a)^{-1}(1 - p_{\sigma_{n_0},a}) = -\sum_{k=0}^{+\infty} \lambda^k (a^{D,\sigma_{n_0}})^{k+1}(1 - p_{\sigma_{n_0},a}),$$

for all $\lambda \in D(0,r) \setminus \sigma_{n_0}$. We deduce that

$$a^{D,\sigma_{n_0}} = \lim_{\substack{\lambda \longrightarrow 0 \\ \lambda \in D(0,r) \setminus \sigma_{n_0}}} (a - \lambda)^{-1}(1 - p_{\sigma_{n_0},a}).$$

□

Remark 2.3. 1. *The last theorem is a revision of [1, Theorem 2.12].*
2. *[1, Theorem 2.12] is valid as shown in the last corollary.*

Theorem 2.4. *Let $0 \in iso\, \sigma_{T,b}(a)$, and let $a^{D,\sigma_{n_0}}$ be a generalized Drazin T-Riesz inverse of a. Then, for all $\lambda_p \in (\sigma_{n_0} \setminus \{0\}) \cap D(0,r)$, there exists $r_{\lambda_p} > 0$ provided that $D(\lambda_p, r_{\lambda_p}) \cap \sigma_{n_0} = \{\lambda_p\}$. Then for all $\mu \in D(\lambda_p, r_{\lambda_p}) \setminus \{\lambda_p\}$, we have*

$$\begin{aligned}(\mu - a)^{-1} = &\sum_{k=1}^{\infty} (\mu - \lambda_p)^{-k}(ap_{\sigma_{n_0},a} - \lambda_p)^{k-1} p_{\sigma_{n_0},a} p_{\{\lambda_p\},a} \\ &- \sum_{k=0}^{\infty} (\mu - \lambda_p)^k ((a - \lambda_p)^{gD})^{k+1} p_{\sigma_{n_0},a} - \sum_{k=0}^{+\infty} \mu^k (a^{D,\sigma_{n_0}})^{k+1}.\end{aligned} \tag{2.2}$$

Proof. Let λ_p and μ be two complex numbers satisfying the conditions of the theorem, following [15, Example 2.1], there exists a radius r_{λ_p}, provided that $((\mu - \lambda_p) - (ap_{\sigma_{n_0},a} - \lambda_p)p_{\{\lambda_p\},a})$ and $(\mu - \lambda_p) - (p_{\{\lambda_p\},a} + ap_{\sigma_{n_0},a} - \lambda_p)$

are invertible for all $0 < |\lambda_p - \mu| < r_{\lambda_p} < \min(\frac{1}{4}, \frac{1}{\|(a+p_{\sigma_{n_0},a})^{-1}\|})$. We have

$$\begin{aligned}(\mu - a)p_{\sigma_{n_0},a} &= (\mu - ap_{\sigma_{n_0},a})p_{\sigma_{n_0},a}\\ &= \Big((\mu-\lambda_p) - (ap_{\sigma_{n_0},a} - \lambda_p)p_{\{\lambda_p\},a}\Big)p_{\{\lambda_p\}} +\\ &\Big((\mu-\lambda_p) - (p_{\{\lambda_p\},a} + ap_{\sigma_{n_0},a} - \lambda_p\Big)(p_{\sigma_{n_0},a} - p_{\{\lambda_p\}}).\end{aligned}$$

Taking into account that $\mu \in D(\lambda_p, r_p) \setminus \{\lambda_p\}$, $p_{\sigma_{n_0},a}p_{\{\lambda_p\},a} = p_{\{\lambda_p\},a}p_{\sigma_{n_0},a} = p_{\{\lambda_p\},a}$, and $(p_{\{\lambda_p\},a} + ap_{\sigma_{n_0},a} - \lambda_p)$ is invertible , we obtain by virtue of [8, Lemma 2.1]

$$\begin{aligned}(\mu - a)^{-1}p_{\sigma_{n_0},a} &= (\mu - ap_{\sigma_{n_0},a})^{-1}p_{\sigma_{n_0},a}\\ &= \Big((\mu-\lambda_p) - (ap_{\sigma_{n_0},a} - \lambda_p)p_{\{\lambda_p\},a}\Big)^{-1}p_{\{\lambda_p\},a} +\\ &\Big((\mu-\lambda_p) - (p_{\{\lambda_p\},a} + ap_{\sigma_{n_0},a} - \lambda_p)\Big)^{-1}(p_{\sigma_{n_0},a} - p_{\{\lambda_p\},a})\\ &= \sum_{k=1}^{\infty}(\mu-\lambda_p)^{-k}(ap_{\sigma_{n_0},a} - \lambda_p)^{k-1}p_{\{\lambda_p\},a}\\ &- \sum_{k=0}^{\infty}(\mu-\lambda_p)^{k}((a-\lambda_p)^{gD})^{k+1}p_{\sigma_{n_0},a}\end{aligned}$$

Now, by Theorem 2.1 and as $\mu \in D(\lambda_p, r_p) \setminus \{\lambda_p\}(\subset D(0, \frac{1}{\|(a+p_{\sigma_{n_0},a})^{-1}\|}) \setminus \sigma_{n_0})$, we get

$$(\mu - a)^{-1}(1 - p_{\sigma_{n_0},a}) = -\sum_{k=0}^{+\infty}\mu^k(a^{D,\sigma_{n_0}})^{k+1}.$$

Finally, another application of [8, Lemma 2.1] allows us to have

$$\begin{aligned}(\mu - a)^{-1} &= \sum_{k=1}^{\infty}(\mu-\lambda_p)^{-k}(ap_{\sigma_{n_0},a} - \lambda_p)^{k-1}p_{\{\lambda_p\},a}\\ &- \sum_{k=0}^{\infty}(\mu-\lambda_p)^{k}((a-\lambda_p)^{gD})^{k+1}p_{\sigma_{n_0},a} - \sum_{k=0}^{+\infty}\mu^k(a^{D,\sigma_{n_0}})^{k+1}.\end{aligned}$$

□

The next theorem states that if a is generalized Drazin T-Riesz invertible and $0 \in acc\,\sigma(a)$, a has infinitely many generalized Drazin-Riesz inverses because there is infinite Riesz points of a lying in σ_{n_0}.

Theorem 2.5. *Let $a \in \mathcal{A}$ be generalized Drazin T-Riesz invertible with $0 \in acc\,\sigma(a)$, and let $n_0, n_1 \in \mathbb{N}$ such that $n_0 < n_1$ and $r_{n_0} < \frac{1}{2}$. Then*

$$a^{D,\sigma_{n_0}} \neq a^{D,\sigma_{n_1}}.$$

Namely, the generalized Drazin T-Riesz inverse of a is not unique.

Proof. It is a direct consequence of [24, Remark 2.4 and Corollary 4.13]. □

We give a new formula that expresses all the generalized Drazin T-Riesz inverses related to their idempotents by virtue of the subclass of generalized Drazin T-Riesz inverses related to spectral sets. We start by the case where 0 is a limit point of the spectrum of the generalized Drazin T-Riesz invertible element.

But first, we shall construct the spectral sets related to p. Suppose that p is an idempotent in $\mathcal{A}$ such that $ap = pa$, $a + p$ is invertible and ap is T-Riesz. Without loss of generalities, there exists $0 < \epsilon < \frac{1}{2}$ such that

$$D(0, \epsilon) \cap \sigma(a + p) = \emptyset,$$

hence

$$(D(0,\epsilon) \setminus \{0\}) \cap \sigma(a(1-p)) = \emptyset.$$

We put

$$\sigma = (\sigma(ap) \cap D(0,\epsilon)) \setminus (\sigma(ap) \cap \sigma(a(1-p)) = \{0, \mu_1, \mu_n, ...\},$$

then σ is a spectral set composed of 0 and a sequence of T-Riesz points of a converging to 0, hence for all $n \in \mathbb{N}$ we consider the following spectral sets

$$\sigma_n = \sigma \setminus \{\mu_1, ..., \mu_n\}.$$

The next theorem will consider the above spectral sets σ_n.

Theorem 2.6. *Let $a \in \mathcal{A}$ be generalized Drazin T-Riesz invertible such that $0 \in acc\,\sigma(a)$. For all $p \in comm(a) \cap Idemp(\mathcal{A})$ such that $a + p$ is invertible and ap is T-Riesz. Then for all $n \in \mathbb{N}$, we have*

$$(a+p)^{-1}(1-p) = a^{D,\sigma_n}(1-p).$$

Proof. For all $\lambda \in \rho(a)$ we have by [8, Lemma 2.1]

$$(\lambda - a)^{-1} = (\lambda - a)^{-1}p + (\lambda - a)^{-1}(1-p).$$

Thus, taking into consideration [4, page 27], for a well chosen Cauchy countour Γ_n for every n we have

$$\begin{aligned}
p_{\sigma_n,a} &= \frac{1}{2\pi i}\int_{\Gamma_n}(\lambda - a)^{-1}d\lambda \\
&= \frac{1}{2\pi i}\int_{\Gamma_n}(\lambda - a)^{-1}pd\lambda + \frac{1}{2\pi i}\int_{\Gamma_n}(\lambda - a)^{-1}(1-p)d\lambda \\
&= \frac{1}{2\pi i}\int_{\Gamma_n}(\lambda - a)^{-1}d\lambda p + \frac{1}{2\pi i}\int_{\Gamma_n}(\lambda - a(1-p))^{-1}(1-p)d\lambda \\
&= \frac{1}{2\pi i}\int_{\Gamma_n}(\lambda - a)^{-1}d\lambda p \\
&= p\frac{1}{2\pi i}\int_{\Gamma_n}(\lambda - a)^{-1}d\lambda \\
&= p_{\sigma_n,a}p = pp_{\sigma_n,a}.
\end{aligned}$$

Thus, we have for all $n \in \mathbb{N}$

$$a + p_{\sigma_n,a} = a + p + (p_{\sigma_n,a} - p) = (a+p)(1 - p_{\sigma_n,a}) + (a+p)p_{\sigma_n,a} + (p_{\sigma_n,a} - p).$$

Consequently

$$(a + p_{\sigma_n,a})(1-p) = (a+p)(1-p)(1-p_{\sigma_n,a}).$$

Therefore, for all $n \in \mathbb{N}$ we obtain

$$(a+p)^{-1}(1-p) = (a + p_{\sigma_n,a})^{-1}(1 - p_{\sigma_n,a})(1-p) = a^{D,\sigma_n}(1-p).$$

□

Question How about the existence of the limit $\lim\limits_{n\to+\infty} a^{D,\sigma_n}$? If it exists, then we can consider the following expression in the last theorem

$$(a+p)^{-1}(1-p) = (\lim_{n\to+\infty} a^{D,\sigma_n})(1-p).$$

We close this section by giving an example that shows that the idempotent related to every generalized Drazin T-Riesz inverse is not necessarily a spectral idempotent. Therefore, we conclude that the class of generalized Drazin T-Riesz inverses does not only contain generalized Drazin T-Riesz inverses related to spectral sets that contain 0 and Riesz points of the given generalized Drazin T-Riesz invertible element.

Following [17], we recall that the quasinilpotent part of a bounded operator A is defined by

$$H_0(A) := \{x \in X \ : \ \lim_{n\to+\infty} \|A^n x\|^{\frac{1}{n}} = 0\}.$$

Example 2.7. *We consider A_1 the bilateral unweighted shift operator defined on $\ell^2(\mathbb{Z})$. We know that*

$$\sigma(A_1) = \{\lambda \in \mathbb{C} \ : \ |\lambda| = 1\}.$$

Let $\mathcal{H}$ be a complex Hilbert space and $\{e_n\}_{n\geq 0}$ be an orthonormal basis in $\mathcal{H}$. Let (λ_n) be a sequence of non-zero complex numbers converging to 0, such that $\sup_n |\lambda_n| < \frac{1}{2}$. Consider A_3 to be the operator defined by $A_3(e_n) = \lambda_n e_n$, A_3 is a compact operator, hence it is a Riesz operator.

Also, consider the operator $A_2 : H_0(\lambda_1 I - A_3) \mapsto H_0(\lambda_1 I - A_3)$, $A_2 = (A_3)_{H_0(\lambda_1 I - A_3)}$.

Hence, if we put $A = (A_1 \oplus A_2) \oplus A_3$ over the space $X = \ell^2(\mathbb{Z}) \oplus H_0(\lambda_1 I - A_3)) \oplus \mathcal{H}$, we obtain that A is generalized Drazin-Riesz invertible, one of its inverses is

$$S = (A_1 \oplus A_2)^{-1} \oplus 0_{\mathcal{H}}.$$

The projection related to S is $P = 0_{\ell^2(\mathbb{Z})} \oplus 0_{H_0(\lambda_1 I - A_3)} \oplus I_{\mathcal{H}}$, which is not a spectral projection of A, since $\sigma(AP) = \sigma(A_3)$, and $\sigma(A(I-P)) = \{\lambda \in \mathbb{C} \ : \ |\lambda| = 1\} \cup \{\lambda_1\}$, with $\sigma(AP) \cap \sigma(A(I-P)) = \{\lambda_1\} \neq \emptyset$.

Exploiting Theorem 2.6, we obtain for all $n \in \mathbb{N}$

$$S = (A+P)^{-1}(I-P) = A^{D,\sigma_n}(I-P), \text{ with } \sigma_n = \{0, \lambda_{n+1}, \lambda_{n+2}, ...\}.$$

3. Uniqueness of generalized Drazin T-Riesz inverses

The aim of this section is to prove when the uniqueness of generalized Drazin T-Riesz inverses holds in the case of Banach algebras with a homomorphism T having the strong Riesz property using (topological) functional analysis tools.

The following definitions represent the cornerstones of the topological tools used to characterize the uniqueness of generalized Drazin T-Riesz inverses in this section, also to give a topological characterization of the uniqueness of generalized Drazin-Riesz inverses in the context of semi-simple Banach algebras in Section 4.

Definition 3.1. *[16, Definition 7.1 and Definition 7.3] Let E be an algebra. Then, by a character of E, we mean a non-zero (complex) morphism of E into $\mathbb{C}$. The set of all characters of E is denoted by $Char(E)$.*

Also, the Gelfand transform of x denoted by $\hat{x}$ is defined to be the map $\hat{x} : Char(E) \mapsto \mathbb{C}, f \mapsto \hat{x}(f) = f(x)$.

Thus, if $F(Char(E), \mathbb{C})$ denotes the set of all complex-valued maps on $Char(E)$, the resulting map $\mathcal{G}_E : E \mapsto F(Char(E), \mathbb{C}), x \mapsto \mathcal{G}_E(x) = \hat{x}$ is called the Gelfand transform of the given algebra E, and $\mathcal{G}_E(E) = E^\wedge$ is called the Gelfand transform algebra of E.

The next theorem is an analogue of Theorem 2.6, in the case where 0 is not an accumulation point of the spectrum of the given generalized Drazin T-Riesz invertible element.

Theorem 3.2. *Let $a \in \mathcal{A}$ be generalized Drazin T-Riesz invertible such that $0 \in iso\,\sigma(a)$. For all $p \in comm(a) \cap Idemp(\mathcal{A})$ such that $a + p$ is invertible and ap is T-Riesz. Then*

$$(a+p)^{-1}(1-p) = a^{gD}(1-p).$$

Proof. For an idempotent $p \in \mathcal{A}$ such that $ap = pa$, $a + p$ is invertible and ap is T-Riesz, we have

$$\sigma(a) = \sigma_{(1-p)\mathcal{A}(1-p)}(a(1-p)) \cup \sigma(ap),$$

with $\sigma(ap) = \{0, \mu_1, ..., \mu_k\}$.

As $\{0\}$ is a spectral set of a, we show that $pp_{\{0\},a} = p_{\{0\},a}p = p_{\{0\},a}$. For all $\lambda \in \rho(a)$, we have

$$(\lambda - a)^{-1} = (\lambda - a)^{-1}p + (\lambda - a)^{-1}(1 - p).$$

With the same argument and for a convenient $\epsilon > 0$ allowing us to take the circle $C(0, \epsilon)$ instead of Γ_n in the proof of Theorem 2.6, we get

$$\begin{aligned}
p_{\{0\},a} &= \frac{1}{2\pi i}\int_{C(0,\epsilon)} (\lambda - a)^{-1} d\lambda \\
&= \frac{1}{2\pi i}\int_{C(0,\epsilon)} (\lambda - a)^{-1} p d\lambda + \frac{1}{2\pi i}\int_{C(0,\epsilon)} (\lambda - a)^{-1}(1-p) d\lambda \\
&= \frac{1}{2\pi i}\int_{C(0,\epsilon)} (\lambda - a)^{-1} d\lambda p + \frac{1}{2\pi i}\int_{C(0,\epsilon)} (\lambda - a(1-p))^{-1}(1-p) d\lambda \\
&= \frac{1}{2\pi i}\int_{C(0,\epsilon)} (\lambda - a)^{-1} d\lambda p \\
&= p\frac{1}{2\pi i}\int_{C(0,\epsilon)} (\lambda - a)^{-1} d\lambda \\
&= p_{\{0\},a}p = pp_{\{0\},a}.
\end{aligned}$$

Now, as $pp_{\{0\},a} = p_{\{0\},a}p = p_{\{0\},a}$, we obtain

$$\begin{aligned}
a + p_{\{0\},a} &= a + p + (p_{\{0\},a} - p) \\
&= (a + p)p_{\{0\},a} + (a + p)(1 - p_{\{0\},a}) + (p_{\{0\},a} - p).
\end{aligned}$$

Therefore $(a + p_{\{0\},a})(1 - p) = (a + p)(1 - p)(1 - p_{\{0\},a})$. Thus we conclude that

$$(a + p)^{-1}(1 - p) = a^{gD}(1 - p).$$

□

Remark 3.3. *Theorem 3.2 gives an important insight in the case where the generalized Drazin T-Riesz inverse may be unique, it is sufficient to determinate when the idempotent p which satisfies conditions of Theorem 3.2 is equal to $p_{\{0\},a}$, in order to find the uniqueness of the generalized Drazin T-Riesz inverse.*

Theorem 3.4. *Let $a \in \mathcal{A}$ be generalized Drazin T-Riesz invertible such that $0 \in \sigma(a)$. If a has a unique generalized Drazin T-Riesz inverse, this implies that $\sigma(a) = \sigma_{T,b}(a) \cup \{0\}$. In this case, the only generalized Drazin T-Riesz inverse is the generalized Drazin inverse.*

Proof. Suppose that a has a unique generalized Drazin T-Riesz inverse. By way of contradiction, suppose that $\sigma(a) \neq \sigma_{T,b}(a) \cup \{0\}$. Then $\pi_{T,00}(a) \neq \emptyset$.

Case 1. $\pi_{T,00}(a) \cup \{0\}$ is finite: then $\mathrm{card}(\pi_{T,00}(a) \cup \{0\}) \geq 2$. Using [24, Theorem 4.10] $0 \notin acc\, \sigma_{T,b}(a)$, we have $\pi_{T,00}(a) \cup \{0\}$ and $\sigma_{T,b}(a) \setminus \{0\}$ are disjoint clopen sets in $\sigma(a)$, with

$$\sigma(a) = (\sigma_{T,b}(a) \setminus \{0\}) \sqcup (\pi_{T,00}(a) \cup \{0\}).$$

Therefore, taking into account that $\sigma_n = \pi_{T,00}(a) \cup \{0\} = \{0, \lambda_1, \lambda_2, ..., \lambda_n\}$, and $\sigma'_n = (\sigma_{T,b}(a) \setminus \{0\})$, we obtain that $a = ap_{\sigma_n,a} + ap_{\sigma'_n,a}$, and the element $s_1 = (a + p_{\sigma_n,a})^{-1}(1 - p_{\sigma_n,a})$ is a generalized Drazin T-Riesz inverse of a. Additionally, as $0 \in iso\, \sigma(a)$, in light of Theorem 3.2, we have $s_1 = a^{gD}(1 - p_{\sigma_n,a})$, set $s_2 = a^{gD}$.
Then

$$\begin{aligned}
s_1 - s_2 &= a^{gD}(1 - p_{\sigma_n,a}) - a^{gD} \\
&= a^{gD}\left((1 - p_{\{0\},a}) + (p_{\{0\},a} - p_{\sigma_n,a})\right) - a^{gD} \\
&= a^{gD} + a^{gD}(p_{\{0\},a} - p_{\sigma_n,a}) - a^{gD} \\
&= a^{gD}(p_{\{0\},a} - p_{\sigma_n,a}) \neq 0
\end{aligned}$$

Hence $s_1 \neq s_2$, which is a contradiction.

Case 2. $\pi_{T,00}(a)$ is infinite: If $0 \in acc\,\pi_{T,00}(a)$, then $0 \in acc\,\sigma(a)$. Since a is generalized Drazin T-Riesz invertible, by Theorem 2.5, we conclude that the generalized Drazin T-Riesz inverse of a is not unique, which forms a contradiction. Next, if $0 \notin acc\,\pi_{T,00}(a)$, then we consider the following spectral sets $\sigma_n = \{0, \lambda_1, \lambda_2, ..., \lambda_n\}$, where (λ_i) is a finite family of non-zero T-Riesz points, and $\sigma_n' = (\sigma_{T,b}(a) \setminus \{0\}) \cup ((\pi_{T,00}(a) \cup \{0\}) \setminus \sigma_n)$. Hence we get the desired decompositions by manipulating σ_n and σ_n' as in the case where $\pi_{T,00}(a)$ is finite, and we obtain different generalized Drazin T-Riesz inverses for a, which is a contradiction.

Finally $\sigma(a) = \sigma_{T,b}(a) \cup \{0\}$. Subsequently, a is generalized Drazin T-Riesz invertible, then by [24, Theorem 4.10], $0 \notin acc\,\sigma_{T,b}(a)$ $(= acc\,\sigma(a))$, thus a is generalized Drazin invertible. □

Remark 3.5. *The last theorem is a revised version of [1, Theorem 2.10].*

Notice that if a has a unique Drazin T-Riesz inverse, this is similar to say that the idempotent p related to the Drazin T-Riesz inverse of a is unique and is equal to $p = p_{\{0\},a}$.

We have to search for necessary and sufficient conditions to find the uniqueness of p.

Lemma 3.6. *Let $a \in \mathcal{A}$ be generalized Drazin T-Riesz invertible such that $\sigma(a) = \sigma_{T,b}(a) \cup \{0\}$, and p is the idempotent related to a generalized Drazin T-Riesz inverse of a. Then, there exists $k \in \mathbb{N}$, such that $\forall i \in \{1, ..., k\}$, $\mu_i \in \pi_{T,00}(ap)$, such that*

$$p = p_{\{0\},a} + \sum_{i=1}^{k} p_{\{\mu_i\},ap}, \text{ and } \sigma_{(1-p)\mathcal{A}(1-p)}(a(1-p)) = \sigma_{T,b}(a) \setminus \{0\}.$$

Proof. We have $\sigma(a) = \sigma_{(1-p)\mathcal{A}(1-p)}(a(1-p)) \cup \sigma_{p\mathcal{A}p}(ap)$. As a is generalized Drazin T-Riesz invertible and $\sigma(a) = \sigma_{T,b}(a) \cup \{0\}$, we get $0 \in iso\,\sigma(a)$. Hence $0 \in iso\,\sigma_{p\mathcal{A}p}(ap)$, and taking in account that ap is T-Riesz, we conclude that there is no accumulation point in $\sigma_{p\mathcal{A}p}(ap)$. Hence, there exists $k \in \mathbb{N}$ such that $\forall i \in \{1, ..., k\}$, $\mu_i \in \pi_{T,00}(ap)$ and $\sigma_{p\mathcal{A}p}(ap) = \sigma(ap) = \{0, \mu_1, ..., \mu_k\}$. Since we have $\sigma(a) = \sigma_{T,b}(a) \cup \{0\}$, it follows that $\pi_{T,00}(a) \subset \{0\}$. Hence, $\mu_i \notin \pi_{T,00}(a)$, for all $i \in \{1, ..., k\}$.

Now, if there exists $i_0 \in \{1, ..., k\}$ such that $\mu_{i_0} \notin \sigma_{(1-p)\mathcal{A}(1-p)}(a(1-p))$, then $\mu_{i_0} \in \rho_{(1-p)\mathcal{A}(1-p)}(a(1-p)) \subset \rho_{T,b,(1-p)\mathcal{A}(1-p)}(a(1-p))$, and $\mu_{i_0} \in \rho_{T,b,p\mathcal{A}p}(ap)$, from which we obtain that $\mu_{i_0} \in \rho_{T,b}(a) \cap \sigma(a) = \pi_{T,00}(a)$, which is absurd. Therefore, for each $i \in \{1, ..., k\}$, $\mu_i \in \sigma_{(1-p)\mathcal{A}(1-p)}(a(1-p))$. This implies that $\sigma(a) = \sigma_{(1-p)\mathcal{A}(1-p)}(a(1-p)) \sqcup \{0\}$, hence

$$\sigma_{(1-p)\mathcal{A}(1-p)}(a(1-p)) = \sigma(a) \setminus \{0\} = \sigma_{T,b}(a) \setminus \{0\}.$$

Finally, as $\sigma_{p\mathcal{A}p}(ap) = \{0, \mu_1, \dots, \mu_k\}$, then, it is straightforward to obtain the following

$$p = p_{\{0\},ap} + \sum_{i=1}^{k} p_{\{\mu_i\},ap}.$$

And since we have $0 \in iso\,\sigma(a)$, with $0 \in \sigma_{p\mathcal{A}p}(ap)$ and $0 \notin \sigma_{(1-p)\mathcal{A}(1-p)}(a(1-p))$, we conclude that $p_{\{0\},ap} = p_{\{0\},a}$. Finally we get

$$p = p_{\{0\},a} + \sum_{i=1}^{k} p_{\{\mu_i\},ap}.$$

□

Remarks 3.7. *The following observations will be of wide use in the sequel.*

(1) For each $i \in \{1, ..., k\}$,

$$p_{\{\mu_i\},ap} = \frac{1}{2\pi i} \int_{\Gamma_i} (\lambda - ap)^{-1} p d\lambda.$$

On one hand, since $\sigma_{\mathcal{B}}(T(ap)) = \{0\}$, we conclude that the Cauchy domain $D_i = D(\mu_i, \epsilon_i)$ surrounded by $\Gamma_i = C(\mu_i, \epsilon_i)$ is in the exterior of $\sigma_{\mathcal{B}}(T(ap))$, hence

$$\frac{1}{2\pi i}\int_{\Gamma_i}(\lambda 1_{\mathcal{B}} - T(ap))^{-1}T(p)d\lambda = 0 \text{ for all } i \in \{1, ..., k\}. \tag{3.1}$$

Indeed, consider the Cauchy domain $K_i = D(\mu_i, r_i) \setminus D(\mu_i, \epsilon_i)$ such that $(D(\mu_i, \epsilon_i) \setminus \{\mu_i\}) \cap \sigma(ap) = \emptyset$, and $0 \in D(\mu_i, r_i)$. Hence, $0 \in K_i$. Now, Considering $C_i = C(\mu_i, r_i)^+ \cup C(\mu_i, \epsilon_i)^-$, with $C(\mu_i, \epsilon_i)^- = \Gamma_i^-$, then, on the Banach subalgebra $T(p)\mathcal{B}T(p)$ of $\mathcal{B}$, and in light of functional calculus applied on $T(ap)$ through the constant holomorphic function 1, we have

$$\begin{aligned} T(p) = 1(T(ap)) &= \frac{1}{2\pi i}\int_{C_i}(\lambda 1_{\mathcal{B}} - T(ap))^{-1}T(p)d\lambda \\ &= \frac{1}{2\pi i}\int_{C(\mu_i, r_i)^+}(\lambda 1_{\mathcal{B}} - T(ap))^{-1}T(p)d\lambda \\ &- \frac{1}{2\pi i}\int_{C(\mu_i, \epsilon_i)^+}(\lambda 1_{\mathcal{B}} - T(ap))^{-1}T(p)d\lambda \\ &= T(p) - \int_{C(\mu_i, \epsilon_i)^+}(\lambda 1_{\mathcal{B}} - T(ap))^{-1}T(p)d\lambda, \end{aligned}$$

for all $i \in \{1, ..., k\}$, as desired.

On another hand, following the proof of Proposition 2.1 [11], let $C_{\mathcal{A}}$ and $C_{\mathcal{B}}$ be the respective maximal commutative subalgebras of $\mathcal{A}$ and $\mathcal{B}$ such that $ap \in C_{\mathcal{A}}$, $T(ap) \in C_{\mathcal{B}}$, and $T(C_{\mathcal{A}}) \subset C_{\mathcal{B}}$. If Φ is a multiplicative linear form on $C_{\mathcal{B}}$, then $\Phi \circ T$ is a continuous multiplicative form on $C_{\mathcal{A}}$. Therefore, for every Cauchy domain D_i containing μ_i and surrounded by Γ_i, due to (3.1), we have

$$\begin{aligned} (\Phi \circ T)(p_{\{\mu_i\},ap}) &= \Phi \circ T(\frac{1}{2\pi i}\int_{\Gamma_i}(\lambda - ap)^{-1}pd\lambda) \\ &= \frac{1}{2\pi i}\int_{\Gamma_i}(\lambda - \Phi \circ T(ap))^{-1}\Phi \circ T(p)d\lambda \\ &= \Phi(\frac{1}{2\pi i}\int_{\Gamma_i}(\lambda - T(ap))^{-1}T(p)d\lambda) = \Phi(0) = 0. \end{aligned}$$

Now, as $Tp_{\{\mu_i\},ap}$ is an idempotent and from the calculcation made above, we find that $Tp_{\{\mu_i\},ap}$ is in the radical of $C_{\mathcal{B}}$, this is equivalent by [19, Corollary 2.3.6] to say that

$$\lim_{n\longrightarrow\infty}\|(Tp_{\{\mu_i\},ap})^n\|_{C_{\mathcal{B}}}^{\frac{1}{n}} = \lim_{n\longrightarrow\infty}\|Tp_{\{\mu_i\},ap}\|_{C_{\mathcal{B}}}^{\frac{1}{n}} = 0,$$

where $\|.\|_{C_{\mathcal{B}}}$ is the norm induced by $\|.\|_{\mathcal{B}}$ in the subalgebra $C_{\mathcal{B}}$. Therefore, $Tp_{\{\mu_i\},ap} = 0$. Hence, we conclude for every $i \in \{1, ..., k\}$ that $p_{\{\mu_i\},ap} \in T^{-1}(0)$.

Thus $p - p_{\{0\},a} = \sum_{i=1}^k p_{\{\mu_i\},ap}$ is in $T^{-1}(0)$. Also, $p_{\{\mu_i\},ap} \in comm(a)$ and $1 - p_{\{0\},a} = 1 - p + \sum_{i=1}^k p_{\{\mu_i\},ap}$ with

$$(1 - p)(p - p_{\{0\},a}) = (p - p_{\{0\},a})(1 - p) = 0;$$

(2) We have $\sum_{i=1}^k p_{\{\mu_i\},ap} = p - p_{\{0\},a} \in (1 - p_{\{0\},a})\mathcal{A}(1 - p_{\{0\},a})$. Thus,

$$p - p_{\{0\},a} \in comm(a) \cap (1 - p_{\{0\},a})\mathcal{A}(1 - p_{\{0\},a});$$

(3) $comm(a) \cap (1 - p_{\{0\},a})\mathcal{A}(1 - p_{\{0\},a})$ is both an algebra and a ring with unit $1 - p_{\{0\},a}$.

The next theorem is a starting point to prove when the uniqueness of the generalized Drazin T-Riesz inverse occurs.

Theorem 3.8. *Let $p \in \mathcal{A}$ an idempotent related to a generalized Drazin T-Riesz inverse of a generalized Drazin T-Riesz invertible element $a \in \mathcal{A}$. The following assertions are equivalent:*

(i) p is unique;
(ii) $\sigma(a) = \sigma_{T,b}(a) \cup \{0\}$ and $Idemp(comm(a) \cap (1 - p_{\{0\},a})\mathcal{A}(1 - p_{\{0\},a})) \cap T^{-1}(0) = \{0\}$.

Proof. (i)⇒(ii) Suppose that p is unique, then by Theorem 3.4 $p = 1 - aa^{gD} = p_{\{0\},a}$, because a^{gD} is the unique generalized Drazin T-Riesz inverse of a. For the sake of a contradiction, suppose that there exists $q \in Idemp(comm(a) \cap (1 - p_{\{0\},a})\mathcal{A}(1 - p_{\{0\},a})) \cap T^{-1}(0)$ such that q is proper in $comm(a) \cap (1 - p_{\{0\},a})\mathcal{A}(1 - p_{\{0\},a})$ (different from 0 or $1 - p_{\{0\},a}$).

We have $1 - p_{\{0\},a} = q + (1 - p_{\{0\},a} - q)$. As a and $1 - p_{\{0\},a} - q$ commute, by [8, Lemma 2.1], we have $a(1 - p_{\{0\},a} - q)$ is invertible in $(1 - p_{\{0\},a} - q)\mathcal{A}(1 - p_{\{0\},a} - q)$. Also, by virtue of [3, R.1.2 Theorem], as $ap_{\{0\},a}$ and aq commute and are T-Riesz elements, therefore, $a(p_{\{0\},a} + q)$ is a T-Riesz element. By uniqueness of p we have $p = p_{\{0\},a} + q = p_{\{0\},a}$, which is absurd. Thus, $Idemp(comm(a) \cap (1 - p_{\{0\},a})\mathcal{A}(1 - p_{\{0\},a})) \cap T^{-1}(0) \subset \{0, 1 - p_{\{0\},a}\}$.
Again, seeking for contradiction, suppose that $q = 1 - p_{\{0\},a}$, and $p_{\{0\},a} \neq 1$. In this case, we have $p_{\{0\},a}$ is an idempotent related to the following generalized Drazin T-Riesz inverse

$$(a(1 - p_{\{0\},a}))^{-1}_{(1-p_{\{0\},a})\mathcal{A}(1-p_{\{0\},a})} = a^{gD} \text{ of } a,$$

and $p_{\{0\},a} + 1 - q = 1$ is related to the generalized Drazin T-Riesz inverse $0_{\mathcal{A}}$ of a. As $a = aq + ap_{\{0\},a}$, we conclude that a is T-Riesz which is a contradiction.
Hence, we conclude that $Idemp(comm(a) \cap (1 - p_{\{0\},a})\mathcal{A}(1 - p_{\{0\},a})) \cap T^{-1}(0) = \{0\}$.
(ii)⇒(i) Suppose that (ii) is satisfied. By way of contradiction, suppose also that there exists an idempotent p related to a generalized Drazin T-Riesz inverse of a such that $p \neq p_{\{0\},a}$.

In light of Lemma 3.6, there exists $k \in \mathbb{N}$ such that for every $i \in \{1, ..., k\}$, $\mu_i \in \pi_{T,00}(ap)$, and $p = p_{\{0\},a} + \sum_{i=1}^{k} p_{\{\mu_i\},ap}$. Hence $p - p_{\{0\},a} \in Idemp(comm(a) \cap (1 - p_{\{0\},a})\mathcal{A}(1 - p_{\{0\},a})) \cap T^{-1}(0) = \{0\}$. Therefore, $p - p_{\{0\},a} \in \{0\}$, a contradiction with $p \neq p_{\{0\},a}$. Finally, we conclude that $p = p_{\{0\},a}$, as desired. □

Remark 3.9. *As $comm(a) \cap (1 - p_{\{0\},a})\mathcal{A}(1 - p_{\{0\},a}) \cap T^{-1}(0)$ is an algebra, it may have a unit, let us denote it by e when it exists. But in the case where $a \in \mathcal{A}$ is generalized Drazin T-Riesz invertible and has a unique idempotent $p = p_{\{0\},a}$ which is related to its generalized Drazin T-Riesz inverse a^{gD}, we show that $comm(a) \cap (1 - p_{\{0\},a})\mathcal{A}(1 - p_{\{0\},a}) \cap T^{-1}(0)$ does not have a unit e. Indeed, suppose that p is unique, and for the sake of a contradiction, suppose also that $comm(a) \cap (1 - p_{\{0\},a})\mathcal{A}(1 - p_{\{0\},a}) \cap T^{-1}(0)$ has a unit e. We have $1 - p_{\{0\},a} = (1 - e - p_{\{0\},a}) + e$, it is easy to verify that $(1 - e - p_{\{0\},a})$ is an idempotent. Thus by Lemma 2.1 [8], as $a(1 - p_{\{0\},a})$ is invertible, e and $1 - p_{\{0\},a} - e$ commute with a and are orthogonal, therefore, ae and $a(1 - p_{\{0\},a} - e)$ are invertible each in their respective subalgebras $e\mathcal{A}e$ and $(1 - p_{\{0\},a} - e)\mathcal{A}(1 - p_{\{0\},a} - e)$. Also, as $e \in T^{-1}(0) \cap comm(a)$, therefore ae is a T-Riesz element in $\mathcal{A}$. Thus, by uniqueness of p we have $e = p = p_{\{0\},a}$, a contradiction, due to $e \in (1 - p_{\{0\},a})\mathcal{A}(1 - p_{\{0\},a})$, and $p_{\{0\},a} \neq 0$ taking into accoun that $0 \in iso\,\sigma(a)$. The same reasoning can be done on the algebra generated by $Idemp(comm(a) \cap (1 - p_{\{0\},a})\mathcal{A}(1 - p_{\{0\},a}) \cap T^{-1}(0))$ to deduce that this algebra does not have a unit in the case where $a \in \mathcal{A}$ is generalized Drazin T-Riesz invertible and has a unique idempotent $p = p_{\{0\},a}$ related to its generalized Drazin inverse a^{gD}.*

The next theorem describes when a has a unique generalized Drazin T-Riesz inverse.

Theorem 3.10. *Let $a \in \mathcal{A}$ be generalized Drazin T-Riesz invertible such that $0 \in \sigma(a)$. The following statements are equivalent:*

(i) a has a unique generalized Drazin T-Riesz inverse;
(ii) $0 \in iso\,\sigma(a)$ and $Idemp(comm(a) \cap (1 - p_{\{0\},a})\mathcal{A}(1 - p_{\{0\},a})) \cap T^{-1}(0) = \{0\}$;
(iii) $\sigma(a) = \sigma_{T,b}(a) \cup \{0\}$ and $Idemp(comm(a) \cap (1 - p_{\{0\},a})\mathcal{A}(1 - p_{\{0\},a})) \cap T^{-1}(0) = \{0\}$.

Proof. (i)⇒(ii). Since a has a unique generalized Drazin T-Riesz inverse, and in light of Theorem 3.4, we get $\sigma(a) = \sigma_{T,b}(a) \cup \{0\}$. As $0 \in iso\,\sigma_{T,b}(a)$, therefore $0 \in iso\,\sigma(a)$. Also, the idempotent p related to the generalized Drazin T-Riesz inverse of a is unique, because there is a unique generalized Drazin T-Riesz inverse of a

which is the generalized Drazin inverse a^{gD}. As $\sigma(a) = \sigma_{T,b}(a) \cup \{0\}$ and a has a unique Drazin T-Riesz inverse, the only idempotent related to a generalized Drazin T-Riesz inverse of a is $p_{\{0\},a}$. We have by Theorem 3.8 that

$$Idemp(comm(a) \cap (1 - p_{\{0\},a})\mathcal{A}(1 - p_{\{0\},a})) \cap T^{-1}(0) = \{0\}.$$

(ii)⇒(i). Suppose that (ii) is satisfied. As $0 \in iso\,\sigma(a)$ then $a = a(1 - p_{\{0\},a}) + ap_{\{0\},a}$, and

$$(a(1 - p_{\{0\},a}))^{-1}_{(1-p_{\{0\},a})\mathcal{A}(1-p_{\{0\},a})} = a^{gD}.$$

Seeking for a contradiction, suppose that there exists $p \in \mathcal{A}$ which is related to a generalized Drazin T-Riesz inverse of a and different from $p_{\{0\},a}$.

We have $\sigma(ap) = \{0, \lambda_1, ..., \lambda_n\}$ (otherwise, $0 \in acc\,\sigma(ap)$, hence $0 \in acc\,\sigma(a)$ and this is absurd with $0 \in iso\,\sigma(a)$). Thus $p = p_{\{0\},a} + \sum_{i=1}^{n} p_{\{\lambda_i\},ap}$ and for every $i \in \{1, ..., n\}$, $p_{\{\lambda_i\},ap} \in T^{-1}(0)$, hence $p - p_{\{0\},a} \in T^{-1}(0)$. On the other hand, we have $p_{\{0\},a}p = pp_{\{0\},a} = p_{\{0\},a}$ and $(1 - p_{\{0\},a})(1 - p) = (1 - p)(1 - p_{\{0\},a}) = 1 - p$, which implies $(p - p_{\{0\},a})(1 - p) = (1 - p)(p - p_{\{0\},a}) = 0$ with $(p - p_{\{0\},a})$ is an idempotent.
Thus $p - p_{\{0\},a} \in Idemp(comm(a) \cap (1 - p_{\{0\},a})\mathcal{A}(1 - p_{\{0\},a})) \cap T^{-1}(0) = \{0\}$, a contradiction. Hence, the only idempotent related to a generalized Drazin T-Riesz inverse of a is $p_{\{0\},a}$. Consequently, the only generalized Drazin T-Riesz inverse is a^{gD}.
(iii)⇔(i). Suppose that $\sigma(a) = \sigma_{T,b}(a) \cup \{0\}$ and $Idemp(comm(a) \cap (1 - p_{\{0\},a})\mathcal{A}(1 - p_{\{0\},a})) \cap T^{-1}(0) = \{0\}$ together hold with the initial condition that a is generalized Drazin T-Riesz invertible, this is equivalent to say that a possesses a unique idempotent p related to the generalized Drazin T-Riesz inverses of a by Theorem 3.8, and having a unique idempotent p related to the generalized Drazin T-Riesz inverse of a is equivalent with a having a unique generalized Drazin T-Riesz inverse. As desired. □

Corollary 3.11. *Let $\mathcal{A}$ be an infinite dimensional Banach algebra with a one-to-one homomorphism T, and let $a \in \mathcal{A}$. The following assertions are equivalent*

(i) a is generalized Drazin invertible;
(ii) a has a unique generalized Drazin T-Riesz inverse.

Proof. Either a is generalized Drazin invertible or a has a unique generalized Drazin T-Riesz inverse, the spectral idempotent $p_{\{0\},a}$ of a related to 0 exists. Also, as T is one-to-one, therefore $T^{-1}(0) = \{0\}$. Hence

$$Idemp(comm(a) \cap (1 - p_{\{0\},a})\mathcal{A}(1 - p_{\{0\},a})) \cap T^{-1}(0) = \{0\},$$

we apply Theorem 3.10 to find the desired equivalence between (*i*) and (*ii*). □

Remark 3.12. *We shall say that a unital algebra E is strongly connected if the only idempotents lying in E are* 0 *and* 1.

Corollary 3.13. *Let $\mathcal{A}$ be an infinite dimensional unital complex Banach algebra. Let a be generalized Drazin T-Riesz invertible in $\mathcal{A}$, with $comm(a) \cap (1 - p_{\{0\},a})\mathcal{A}(1 - p_{\{0\},a})$ is a commutative and strongly connected Banach algebra, and $(1 - p_{\{0\},a}) \notin T^{-1}(0)$. The following assertions are equivalent:*

(i) a has a unique generalized Drazin T-Riesz inverse;
(ii) a is generalized Drazin invertible;
(iii) $\sigma(a) = \sigma_{T,b}(a) \cup \{0\}$.

Proof. As $(1 - p_{\{0\},a}) \notin T^{-1}(0)$ and $Q = comm(a) \cap (1 - p_{\{0\},a})\mathcal{A}(1 - p_{\{0\},a})$ is a commutative and connected Banach algebra, therefore the only idempotents lying in Q are 0 and $(1 - p_{\{0\},a})$, hence the only idempotent lying in $Q \cap T^{-1}(0)$ is 0. Finally, in order to find all the equivalences between (i), (ii) and (iii), we apply Theorem 3.10. □

For a generalized Drazin invertible element $a \in \mathcal{A}$, we consider the algebra $\mathcal{Z}$ generated by

$$Q_1 = Idemp(comm(a) \cap (1 - p_{\{0\},a})\mathcal{A}(1 - p_{\{0\},a}) \cap T^{-1}(0)).$$

$\mathcal{Z}$ is not unital in our case (if so, the generalized Drazin T-Riesz inverse of a will not be unique as shown in Remark 3.9), so we have to consider $\mathcal{Z}^\sharp = \mathcal{Z} \times \mathbb{C}$ with its respective laws for all $x, y \in \mathcal{Z}$, and for all $\alpha, \beta, \lambda \in \mathbb{C}$ such that

$$\begin{cases} (x, \alpha) + \lambda(y, \beta) = (x + \lambda y, \alpha + \lambda\beta), \\ (x, \alpha).(y, \beta) = (\alpha y + \beta x + xy, \alpha\beta). \end{cases}$$

$\mathcal{Z}$ is naturally embedded in $\mathcal{Z}^\sharp$, also $\overline{\mathcal{Z}}^\sharp = \overline{\mathcal{Z}} \times \mathbb{C}$ is a complex unital Banach algebra.

Theorem 3.14. *Let a be a generalized Drazin invertible element in $\mathcal{A}$. The following assertions are equivalent:*

(i) a has a unique generalized Drazin T-Riesz inverse;

(ii) $\overline{\mathcal{Z}} = \{0\}$;

(iii) $Char(\overline{\mathcal{Z}}^\sharp)$ is connected (i.e: does not split into two disjoint closed sets under the weak topology $\sigma((\overline{\mathcal{Z}}^\sharp)', \overline{\mathcal{Z}}^\sharp)$);

(iv) $\overline{\mathcal{Z}}^\sharp$ is strongly connected (i.e: The only idempotents lying in $\overline{\mathcal{Z}}^\sharp$ are $(0, 0)$ and $(0, 1)$).

Proof. (i)⇒(ii). a has a unique generalized Drazin T-Riesz inverse implies by Theorem 3.8 that $Q_1 = \{0\}$, hence $\mathcal{Z} = \{0\}$, therefore $\overline{\mathcal{Z}} = \{0\}$.
(ii)⇒(iii). As $\overline{\mathcal{Z}} = \{0\}$, therefore $Char(\overline{\mathcal{Z}}^\sharp)$ is connected, by virtue of the negation of [12, (1) ⇔ (5) Theorem 3.12].
(iii)⇒(iv). By the negation of [12, (3) ⇒ (6) Theorem 3.12], we find the desired result.
(iv)⇒(i). As $(0, 0)$ and $(0, 1)$ are the only idempotents lying in $\overline{\mathcal{Z}}^\sharp$, therefore the only idempotent lying in $\mathcal{Z}$ is 0. Hence $Q_1 = \{0\}$. So, by Theorem 3.10, we conclude that a has a unique generalized Drazin T-Riesz inverse. □

Corollary 3.15. Let $\mathcal{A}$ be a Banach algebra with a homomorphism T having the strong Riesz property. Let a be generalized Drazin invertible in $\mathcal{A}$, $(1-p_{\{0\},a})\mathcal{A}(1-p_{\{0\},a})$ is a Banach space with $Char((1-p_{\{0\},a})\mathcal{A}(1-p_{\{0\},a}))$ is connected, and $1 - p_{\{0\},a} \notin T^{-1}(0)$. Then we have

(i) a has a unique generalized Drazin T-Riesz inverse.

(ii) $Char(\overline{\mathcal{Z}}^\sharp)$ is connected.

Moreover, (i) and (ii) are equivalent. Hence, under these conditions, every generalized Drazin invertible element in $\mathcal{A}$ has a unique generalized Drazin-Riesz inverse which is its generalized Drazin inverse.

Proof. Since $Char((1-p_{\{0\},a})\mathcal{A}(1-p_{\{0\},a}))$ is connected and $(1-p_{\{0\},a})\mathcal{A}(1-p_{\{0\}})$ is a Banach algebra, by applying the negation of the implication (1) ⇒ (3) [12, Theorem 3.12], we find that $(1 - p_{\{0\},a})\mathcal{A}(1 - p_{\{0\},a})$ is connected in the sense that the only idempotents are 0 and $(1 - p_{\{0\},a})$. As $1 - p_{\{0\},a} \notin T^{-1}(0)$, we conclude that $Q_1 = \{0\}$, hence $\overline{\mathcal{Z}} = \{0\}$. Therefore by applying Theorem 3.14 we find that a has a unique generalized Drazin T-Riesz inverse.

The equivalence between (i) and (ii) is also found by applying Theorem 3.14.
Hence, under the conditions of this corollary, every generalized Drazin invertible element has a unique generalized Drazin T-Riesz inverse which is of course its generalized Drazin inverse. □

4. The case of semi-simple Banach algebras

This section is meant to show when the uniqueness of generalized Drazin-Riesz inverses holds in the case of semisimple Banach algebras, using both algebraic geometry tools and functional analysis (topological) tools. This will allow us to also construct non trivial examples.
Recall the following algebraic geometry notions.

Let $\mathcal{R}$ be an associative unital ring and $Spec(\mathcal{R})$ be the set of all prime ideals of $\mathcal{R}$, recall that a proper (two-sided) ideal P is a *prime ideal* of $\mathcal{R}$ if for all $a, b \in \mathcal{R}$, $a\mathcal{R}b \subset P$, implies that $a \in P$ or $b \in P$. We say that a commutative ring $\mathcal{R}$ is *connected* if its spectrum is connected under the Zariski topology [21].

An idempotent e in $\mathcal{R}$ is said to be clopen if $e \notin P$ implies $1 - e \in P$ for any prime ideal P of $\mathcal{R}$. In the case where $\mathcal{R}$ is a commutative ring, every idempotent in $\mathcal{R}$ is a clopen idempotent [21].

In order to characterize the connectedness in an algebraic sense of a given associative ring, we shall use the next theorem.

Theorem 4.1. *[21, Theorem 3.11] Let $\mathcal{R}$ be an associative unital ring. The following statements are equivalent:*

1. *$\mathcal{R}$ is connected;*
2. *The only clopen idempotents in $\mathcal{R}$ are 0 and 1.*

Remark 4.2. *Theorem 4.1 allows us to define the connectedness of a given unital algebra E by saying that the only clopen idempotents lying in E are 0 and 1, hence we do not need the Zariski Topology to define the connectedness of E.*

The next theorem will enable us to construct several examples in the case of semi-simple Banach algebras.

Theorem 4.3. *Let $\mathcal{A}$ be an infinite dimensional unital complex semi-simple Banach algebra. Let a be generalized Drazin-Riesz invertible in $\mathcal{A}$, with $Q = comm(a) \cap (1 - p_{\{0\},a})\mathcal{A}(1 - p_{\{0\},a})$ is an infinite dimensional commutative and connected Banach algebra. The following assertions are equivalent:*

(i) a has a unique generalized Drazin-Riesz inverse;
(ii) a is generalized Drazin invertible;
(iii) $\sigma(a) = \sigma_b(a) \cup \{0\}$.

Proof. As in the proof of Corollary 3.13, we conclude that Q has only 0 and $(1 - p_{\{0\},a})$ as idempotents, as Q is an infinite dimensional Banach algebra, $1 - p_{\{0\},a}$ is not therefore a minimal idempotent of $\mathcal{A}$ (otherwise $(1 - p_{\{0\},a})\mathcal{A}(1 - p_{\{0\},a}) = \mathbb{C}(1 - p_{\{0\},a})$, absurd). Hence

$$Q \cap S_{\mathcal{A}} = \{0\}.$$

Finally, taking into account that $T = \pi$, we apply Theorem 3.10 to find the desired equivalences. □

Remark 4.4. *Staying in the case of semi-simplicity, we can say that if a is generalized Drazin-Riesz invertible and has a unique generalized Drazin-Riesz inverse, this is equivalent to say that the only idempotent of finite rank lying in $(1 - p_{\{0\},a})\mathcal{A}(1 - p_{\{0\},a})$ which commute with a is 0. This is similar to have $S_{comm(a)\cap(1-p_{\{0\},a})\mathcal{A}(1-p_{\{0\},a})} = \{0\}$, since $S_{comm(a)\cap(1-p_{\{0\},a})\mathcal{A}(1-p_{\{0\},a})}$ is formed by the minimal idempotents of $comm(a) \cap (1 - p_{\{0\},a})\mathcal{A}(1 - p_{\{0\},a})$ which form the idempotents of finite rank that are in $(1 - p_{\{0\},a})\mathcal{A}(1 - p_{\{0\},a})$ and that commute with a. Hence for X an infinite dimensional Banach space, if $A \in \mathcal{L}(X)$ has a unique generalized Drazin-Riesz inverse, this is equivalent to say that the only finite rank operator on $comm(A) \cap (I - P_{\{0\},A})\mathcal{L}(X)(I - P_{\{0\},A})$ is 0, where $P_{\{0\},A}$ is the spectral projection of A related to $\{0\}$.*

The last remark combined with the negation of Theorem 3.10 allow us to state when a generalized Drazin invertible element does have at least two generalized Drazin-Riesz invertible elements, notice that in the case where an element a is not generalized Drazin invertible, $0 \in acc\,\sigma(a)$, and a is generalized Drazin-Riesz invertible, then by virtue of Theorem 2.5, a has infinite generalized Drazin-Riesz inverses. We say that an element $f \in \mathcal{A}$ is a finite rank element if $f \in S_{\mathcal{A}}$.

Theorem 4.5. *Let $\mathcal{A}$ be an infinite dimensional complex unital semi-simple Banach algebra, and $a \in \mathcal{A}$ such that $0 \in iso\,\sigma(a)$. The following statements are equivalent:*

(i) a has at least a generalized Drazin-Riesz inverse that is different from the generalized Drazin inverse of a;
(ii) there exists a finite rank element $f \in (1 - p_{\{0\},a})\mathcal{A}(1 - p_{\{0\},a})$ such that $af = fa \neq 0$;

Proof. (i)⇒(ii). Let a be generalized Drazin invertible having b as a generalized Drazin-Riesz inverse which is different from a^{gD}.
Hence, there exists an idempotent $p \in \mathcal{A}$ different from $p_{\{0\},a}$ related with b.
We have

$$\begin{aligned}\sigma(a) &= \sigma_{(1-p_{\{0\},a})\mathcal{A}(1-p_{\{0\},a})}(a(1 - p_{\{0\},a})) \cup \sigma_{p_{\{0\},a}\mathcal{A}p_{\{0\},a}}(ap_{\{0\},a})\\ &= \sigma_{(1-p)\mathcal{A}(1-p)}(a(1 - p)) \cup \sigma_{p\mathcal{A}p}(ap),\end{aligned}$$

where $p = p_{\{0\},a} + q$, with $q \in S_{\mathcal{A}}$ such that $q^2 = q \neq 0$, $aq = qa \neq 0$, and $p_{\{0\},a}q = qp_{\{0\},a} = 0$.

This implies that $q \in (1 - p_{\{0\},a})\mathcal{A}(1 - p_{\{0\},a}) \cap S_{\mathcal{A}} \cap comm(a)$, q is then a finite rank element that commutes with a and $aq = qa \neq 0$.
(ii)⇒(i). Now suppose that there exists a finite rank element $f \in (1 - p_{\{0\},a})\mathcal{A}(1 - p_{\{0\},a}) \setminus \{0\}$ such that $af = fa \neq 0$.
As $f \in S_{\mathcal{A}} \cap (1 - p_{\{0\},a})\mathcal{A}(1 - p_{\{0\},a})$, there exists minimal idempotents $q_1, ..., q_n \in (1 - p_{\{0\},a})\mathcal{A}(1 - p_{\{0\},a})$ such that $f = \sum_1^n \lambda_i q_i$.
Hence, as $af = fa \neq 0$, there exists $k \in \{1, ..., n\}$ such that $aq_k = q_k a \neq 0$.
Thus by the minimality of q_k and $aq_k = q_k a \neq 0$, there exists $\lambda_k \in \mathbb{C} \setminus \{0\}$ such that $aq_k = q_k a = \lambda_k q_k$.
We put $p = p_{\{0\},a} + q_k$ to find that $ap = pa$, ap is a Riesz element, and by Lemma 2.1 [8], $a(1 - p)$ is invertible in $(1 - p)\mathcal{A}(1 - p)$.

Finally, we have $0 \in iso\,\sigma(a)$ and $Idemp(comm(a) \cap (1 - p_{\{0\},a})\mathcal{A}(1 - p_{\{0\},a})) \cap S_{\mathcal{A}} \neq \{0\}$, and by applying the negation of Theorem 3.10, we find that a has at least two different generalized Drazin-Riesz inverses, hence a^{gD} is not the only generalized Drazin-Riesz inverse for a. □

The last theorem gives a practical way to see if a does not have a unique generalized Drazin-Riesz inverse, it suffices to find one finite rank element f in $(1 - p_{\{0\},a})\mathcal{A}(1 - p_{\{0\},a})$ such that $af = fa \neq 0$.

Example 4.6. *Let a be a quasinilpotent element in $\mathcal{A}$. a has a unique generalized Drazin-Riesz inverse which is 0. We have $p_{\{0\},a} = 1_{\mathcal{A}}$, thus $1 - p_{\{0\},a} = 0$, hence $comm(a) \cap (1 - p_{\{0\},a})\mathcal{A}(1 - p_{\{0\},a}) \cap S_{\mathcal{A}} = \{0\}$, with $\sigma(a) = \{0\}$, as desired.*

We consider the algebra (it is also a ring) $R = comm(a) \cap (1 - p_{\{0\},a})\mathcal{A}(1 - p_{\{0\},a}) \cap S_{\mathcal{A}}$, notice that $\mathcal{Z} = R$ in this case, because $T = \pi$, and hence, $S_{\mathcal{A}} = T^{-1}(0)$.
Therefore, R does not have a unit. Then, we consider the unitization $R^\sharp = R \times \mathbb{C}$ of R as defined in the last part of Section 3. Notice that $\overline{R}^\sharp = \overline{R} \times \mathbb{C}$ is a complex semi-simple unital Banach algebra.
We can confuse every idempotent $p \in R$ with $(p, 0) \in R^\sharp$ ($(p, 0).(p, 0) = (p, 0)$). Also if the only clopen idempotents of $R^\sharp$ are $(0, 0)$ and $(0, 1)$, then the only clopen idempotent lying in R is 0.

The next result shows that for every minimal idempotent e of $\mathcal{A}$ in R, the idempotent $(e, 0)$ is clopen in $R^\sharp$.

Lemma 4.7. *Let $e \in \mathcal{A}$ be a minimal idempotent such that $e \in R$. Then $(e, 0)$ is a clopen idempotent in $R^\sharp$.*

Proof. Let $(x, \alpha) \in R^\sharp$, and $e \in R$ such that e is a minimal idempotent of $\mathcal{A}$. Consider $(e, 0) \notin P^\sharp$ for an arbitrary two-sided prime ideal $P^\sharp$ of $R^\sharp$.

We have

$$\begin{aligned}(e, 0).(x, \alpha).(-e, 1) &= (e, 0).(-\alpha e + x - xe, \alpha)\\ &= (\alpha e - \alpha e + ex - exe, 0)\\ &= (ex - exe, 0)\end{aligned}$$

As $x \in S_{\mathcal{A}}$, there exists $\lambda_i \in \mathbb{C}$ and e_i minimal idempotents such that $x = \sum_{i=1}^{n} \lambda_i e_i$, hence by minimality of e and e_i, as well as $e_i e_j = 0$ for $i \neq j$, it is easy to conclude that $(e, 0).(x, \alpha).(-e, 1) = (0, 0) \in P^\sharp$. As $P^\sharp$ is a two sided prime ideal of $R^\sharp$ and $(e, 0) \notin P^\sharp$, then $(-e, 1) \in P^\sharp$. Hence $(e, 0)$ is a clopen idempotent of $R^\sharp$. □

The last lemma allows us to state that if $R^\sharp$ is connected, this is equivalent to say, using Theorem 4.1, that the only clopen idempotents of $R^\sharp$ are $(0, 0)$ and $(0, 1)$. Consequently, the only minimal idempotent of $\mathcal{A}$ in R is 0.
Thus $Idemp(R) = \{0\}$ (this is equivalent to say that $R^\sharp$ is connected in the sense that the only idempotents of $R^\sharp$ are $(0, 0)$ and $(0, 1)$) because $R \subset S_{\mathcal{A}}$. The latter will enable us to give the next characterizations about the uniqueness of generalized Drazin-Riesz inverse by mean of connectedness in the algebraic geometry sense characterized in Theorem 4.1.

Theorem 4.8. *Let $\mathcal{A}$ be an infinite dimensional unital complex semi-simple Banach algebra. Let a be generalized Drazin-Riesz invertible in $\mathcal{A}$ with $0 \in \sigma(a)$. The following assertions are equivalent:*

(i) a has a unique generalized Drazin-Riesz inverse;
(ii) a is generalized Drazin invertible and $R^\sharp$ is connected;
(iii) $\sigma(a) = \sigma_b(a) \cup \{0\}$ and $R^\sharp$ is connected;
(iv) $\sigma(a) = \sigma_b(a) \cup \{0\}$ and $\overline{R} = \{0\}$;
(v) a is generalized Drazin invertible and $\overline{R} = \{0\}$.

Proof. (i)⇒(ii). as a has a unique generalized Drazin-Riesz inverse, then a is generalized Drazin invertible and by Theorem 3.8 $Idemp(R) = \{0\}$, therefore, $R^\sharp$ is connected since the only idempotents of $R^\sharp$ are $(0, 0)$ and $(0, 1)$.
(ii)⇒(iii). As a is generalized Drazin invertible and $R^\sharp$ is connected, then a is generalized Drazin invertible and $Idemp(R) = \{0\}$. Thus, we have by Theorem 3.10 $\sigma(a) = \sigma_b(a) \cup \{0\}$ and $Idemp(R) = \{0\}$, which is equivalent to say that $\sigma(a) = \sigma_b(a) \cup \{0\}$ and $R^\sharp$ is connected.
(iii)⇒(iv). Since $R^\sharp$ is connected, and since all the idempotents of R are a sum of minimal idempotents which are clopen by Lemma 4.7, we conclude that the only idempotents of $R^\sharp$ are $0_{R^\sharp} = (0, 0)$ and $1_{R^\sharp} = (0, 1)$. Hence the only idempotent of R is 0, therefore $R = \{0\}$, because every element of R is a finite linear combination of minimal idempotents. Finally, $\overline{R} = \{0\}$.
(iv)⇒(v). It suffices to see that $0 \notin acc\,\sigma_b(a)$, because a is generalized Drazin-Riesz invertible, hence $0 \in iso\,\sigma(a)$, because $\sigma(a) = \sigma_b(a) \cup \{0\}$, and by hypotheses of (iv), $\overline{R} = \{0\}$.
(v)⇒(i). As a is generalized Drazin invertible and $\overline{R} = \{0\}$, we get that a is generalized Drazin invertible and $Idemp(comm(a) \cap (1 - p_{\{0\},a})\mathcal{A}(1 - p_{\{0\},a})) \cap S_{\mathcal{A}} = \{0\}$. Hence, by virtue of Theorem 3.10 we conclude that a has a unique generalized Drazin-Riesz inverse. □

The next Theorem gives characterizations of the uniqueness of generalized Drazin T-Riesz inverses following notions that are topological using the connectedness of the set of Characters.

Theorem 4.9. *Let $\mathcal{A}$ be an infinite dimensional unital complex semi-simple Banach algebra. Let a be generalized Drazin-Riesz invertible in $\mathcal{A}$ with $0 \in \sigma(a)$. The following assertions are equivalent:*

(i) a has a unique generalized Drazin-Riesz inverse;
(ii) a is generalized Drazin invertible and $Char(\overline{R}^\sharp)$ is connected (i.e: does not split into two disjoint closed sets under the weak topology $\sigma((\overline{R}^\sharp)', \overline{R}^\sharp)$);
(iii) a is generalized Drazin invertible and $(\overline{R}^\sharp)^\wedge$ is strongly connected;
(iv) $\sigma(a) = \sigma_b(a) \cup \{0\}$ and $(\overline{R}^\sharp)^\wedge$ is strongly connected.

Proof. (i) ⇒ (ii). By the equivalence (i) ⇔ (v) of Theorem 4.8, we have $\overline{R} = \{0\}$, hence $\overline{R}^\sharp = \{0\} \times \mathbb{C}$ is a semi-simple unital and finite dimensional algebra, therefore its Gelfand map $\mathcal{G}_{\overline{R}^\sharp}$ is continuous (because

$\overline{R}^\sharp$ is finite dimensional, also it is a unital semisimple Banach algebra and this is equivalent to say that $\mathcal{G}_{\overline{R}^\sharp}$ is one-to-one, for more details see [16, Definitions 3.2 and 6.2, Corollary 7.3]), and its Gelfand transform algebra $(\overline{R}^\sharp)^\wedge$ is complete ($(\overline{R}^\sharp)^\wedge \subset (\overline{R}^\sharp)'$ and $(\overline{R}^\sharp)'$ is finite dimensional, hence $(\overline{R}^\sharp)^\wedge$ is finite dimensional, thus complete), we are then able to apply the negation of [12, (1) $\Leftrightarrow$ (5) Corollary 3.14] to find that $Char(\overline{R}^\sharp)$ is connected.
(ii)$\Rightarrow$(iii). We apply on $Char(\overline{R}^\sharp)$ the negation of the implication (3) $\Rightarrow$ (6) [12, Theorem 3.12] to find that $(\overline{R}^\sharp)^\wedge$ is strongly connected.
(iii)$\Rightarrow$(iv). As $0 \in \sigma(a)$ and a is generalized Drazin invertible, then $0 \in iso\,\sigma(a)$, also by assumptions of (iii), $(\overline{R}^\sharp)^\wedge$ is connected, and by the negation of the implications [12, (1) $\Rightarrow$ (3) $\Leftrightarrow$ (4) Theorem 3.12], we conclude that $\overline{R}^\sharp$ is strongly connected, which implies that $Idemp(R) = \{0\}$. Now, as $0 \in iso\,\sigma(a)$ and $Idemp(R) = \{0\}$, we conclude by Theorem 3.10 that $\sigma(a) = \sigma_b(a) \cup \{0\}$, and by assumptions of (iii), we already have that a is generalized Drazin invertible.
(iv)$\Rightarrow$(i). $(\overline{R}^\sharp)^\wedge$ is strongly connected implies through the negation of the implication (1) $\Rightarrow$ (3) of [12, Theorem 3.12] that $\overline{R}^\sharp$ is strongly connected. This is equivalent to say that $Idemp(\overline{R}) = \{0\}$, hence $Idemp(R) = \{0\}$ (R does not possess a unit and $(0,0)$ is identified with 0), combined with $\sigma(a) = \sigma_b(a) \cup \{0\}$, we get by applying Theorem 3.10 that a has a unique generalized Drazin-Riesz inverse. □

The importance of having these equivalences considering the connectedness of $Char(\overline{R}^\sharp)$ lies in the next corollary.

Corollary 4.10. *Let $\mathcal{A}$ be a semi-simple Banach algebra. Let a be generalized Drazin-Riesz invertible in $\mathcal{A}$ such that $0 \in iso\,\sigma(a)$, and $(1-p_{\{0\},a})\mathcal{A}(1-p_{\{0\},a})$ is an infinite dimensional Banach space with $Char((1-p_{\{0\},a})\mathcal{A}(1-p_{\{0\},a}))$ is connected. Then we have*

(i) a has a unique generalized Drazin-Riesz inverse.

(ii) $Char(\overline{R}^\sharp)$ is connected.

Moreover, (i) and (ii) are equivalent. Also, every generalized Drazin-Riesz invertible element in $\mathcal{A}$ has a unique generalized Drazin-Riesz inverse which is its generalized Drazin inverse.

Proof. Since $Char((1-p_{\{0\},a})\mathcal{A}(1-p_{\{0\},a}))$ is connected and $(1-p_{\{0\},a})\mathcal{A}(1-p_{\{0\},a})$ is a Banach algebra, by applying the negation of the implication (1) $\Rightarrow$ (3) [12, Theorem 3.12], we find that $(1-p_{\{0\},a})\mathcal{A}(1-p_{\{0\},a})$ in the sense that the only idempotents are 0 and $(1-p_{\{0\},a})$, as $(1-p_{\{0\},a})\mathcal{A}(1-p_{\{0\},a})$ is infinite dimensional, then $1-p_{\{0\},a} \notin S_{\mathcal{A}}$, hence $Idemp(R) = \{0\}$, hence $R^\sharp$ is connected, therefore by applying Theorem 4.8 we find that a has a unique generalized Drazin-Riesz inverse.

The equivalence between (*i*) and (*ii*) is also found by applying Theorem 4.9. Hence, in this case, every generalized Drazin-Riesz invertible element has a unique generalized Drazin-Riesz inverse which is of course its generalized Drazin inverse. □

Now, we propose some examples in the case of semi-simple Banach algebras that are generated by Theorem 4.3.

Example 4.11. *Consider the commutative semi-simple Banach algebra $Ea[-1,1]$ called extremal algebra [5, Section 24 p. 53]. We have $char(Ea[-1,1]) = [-1,1]$ (see [10]), hence it is connected, therefore the only idempotents of $Ea[-1,1]$ are 0 and $1_{Ea[-1,1]}$.*
Now consider the semi-simple Banach algebra $\mathcal{A} = Ea[-1,1] \oplus \mathcal{L}(\ell^2(\mathbb{N}))$.
Take $a = 1_{Ea[-1,1]} \oplus B$ such that

$$B(x_1, x_2, x_3, \dots) = (0, \frac{x_1}{2}, \frac{x_2}{2^2}, \frac{x_3}{2^3}, \dots),\ \forall (x_1, x_2, x_3, \dots) \in \ell^2(\mathbb{N}).$$

a is generalized Drazin invertible, because $1_{Ea[-1,1]}$ is invertible in $Ea[-1,1]$ and B is quasinilpotent in $\mathcal{L}(\ell^2(\mathbb{N}))$, thus it is generalized Drazin-Riesz invertible.

As the only idempotents of $Ea[-1,1]$ are 0 and $1_{Ea[-1,1]}$ and $\sigma(a) = \{0;1\} = \sigma_{T,b}(a) \cup \{0\}$, also, $1_{Ea[-1,1]} \notin S_{\mathcal{A}}$, because $Ea[-1,1]$ is infinite dimensional, we have $Idemp(Ea[-1,1]) \cap S_{\mathcal{A}} = \{0\}$. Consequently, a has a unique generalized Drazin-Riesz inverse which is $b = 1_{Ea[-1,1]} \oplus 0$.

Example 4.12. *We consider the infinite dimensional semi-simple commutative Banach algebra $C([-1,1])$ of all continuous functions on $[-1,1]$ over $\mathbb{C}$, by [16, Corollary 1.2 p. 221], $Char(C([-1,1])) = [-1,1]$, thus $Char(C[-1,1])$ is connected, as $C([-1,1])$ is a commutative algebra, then for every $a \in C([-1,1])$, $comm(a) \cap C([-1,1]) = C([-1,1])$, $C([-1,1])$ being infinite dimensional, we conclude that $1_{C([-1,1])}$ does not belong to $S_{C([-1,1])}$, hence the only idempotent lying in $S_{C([-1,1])}$ is 0, therefore $S_{C([-1,1])} = \{0\}$ (this is already known, but we give a new demonstration using the tools we have employed in the last theorem). Consider B the operator defined in the last example in $\mathcal{L}(\ell^1(\mathbb{N}))$, and consider $\mathcal{A} = C([-1,1]) \oplus \mathcal{L}(\ell^1(\mathbb{N}))$, and $a = f \oplus B$ where $\forall x \in [-1,1]$, $f(x) \neq 0$, a is generalized Drazin-Riesz invertible, in particular it is generalized Drazin invertible.*
Thus, by virtue of Theorem 4.8 considering that $S_{C([-1,1])} = \{0\}$ or by virtue of Theorem 4.9 taking into account that $Char(C[-1,1])$ is connected, we have a has a unique generalized Drazin-Riesz inverse which is $a^{gD} = f^{-1} \oplus 0$.

We also conclude by Corollary 4.10 that every generalized Drazin-Riesz invertible element on $C([-1,1])$ has a unique generalized Drazin-Riesz inverse, that is f is generalized Drazin-Riesz invertible in $C([-1,1])$ if and only if f is generalized Drazin invertible (this is easily seen without using Corollary 4.10, because $S_{C([-1,1])} = \{0\}$).

5. Application in the case of bounded operators

Theorem 5.1. *Let X be an infinite dimensional Banach space, let A be bounded operator $A : X \to X$, such that A is generalized Drazin-Riesz invertible and $0 \in \sigma(A)$. The following assertions are equivalent:*

(i) *A has a unique generalized Drazin-Riesz inverse;*
(ii) *$0 \in iso\,\sigma(A)$ and there exists no non-zero finite rank operator in $\mathcal{L}(\mathcal{R}(I-P_{\{0\},A}))$ which commutes with $A_{|\mathcal{R}(I-P_{\{0\},A})}$;*
(iii) *$0 \in iso\,\sigma(A)$ and there exists no non-zero finite rank idempotent in $\mathcal{L}(\mathcal{R}(I-P_{\{0\},A}))$ which commutes with $A_{|\mathcal{R}(I-P_{\{0\},A})}$;*
(iv) *$0 \in iso\,\sigma(A)$, $dim(\mathcal{R}(I-P_{\{0\},A})) = \infty$, and for every non-trivial idempotent P in $\mathcal{L}(\mathcal{R}(I-P_{\{0\},A}))$ which commutes with $A_{|\mathcal{R}(I-P_{\{0\},A})}$, $dim(\mathcal{R}(P)) = dim(\mathcal{N}(P)) = \infty$.*

Proof. (i)⇒(ii). As A has a unique generalized Drazin-Riesz inverse and by virtue of Remark 4.4, we have $\sigma_b(A) \cup \{0\} = \sigma(A)$ and $comm(A) \cap (I-P_{\{0\},A})\mathcal{L}(X)(I-P_{\{0\},A}) \cap \mathcal{F}(X) = \{0\}$, hence (2) is satisfied.
(ii)⇒(iii) and (iii)⇒(iv) are obvious.
(iv)⇒(i). Suppose that (iv) holds. $I - P_{\{0\},A}$ is not a finite rank projection because $dim(\mathcal{R}(I-P_{\{0\},A})) = \infty$. We have $A = A(I-P_{\{0\},A}) + AP_{\{0\},A} = A_{|\mathcal{R}(I-P_{\{0\},A})} \oplus A_{|\mathcal{R}(P_{\{0\},A})}$, with $A_{|\mathcal{R}(I-P_{\{0\},A})}$ is invertible, let P be an arbitrary idempotent described in (iv). Hence

$$A_{|\mathcal{R}(I-P_{\{0\},A})} = (A_{|\mathcal{R}(I-P_{\{0\},A})})_{|\mathcal{R}(P)} \oplus (A_{|\mathcal{R}(I-P_{\{0\},A})})_{|\mathcal{N}(P)},$$

as $A_{|\mathcal{R}(I-P_{\{0\},A})}$ is invertible, this implies that

$$A^{-1}_{|\mathcal{R}(I-P_{\{0\},A})} = (A_{|\mathcal{R}(I-P_{\{0\},A})})^{-1}_{|\mathcal{R}(P)} \oplus (A_{|\mathcal{R}(I-P_{\{0\},A})})^{-1}_{|\mathcal{N}(P)}.$$

Taking into account that $dim(\mathcal{R}(P)) = dim(\mathcal{N}(P)) = \infty$, we conclude that $(A_{|\mathcal{R}(I-P_{\{0\},A})})_{|\mathcal{R}(P)}$ and $(A_{|\mathcal{R}(I-P_{\{0\},A})})_{|\mathcal{N}(P)}$ cannot be Riesz operators, because if it is the case, consider for example that $(A_{|\mathcal{R}(I-P_{\{0\},A})})_{|\mathcal{R}(P)}$ is a Riesz operator, there exists $\mu \neq 0 \in \sigma((A_{|\mathcal{R}(I-P_{\{0\},A})})_{|\mathcal{R}(P)})$ such that $(P_{\{\mu\},(A_{|\mathcal{R}(I-P_{\{0\},A})})_{|\mathcal{R}(P)}})$ is a finite rank projection which commutes with $A_{|\mathcal{R}(I-P_{\{0\},A})}$.
Hence $dim(\mathcal{R}((P_{\{\mu\},(A_{|\mathcal{R}(I-P_{\{0\},A})})_{|\mathcal{R}(P)}}))) < \infty$, absurd with the assumption made in (iv). □

As an application of Theorem 4.5, we give its analogue in the case of $\mathcal{L}(X)$.

Corollary 5.2. *Let X be an infinite dimensional Banach space, let A be a bounded operator $A : X \to X$, such that A is generalized Drazin invertible. The following assertions are equivalent:*

(i) A possesses at least two distinct generalized Drazin-Riesz inverses;

(ii) There exists a finite rank operator F such that $AF = FA$, and $AF(I - P_{\{0\},A}) = FA(I - P_{\{0\},A}) \neq 0$.

Proof. (i)⇒(ii) we apply directly the implication $(i) \Rightarrow (ii)$ of Theorem 4.5 by considering $\mathcal{A} = \mathcal{L}(X)$.
(ii)⇒(i) Taking into account that $P_{\{0\},A} \in comm^2(A)$ and $AF = FA$, then $Q = F(1 - P_{\{0\},A}) \in (I - P_{\{0\},A})\mathcal{L}(X)(I - P_{\{0\},A})$, and we have $AQ = QA \neq 0$, with Q a finite rank operator in $(I - P_{\{0\},A})\mathcal{L}(X)(I - P_{\{0\},A}) = \mathcal{L}(\mathcal{R}(I - P_{\{0\},A}))$. Thus by applying the implication $(ii) \Rightarrow (i)$ of Theorem 4.5, we obtain the desired result. □

We give a specific characterization for the uniqueness of generalized Drazin-Riesz inverses of bounded operators which act on a Hilbert space H. We denote by $\mathcal{F}(H)$, the set of finite rank operators.

Theorem 5.3. *Let H be a Hilbert space and $A \in \mathcal{L}(H)$ such that A is generalized Drazin-Riesz invertible and $0 \in \sigma(A)$. T has a unique generalized Drazin-Riesz inverse if and only if A is generalized Drazin invertible and there exists no non-zero eigenvalue $\lambda \in \sigma(A(I - P_{\{0\},A}))$ related to the projection $I - Q$ defined to be $\mathcal{R}(I - Q) = \mathcal{N}((\lambda I - A)(I - P_{\{0\},A}))$ and $\mathcal{N}(I - Q) = \mathcal{N}((\lambda I - A)(I - P_{\{0\},A}))^{\perp}$ such that $A(I - P_{\{0\},A})(I - Q)$ is a finite rank operator.*

Proof. Theorem 3.4 ensures that if A has a unique generalized Drazin-Riesz inverse, then A is generalized Drazin invertible. By way of contradiction, we suppose that $A(I - P_{\{0\},A})$ has a non-zero eigenvalue $\lambda \in \sigma(A(I - P_{\{0\},A}))$ related to the projection $I - Q$ defined to be $\mathcal{R}(I - Q) = \mathcal{N}((\lambda I - A)(I - P_{\{0\},A}))$ and $\mathcal{N}(I - Q) = \mathcal{N}((\lambda I - A)(I - P_{\{0\},A}))^{\perp}$ such that $A(I - P_{\{0\},A})(I - Q)$ is a finite rank operator. Without loss of generality, we have $AQ = QA$ so $(I - P_{\{0\},A})Q = Q(I - P_{\{0\},A})$ and $A = A(I - P_{\{0\},A})Q + A(I - P_{\{0\},A})(I - Q) + AP_{\{0\},A}$. Hence, by considering $A_1 = A_{\mathcal{R}((I-P_{\{0\},A})Q)}$, which is invertible, $A_2 = A_{\mathcal{R}((I-P_{\{0\},A})(I-Q))}$ which is invertible and of finite rank at the same time, and $A_3 = A_{\mathcal{R}(P_{\{0\},A})}$ which is Riesz, we obtain $A = A_1 \oplus A_2 \oplus A_3$. From this reduction of A, we get two distinct generalized Drazin-Riesz inverses $S_1 = A_1^{-1} \oplus A_2^{-1} \oplus 0$ and $S_2 = A_1^{-1} \oplus 0 \oplus 0$, which is a contradiction.

Conversely, suppose that $0 \in iso\,\sigma(A)$ and there exists no non-zero eigenvalue $\lambda \in \sigma(A(I - P_{\{0\},A}))$ related to the projection $I - Q$ defined to be $\mathcal{R}(I - Q) = \mathcal{N}((\lambda I - A)(I - P_{\{0\},A}))$, and $\mathcal{N}(I - Q) = \mathcal{N}((\lambda I - A)(I - P_{\{0\},A}))^{\perp}$, such that $A(I - P_{\{0\},A})(I - Q)$ is a finite rank operator. By way of contradiction, suppose that there exists $P \neq P_{\{0\},A}$ a projection such that $AP = PA$, $A + P$ is invertible and AP is Riesz. Thus there exists $\mu \in \sigma(A)$ provided that μ is a Riesz point (hence an eigenvalue) of AP. Consequently, there exists a non-zero vector $x \in \mathcal{R}(P)$ such that $APx = \mu x = \mu Px$. Hence Px is an eigenvector of A, therefore μ is an eigenvalue of A. Now, we prove that $A(I - P_{\{0\},A})P_{\{\mu\},AP}$ is a finite rank operator which will lead us to the contradiction. Indeed, $P_{\{\mu\},AP} \in \mathcal{F}(H)$, as it commutes with $I - P_{\{0\},A}$ and A, therefore $A(I - P_{\{0\},A})P_{\{\mu\},AP} \in \mathcal{F}(H)$, notice that $\mathcal{R}(P_{\{\mu\},AP}) = \mathcal{N}((\mu I - A)(I - P_{\{0\},A}))$, and $\mathcal{N}(P_{\{\mu\},AP}) = \mathcal{N}((\mu I - A)(I - P_{\{0\},A}))^{\perp}$, a contradiction (put $P_{\{\mu\},AP} = I - Q$).

We conclude that A has a unique generalized Drazin-Riesz inverse, as desired. □

As a direct consequence of the last theorem, we give the following result.

Corollary 5.4. *Let H be a Hilbert space and $A \in \mathcal{L}(H)$ such that A is generalized Drazin-Riesz invertible, A has no non-zero eigenvalue and $0 \in \sigma(A)$.*

A has a unique generalized Drazin-Riesz inverse if and only if A is generalized Drazin invertible.

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