Filomat 39:26 (2025), 9035–9051
https://doi.org/10.2298/FIL2526035L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some properties of SEP matrices

Sijia Liu[a], Yun Zhang[b]

[a] School of Mathematical Sciences, Nanjing Normal University, Nanjing, Jiangsu 210023, P.R.China
[b] School of Mathematical Sciences, Yangzhou University, Yangzhou, Jiangsu 225002, P.R.China

Abstract. In this paper, we mainly introduce some equivalent conditions for SEP matrices. Firstly, we provide characterizations of SEP matrices in terms of projections. Secondly, we characterize SEP matrices by the representations of group inverses and Moore-Penrose inverses. Finally, we propose characterizations of SEP matrices, specifically by constructing three matrix equations and discussing whether they have solutions in given sets to determine whether a group invertible matrix is a SEP matrix respectively.

1. Introduction

The present article concerns matrices belonging to the ring $\mathbb{C}^{n\times n}$, composed of square matrices of order n with complex entries. Denotes the conjugate transpose matrix of A by A^H. $A \in \mathbb{C}^{n\times n}$ is called a group invertible matrix if there exists a matrix $X \in \mathbb{C}^{n\times n}$ such that

$$AXA = A,\ XAX = X,\ AX = XA.$$

If such X exists, then it is unique and is denoted by $A^\#$. $A^\#$ is called the group inverse of A [3]. It is well known that A is group invertible if and only if $rank(A) = rank(A^2)$ [11].

$A \in \mathbb{C}^{n\times n}$ is said to be Moore-Penrose invertible if there exists $X \in \mathbb{C}^{n\times n}$ such that

$$AXA = A,\ XAX = X,\ (AX)^H = AX,\ (XA)^H = XA.$$

Such X always exists and is uniquely determined [1, 2]. It is denoted by A^+ and is called the Moore-Penrose inverse of A.

$A \in \mathbb{C}^{n\times n}$ is called an EP matrix if A is group invertible and $A^\# = A^+$ [13]. It is known that A is EP if and only if $AA^+ = A^+A$ [4]. For the study of EP matrices, we can refer to [2, 5, 6, 12]. A is called a SEP matrix if $A^\# = A^+ = A^H$ [8, 9]. A is called partial isometry (or PI for simplicity) if $A = AA^HA$ [6, 10]. Clearly, A is PI if and only if $A^+ = A^H$, while A is SEP if and only if A is both EP and PI. A is called a projection if $A^2 = A = A^H$. Obviously, AA^+ and A^+A are both projections.

Throughout of this paper, $G_n(\mathbb{C})$ stands for the set of all group invertible matrices in $\mathbb{C}^{n\times n}$ and $\mathbb{C}_n^{\mathrm{SEP}}$ stands for the set of all SEP matrices in $\mathbb{C}^{n\times n}$.

2020 *Mathematics Subject Classification*. Primary 15A09; Secondary 16W10, 16U80.
Keywords. Moore-Penrose inverse; Group inverse; Conjugate transpose; Projection; EP matrix; SEP matrix.
Received: 15 November 2024; Revised: 22 July 2025; Accepted: 30 July 2025
Communicated by Dijana Mosić
* Corresponding author: Yun Zhang
Email addresses: 2594639292@qq.com (Sijia Liu), 862652675@qq.com (Yun Zhang)
ORCID iDs: https://orcid.org/0009-0001-2759-1344 (Sijia Liu), https://orcid.org/0009-0006-9234-5744 (Yun Zhang)

In this paper, we continue to study SEP matrices. In Section 2, we intend to develop some properties of SEP matrices. More specifically, in Subsection 2.1 we propose characterizations of SEP matrices in terms of projections; in Subsection 2.2 we produce a characterization of SEP matrices by proving that any $A \in G_n(\mathbb{C})$ is SEP if and only if

$$(A^H A^+ (A^+)^H (A^\#)^H)^\# = A^\# A^H A (A^\#)^H;$$

while in Subsection 2.3 we provide characterizations of SEP matrices by demonstrating that for any $A \in G_n(\mathbb{C})$, A is SEP if and only if

$$(A^\# A^H A (A^\#)^H + E_n - AA^+)^{-1} = A^+ A^+ (A^+)^H A^2 A^+ + E_n - AA^+$$

and A is SEP if and only if

$$(A^+ (A^\#)^H + E_n - AA^+)^{-1} = AA^+ A^H A^+ (A^+)^H A^2 A^+ + E_n - AA^+.$$

In Section 3, we are dedicated to proposing more characterizations of SEP matrices by constructing some specific matrix equations involving A and A^+, and discussing respectively whether they have solutions in given sets to determine whether A is SEP.

2. Some properties of SEP matrices

In this section, we give new characterizations of SEP matrices. More specifically, in Subsection 2.1, we characterize SEP matrices in terms of projections. In Subsection 2.2, we characterize SEP matrices by the representations of group inverses and Moore-Penrose inverses of some matrices involving A, A^H, $A^\#$ and A^+. And in Subsection 2.3, we characterize SEP matrices by the representations of inverses of two matrices involving A, A^H, $A^\#$ and A^+.

2.1. Using projections to characterize SEP matrices

In this subsection, we intend to propose characterizations of SEP matrices in terms of projections. We begin with some auxiliary lemmas.

It is known that $A \in \mathbb{C}^{n\times n}$ is partial isometry if and only if AA^H is idempotent and A is EP if and only if $AA^\#$ is a projection [12, Theorem 1.1.3]. Hence we have the following lemma, the proof of which is routine.

Lemma 2.1.1. *Let $A \in G_n(\mathbb{C})$. Then the following conditions are equivalent:*
(1) $A \in \mathbb{C}_n^{\mathrm{SEP}}$;
(2) $AA^\# - AA^H$ *is a projection;*
(3) $(AA^\#)^H - AA^H$ *is a projection;*
(4) $AA^\# - A^H A$ *is a projection;*
(5) $(AA^\#)^H - A^H A$ *is a projection.*

Note that $A \in \mathbb{C}^{n\times n}$ is a projection if and only if A is idempotent and Hermitian. This induces the following evident characterization.

Lemma 2.1.2. *Let $A \in \mathbb{C}^{n\times n}$. Then the following conditions are equivalent:*
(1) *A is a projection;*
(2) *A is idempotent and $A - AA^H$ is Hermitian;*
(3) *A is idempotent and $A - A^H A$ is Hermitian;*
(4) $A = AA^H$;
(5) $A = A^H A$.

Lemma 2.1.3. *[13] Let $A \in \mathbb{C}^{n\times n}$. Then $A \in G_n(\mathbb{C})$ if and only if $rank(A) = rank(A^2)$.*

The following lemma comes from [9].

Lemma 2.1.4. *Let $A \in G_n(\mathbb{C})$. Then the following conditions are equivalent:*
(1) $A \in \mathbb{C}_n^{\mathrm{SEP}}$;
(2) $AA^{\#} = A^H A$;
(3) $AA^H = AA^{\#}$.

Lemma 2.1.5. *[12, Theorem 1.5.3] Let $A \in \mathbb{C}^{n\times n}$. Then $A \in \mathbb{C}_n^{\mathrm{SEP}}$ if and only if $A^+ = A^H A^+ A$.*

Lemma 2.1.6. *Let $A \in \mathbb{C}^{n\times n}$. Then $A \in \mathbb{C}_n^{\mathrm{SEP}}$ if and only if $A^H = A^H A^+ (A^+)^H$.*

PROOF. " $\Rightarrow$ " One has $A^{\#} = A^+ = A^H$ by the assumption. Thus,

$$A^H A^+ (A^+)^H = A^{\#} A^+ A = A^{\#} = A^H.$$

We are done.

" $\Leftarrow$ " From the assumption, one has $A^H = A^H A^+ (A^+)^H$. Since $(A^+)^H = (A^+)^H A^+ A$, one gets

$$A^H A^+ A = A^H A^+ (A^+)^H A^+ A = A^H A^+ (A^+)^H = A^H, \tag{2.1}$$

which implies that

$$rank(A) = rank(A^H A) = rank(A^H A^+ A^2) \le rank(A^2) \le rank(A).$$

Hence, $rank(A) = rank(A^2)$. By Lemma 2.1.3, A is group invertible. And so, by [12, Theorem 1.2.1], A is EP. Now, multiplying the equality $A^H A^+ A = A^H A^+ (A^+)^H$ on the right by A^H, one obtains $A^H A^H = A^H A^+$. By [9, Corollary 2.10], A is partial isometry. Thus, $A \in \mathbb{C}_n^{\mathrm{SEP}}$. ∎

Theorem 2.1.7. *Let $A \in G_n(\mathbb{C})$. Then $A \in \mathbb{C}_n^{\mathrm{SEP}}$ if and only if $A^H A^+ (A^+)^H (A^{\#})^H$ is a projection.*

PROOF. " $\Rightarrow$ " It follows from the assumption and Lemma 2.1.6 that $A^H = A^H A^+ (A^+)^H$. This gives $A^H A^+ (A^+)^H (A^{\#})^H = (AA^{\#})^H$ is a projection by [12, Theorem 1.1.3].

" $\Leftarrow$ " From the assumption, $B = A^H A^+ (A^+)^H (A^{\#})^H$ is a projection. Thus, $B = BB^H$ by Lemma 2.1.2. Since

$$\begin{aligned} B^H B &= A^{\#} A^+ (A^+)^H A A^H A^+ (A^+)^H (A^{\#})^H \\ &= A^{\#} A^+ [AA^{\#} (A^+)^H] A A^H A^+ (A^+)^H (A^{\#})^H \\ &= A^{\#} A^{\#} (A^+)^H A A^H A^+ (A^+)^H (A^{\#})^H, \end{aligned}$$

one gets

$$A^{\#} A^{\#} (A^+)^H A A^H A^+ (A^+)^H (A^{\#})^H = A^H A^+ (A^+)^H (A^{\#})^H. \tag{2.2}$$

Multiplying the equality (2.2) on the left by A^2 and on the right by $A^H A^H A$ yields

$$(A^+)^H A A^H A^+ A = A^2 A^H A^+ A.$$

Multiplying the equality on the right by $(A^+ A^{\#} A)^H A^{\#}$ yields $(A^+)^H = A$, that is, $A^+ = A^H$. It follows by (2.2) that

$$A^{\#} A^{\#} A A A^H A^+ A (A^{\#})^H = A^H A^+ A (A^{\#})^H,$$

namely,

$$A^{\#} A A^+ (A^{\#})^H = A^H (A^{\#})^H.$$

Multiplying the equality on the right by $A^+ A$ yields $A^{\#} (A^+)^H = A^+ A$, that is, $A^{\#} A = A^H A$ since $A^+ = A^H$. It follows from [12, Theorem 1.5.3] that $A \in \mathbb{C}_n^{\mathrm{SEP}}$. ∎

Theorem 2.1.8. *Let $A \in G_n(\mathbb{C})$. Then $A \in \mathbb{C}_n^{\mathrm{SEP}}$ if and only if $AA^+ A^+ (A^+)^H$ is a projection.*

PROOF. " $\Rightarrow$ " By the assumption and Lemma 2.1.6, $A^H = A^HA^+(A^+)^H$. This gives $AA^+A^+(A^+)^H = (A^+)^HA^HA^+(A^+)^H = (A^+)^HA^H = AA^+$ is a projection.
" $\Leftarrow$ " From the assumption and Lemma 2.1.2, one gets

$$[AA^+A^+(A^+)^H][AA^+A^+(A^+)^H]^H = AA^+A^+(A^+)^H,$$

that is,

$$AA^+A^+(A^+)^HA^+(A^+)^HAA^+ = AA^+A^+(A^+)^H. \tag{2.3}$$

Multiplying the equality (2.3) on the right by $AA^\#$ yields

$$AA^+A^+(A^+)^HA^+(A^+)^H = AA^+A^+(A^+)^H.$$

Multiplying the equality on the left by $(AA^\#)^H$ and on the right by A^H respectively yields

$$A^+(A^+)^HA^+ = A^+.$$

Hence,

$$A = AA^+A = AA^+[(A^+)^HA^+A] = AA^+(A^+)^H = (A^+)^H,$$

that is, $A^H = A^+$. Now, one obtains by (2.3) that

$$AA^+A^+(AA^+A)AA^+ = AA^+A^+A,$$

namely,

$$AA^+A^+AAA^+ = AA^+A^+A.$$

Multiplying the equality on the left by $AA^\#(AA^\#)^H$ yields $AA^+ = AA^\#$. Thus, $A \in \mathbb{C}_n^{\mathrm{SEP}}$. ∎

Observe that A is a projection if and only if A^H is. Since $(AA^+A^+(A^+)^H)^H = A^+(A^+)^HAA^+$, one can get the following corollary by Theorem 2.1.8.

Corollary 2.1.9. *Let $A \in G_n(\mathbb{C})$. Then $A \in \mathbb{C}_n^{\mathrm{SEP}}$ if and only if $A^+(A^+)^HAA^+$ is a projection.*

Lemma 2.1.10. *Let $A \in \mathbb{C}^{n\times n}$. If A is a projection, then $A^+ = A^\#$ and both of them are projections.*

PROOF. By assumption, one has $A^2 = A = A^H$, which implies that A is group invertible by Lemma 2.1.3. Now, one has

$$(AA^\#)^H = (A^2A^\#)^H = A^H = A = AA^\#,$$

which implies that A is EP by [12, Theorem 1.1.3], and so $A^+ = A^\#$. Furthermore, one has

$$(A^\#)^2 = A^\#(A^\#AA^\#) = (A^\#)^3A = (A^\#)^3A^2 = A^\# = (A^H)^\# = (A^\#)^H,$$

which implies that $A^\#(= A^+)$ is a projection. ∎

Lemma 2.1.11. *Let $A \in G_n(\mathbb{C})$. Then*
(1) $(A^HA^+(A^+)^H(A^\#)^H)^\# = A^HA^HA(A^\#)^H$;
(2) $(A^HA^+(A^+)^H(A^\#)^H)^+ = AA^+A^HA^HA(A^\#)^HA^+A$.

PROOF. (1) For simplicity, let $M = A^HA^+(A^+)^H(A^\#)^H$ and $B = A^HA^HA(A^\#)^H$. We prove that $M^\# = B$ as follows. Firstly, since

$$\begin{aligned} MB &= A^HA^+(A^+)^H[(A^\#)^HA^HA^H]A(A^\#)^H \\ &= A^HA^+(A^+)^HA^HA(A^\#)^H \\ &= A^HA^+A(A^\#)^H \\ &= (A^\#A)^H \end{aligned}$$

and

$$\begin{aligned} BM &= A^H A^H A[(A^\#)^H A^H A^+](A^+)^H (A^\#)^H \\ &= A^H A^H [AA^+(A^+)^H](A^\#)^H \\ &= A^H A^H (A^+)^H (A^\#)^H \\ &= (A^\# A)^H, \end{aligned}$$

one has $BM = MB$.

Secondly, one has

$$\begin{aligned} MBM &= (A^\# A)^H A^H A^+(A^+)^H (A^\#)^H \\ &= A^H A^+(A^+)^H (A^\#)^H \\ &= M \end{aligned}$$

and

$$\begin{aligned} BMB &= (A^\# A)^H A^H A^H A(A^\#)^H \\ &= A^H A^H A(A^\#)^H \\ &= B \end{aligned}$$

Consequently, $M^\# = B$ as desired.

(2) For simplicity, let $D = AA^+A^H A^H A(A^\#)^H A^+A$ and M as in (1). It suffices to prove that $M^+ = D$.

Firstly, since

$$\begin{aligned} MD &= A^H A^+(A^+)^H (A^\#)^H AA^+A^H A^H A(A^\#)^H A^+A \\ &= A^H A^+(A^+)^H (A^2 AA^+A^\#)^H A(A^\#)^H A^+A \\ &= A^H [A^+(A^+)^H A^H] A(A^\#)^H A^+A \\ &= A^H [A^+A(A^\#)^H] A^+A \\ &= A^H (A^\#)^H A^+A \\ &= A^+A, \end{aligned}$$

one has $MD = (MD)^H$.

Secondly, since

$$\begin{aligned} DM &= AA^+A^H A^H A(A^\#)^H [A^+AA^H] A^+(A^+)^H (A^\#)^H \\ &= AA^+A^H A^H A[(A^\#)^H A^H A^+](A^+)^H (A^\#)^H \\ &= AA^+A^H A^H A[(AA^\#)^H A^+](A^+)^H (A^\#)^H \\ &= AA^+A^H [A^H AA^+](A^+)^H (A^\#)^H \\ &= AA^+A^H A^H (A^+)^H (A^\#)^H \\ &= AA^+A^H (A^\#)^H \\ &= AA^+, \end{aligned}$$

one gets $DM = (DM)^H$.

Finally, one has

$$DMD = AA^+AA^+A^H A^H A(A^\#)^H A^+A = AA^+A^H A^H A(A^\#)^H A^+A = D$$

and

$$MDM = A^+AA^H A^+(A^+)^H (A^\#)^H = A^H A^+(A^+)^H (A^\#)^H = M.$$

Consequently, $M^+ = D$ as desired. ∎

By Lemma 2.1.10, 2.1.11 and Theorem 2.1.7, one can get the following corollary.

Corollary 2.1.12. *Let $A \in G_n(\mathbb{C})$. Then the following conditions are equivalent.*

(1) $A \in \mathbb{C}_n^{\mathrm{SEP}}$;
(2) $A^H A^H A(A^\#)^H$ *is a projection;*
(3) $AA^+A^H A^H A(A^\#)^H A^+A$ *is a projection;*
(4) $A^\# A^H A^2$ *is a projection;*
(5) $A^+A^H A^3 A^+$ *is a projection.*

PROOF. It suffices to prove that (1) implies (5) and (5) implies (1), since the others are followed directly by Lemma 2.1.10, 2.1.11 and Theorem 2.1.7.

"(1) $\Rightarrow$ (5)" By the assumption, one has $A^+ = A^\# = A^H$. Hence, $A^+A^HA^3A^+ = A^\#A^\#A^3A^\# = A^\#A = A^+A$ is a projection.

"(5) $\Rightarrow$ (1)" By assumption, one has

$$A^+A^HA^3A^+A^+A^HA^3A^+ = (A^+A^HA^3A^+)^2 = A^+A^HA^3A^+.$$

Multiplying the last equality on the right by A, $(A^\#)^2$, A^+ and $(A^\#)^H$ successively yields

$$A^+A^HA^3A^+A^+ = A^+.$$

Hence,

$$(A^+)^H = (A^+A^HA^3A^+A^+)^H = (A^+)^H(A^+A^HA^3A^+)^H = (A^+)^H(A^+A^HA^3A^+).$$

Multiplying the equality on the right by $A^\#$ and on the left by A^H respectively yields

$$A^+AA^\# = A^+A^HA. \tag{2.4}$$

It follows that

$$(A^+)^H = (A^+)^HA^+A^HA^3A^+ = (A^+)^HA^+AA^\#A^2A^+ = (A^+)^HA^+A^2A^+ = (AA^+A^+)^H,$$

that is, $A^+ = AA^+A^+$, which implies that A is EP. By (2.4), one gets $A^\# = A^+A^HA$. As a result, $A^+A^\# = A^\#A^+ = A^+A^HAA^+ = A^+A^H$. It follows from [12, Theorem 1.5.3] that $A \in \mathbb{C}_n^{\mathrm{SEP}}$. ∎

2.2. Using the representations of group inverses and Moore-Penrose inverses to characterize SEP matrices

Inspired by Lemma 2.1.11, we propose the following theorem.

Theorem 2.2.1. *Let $A \in G_n(\mathbb{C})$. Then $A \in \mathbb{C}_n^{\mathrm{SEP}}$ if and only if $(A^HA^+(A^+)^H(A^\#)^H)^\# = A^\#A^HA(A^\#)^H$.*

PROOF. " $\Rightarrow$ " It follows directly by Lemma 2.1.11.

" $\Leftarrow$ " By Lemma 2.1.11 and the assumption, one has $A^\#A^HA(A^\#)^H = A^HA^HA(A^\#)^H$. Multiplying the equality on the right by $(A^+AA)^H$, A^+ and $(A^+)^H$ successively yields $A^\#A^+A = A^HA^+A$, that is, $A^\# = A^HA^+A$. Hence, $A \in \mathbb{C}_n^{\mathrm{SEP}}$ by [12, Theorem 1.5.3]. ∎

Note that $A^\# = B$ if and only if $B^\# = A$. Hence, one gets the following corollary.

Corollary 2.2.2. *Let $A \in G_n(\mathbb{C})$. Then $A \in \mathbb{C}_n^{\mathrm{SEP}}$ if and only if $(A^\#A^HA(A^\#)^H)^\# = A^HA^+(A^+)^H(A^\#)^H$.*

Lemma 2.2.3. *Let $A \in G_n(\mathbb{C})$. Then $A^\#A^HA(A^\#)^H$ is EP and $(A^\#A^HA(A^\#)^H)^\# = (A^\#A^HA(A^\#)^H)^+ = AA^+A^HA^+(A^+)^HA^2A^+$.*

PROOF. Let $D = A^\#A^HA(A^\#)^H$ and $M = AA^+A^HA^+(A^+)^HA^2A^+$ for simplicity. It suffices to prove that $D^\# = D^+ = M$.

Firstly,

$$\begin{aligned} MD &= AA^+A^HA^+(A^+)^HA(AA^+A^\#)A^HA(A^\#)^H \\ &= AA^+A^HA^+[(A^+)^HAA^\#]A^HA(A^\#)^H \\ &= AA^+A^H[A^+(A^+)^HA^H]A(A^\#)^H \\ &= AA^+A^H[A^+A(A^\#)^H] \\ &= AA^+A^H(A^\#)^H \\ &= AA^+, \end{aligned}$$

which tells us that $(MD)^H = MD$.

Secondly,

$$DMD = A^\#A^HA[(A^\#)^HAA^+] = A^\#A^HA(A^\#)^H = D$$

and

$$MDM = AA^+AA^+A^HA^+(A^+)^HA^2A^+ = AA^+A^HA^+(A^+)^HA^2A^+ = M.$$

Finally, since

$$\begin{aligned} DM &= A^\#A^HA[(A^\#)^HAA^+]A^HA^+(A^+)^HA^2A^+ \\ &= A^\#A^HA[(A^\#)^HA^HA^+](A^+)^HA^2A^+ \\ &= A^\#A^H[AA^+(A^+)^H]A^2A^+ \\ &= [A^\#A^H(A^+)^H]A^2A^+ \\ &= AA^+, \end{aligned}$$

one has $(DM)^H = DM = MD$.

Consequently, $D^\# = D^+ = M$ as desired. ∎

The following corollary is a direct result of Lemma 2.2.3.

Corollary 2.2.4. *Let $A \in \mathbb{C}_n^{\mathrm{SEP}}$. Then $A^\#A^HA(A^\#)^H \in \mathbb{C}_n^{\mathrm{SEP}}$.*

The following corollary follows directly by Corollary 2.2.2 and Lemma 2.2.3.

Corollary 2.2.5. *Let $A \in G_n(\mathbb{C})$. Then $A \in \mathbb{C}_n^{\mathrm{SEP}}$ if and only if $AA^+A^HA^+(A^+)^HA^2A^+ = A^HA^+(A^+)^H(A^\#)^H$.*

Inspired by Corollary 2.2.5, we propose the following theorem.

Theorem 2.2.6. *Let $A \in G_n(\mathbb{C})$. Then $A \in \mathbb{C}_n^{\mathrm{SEP}}$ if and only if $A^\#(A^+)^HA^2A^+ = A^+(A^+)^H(A^\#)^H$.*

PROOF. " ⇒ " By the assumption, one has $A^\# = A^+ = A^H$. Thus,

$$A^\#(A^+)^HA^2A^+ = A^+(A^+)^HA^2A^\# = A^+(A^+)^HA = A^+(A^+)^H(A^\#)^H.$$

" ⇐ " From the assumption, one obtains $A^\#(A^+)^HA^2A^+ = A^+(A^+)^H(A^\#)^H$. Multiplying the equality on the left by A yields

$$(A^+)^HA^2A^+ = (A^+)^H(A^\#)^H. \tag{2.5}$$

Thus,

$$A^\#(A^+)^HA^2A^+ = A^+(A^+)^HA^2A^+.$$

Multiplying the last equality on the right by $A^\#$, A^H and A successively, one obtains $A^\#A = A^+A$, which tells that A is EP. Hence, $(A^+)^HA = (A^\#A^\#)^H$ by (2.5). Multiplying the equality on the left by A^H yields $A = (A^\#)^H$, that is, $A^\# = A^H$. It follows that $A \in \mathbb{C}_n^{\mathrm{SEP}}$. ∎

The following corollary is induced by Theorem 2.2.6.

Corollary 2.2.7. *Let $A \in G_n(\mathbb{C})$. Then the following conditions are equivalent.*
(1) $A \in \mathbb{C}_n^{\mathrm{SEP}}$;
(2) $A^\#(A^+)^HA = A^+(A^+)^H(A^\#)^H$;
(3) $(A^+)^HA^2A^+ = (A^\#)^H(A^\#)^H$;
(4) $AA^+A^HA^+ = A^\#A^\#$.

PROOF. "(1) ⇒ (2)" It is evident since $A^H = A^\# = A^+$.

"(2) ⇒ (3)" By the assumption one has $A^\#(A^+)^HA = A^+(A^+)^H(A^\#)^H$. Multiplying the equality on the left by A and on the right by AA^+ respectively yields $(A^+)^HA^2A^+ = (A^+)^H(A^\#)^H$. Note that

$$A^\#(A^+)^HA = A^+(A^+)^H(A^\#)^H = A^+A(A^+(A^+)^H(A^\#)^H) = A^+AA^\#(A^+)^HA = A^+(A^+)^HA.$$

Thus,

$$A^{\#}AA^{+} = A^{\#}(A^{+})^{H}A^{H} = A^{\#}(A^{+})^{H}AA^{\#}A^{H} = A^{+}(A^{+})^{H}AA^{\#}A^{H} = A^{+}(A^{+})^{H}A^{H} = A^{+},$$

which implies that A is EP by [12, Theorem 1.2.1]. Hence, $(A^{+})^{H}A^{2}A^{+} = (A^{\#})^{H}(A^{\#})^{H}$.

"(3) ⇒ (4)" From the assumption, one has $(A^{+})^{H}A^{2}A^{+} = (A^{\#})^{H}(A^{\#})^{H}$. Applying the involution on the equality yields $AA^{+}A^{H}A^{+} = A^{\#}A^{\#}$.

"(4) ⇒ (1)" The condition $AA^{+}A^{H}A^{+} = A^{\#}A^{\#}$ gives

$$A^{\#}A^{\#} = AA^{+}A^{H}A^{+} = (AA^{+}A^{H}A^{+})AA^{+} = A^{\#}A^{\#}AA^{+} = A^{\#}A^{+}.$$

Hence, A is EP by [12, Theorem 1.2.1]. It follows that $A^{+}A^{+} = A^{\#}A^{\#} = AA^{+}A^{H}A^{+} = A^{H}A^{+}$. By [9, Corollary 2.10], A is partial isometry. Thus, $A \in \mathbb{C}_n^{\mathrm{SEP}}$. ∎

2.3. *Using the inverses of matrices involving group inverses and Moore-Penrose inverses to characterize SEP matrices*

Inspired by Lemma 2.2.3, we propose the following lemma.

Lemma 2.3.1. *Let $A \in G_n(\mathbb{C})$. Then $A^{\#}A^{H}A(A^{\#})^{H} + E_n - AA^{+}$ is invertible and*

$$(A^{\#}A^{H}A(A^{\#})^{H} + E_n - AA^{+})^{-1} = AA^{+}A^{H}A^{+}(A^{+})^{H}A^{2}A^{+} + E_n - AA^{+}.$$

PROOF. Let $D = A^{\#}A^{H}A(A^{\#})^{H}$ and $M = AA^{+}A^{H}A^{+}(A^{+})^{H}A^{2}A^{+}$. Then $D^{\#} = D^{+} = M$ by Lemma 2.2.3. Since

$$DAA^{+} = A^{\#}A^{H}A[(A^{\#})^{H}AA^{+}] = A^{\#}A^{H}A^{H}A(A^{\#})^{H} = D,$$
$$AA^{+}M = AA^{+}AA^{+}A^{H}A^{+}A^{+}(A^{+})^{H}A^{2}A^{+} = AA^{+}A^{H}A^{+}(A^{+})^{H}A^{2}A^{+} = M,$$

and $DM = AA^{+}$ by the proof of Lemma 2.2.3, one gets

$$\begin{aligned} & (D + E_n - AA^{+})(M + E_n - AA^{+}) \\ = \ & DM + D - DAA^{+} + M + E_n - AA^{+} - AA^{+}M - AA^{+} + AA^{+}AA^{+} \\ = \ & AA^{+} + D - D + M + E_n - AA^{+} - M - AA^{+} + AA^{+} \\ = \ & E_n, \end{aligned}$$

which implies that $(D + E_n - AA^{+})^{-1} = M + E_n - AA^{+}$. That is,

$$(A^{\#}A^{H}A(A^{\#})^{H} + E_n - AA^{+})^{-1} = AA^{+}A^{H}A^{+}(A^{+})^{H}A^{2}A^{+} + E_n - AA^{+}.$$

∎

The following theorem is inferred by Lemma 2.3.1.

Theorem 2.3.2. *Let $A \in G_n(\mathbb{C})$. Then $A \in \mathbb{C}_n^{\mathrm{SEP}}$ if and only if*

$$(A^{\#}A^{H}A(A^{\#})^{H} + E_n - AA^{+})^{-1} = A^{+}A^{+}(A^{+})^{H}A^{2}A^{+} + E_n - AA^{+}.$$

PROOF. " ⇒ " By the assumption, $A^{\#} = A^{+} = A^{H}$. Thus

$$AA^{+}A^{H}A^{+}(A^{+})^{H}A^{2}A^{+} = (AA^{\#}A^{\#})A^{+}(A^{+})^{H}A^{2}A^{+} = A^{+}A^{+}(A^{+})^{H}A^{2}A^{+}.$$

It follows by Lemma 2.3.1 that

$$(A^{\#}A^{H}A(A^{\#})^{H} + E_n - AA^{+})^{-1} = A^{+}A^{+}(A^{+})^{H}A^{2}A^{+} + E_n - AA^{+}.$$

" ⇐ " From the assumption, one has

$$(A^{\#}A^{H}A(A^{\#})^{H} + E_n - AA^{+})^{-1} = A^{+}A^{+}(A^{+})^{H}A^{2}A^{+} + E_n - AA^{+}.$$

Thus, by Lemma 2.3.1, one gets

$$AA^+A^HA^+(A^+)^HA^2A^+ = A^+A^+(A^+)^HA^2A^+.$$

Multiplying the equality on the right by $A^\#A^H$ yields $AA^+A^HA^+ = A^+A^+$. Hence,

$$AA^+A^H = (AA^+A^HA^+)AA^H(A^\#)^H = A^+[A^+AA^H(A^\#)^H] = A^+(A^\#A)^H = A^+,$$

which implies that $A \in \mathbb{C}_n^{\mathrm{SEP}}$ as desired.

∎

Lemma 2.3.3. *Let $A \in G_n(\mathbb{C})$. Then $A \in \mathbb{C}_n^{\mathrm{SEP}}$ if and only if $A^\#A^HA(A^\#)^H = A^+(A^\#)^H$.*

PROOF. " $\Rightarrow$ " It is evident.
" $\Leftarrow$ " By the assumption, one has $A^\#A^HA(A^\#)^H = A^+(A^\#)^H$. Hence,

$$\begin{aligned} A^+ &= [A^+(A^\#)^H]A^H = A^\#A^HA(A^\#)^HA^H = AA^+[A^\#A^HA(A^\#)^H]A^H \\ &= AA^+[A^+(A^\#)^HA^H] = AA^+A^+, \end{aligned}$$

which implies that A is EP. It follows that

$$A^+ = A^+(AA^\#)^H = [A^+(A^\#)^H]A^H = A^\#A^HA[(A^\#)^HA^H] = A^\#A^HA.$$

Therefore, $A \in \mathbb{C}_n^{\mathrm{SEP}}$ by [12, Theorem 1.5.3]. ∎

Theorem 2.3.4. *Let $A \in G_n(\mathbb{C})$. Then $A \in \mathbb{C}_n^{\mathrm{SEP}}$ if and only if $A^+(A^\#)^H + E_n - AA^+$ is invertible and*

$$(A^+(A^\#)^H + E_n - AA^+)^{-1} = AA^+A^HA^+(A^+)^HA^2A^+ + E_n - AA^+.$$

PROOF. " $\Rightarrow$ " Assume that $A \in \mathbb{C}_n^{\mathrm{SEP}}$, then $A^\#A^HA(A^\#)^H = A^+(A^\#)^H$ by Lemma 2.3.3. As in the proof of Lemma 2.3.1, we denote $A^\#A^HA(A^\#)^H = A^+(A^\#)^H$ by D and denote $AA^+A^HA^+(A^+)^HA^2A^+$ by M. Then $D^\# = D^+ = M$, $DM = AA^+$, $DAA^+ = D$ and $AA^+M = M$. Hence,

$$\begin{aligned} & (D + E_n - AA^+)(M + E_n - AA^+) \\ = \ & DM + D - DAA^+ + M + E_n - AA^+ - AA^+M - AA^+ + AA^+AA^+ \\ = \ & AA^+ + D - DAA^+ + M + E_n - AA^+ - AA^+M - AA^+ + AA^+ \\ = \ & E_n, \end{aligned}$$

which implies that $(D + E_n - AA^+)^{-1} = M + E_n - AA^+$. Namely, $A^+(A^\#)^H + E_n - AA^+$ is invertible and

$$(A^+(A^\#)^H + E_n - AA^+)^{-1} = AA^+A^HA^+(A^+)^HA^2A^+ + E_n - AA^+.$$

" $\Leftarrow$ " Using the assumption and Lemma 2.3.1, one has

$$A^\#A^HA(A^\#)^H + E_n - AA^+ = A^+(A^\#)^H + E_n - AA^+,$$

which implies that $A^\#A^HA(A^\#)^H = A^+(A^\#)^H$. Hence, $A \in \mathbb{C}_n^{\mathrm{SEP}}$ by Lemma 2.3.3.

∎

Lemma 2.3.5. *Let $A \in G_n(\mathbb{C})$. Then $A^+(A^\#)^H + E_n - (AA^\#)^H$ is invertible and*

$$(A^+(A^\#)^H + E_n - (AA^\#)^H)^{-1} = A^HA(AA^\#)^H + E_n - (AA^\#)^H.$$

PROOF. It is followed by the fact that

$$\begin{aligned} & [A^+(A^\#)^H + E_n - (AA^\#)^H][A^HA(AA^\#)^H + E_n - (AA^\#)^H] \\ = \ & (A^\#A)^H + A^+(A^\#)^H - A^+(A^\#)^H + A^HA(AA^\#)^H + E_n - (AA^\#)^H - A^HA(AA^\#)^H \\ & -(AA^\#)^H + (AA^\#)^H \\ = \ & E_n. \end{aligned}$$

∎

The following theorem is induced by Theorem 2.3.4 and Lemma 2.3.5.

Theorem 2.3.6. *Let $A \in G_n(\mathbb{C})$. Then $A \in \mathbb{C}_n^{\text{SEP}}$ if and only if*

$$(AA^+A^HA^+(A^+)^HA^2A^+ + E_n - AA^+)^{-1} = A^+(A^+)^H + E_n - (AA^\#)^H.$$

PROOF. " $\Rightarrow$ " It is an immediate result of Lemma 2.3.4 and Lemma 2.3.5.
" $\Leftarrow$ " By the assumption one obtains

$$(AA^+A^HA^+(A^+)^HA^2A^+ + E_n - AA^+)^{-1} = A^+(A^+)^H + E_n - (AA^\#)^H.$$

Thus,

$$E_n = [AA^+A^HA^+(A^+)^HA^2A^+ - AA^+ + E_n][A^+(A^+)^H + E_n - (AA^\#)^H].$$

It follows that

$$AA^+A^HA^+(A^+)^HA^2A^+A^+(A^+)^H + A^+(A^+)^H = (AA^\#)^H + AA^+A^+(A^+)^H. \tag{2.6}$$

Multiplying the equality on the right by $AA^\#$, one yields $(AA^\#)^H = (AA^\#)^HAA^\#$. Hence, A is EP, which implies $A^2A^+A^+(A^+)^H = AA^\#(A^+)^H = (A^+)^H$, $AA^+A^H = A^H$ and $AA^+A^+ = A^+$. It follows by (2.6) that

$$A^HA^+(A^+)^H(A^+)^H = (AA^\#)^H.$$

Applying the involution, one gets $A^+A^+(A^+)^HA = AA^\#$, which further implies that $A^2 = A^3A^\# = A^2A^+A^+(A^+)^HA = (A^+)^HA$. Hence, A is partial isometry and so $A \in \mathbb{C}_n^{\text{SEP}}$. ∎

3. Characterizations of SEP matrices by the general solutions of matrix equations

In [7], some equivalent conditions for SEP matrix are proposed by constructing some specific matrix equations and discussing whether these matrix equations have solutions in given sets to determine whether a group invertible matrix A is in $\mathbb{C}_n^{\text{SEP}}$. Inspired by [7], we intend to construct several matrix equations involving but not limit to A, A^+ or $A^\#$ and either prove that they have solutions in given sets if and only if $A \in \mathbb{C}_n^{\text{SEP}}$, or prove that $A \in \mathbb{C}_n^{\text{SEP}}$ if and only if some of these matrix equations have general solutions of specific forms.

3.1. *Characterizations of SEP matrices by the general solutions of matrix equation of the type $AXB = C$.*

The following lemma is well known [11].

Lemma 3.1.1. *Let A, B and C be matrices in $\mathbb{C}^{n\times n}$. Then the following Eq.(3.1) has solutions if and only if $C = AA^+CB^+B$.*

$$AXB = C. \tag{3.1}$$

In the case that Eq.(3.1) is consistent, the general solution is given by

$$X = A^+CB^+ + U - A^+AUBB^+, \text{ where } U \in \mathbb{C}^{n\times n}. \tag{3.2}$$

Inspired by Lemma 2.1.3, we construct the following matrix equation:

$$A^HX(A^+)^H = A^H. \tag{3.3}$$

Theorem 3.1.2. *Let $A \in G_n(\mathbb{C})$. Then $A \in \mathbb{C}_n^{\text{SEP}}$ if and only if Eq.(3.3) is consistent and the general solution is given by*

$$X = A^+ + U - AA^+UAA^+, \ U \in \mathbb{C}^{n\times n}. \tag{3.4}$$

PROOF. " $\Rightarrow$ " By the assumption and Lemma 2.1.6, one has $A^HA^+(A^+)^H = A^H$, which says that A^+ is a solution of Eq.(3.3). We prove that the general solution of Eq.(3.3) is given by (3.4) as follows.

On one hand, $\forall U \in \mathbb{C}^{n\times n}$, since $A \in \mathbb{C}_n^{\text{SEP}}$, one has

$$\begin{aligned} & A^H(A^+ + U - AA^+UAA^+)(A^+)^H \\ = \; & A^HA^+(A^+)^H + A^HU(A^+)^H - A^HAA^+UAA^+(A^+)^H \\ = \; & A^H + A^HU(A^+)^H - A^HU(A^+)^H \\ = \; & A^H, \end{aligned}$$

which implies that any matrix given by (3.4) is a solution of Eq.(3.3).

On the other hand, let $X = X_0$ be any solution of Eq.(3.3), then $A^HX_0(A^+)^H = A^H$. Thus, $AA^+X_0AA^+ = (A^+)^HA^HX_0(A^+)^HA^H = (A^+)^HA^HA^H = A^H = A^+$, since $A \in \mathbb{C}_n^{\text{SEP}}$. As a result,

$$X_0 = A^+ + X_0 - AA^+X_0AA^+,$$

which tells that any solution of Eq.(3.3) is given by (3.4).

Consequently, the general solution of Eq.(3.3) is given by (3.4) provided that A is SEP.

" $\Leftarrow$ " By the assumption, any matrix given by (3.4) is a solution of Eq.(3.3). In particular, let $U = -A^+$. Then $X_0 = AA^+A^+AA^+ = AA^+A^+$ is a solution of Eq.(3.3), that is,

$$A^H = A^HA^+(A^+)^H. \tag{3.5}$$

By Lemma 2.1.6, $A \in \mathbb{C}_n^{\text{SEP}}$.

∎

Now, we construct a new equation as follows, which has the general solution given by (3.4).

$$A^HXA(AA^\#)^H = A^H. \tag{3.6}$$

Lemma 3.1.3. *Let $A \in G_n(\mathbb{C})$. Then Eq.(3.6) is consistent and the general solution is given by (3.4).*

PROOF. $\forall U \in \mathbb{C}^{n\times n}$, since

$$\begin{aligned} & A^H(A^+ + U - AA^+UAA^+)A(AA^\#)^H \\ = \; & A^HA^+A(AA^\#)^H + A^HUA(AA^\#)^H - A^HAA^+UAA^+A(AA^\#)^H \\ = \; & A^H + A^HUA(AA^\#)^H - A^HUA(AA^\#)^H \\ = \; & A^H, \end{aligned}$$

any matrix given by (3.4) is a solution of Eq.(3.6).

On the other hand, let X_0 be any solution of Eq.(3.6), that is, $A^HX_0A(AA^\#)^H = A^H$. Hence,

$$\begin{aligned} & AA^+X_0AA^+ \\ = \; & (A^+)^HA^HX_0(AA^+)AA^+ \\ = \; & (A^+)^HA^HX_0A(A^+A)^HA^+ \\ = \; & (A^+)^H[A^HX_0A(AA^\#)^H]A^H(A^+)^HA^+ \\ = \; & (A^+)^HA^HA^H(A^+)^HA^+ \\ = \; & (A^+)^HA^HA^+ \\ = \; & AA^+A^+. \end{aligned}$$

Let $U_0 = X_0 - A^+$. Then one has

$$AA^+U_0AA^+ = AA^+X_0AA^+ - AA^+A^+AA^+ = AA^+A^+ - AA^+A^+ = 0,$$

which implies that

$$X_0 = A^+ + X_0 - A^+ = A^+ + U_0 = A^+ + U_0 - AA^+U_0AA^+.$$

Consequently, the general solution of Eq.(3.6) is given by (3.4). ∎

The following theorem can be induced by Theorem 3.1.2 and Lemma 3.1.3.

Theorem 3.1.4. *Let $A \in G_n(\mathbb{C})$. Then $A \in \mathbb{C}_n^{\mathrm{SEP}}$ if and only if Eq.(3.3) and Eq.(3.6) share the same solutions.*

PROOF. " $\Rightarrow$ " It is evident since Eq.(3.3) and Eq.(3.6) are exactly the same under the assumption.
" $\Leftarrow$ " It is a direct result of Theorem 3.1.2 and Lemma 3.1.3. ∎

Now we construct the following matrix equation:

$$A^H X A A^+ = A^H A^+. \tag{3.7}$$

Lemma 3.1.5. *Let $A \in G_n(\mathbb{C})$. Then Eq.(3.6) and Eq.(3.7) share the same solutions.*

PROOF. It suffices to prove that the general solution of Eq.(3.7) is also given by (3.4).
On one hand, for any $U \in \mathbb{C}^{n\times n}$, one has

$$\begin{aligned} & A^H(A^+ + U - AA^+UAA^+)AA^+ \\ = \ & A^HA^+AA^+ + A^HUAA^+ - A^HAA^+UAA^+AA^+ \\ = \ & A^HA^+ + A^HUAA^+ - A^HUAA^+ \\ = \ & A^HA^+, \end{aligned}$$

which tells that any matrix given by (3.4) is a solution of Eq.(3.7).
On the other hand, let X_0 be any solution of Eq.(3.7), that is, $A^HX_0AA^+ = A^HA^+$. Thus, one has

$$AA^+X_0AA^+ = (A^+)^HA^HX_0AA^+ = (A^+)^HA^HA^+ = AA^+A^+.$$

Let $U_0 = X_0 - A^+$. Then one obtains

$$AA^+U_0AA^+ = AA^+X_0AA^+ - AA^+A^+AA^+ = AA^+X_0AA^+ - AA^+A^+ = 0.$$

It follows that $X_0 = A^+ + X_0 - A^+ = A^+ + U_0 - AA^+U_0AA^+$, namely, any solution of Eq.(3.7) is of the form given by (3.4). Consequently, Eq.(3.6) and Eq.(3.7) share the same solutions. ∎

The following theorem is an immediate result of Theorem 3.1.4 and Lemma 3.1.5.

Theorem 3.1.6. *Let $A \in G_n(\mathbb{C})$. Then $A \in \mathbb{C}_n^{\mathrm{SEP}}$ if and only if Eq.(3.3) and Eq.(3.7) have exactly the same solutions.*

3.2. Characterizations of SEP matrices by univariate matrix equations involving A^+ and $A^\#$

To begin with, we construct the following matrix equation:

$$A^+X(A^+)^H = X. \tag{3.8}$$

Theorem 3.2.1. *Let $A \in G_n(\mathbb{C})$. Then $A \in \mathbb{C}_n^{\mathrm{SEP}}$ if and only if Eq.(3.8) has at least one solution in the following set*

$$\zeta_A = \{A, A^\#, A^+, A^H, (A^+)^H, (A^\#)^H, (A^\#)^+, (A^+)^\#, AA^+, AA^H, AA^\#, (AA^\#)^H\}.$$

PROOF. " $\Rightarrow$ " Assume that $A \in \mathbb{C}_n^{\mathrm{SEP}}$, then $A^+A(A^+)^H = A^\#AA = A$, that is, $X = A$ is a solution of Eq.(3.8).
" $\Leftarrow$ " The proof is divided into several cases as follows.
Case 1. If any $X \in \{A, A^\#, A^+, A^H, (A^+)^H, (A^\#)^H\}$ is a solution of Eq.(3.8), then A is SEP by [8, Theorem 2.7].
Case 2. If $X = (A^\#)^+ = A^+A^3A^+$ is a solution of Eq.(3.8), then

$$A^+A^+A^3A^+(A^+)^H = A^+A^3A^+. \tag{3.9}$$

Multiplying the equality (3.9) on the right by $AA^\#$, one gets

$$A^+A^3A^+ = A^+A^2.$$

This gives $AA^{+} = A^{\#}A^{+}A^{3}A^{+} = A^{\#}A^{+}A^{2} = AA^{\#}$. Hence, A is EP, which implies that $X = (A^{\#})^{+} = (A^{+})^{+} = A$ is a solution. By Case 1, A is SEP.

Case 3. If $X = (A^{+})^{\#} = (AA^{\#})^{H}A(AA^{\#})^{H}$ is a solution of Eq.(3.8), then

$$A^{+}(AA^{\#})^{H}A(AA^{\#})^{H}(A^{+})^{H} = (AA^{\#})^{H}A(AA^{\#})^{H},$$

that is,

$$(AA^{\#})^{H}(A^{+})^{H} = (AA^{\#})^{H}A(AA^{\#})^{H}.$$

Multiplying the equality on the left by A^{+} yields $A^{+}(A^{+})^{H} = (AA^{\#})^{H}$, which implies $AA^{\#} = (AA^{\#})^{H}$. Hence A is EP by [12, Theorem 1.1.3], which induces that $X = (A^{+})^{\#} = (A^{\#})^{\#} = A$ is a solution. Thus, A is SEP by Case 1.

Case 4. If $X = AA^{+}$ is a solution of Eq.(3.8), then $A^{+}(AA^{+})(A^{+})^{H} = AA^{+}$, that is,

$$A^{+}(A^{+})^{H} = AA^{+}. \tag{3.10}$$

This gives

$$AA^{\#} = AA^{+}AA^{\#} = A^{+}(A^{+})^{H}AA^{\#} = A^{+}(A^{+})^{H} = AA^{+}.$$

Hence, A is EP and so $A = A^{2}A^{+} = AA^{+}(A^{+})^{H} = (A^{+})^{H}$ by (3.10). Therefore, A is SEP.

Case 5. If $X = AA^{H}$ is a solution of Eq.(3.8), then $A^{+}(AA^{H})(A^{+})^{H} = AA^{H}$, that is, $A^{+}A = AA^{H}$. Thus, A is SEP by [12, Theorem 1.5.3].

Case 6. If $X = AA^{\#}$ is a solution of Eq.(3.8), then $A^{+}(AA^{\#})(A^{+})^{H} = AA^{\#}$, that is, $A^{+}(A^{+})^{H} = AA^{\#}$. This gives $AA^{\#} = (AA^{\#})^{H}$. Hence, A is EP and $X = AA^{\#} = AA^{+}$. By Case 4, A is SEP.

Case 7. If $X = (AA^{\#})^{H}$ is a solution of Eq.(3.8), then $A^{+}(AA^{\#})^{H}(A^{+})^{H} = (AA^{\#})^{H}$, that is, $A^{+}(A^{+})^{H} = (AA^{\#})^{H}$. Hence, A is EP and $X = (AA^{\#})^{H} = AA^{\#}$. By Case 6, A is SEP.

∎

Now we construct a matrix equation similar to Eq.(3.8) as follows.

$$A^{+}XAA^{\#} = XA^{H}. \tag{3.11}$$

Theorem 3.2.2. *Let $A \in G_n(\mathbb{C})$. Then $A \in \mathbb{C}_n^{\mathrm{SEP}}$ if and only if Eq.(3.11) has at least one solution in ζ_A.*

PROOF. " $\Rightarrow$ " From the hypothesis, one gets $A^{+} = A^{H} = A^{\#}$. Thus, $A^{+}AAA^{\#} = A^{+}A = AA^{H}$, which tells that $X = A$ is a solution of Eq.(3.11).

" $\Leftarrow$ " Suppose that some $X \in \zeta_A$ is a solution of Eq.(3.11). It is divided into twelve cases as follows.

Case 1. If $X = A$, then $A^{+}AAA^{\#} = AA^{H}$, that is, $A^{+}A = AA^{H}$. Hence A is SEP by [12, Theorem 1.5.3].

Case 2. If $X = A^{\#}$, then $A^{+}A^{\#}AA^{\#} = A^{\#}A^{H}$, that is, $A^{+}A^{\#} = A^{\#}A^{H}$. Hence, $AA^{H} = A^{2}A^{\#}A^{H} = A^{2}A^{+}A^{\#} = AA^{\#}$, which implies that A is SEP by [12, Theorem 1.5.3].

Case 3. If $X = A^{+}$, then $A^{+}A^{+}AA^{\#} = A^{+}A^{H}$. By [9, Lemma 2.11], one gets $A^{+}AA^{\#} = A^{H}$. Hence, A is SEP by [12, Theorem 1.5.3].

Case 4. If $X = A^{H}$, then $A^{+}A^{H}AA^{\#} = A^{H}A^{H}$. Hence,

$$\begin{aligned} A^{H} &= A^{H}A^{H}(A^{+})^{H}A^{H}(A^{\#})^{H} \\ &= A^{+}A^{H}AA^{\#}(A^{+})^{H}A^{H}(A^{\#})^{H} \\ &= A^{+}[A^{H}(A^{+})^{H}A^{H}(A^{\#})^{H}] \\ &= A^{+}(A^{\#}A)^{H} \\ &= A^{+}. \end{aligned}$$

Therefore, $A \in \mathbb{C}_n^{\mathrm{SEP}}$ by Case 3.

Case 5. If $X = (A^{+})^{H}$, then $A^{+}(A^{+})^{H}AA^{\#} = (A^{+})^{H}A^{H}$, that is, $A^{+}(A^{+})^{H} = AA^{+}$. Hence, A is SEP by Case 4 of Theorem 3.2.1.

Case 6. If $X = (A^{\#})^{H}$, then

$$A^{+}(A^{\#})^{H}AA^{\#} = (A^{\#})^{H}A^{H}. \tag{3.12}$$

Multiplying the equality on the right by $(A^+)^H$ and $A^H A^H$ successively yields $A^+ = A^H$. Thus, one has $A^H(A^\#)^H AA^\# = (A^\#)^H A^H$ by (3.12). It follows that

$$AA^H = AA^H(A^\#)^H A^H = AA^H A^H(A^\#)^H AA^\# = AA^H AA^\# = AA^+AA^\# = AA^\#,$$

which tells that A is SEP by [12, Theorem 1.5.3].

Case 7. If $X = (A^\#)^+ = A^+A^3A^+$, then $A^+A^+A^3A^+AA^\# = A^+A^3A^+A^H$. As a result,

$$A^+A^+A^2 = A^+A^3A^+A^H = A^+A^3A^+A^H AA^+ = A^+A^+A^3A^+.$$

By [9, Lemma 2.11], one yields $A^+A^2 = A^+A^3A^+$. By the proof of Case 2 of Theorem 3.2.1, A is EP, which implies $X = (A^\#)^+ = A$. Hence, A is SEP by Case 1.

Case 8. If $X = (A^+)^\# = (AA^\#)^H A(AA^\#)^H$, then

$$A^+(AA^\#)^H A(AA^\#)^H AA^\# = (AA^\#)^H A(AA^\#)^H A^H$$

that is,

$$(AA^\#)^H AA^\# = (AA^\#)^H AA^H.$$

Thus,

$$A^+AA^\# = A^+(AA^\#)^H AA^\# = A^+(AA^\#)^H AA^H = A^+AA^H = A^H.$$

Hence, A is SEP by [12, Theorem 1.5.3].

Case 9. If $X = AA^+$, then $A^+(AA^+)AA^\# = AA^+A^H$, that is, $A^+AA^\# = AA^+A^H$. Multiplying the equality on the right by AA^+ yields $A^+ = AA^+A^H$. Hence A is SEP by [12, Theorem 1.5.3].

Case 10. If $X = AA^H$, then $A^+AA^H AA^\# = AA^H A^H$. Multiplying the equality on the right by $(A^+)^H$ and $(AA^\#)^H$ successively yields $(A^\#A)^H = AA^H$, that is, $AA^\# = AA^H$. Hence, A is SEP by Lemma 2.1.4.

Case 11. If $X = AA^\#$, then $A^+(AA^\#)(AA^\#) = AA^\#A^H$, that is, $A^+AA^\# = AA^\#A^H$. Multiplying the equality on the left by A yields $AA^\# = AA^H$, which implies that A is SEP by Lemma 2.1.4.

Case 12. If $X = (AA^\#)^H$, then $A^+(AA^\#)^H AA^\# = (AA^\#)^H A^H$, that is, $A^+AA^\# = A^H$, which further implies that $AA^\# = AA^H$. Hence, A is SEP by Lemma 2.1.4.

∎

Now we construct the following matrix equation:

$$XA^+(A^+)^H + A^\# = X + A^+. \tag{3.13}$$

Theorem 3.2.3. *Let $A \in \mathbb{C}^{n\times n}$. Then $A \in \mathbb{C}_n^{\mathrm{SEP}}$ if and only if Eq.(3.13) has at least one solution in the set $\chi_A = \{A, A^\#, A^+, A^H, (A^+)^H, (A^\#)^H\}$.*

PROOF. " ⇒ " By the assumption, $A^\# = A^+ = A^H$. Thus, $AA^+(A^+)^H + A^\# = AA^+A + A^\# = A + A^+$, which tells that $X = A$ is a solution of Eq.(3.13).

" ⇐ " Suppose that some $X \in \chi_A$ is a solution of Eq.(3.13).

Case 1. If $X = A$, then $AA^+(A^+)^H + A^\# = A + A^+$, that is,

$$(A^+)^H + A^\# = A + A^+. \tag{3.14}$$

Multiplying the equality on the left by $AA^\#$ yields $(A^+)^H + A^\# = A + AA^\#A^+$. By (3.14), one has $A^+ = AA^\#A^+$. By [12, Theorem 1.2.1], A is EP. It follows by (3.14) that $(A^+)^H = A$. Consequently, A is SEP.

Case 2. If $X = A^\#$, then $A^\#A^+(A^+)^H + A^\# = A^\# + A^+$, that is,

$$A^\#A^+(A^+)^H = A^+. \tag{3.15}$$

Multiplying the equality on the left by A^2 and on the right by $AA^\#$ respectively yields $(A^+)^H = A$, namely, $A^+ = A^H$. Hence, one has $A^\# = A^\#A^+A = A^+$ by (3.15). It follows that A is SEP.

Case 3. If $X = A^+$, then

$$A^+A^+(A^+)^H + A^\# = 2A^+. \tag{3.16}$$

Multiplying the equality on the right by $AA^\#$ yields $A^+A^+(A^+)^H + A^\# = 2A^+AA^\#$. Hence, one has $A^+ = A^+AA^\#$ by (3.16), which implies that A is EP, and so $X = A^+ = A^\#$. It follows by Case 2 that A is SEP.

Case 4. If $X = A^H$, then

$$A^HA^+(A^+)^H + A^\# = A^H + A^+. \tag{3.17}$$

Multiplying the equality on the left by A^+A yields $A^HA^+(A^+)^H + A^+AA^\# = A^H + A^+$. Hence, one obtains $A^\# = A^+AA^\#$. As a result, $A^\#A = A^+AA^\#A = A^+A$, which tells that A is EP. By (3.17) one gets $A^HA^+(A^+)^H = A^H$. Hence, A is SEP by Lemma 2.1.6.

Case 5. If $X = (A^+)^H$, then

$$(A^+)^HA^+(A^+)^H + A^\# = (A^+)^H + A^+. \tag{3.18}$$

Multiplying the equality one the right by $AA^\#$ yields

$$(A^+)^HA^+(A^+)^H + A^\# = (A^+)^H + A^+AA^\#.$$

Hence, $A^+ = A^+AA^\#$, which tells that A is EP. As a result, one has $(A^+)^HA^+(A^+)^H = (A^+)^H$ by (3.18). It follows that $A^+(A^+)^HA^+ = A^+ = A^+AA^\#$. Multiplying the equality on both sides by A respectively yields $A = (A^+)^H$, namely, $A^H = A^+$. Consequently, A is SEP.

Case 6. If $X = (A^\#)^H$, then $(A^\#)^HA^+(A^+)^H + A^\# = (A^\#)^H + A^+$. Pre-multiplying the equality by A^+A yields $A^\# = A^+AA^\#$. Hence A is EP, which implies $X = (A^\#)^H = (A^+)^H$. By Case 5, A is SEP.

∎

3.3. *Characterizations of SEP matrix by general solutions of bivariate matrix equations involving A^+ and $A^\#$*

To begin with we construct the following matrix equation

$$A^+X(A^+)^H = Y. \tag{3.19}$$

Theorem 3.3.1. *Let $A \in G_n(\mathbb{C})$. Then the general solution of Eq.(3.19) is given as follows:*

$$\begin{cases} X = P + U - AA^+UAA^+, \\ Y = A^+P(A^+)^H. \end{cases} \quad P, U \in \mathbb{C}^{n\times n}. \tag{3.20}$$

PROOF. On one hand, since $\forall P, U \in \mathbb{C}^{n\times n}$, one has

$$\begin{aligned} & A^+(P + U - AA^+UAA^+)(A^+)^H \\ = \; & A^+P(A^+)^H + A^+U(A^+)^H - A^+U(A^+)^H \\ = \; & A^+P(A^+)^H, \end{aligned}$$

any (X, Y) given by (3.20) is a solution of Eq.(3.19).

On the other hand, suppose that (X_0, Y_0) is any solution of Eq.(3.19), then $A^+X_0(A^+)^H = Y_0$. Let $U = X_0, P = AA^+X_0AA^+$. Then one has

$$\begin{cases} X_0 = P + U - AA^+UAA^+; \\ \quad Y_0 = A^+X_0(A^+)^H = A^+AA^+X_0AA^+(A^+)^H = A^+P(A^+)^H. \end{cases}$$

Consequently, the general solution of Eq.(3.19) is given by (3.20).

∎

Theorem 3.3.2. *Let $A \in G_n(\mathbb{C})$. Then $A \in \mathbb{C}_n^{\mathrm{SEP}}$ if and only if the general solution of Eq.(3.19) is given as follows:*

$$\begin{cases} X = P + U - AA^+UAA^+; \\ Y = A^\# PA. \end{cases} \quad P, U \in \mathbb{C}^{n\times n}. \tag{3.21}$$

PROOF. " $\Rightarrow$ " It follows directly by Theorem 3.3.1.

" $\Leftarrow$ " By the assumption, one has $A^+P(A^+)^H = A^\# PA, \forall P \in \mathbb{C}^{n\times n}$. In particular, let $P = A$ and one obtains $A^+A(A^+)^H = A$. Thus, $A = A^\# AA = A^\# AA^+A(A^+)^H = A^\# A(A^+)^H = (A^+)^H$, namely, $A^+ = A^H$. It follows that $A^+A^2 = A^+A(A^+)^H = A$. As a result, A is EP. Consequently, A is SEP. ∎

Theorem 3.3.2 tells that the general solution of Eq.(3.19) is given by (3.21) provided that A is SEP. One may be curious to know in the case that A is not SEP, which equation has the general solution given by (3.21). Motivated by this, we construct a new matrix equation as follows.

$$A^\#(AA^\#)^H XA = Y. \tag{3.22}$$

Theorem 3.3.3. *Let $A \in G_n(\mathbb{C})$. Then the general solution of Eq.(3.22) is given as follows.*

$$\begin{cases} X = P + U - AA^+UAA^+; \\ Y = A^\# PA. \end{cases} \quad P, U \in \mathbb{C}^{n\times n} \quad \text{and} \quad A^+P = A^+A^+AP. \tag{3.23}$$

PROOF. On one hand, we prove that any (X, Y) given by (3.23) is a solution of Eq.(3.22). In fact, one has

$$\begin{aligned} & A^\#(AA^\#)^H(P + U - AA^+UAA^+)A \\ = \; & A^\#(AA^\#)^H PA + A^\#(AA^\#)^H UA - A^\#(AA^\#)^H AA^+UAA^+A \\ = \; & A^\#(AA^\#)^H PA \\ = \; & A^\#(A^\#)^H(AA^+A)^H PA \\ = \; & A^\#(A^\#)^H A^H A(A^+P)A \\ = \; & A^\#(AA^\#)^H A(A^+A^+AP)A \\ = \; & A^\#(A^+AAA^+AA^\#)^H PA \\ = \; & A^\# A^+APA \\ = \; & A^\# PA \end{aligned}$$

as desired.

On the other hand, let (X_0, Y_0) be any solution of Eq.(3.22), that is, $A^\#(AA^\#)^H X_0A = Y_0$. Let $P = (AA^\#)^H X_0AA^+$ and $U = X_0 + AA^+X_0AA^+ - P$. Then, one has

$$A^+P = A^+(AA^\#)^H X_0AA^+ = A^+X_0AA^+$$

and

$$\begin{aligned} A^+A^+AP = \; & A^+A^+A(AA^\#)^H X_0AA^+ \\ = \; & A^+(AA^\# A^+A)^H X_0AA^+ \\ = \; & A^+(A^\# A)^H X_0AA^+ \\ = \; & A^+X_0AA^+, \end{aligned}$$

which implies that $A^+P = A^+A^+AP$. Since

$$A^\# PA = A^\#(AA^\#)^H X_0AA^+A = A^\#(AA^\#)^H X_0A = Y_0$$

and

$$\begin{aligned} AA^+UAA^+ = \; & AA^+(X_0 + AA^+X_0AA^+ - P)AA^+ \\ = \; & 2AA^+X_0AA^+ - AA^+PAA^+ \\ = \; & 2AA^+X_0AA^+ - AA^+[(AA^\#)^H X_0AA^+]AA^+ \\ = \; & 2AA^+X_0AA^+ - AA^+X_0AA^+ \\ = \; & AA^+X_0AA^+, \end{aligned}$$

one obtains

$$\begin{cases} X_0 = P + U - AA^+UAA^+; \\ Y_0 = A^\# PA, \end{cases} \quad where \quad A^+P = A^+A^+AP.$$

Consequently, the general solution of Eq.(3.22) is given by (3.23). ∎

Theorem 3.3.4. *Let $A \in G_n(\mathbb{C})$. Then $A \in \mathbb{C}_n^{\mathrm{SEP}}$ if and only if Eq.(3.22) and (3.19) share the same solutions.*

PROOF. " $\Rightarrow$ " It is evident since Eq.(3.22) and (3.19) are exactly the same equation under the assumption that A is SEP.

" $\Leftarrow$ " By the assumption, any (X, Y) given by (3.20) is a solution of Eq.(3.22). Hence, $\forall P, U \in \mathbb{C}^{n\times n}$, one has

$$A^\#(AA^\#)^H(P + U - AA^+UAA^+)A = A^+P(A^+)^H,$$

namely,

$$A^\#(AA^\#)^H PA = A^+P(A^+)^H. \tag{3.24}$$

In particular, letting P be A^H yields $A^\#A^HA = A^+A^H(A^+)^H = A^+A^+A$. Hence, $A^\#A^H = A^\#A^HAA^+ = A^+A^+AA^+ = A^+A^+$. Letting P be A in (3.24), one obtains

$$A^\#(AA^\#)^H A^2 = A^+A(A^+)^H. \tag{3.25}$$

Multiplying the equality on the left by $AA^\#$ yields $A^\#(AA^\#)^HA^2 = (A^+)^H$. As a result, $A^+A(A^+)^H = (A^+)^H$, that is, $A^+A^+A = A^+$. Hence, A is EP. It follows by (3.25) that $A = (A^+)^H$, namely, $A^+ = A^H$. Consequently, A is SEP.

∎

Acknowledgements

The authors are thankful to the anonymous referees for their valuable comments on earlier version of this article. Special thanks to Professor Dijana Mosić.

References

[1] O. M. Baksalary, G. Trenkler. *Characterizations of EP, normal and Hermitian matrices*. Linear Multil.Algebra. 56(2006), 299-304.
[2] E. Boasso. *On the Moore-Penrose inverse in C*-algebras*. Extracta Math. 21(2)(2006), 93-106.
[3] R. Bru, N. Thome. *Group inverse and group involutory matrices*. Linear Multil. Algebra. 45(1998), 207-218.
[4] J. J. Koliha, P. Patrićio. *Elements of rings with equal spectral idempotents*. J. Aust. Math. Soc. 72(1)(2002), 137-152.
[5] D. Mosić, D. S. Djordjević. *Partial isometries and EP elements in rings with involution*. Electron. J. Linear Algebra 18(2009), 761-772.
[6] D. Mosić, D. S. Djordjević. *Further results on partial isometries and EP elements in rings with involution*. Math. Comput. Model. 54(2011), 460-465.
[7] X. R. Wang, J. C. Wei. *EP matrix and the solution of matrix equation*. Filomat 38(8)(2024), 2747-2753.
[8] Z. C. Xu, R. J. Tang, J. C. Wei. *Strongly EP elements in a ring with involution*. Filomat 34(6)(2020), 2101-2107.
[9] D. D. Zhao, J. C. Wei. *Strongly EP elements in rings with involution*. J. Algebra Appl. (2022), 2250088, 10 pages.
[10] R. J. Zhao, H. Yao, J. C. Wei. *Characterizations of partial isometries and two special kinds of EP elements*. Czecho. Math. J. 70(2)(2020), 539-551.
[11] A. Ben-Israel, T. N. E. Greville. *Generalized Inverses: Theory and Applications*, 2nd edn. Springer, New York, 2003.
[12] D. Mosić. *Generalized inverses*. Faculty of Sciences and Mathematics, University of Niš, Niš, 2018.
[13] C. R. Rao, S. K. Mitra. *Generalized inverse of matrices and its applications*. Wiley, New York, 1971.