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On topological properties of Argmin multifunction

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Abstract. Let X be a Tychonoff topological space, $C(X,\mathbb{R})$ be the space of continuous real-valued functions defined on X and $K(X)$ be the space of all nonempty compact subsets of X. The multifunction argmin : $C(X,\mathbb{R}) \times K(X) \to X$ is defined as follows: $\operatorname{argmin}(f,K) = \{x \in K : f(x) = \min\{f(y) : y \in K\}\}$. We present topologies on $C(X,\mathbb{R}) \times K(X)$ under which argmin : $C(X,\mathbb{R}) \times K(X) \to X$ has a closed graph. We also extend a generic optimization theorem of G. Beer from Čech-complete spaces to locally Čech-complete ones.

1. Introduction

Let X be a Tychonoff topological space and let $C(X,\mathbb{R})$ denote the space of continuous real-valued functions defined on X. It is well known ([2, 5, 6, 14, 20]) that if X is a compact metric space, then with respect to the topology of uniform convergence on $C(X,\mathbb{R})$, most functions (in the sense of Baire category) have a unique minimum.

In [19, 20] Kenderov initiated a study of constrained minimization problems. Let $K(X)$ be the space of all nonempty compact subsets of X. Each pair (f,K), where $f \in C(X,\mathbb{R})$ and $K \in K(X)$ determines a (constrained) minimization problem: find $x \in K$ at which f attains its minimum over K.

Define the multifunction argmin : $C(X,\mathbb{R}) \times K(X) \to X$ as follows:

$$\operatorname{argmin}(f,K) = \{x \in K : f(x) = \min\{f(y) : y \in K\}\}.$$

We are interested in topological properties of argmin multifunction.
Any reasonable topology τ on $C(X,\mathbb{R}) \times K(X)$ should have the property that the graph of argmin is closed, i.e., if $\{(f_\lambda, K_\lambda)\}$ is τ convergent to (f,K) and $x_\lambda \in \operatorname{argmin}(f_\lambda, K_\lambda)$ for every λ, then the convergence of $\{x_\lambda\}$ to x implies $x \in \operatorname{argmin}(f,K)$. This property is very important in optimization.

There is a rich literature on argmin multifunction either on the space of lower semicontinuous functions [2, 4, 24] or on the space of continuous functions [3, 9, 14, 19, 20].

In our paper we present topologies on $C(X,\mathbb{R}) \times K(X)$ under which argmin : $C(X,\mathbb{R}) \times K(X) \to X$ has a closed graph. We also extend a generic optimization theorem of G. Beer from Čech-complete spaces to locally Čech-complete ones.

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2. Preliminaries

In the sequel, all spaces are assumed to be Hausdorff, $\mathbb{N}$ denotes the space of all positive integers and $\mathbb{R}$ denotes the Euclidean space of real numbers with the usual Euclidean metric.

The Vietoris (or finite) topology τ_V [21] on the collection of nonempty compact subsets $K(X)$ of a topological space X has a subbase all sets of the form

$$V^- = \{A : A \in K(X) \text{ and } A \cap V \neq \emptyset\}$$

and

$$V^+ = \{A : A \in K(X) \text{ and } A \subset V\},$$

where V runs over the open subsets of X.

The Fell topology τ_F [3] on the collection of nonempty compact subsets $K(X)$ of a space X has a subbase all sets of the form

$$V^- = \{A : A \in K(X) \text{ and } A \cap V \neq \emptyset\}$$

and

$$(K^c)^+ = \{A : A \in K(X) \text{ and } A \cap K = \emptyset\},$$

where V runs over the open subsets of X and K runs over compact subsets of X.

The upper Vietoris topology τ_{V^+} [3, 13] on the collection of nonempty compact subsets $K(X)$ of a space X has a base all sets of the form

$$V^+ = \{A : A \in K(X) \text{ and } A \subset V\},$$

where V runs over the open subsets of X and the lower Vietoris topology τ_{V^-} [3] on the collection of nonempty compact subsets $K(X)$ of a space X has a subbase all sets of the form

$$V^- = \{A : A \in K(X) \text{ and } A \cap V \neq \emptyset\},$$

where V runs over the open subsets of X.

A base for the Vietoris topology is the family of sets of the form

$$[V_1, \dots, V_n] = \bigcap_{i=1}^{n} V_i^- \cap (\bigcup_{i=1}^{n} V_i)^+,$$

where $V_1, \dots, V_n$ are open sets in X.

Let X be a topological space and (Y, d) be a metric one. We will remind the topologies of uniform convergence and the topology of uniform convergence on compacta on Y^X.

The topology τ_U of uniform convergence on Y^X is induced by the uniformity $\mathfrak{U}_U$ of uniform convergence which has a base consisting of all sets of the form

$$W(\varepsilon) = \{(f, g) : \forall x \in X,\ d(f(x), g(x)) < \varepsilon\},$$

where $\varepsilon > 0$. The general τ_U-basic neighborhood of $f \in Y^X$ is the set $W(f, \varepsilon) = \{g \in Y^X : d(f(x), g(x)) < \varepsilon \text{ for every } x \in X\}$.

The topology τ_{UC} of uniform convergence on compacta on Y^X is induced by the uniformity $\mathfrak{U}_{UC}$ of uniform convergence on compacta which has a base consisting of all sets of the form

$$W(K, \varepsilon) = \{(f, g) : \forall x \in K,\ d(f(x), g(x)) < \varepsilon\},$$

where $K \in K(X)$ and $\varepsilon > 0$. The general τ_{UC}-basic neighborhood of $f \in Y^X$ is the set $W(f, K, \varepsilon) = \{g \in Y^X : d(f(x), g(x)) < \varepsilon \text{ for every } x \in K\}$.

A set-valued map, or a multifunction from X to Y is a function that assigns to each element of X a subset of Y. Following [7] the term map is reserved for a set-valued map. If F is a map from X to Y, then its graph is the set $\{(x, y) \in X \times Y : y \in F(x)\}$. In our paper we will identify maps with their graphs.

Notice that if $f : X \to Y$ is a single-valued function, we will use the symbol f also for the graph of f.

Let X and Y be topological spaces. A map $F : X \to Y$ is upper semicontinuous at a point $x \in X$ if for every open set V containing $F(x)$, there exists an open set U such that $x \in U$ and

$$F(U) = \bigcup\{F(u) : u \in U\} \subset V.$$

F is upper semicontinuous if it is upper semicontinuous at each point of X. Following Christensen [17] we say, that a map F is usco if it is upper semicontinuous and takes nonempty compact values. Finally, a map F from a topological space X to a topological space Y is said to be minimal usco if it is a minimal element in the family of all usco maps (with the domain X and the range Y); that is, if it is usco and does not contain properly any other usco map from X into Y. Using the Kuratowski-Zorn principle we can guarantee that every usco map from X to Y contains a minimal usco map from X to Y (see [7]).

A map $F : X \to Y$ is subcontinuous at $x \in X$ [16], iff for every net $\{x_\lambda : \lambda \in \Lambda\}$ converging to x and for every $y_\lambda \in F(x_\lambda)$, the net $\{y_\lambda : \lambda \in \Lambda\}$ has a cluster point. F is subcontinuous if it is subcontinuous at each point of X.

In the last few decades, the minimal usco maps have found many applications in different areas of mathematics, for example, in optimization, in the study of differentiability of Lipschitz functions, or in selection theorems. The recent book [11] Usco and quasicontinuous mappings contains some applications of minimal usco and minimal cusco maps in the above mentioned areas.

3. When has argmin multifunction closed graph?

We prove the following theorem:

Theorem 3.1. *Let X be a locally compact Hausdorff space. Then* argmin : $(C(X,\mathbb{R}),\tau_{UC}) \times (K(X),\tau_F) \to X$ *has a closed graph.*

Proof. Let $\{(f_\lambda, K_\lambda) : \lambda \in \Lambda\}$ converge to (f, K) in $(C(X,\mathbb{R}),\tau_{UC}) \times (K(X),\tau_F)$. Let $x_\lambda \in \operatorname{argmin}(f_\lambda, K_\lambda)$ for every $\lambda \in \Lambda$ and let $\{x_\lambda : \lambda \in \Lambda\}$ converge to $x \in X$. We will prove that $x \in \operatorname{argmin}(f, K)$. First we prove that $x \in K$. Suppose $x \notin K$. There is an open set U in X such that $x \in U$, $\overline{U}$ is compact and $\overline{U} \cap K = \emptyset$. Since $\{K_\lambda : \lambda \in \Lambda\}$ converges to K in $(K(X),\tau_F)$, there is λ_0 such that $\overline{U} \cap K_\lambda = \emptyset$ for every $\lambda \geq \lambda_0$. A contradiction since $\{x_\lambda : \lambda \in \Lambda\}$ converge to x. Thus $x \in K$. Suppose there is $y \in K$ such that

$$f(y) < f(x).$$

There is $\epsilon > 0$ such that $f(y) + 6\epsilon < f(x)$. The continuity of the function f at x and y and the local compactness of X imply the existence of open sets U and V such that $x \in U$, $y \in V$, $\overline{U}$, $\overline{V}$ are compact sets and $\overline{U} \cap \overline{V} = \emptyset$ and for every $z \in U$

$$|f(z) - f(x)| < \epsilon,$$

for every $s \in V$

$$|f(s) - f(y)| < \epsilon.$$

The sequence $\{x_\lambda : \lambda \in \Lambda\}$ converges to $x \in X$. Thus there is λ_0 such that $x_\lambda \in U$ for every $\lambda \geq \lambda_0$. Since $\{K_\lambda : \lambda \in \Lambda\}$ converges to K in $(K(X),\tau_F)$, there is λ_1 such that $K_\lambda \cap V \neq \emptyset$ for every $\lambda \geq \lambda_1$. For every $\lambda \geq \lambda_1$ choose $y_\lambda \in K_\lambda \cap V$. The sequence $\{f_\lambda : \lambda \in \Lambda\}$ converges to f in $(C(X,\mathbb{R}),\tau_{UC})$. Thus there is λ_2 such that for every $\lambda \geq \lambda_2$ we have

$$|f_\lambda(z) - f(z)| < \epsilon$$

for every $z \in \overline{U} \cup \overline{V}$. Let $\lambda^* \geq \lambda_i$ for every $i \in \{0, 1, 2\}$. Then for $\lambda \geq \lambda^*$ we have

$$|f_\lambda(y_\lambda) - f(y)| \leq |f_\lambda(y_\lambda) - f(y_\lambda)| + |f(y_\lambda) - f(y)| < 2\epsilon,$$

$$|f_\lambda(x_\lambda) - f(x)| \leq |f_\lambda(x_\lambda) - f(x_\lambda)| + |f(x_\lambda) - f(x)| < 2\epsilon.$$

We received the following inequalities:

$$f(x) - 2\epsilon < f_\lambda(x_\lambda) < f(x) + 2\epsilon, f(y) - 2\epsilon < f_\lambda(y_\lambda) < f(y) + 2\epsilon.$$

$$f_\lambda(x_\lambda) - f_\lambda(y_\lambda) > f(x) - f(y) - 4\epsilon > 2\epsilon,$$

for $\lambda \geq \lambda^*$. It is a contradiction, since $x_\lambda \in \operatorname{argmin}(f_\lambda, K_\lambda)$ for every $\lambda \in \Lambda$. □

We have the following characterization of local compactness.

Theorem 3.2. *Let X be a Tychonoff topological space. The following are equivalent:*
(1) X is locally compact,
(2) The argmin : $(C(X,\mathbb{R}),\tau_{UC})\times(K(X),\tau_F)\to X$ *has a closed graph.*

Proof. It is sufficient to prove that (2) ⇒ (1). Suppose that argmin : $(C(X,\mathbb{R}),\tau_{UC})\times(K(X),\tau_F)\to X$ has a closed graph and X is not a locally compact space. Let x_0 be a point from X which fails to have a compact neighbourhood. Let $\mathcal{U}(x_0)$ be the family of all open neighbourhoods of x_0. For every $U\in\mathcal{U}(x_0)$ and for every $K\in K(X)$ there is $x_{U,K}\in U\setminus(K\cup\{x_0\})$. Let $y\in X$ be a point different from x_0. There is a continuous function $f:X\to[0,1]$ such that $f(x_0)=1$ and $f(y)=0$. For every $U\in\mathcal{U}(x_0)$ and for every $K\in K(X)$ put $C_{U,K}=\{x_{U,K},x_0,y\}$ and let $f_{U,K}$ be a continuous function from X to $[0,1]$ such that $f_{U,K}(x_{U,K})=0$ and $f_{U,K}|K=f|K$. Such an extension exists since X is a Tychonoff space.

Let $K(X)$ be directed by the inclusion and $\mathcal{U}(x_0)$ by the reverse inclusion and consider the natural product direction on $\mathcal{U}(x_0)\times K(X)$. The net

$$\{(f_{U,K},C_{U,K}):(U,K)\in\mathcal{U}(x_0)\times K(X)\}$$

converges to $(f,\{x_0,y\})$ in $(C(X,\mathbb{R}),\tau_{UC})\times(K(X),\tau_F)$. First we prove that $\{C_{U,K}:(U,K)\in\mathcal{U}(x_0)\times K(X)\}$ converges to $\{x_0,y\}$ in $(K(X),\tau_F)$. Let $C\in K(X)$ be such that $\{x_0,y\}\in(C^c)^+$. Let V be a fixed open neighbourhood of x_0. Then for every $(U,K)\geq(V,C)$ we have $C_{U,K}\in(C^c)^+$, since $x_{U,K}\in U\setminus K\subset V\setminus C$.

It is easy to verify that $\{f_{U,K}:(U,K)\in\mathcal{U}(x_0)\times K(X)\}$ converges to f in $(C(X,\mathbb{R}),\tau_{UC})$ and $\{x_{U,K}:(U,K)\in\mathcal{U}(x_0)\times K(X)\}$ converges to x_0. For every $(U,K)\in\mathcal{U}(x_0)\times K(X)$, $x_{U,K}\in\operatorname{argmin}(f_{U,K},C_{U,K})$, but $x_0\notin\operatorname{argmin}(f,\{x_0,y\})$, a contradiction. □

Theorem 3.3. *Let X be a Tychonoff topological space. Then* argmin : $(C(X,\mathbb{R}),\tau_U)\times(K(X),\tau_V)\to X$ *has a closed graph.*

Proof. Let $\{(f_\lambda,K_\lambda):\lambda\in\Lambda\}$ converge to (f,K) in $(C(X,\mathbb{R}),\tau_U)\times(K(X),\tau_V)$. Let $x_\lambda\in\operatorname{argmin}(f_\lambda,K_\lambda)$ for every $\lambda\in\Lambda$ and let $\{x_\lambda:\lambda\in\Lambda\}$ converge to $x\in X$. We will prove that $x\in\operatorname{argmin}(f,K)$. First we prove that $x\in K$. Suppose $x\notin K$. There is an open set U in X such that $x\in U$ and $\overline{U}\cap K=\emptyset$. Since $\{K_\lambda:\lambda\in\Lambda\}$ converges to K in $(K(X),\tau_V)$, there is λ_0 such that $\overline{U}\cap K_\lambda=\emptyset$ for every $\lambda\geq\lambda_0$, a contradiction since $\{x_\lambda:\lambda\in\Lambda\}$ converge to x. Thus $x\in K$. Suppose there is $y\in K$ such that

$$f(y)<f(x).$$

There is $\epsilon>0$ such that $f(y)+6\epsilon<f(x)$. The continuity of the function f at x and y implies the existence of open sets U and V such that $x\in U$, $y\in V$, and $U\cap V=\emptyset$ and for every $z\in U$

$$|f(z)-f(x)|<\epsilon,$$

for every $s\in V$

$$|f(s)-f(y)|<\epsilon.$$

The sequence $\{x_\lambda:\lambda\in\Lambda\}$ converges to $x\in X$. Thus there is λ_0 such that $x_\lambda\in U$ for every $\lambda\geq\lambda_0$. Since $\{K_\lambda:\lambda\in\Lambda\}$ converges to K in $(K(X),\tau_V)$, there is λ_1 such that $K_\lambda\cap V\neq\emptyset$ for every $\lambda\geq\lambda_1$. For every $\lambda\geq\lambda_1$ choose $y_\lambda\in K_\lambda\cap V$. The sequence $\{f_\lambda:\lambda\in\Lambda\}$ converges to f in $(C(X),\tau_U)$. Thus there is λ_2 such that for every $\lambda\geq\lambda_2$ we have

$$|f_\lambda(z)-f(z)|<\epsilon$$

for every $z\in X$. Let $\lambda^*\geq\lambda_i$ for every $i\in\{0,1,2\}$. As in the proof of Theorem 3.1 we obtain that for $\lambda\geq\lambda^*$ we have

$$f_\lambda(x_\lambda)-f_\lambda(y_\lambda)>f(x)-f(y)-4\epsilon>2\epsilon.$$

It is a contradiction, since $x_\lambda\in\operatorname{argmin}(f_\lambda,K_\lambda)$ for every $\lambda\in\Lambda$. □

Notice that Theorem 3.3 can be deduced from the proof of Theorem 3.2 and Lemma 4.1 in [2], however we offered a direct proof.

Theorem 3.4. *Let X be a Tychonoff topological space. Then the multifunction* argmin : $(C(X,\mathbb{R}),\tau_U)\times(K(X),\tau_{V^+}) \to X$ *is subcontinuous.*

Proof. Let $\{(f_\lambda, K_\lambda) : \lambda \in \Lambda\}$ converge to (f, K) in $(C(X,\mathbb{R}),\tau_U)\times(K(X),\tau_{V^+})$. Let $x_\lambda \in \text{argmin}(f_\lambda, K_\lambda)$ for every $\lambda \in \Lambda$. We will show that $\{x_\lambda : \lambda \in \Lambda\}$ has a cluster point in K. Suppose that no point of K is a cluster point of $\{x_\lambda : \lambda \in \Lambda\}$. For every $x \in K$ there is an open neighbourhood O_x of x and $\lambda_x \in \Lambda$ such that $x_\lambda \notin O_x$ for all $\lambda \geq \lambda_x$. Since K is compact there exist points $x_1, x_2, ..., x_n$ from K such that

$$K \subset \cup_{i=1}^n O_{x_i}.$$

Let $\lambda^* \in \Lambda$ be such that $\lambda^* \geq \lambda_{x_i}$ for every $i \in \{1, 2, ...n\}$. Then for every $\lambda \geq \lambda^*$, $x_\lambda \notin \cup_{i=1}^n O_{x_i}$, a contradiction since $\{K_\lambda; \lambda \in \Lambda\}$ converges to K in $(K(X),\tau_{V^+})$. □

Theorem 3.5. ([19] for X compact) *Let X be a Tychonoff topological space. Then* argmin : $(C(X,\mathbb{R}),\tau_U)\times(K(X),\tau_V) \to X$ *is an usco map.*

Proof. By Theorem 3.3 argmin : $(C(X,\mathbb{R}),\tau_U)\times(K(X),\tau_V) \to X$ has a closed graph. By Theorem 3.4 argmin : $(C(X,\mathbb{R}),\tau_U)\times(K(X),\tau_{V^+}) \to X$ is subcontinuous, thus also argmin : $(C(X,\mathbb{R}),\tau_U)\times(K(X),\tau_V) \to X$ is subcontinuous. By Corollary 3.12 in [12] every subcontinuous multifunction with a closed graph and values in a regular space is usco. Thus argmin : $(C(X,\mathbb{R}),\tau_U)\times(K(X),\tau_V) \to X$ is usco. □

Now we show that argmin : $(C(\mathbb{R},\mathbb{R}),\tau_{UC})\times(K(\mathbb{R}),\tau_F) \to \mathbb{R}$ is not upper semicontinuous.

Example 3.6. Let $f : \mathbb{R} \to \mathbb{R}$ be the function identically equal to 0. For every $n \in \mathbb{N}$ let $f_n : \mathbb{R} \to \mathbb{R}$ be defined as follows: $f_n(x) = 1/n$ for $x \leq 0$, $f_n(x) = 0$, $x \geq n$ and f_n is linear on the interval $[0, n]$. For every $n \in \mathbb{N}$ put $K_n = \{0, n\}$. Then $\{(f_n, K_n) : n \in \mathbb{N}\}$ converges to $(f, \{0\})$ in $(C(\mathbb{R},\mathbb{R}),\tau_{UC})\times(K(\mathbb{R}),\tau_F)$. Notice that $\text{argmin}(f, \{0\}) = \{0\}$ and for every $n \in \mathbb{N}$, $\text{argmin}(f_n, K_n) = \{n\}$. Thus argmin : $(C(\mathbb{R},\mathbb{R}),\tau_{UC})\times(K(\mathbb{R}),\tau_F) \to \mathbb{R}$ is not upper semicontinuous.

Notice that Example 3.6 shows that argmin : $(C(\mathbb{R},\mathbb{R}),\tau_U)\times(K(\mathbb{R}),\tau_F) \to \mathbb{R}$ is not upper semicontinuous too.

The following result was proved in [15].

Theorem 3.7. ([15]) *Let X be a Tychonoff topological space. Then* argmin : $(C(X,\mathbb{R}),\tau_U)\times(K(X),\tau_V) \to X$ *is a minimal usco map.*

If X is compact, then $(C(X,\mathbb{R}),\tau_U)\times(K(X),\tau_V)$ is a Baire space, so the question do most constrained minimization problems have a unique solution makes sense. Kenderov in [19, 20] obtained an affirmative answer for a large class of compact spaces, including not only metrizable ones, but also Stegall spaces. Beer in [2] extended Kenderov's generic optimization theorem to Čech-complete spaces and Holá in [15] to Tychonoff almost Čech-complete spaces.

4. Results of Beer

In this part we improve a generic optimization theorem of G. Beer concerning partial maps from Čech-complete spaces to locally Čech-complete ones.

We denote the space of all partial maps from a Hausdorff topological space X to $\mathbb{R}$ by $\widetilde{C}(X,\mathbb{R})$. Formally,

$$\widetilde{C}(X,\mathbb{R}) = \bigcup\{C(K,\mathbb{R}) : K \in K(X)\}.$$

If $f \in \widetilde{C}(X,\mathbb{R})$ we write $D(f)$ for its domain. We will identify elements of $\widetilde{C}(X,\mathbb{R})$ with their graphs and consider $\widetilde{C}(X,\mathbb{R})$ as a subspace of $K(X\times\mathbb{R})$ equipped with the Vietoris topology τ_V.

In [2] argmin multifunction from $\widetilde{C}(X,\mathbb{R})$ to X was studied, where

$$\text{argmin}(f) = \{x \in D(f) : f(x) = min\{f(y) : y \in D(f)\}\}.$$

The following Lemma was proved in [2] with the additional assumption that X is regular. However, this assumption is not used in the proof.

Lemma 4.1. *Let X be a Hausdorff topological space. Then* argmin : $(\widetilde{C}(X,\mathbb{R}),\tau_V) \to X$ *is usco map.*

We will prove that argmin is a minimal usco map.

Theorem 4.2. *Let X be a Hausdorff topological space. Then* argmin : $(\widetilde{C}(X,\mathbb{R}),\tau_V) \to X$ *is a minimal usco map.*

Proof. Suppose that argmin is not minimal usco. Since argmin is usco, there is a minimal usco map $L : (\widetilde{C}(X,\mathbb{R}),\tau_V) \to X$ such that $L \subset$ argmin. Let $f \in \widetilde{C}(X,\mathbb{R})$ be such that $L(f) \neq \text{argmin}(f)$. Let $a \in \text{argmin}(f) \setminus L(f)$. Let U, V be two disjoint open sets in X such that $a \in U$ and $L(f) \subset V$. Let G be an open set in $(\widetilde{C}(X,\mathbb{R}),\tau_V)$ such that $f \in G$ and $L(G) \subset V$. There are open sets $S_1,\ldots,S_n$ in X and open intervals $I_1,\ldots,I_n$ in $\mathbb{R}$ such that

$$f \in [S_1 \times I_1,\ldots,S_n \times I_n] \subset G.$$

Let $j \in \{1,\ldots,n\}$ be such that $(a, f(a)) \in S_j \times I_j$ and let ε be such that $(f(a) - 2\varepsilon, f(a) + 2\varepsilon) \subset I_j$. For every $i \in \{1,\ldots,n\}$, $i \neq j$ choose $(x_i, f(x_i)) \in S_i \times I_i$. Without loss of generality we can suppose that all points a and x_i, $i \in \{1,\ldots,n\}$, $i \neq j$ are different (otherwise we will consider only different points which contain a).

Put $M = \{a\} \cup \{x_i : i = 1,\ldots,n, i \neq j\}$. Define the function $g : M \to \mathbb{R}$ as follows: $g(a) = f(a) - \varepsilon$ and $g(x_i) = f(x_i)$ for every $i \in \{1,\ldots,n\}$, $i \neq j$. It is easy to verify that

$$g \in [S_1 \times I_1,\ldots,S_n \times I_n] \subset G,$$

a contradiction, since argmin$(g) = \{a\}$, $L(g) \subset V$ and $a \notin V$. □

We also have the following result.

Proposition 4.3. *Let X be a Hausdorff topological space. Then the set $\{f \in \widetilde{C}(X,\mathbb{R}) : |\text{argmin}(f)| = 1\}$ is dense in $(\widetilde{C}(X,\mathbb{R}),\tau_V)$.*

Proof. Let $f \in \widetilde{C}(X,\mathbb{R})$. Let $S_1,\ldots,S_n$ be open sets in X and $I_1 \ldots, I_n$ be open intervals such that

$$f \in [S_1 \times I_1,\ldots,S_n \times I_n].$$

Let $(x_i, f(x_i)) \in S_i \times I_i$ for every $i \in \{1,\ldots,n\}$. Without loss of generality we can suppose that all x_i are different. Put $b = min\{f(x_i) : i = 1,\ldots,n\}$. Let $j \in \{1,2,...,n\}$ be such that $f(x_j) = b$ and $\varepsilon > 0$ be such that $(b - 2\varepsilon, b + 2\varepsilon) \subset I_j$. Put $M = \{x_1,\ldots,x_n\}$. Define the function $g : M \to \mathbb{R}$ such that $g(x_j) = b - \varepsilon$ and $g(x_i) = f(x_i)$ if $i \neq j$. Then the function g has the unique minimum, $g \in \widetilde{C}(X,\mathbb{R})$ and

$$g \in [S_1 \times I_1,\ldots,S_n \times I_n].$$

□

Beer in his paper [2] proved that if X is a Čech-complete space, then also $(\widetilde{C}(X,\mathbb{R}),\tau_V)$ is Čech-complete. Beer also proved the following result.

Lemma 4.4. *Let X be a Tychonoff topological space. Then $\widetilde{C}(X,\mathbb{R})$ is a G_δ set in $(K(X \times \mathbb{R}),\tau_V)$.*

We will generalize Beer's result for locally Čech-complete spaces. A Tychonoff topological space X is called *locally Čech-complete* [8] if every point $x \in X$ has a Čech-complete neighbourhood.

Lemma 4.5. ([1]) *Let X be a locally Čech complete space. Then every G_δ subset of X is a locally Čech complete space.*

Lemma 4.6. ([14]) *Let X be a locally Čech-complete space. Then $(K(X),\tau_V)$ is locally Čech-complete too.*

Proposition 4.7. *Let X be a locally Čech-complete space. Then also $(\widetilde{C}(X,\mathbb{R}),\tau_V)$ is locally Čech-complete.*

Proof. Since X is a locally Čech complete space and $\mathbb{R}$ is completely metrizable, $X \times \mathbb{R}$ is locally Čech complete and by Lemma 4.6 $(K(X \times \mathbb{R}),\tau_V)$ is a locally Čech complete space. By Lemma 4.4 $\widetilde{C}(X,\mathbb{R})$ is a G_δ set in $(K(X \times \mathbb{R}),\tau_V)$. Thus by Lemma 4.5 $(\widetilde{C}(X,\mathbb{R}),\tau_V)$ is locally Čech-complete. □

5. On generic optimization theorem of Petar Kenderov

We now state the conditions under which most constrained minimization problems have a unique solution.

Recall that a topological space Y is *Stegall* [9], if for every Baire space X, every minimal usco map from X to Y, is single-valued at points of a residual set in X. A residual set in X is any set whose complement is the set of the first Baire category in X.

A prime example of a Stegall space is a fragmentable space, in particular, any metric space, see [23].

Proposition 5.1. *Let Y be a regular locally metrizable space. Then Y is a Stegall space.*

Proof. Let X be a Baire topological space and $F : X \to Y$ be a minimal usco map. Put $L = \{x \in X : |F(x)| = 1\}$. We will prove that the set $X \setminus L$ is of the first Baire category. Let U be a non-empty open set in X. Let $x \in U$. For every $y \in F(x)$ there is a neighborhood O of y such that O is a metrizable space with the induced topology from Y. Since Y is regular we can suppose that O is closed. $F(x)$ is compact, thus there are metrizable closed sets $O_1, O_2, ...O_n$ such that $F(x) \subset \cup_{1\le i\le n} Int(O_i)$. By Theorem 4.4.19 in [8] $\cup_{1\le i\le n} O_i$ is a metrizable space. Since F is upper semicontinuous, there is an open neighbourhood V of x such that $V \subset U$ and

$$F(V) \subset \cup_{1\le i\le n} Int(O_i).$$

$F|V : V \to \cup_{1\le i\le n} O_i$ is a minimal usco map from a Baire space V to a metrizable space $\cup_{1\le i\le n} O_i$. Thus there is a residual set in V in the points of which $F|V$ is single-valued. The set $V \cap (X \setminus L)$ is of the first Baire category in V, thus also in X. By the Banach Category Theorem [10, Theorem 1.7.] the set $X \setminus L$ is of the first Baire category in X. Thus L is a residual set in X. □

Theorem 5.2. *Let X be a locally Čech-complete Stegall space. Then the set $\{f \in \widetilde{C}(X, \mathbb{R}) : |\text{argmin}(f)| = 1\}$ is a residual set in $(\widetilde{C}(X, \mathbb{R}), \tau_V)$.*

Proof. By Proposition 4.7 $(\widetilde{C}(X, \mathbb{R}), \tau_V)$ is locally Čech-complete. Every locally Čech-complete space is an open continuous image of a Čech-complete space, see 3.12.19(d) in [8]. Thus $(\widetilde{C}(X, \mathbb{R}), \tau_V)$ is a Baire space. By Theorem 4.2 argmin: $(\widetilde{C}(X, \mathbb{R}), \tau_V) \to X$ is a minimal usco map. Since X is a Stegall space, the set $\{f \in \widetilde{C}(X, \mathbb{R}) : |\text{argmin}(f)| = 1\}$ is a residual set in $(\widetilde{C}(X, \mathbb{R}), \tau_V)$. □

The following definition was used in [2, 19, 20].

Definition 5.3. Let X and Y be topological spaces. A map $F : X \to Y$ is said to be almost lower semicontinuous at $x_1 \in X$ if there exists a point $y_1 \in F(x_1)$ such that for each neighbourhood V of y_1, there exists an open neighbourhood U of x_1 such that $F(z) \cap V \neq \emptyset$ for every $z \in U$.

Kenderov in [19, 20] introduced the class $\mathcal{L}$ of all spaces Y such that for each Čech-complete space X, every usco map from X to Y is almost lower semicontinuous on some dense G_δ subset of X.

Beer in his paper [2] proved the following theorem.

Theorem 5.4. *Let X be a Čech-complete space from the class $\mathcal{L}$. Then there is a dense G_δ subset of $(\widetilde{C}(X, \mathbb{R}), \tau_V)$ each of whose elements has a unique minimum.*

We generalize Theorem 5.4 for locally Čech-complete spaces from the class $\mathcal{L}$.

We will need the following auxiliary result:

Lemma 5.5. ([15] *Let X and Y be topological spaces and Y be Hausdorff. Let $F : X \to Y$ be minimal usco. If F is almost lower semicontinuous at x_1, then F is single-valued at x_1.*

Theorem 5.6. *Let X be a locally Čech-complete space from the class $\mathcal{L}$. Then there is a dense G_δ subset of $(\widetilde{C}(X, \mathbb{R}), \tau_V)$ each of whose elements has a unique minimum.*

Proof. Put $L = \{f \in \widetilde{C}(X,\mathbb{R}) : |\mathrm{argmin}(f)| = 1\}$. It suffices to prove that the set $\widetilde{C}(X,\mathbb{R}) \setminus L$ is of the first Baire category in $(\widetilde{C}(X,\mathbb{R}), \tau_V)$: let G be any nonempty open subset of $\widetilde{C}(X,\mathbb{R})$. Since by Proposition 4.7 $(\widetilde{C}(X,\mathbb{R}), \tau_V)$ is locally Čech-complete, there exists an open Čech-complete set V such that $V \subset G$. Since $X \in \mathcal{L}$, argmin $\restriction_V \to X$ is almost lower semicontinuous on a dense G_δ set $H \subset V$. By Lemma 5.5 argmin $\restriction_V$ is single-valued at every $f \in H$. Thus $V \cap (\widetilde{C}(X,\mathbb{R}) \setminus L)$ is of the first Baire category, since $(\widetilde{C}(X,\mathbb{R}), \tau_V)$ is a Baire space, which in turn implies, by the Banach Category Theorem [10, Theorem 1.7.], that $\widetilde{C}(X,\mathbb{R}) \setminus L$ is of the first Baire category. Then the result follows, since in a Baire space every residual set contains a dense G_δ subset. □

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