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A study of $\mathcal{S}-$curvature tensor on semi-Riemannian manifolds with relativistic applications

Abdallah Abdelhameed Syied[a,*], Uday Chand De[b], Mohamed F. Ismail[c], Nasser Bin Turki[d]

[a]*Department of Mathematics, Faculty of Science, Zagazig University, P.O. Box 44519, Zagazig, Egypt*
[b]*Department of Pure Mathematics, University of Calcutta 35, Ballygaunge Circular Road Kolkata 700019, West Bengal, India*
[c]*Faculty of computer and information system, Egyptian Chinese University*
[d]*Department of Mathematics, College of Science, King Saud University, P.O. Box 2455,Riyadh 11451, Saudi Arabia*

Abstract. This article introduces a new curvature tensor, the $\mathcal{S}-$curvature tensor, which is seen as a comprehensive extension of various curvature tensors. It is demonstrated that a semi-Riemannian manifold with traceless $\mathcal{S}-$curvature tensor is Einstein. It is proved that a $\mathcal{S}-$curvature flat semi-Riemannian manifold is of constant sectional curvature. Moreover, we show that a perfect fluid $\mathcal{S}-$curvature flat space-time represents dark matter era. It is shown that a perfect fluid spacetime with $\nabla_h \mathcal{S}^h_{jkl} = 0$ is expansion-free and shear-free and its flow is geodesic, but not necessary vorticity-free. We show that a pseudo $\mathcal{S}-$symmetric manifold is reduced to pseudo symmetric manifold if and only if the scalar curvature is constant. Finally, a concrete example of pseudo $\mathcal{S}-$symmetric manifolds is introduced.

1. Introduction

Curvature invariants are essential tools in both differential geometry and general relativity, providing valuable geometric and physical insights into the curvature of spacetime, the structure of manifolds, and the behavior of matter and energy in gravitational fields. Their coordinate independence and geometric significance make them indispensable for studying the fundamental properties of space and the nature of gravity. In semi-Riemannian geometry, these invariants are scalar values created from a variety of curvature tensors, with the most well-known ones being the Riemann, Ricci, and Weyl tensors (for example see [4]).

Inspired by the significance of the curvature tensors in differential geometry and general relativity, we propose a novel curvature tensor known as the $\mathcal{S}-$tensor, which is defined as follows:

$$\begin{aligned} \mathcal{S}_{ijkl} &= b_0 R_{ijkl} + b_1 g_{ij} R_{kl} + b_2 g_{ik} R_{jl} + b_3 g_{il} R_{jk} + b_4 g_{jk} R_{il} \\ &\quad + b_5 g_{jl} R_{ik} + b_6 g_{kl} R_{ij} + b_7 R \left(g_{il} g_{jk} - g_{jl} g_{ik} \right), \end{aligned} \tag{1}$$

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* Corresponding author: Abdallah Abdelhameed Syied
Email addresses: a.a_syied@yahoo.com (Abdallah Abdelhameed Syied), uc_de@yahoo.com (Uday Chand De), m.fekry2015@yahoo.com (Mohamed F. Ismail), nassert@ksu.edu.sa (Nasser Bin Turki)
ORCID iDs: https://orcid.org/0000-0001-5568-7661 (Abdallah Abdelhameed Syied), https://orcid.org/0000-0002-8990-4609 (Uday Chand De), https://orcid.org/0000-0001-8537-2222 (Mohamed F. Ismail), https://orcid.org/0000-0002-9738-026X (Nasser Bin Turki)

with b_i being constants, R_{ijkl}, R_{ij}, and R indicate the curvature tensor with a type of $(0,4)$, the Ricci tensor, and the scalar curvature in that order. The appeal of this curvature tensor lies in its essence of:

1. Weyl curvature tensor [21, 28] for $b_0 = 1, b_3 = b_4 = -b_2 = -b_5 = \frac{1}{n-2},\ b_7 = \frac{-1}{(n-1)(n-2)},\ b_1 = b_6 = 0,$
2. Conharmonic curvature tensor [17, 32] for $b_0 = 1,\ b_2 = b_5 = -b_3 = -b_4 = \frac{1}{n-2},\ b_1 = b_6 = b_7 = 0,$
3. Concircular curvature tensor [12, 36] for $b_0 = 1,\ b_7 = \frac{-1}{n(n-1)},\ b_1 = b_2 = b_3 = b_4 = b_5 = b_6 = 0,$
4. Semi-conformal curvature tensor [3, 18] for $b_2 = b_5 = -b_3 = -b_4 = \frac{b_0}{n-2},\ b_7 = \frac{-b_8}{n-1},\ b_1 = b_6 = 0,$
5. Projective curvature tensor [27, 33] for $b_0 = 1,\ b_5 = -b_3 = \frac{1}{n-1},\ b_1 = b_2 = b_4 = b_6 = b_7 = 0,$
6. W_2-curvature tensor [24, 34] for $b_0 = 1,\ b_2 = -b_4 = \frac{1}{n-1},\ b_1 = b_3 = b_5 = b_6 = b_7 = 0,$
7. Quasi-conformal curvature tensor [5, 14] for $b_3 = b_4 = -b_2 = -b_5,\ b_7 = \frac{-1}{n}\left(\frac{b_0}{n-1} + 2b_3\right),\ b_1 = b_6 = 0,$
8. $M-$Projective curvature tensor [7, 9] for $b_0 = 1,\ b_2 = b_5 = -b_3 = -b_4 = \frac{1}{2(n-1)},\ b_1 = b_6 = b_7 = 0,$ and
9. Pseudo-projective curvature tensor [15, 26] for $b_3 = -b_5 = \frac{1}{n-1},\ b_7 = \frac{-1}{n}\left(\frac{b_0}{n-1} + b_3\right),\ b_1 = b_2 = b_4 = b_6 = 0.$

According to the classification introduced by the author in [2], the manifold M falls into the category of pseudo symmetric manifolds $(\mathrm{PS})_n$ if its Riemann curvature tensor satisfies the following condition:

$$\nabla_h\mathcal{R}_{ijkl} = 2\beta_h\mathcal{R}_{ijkl} + \beta_i\mathcal{R}_{hjkl} + \beta_j\mathcal{R}_{ihkl} + \beta_k\mathcal{R}_{ijhl} + \beta_l\mathcal{R}_{ijkh},$$

where β_i denotes a non-zero $1-$form known as the "*associated covector*" and ∇_h indicates the covariant derivative relative to the metric tensor g. This concept has been generalized by many researchers, for example see [10, 19, 25, 37, 38] and many others.

This work introduces a new extension of pseudo symmetric manifolds called pseudo $\mathcal{S}-$symmetric manifolds, denoted as $(\mathrm{PSS})_n$. These manifolds are characterized by the following condition:

$$\nabla_h\mathcal{S}_{ijkl} = 2\beta_h\mathcal{S}_{ijkl} + \beta_i\mathcal{S}_{hjkl} + \beta_j\mathcal{S}_{ihkl} + \beta_k\mathcal{S}_{ijhl} + \beta_l\mathcal{S}_{ijkh}. \tag{2}$$

In the domain of physics, a space-time can be defined as an $n-$dimensional Lorentzian manifold furnished with a Lorentzian metric. A Lorentzian manifold M is referred to as a pseudo $\mathcal{S}-$symmetric spacetime when the $\mathcal{S}-$tensor satisfies equation (2). For a perfect fluid space-time (PFS), the energy-momentum tensor T_{jk} exhibits the following arrangement [6, 11]:

$$T_{jk} = (\sigma + p)\,\lambda_j\lambda_k + pg_{jk}, \tag{3}$$

where σ, p, and λ_k correspond to the energy density, the isotropic pressure, and a unit time-like vector field or the velocity vector respectively. Moreover, σ and p are linked through the state equation represented as $p = p(\sigma)$, with the PFS recognized as isentropic. Also, the PFS is classified as stiff matter when $p = \sigma$. When $p + \sigma = 0$, the PFS signifies dark energy era. The PFS represents radiation era if $\sigma - 3p = 0$ whereas it represents quintessence era if $\sigma - 3p = 0$ [8].

Multiplying Eq. (3) to g^{jk}, we infer that

$$T = (n-1)\,p - \sigma, \tag{4}$$

where $T = g^{jk}T_{jk}$.

Einstein's field equations EFEs with vanishing cosmological term is given as follows [28]:

$$R_{jk} - \frac{1}{2}g_{jk}R = \kappa T_{jk}, \tag{5}$$

with κ being the gravitational constant. EFEs suggest that T_{jk} is divergence-free, that is, $\nabla_m T^m_k = 0$. These equations establish a relationship between the geometry of spacetime and its matter, meaning that matter defines the geometry of spacetime, and vice versa.

Transvecting Eq. (5) with g^{jk}, one detects that

$$R = \frac{2\kappa T}{2-n}. \tag{6}$$

The generalized Robertson-Walker (GRW) space-time is derived from the warped product construction, which merges a one-dimensional base manifold (symbolizing cosmic time) with $g_{tt} = -1$, along with a $(n-1)$-dimensional fiber manifold (representing space). The base manifold is commonly described using a temporal coordinate, whereas the fiber manifold embodies the spatial slices at fixed intervals of time. In mathematical terms, the GRW space-time can be represented as $M = I \times_f \tilde{M}$, where I denotes an open connected interval of $\mathbb{R}$ [22]. A Robertson-Walker (RW) space-time is a specific type of GRW space-time in which the fiber manifold exhibits a constant sectional curvature [13].

This paper is structured as outlined below: Firstly, we explore characterizations of semi-Riemannian manifolds when the $\mathcal{S}-$tensor has a traceless decomposition. Following that, we investigate semi-Riemannian manifolds with divergence-free $\mathcal{S}-$tensor. Subsequently, we present pseudo $\mathcal{S}-$symmetric semi-Riemannian manifolds. Also, we provide several interesting results in spacetimes. Lastly, we introduce a concrete example of pseudo $\mathcal{S}-$symmetric semi-Riemannian manifolds.

2. Semi-Riemannian manifolds with traceless decomposition of the $\mathcal{S}-$tensor

The current section discusses the properties of semi-Riemannian manifolds admitting traceless $\mathcal{S}-$tensor. The mathematical structure of a $(1,3)$ $\mathcal{S}-$tensor is as follows:

$$\begin{aligned} \mathcal{S}^h_{jkl} &= b_0 R^h_{jkl} + b_1 \delta^h_j R_{kl} + b_2 \delta^h_k R_{jl} + b_3 \delta^h_l R_{jk} + b_4 g_{jk} R^h_l \\ &\quad + b_5 g_{jl} R^h_k + b_6 g_{kl} R^h_j + b_7 R \left(\delta^h_l g_{jk} - \delta^h_k g_{jl} \right). \end{aligned}$$

It is easy to verify that

$$\begin{aligned} \mathcal{S}^t_{lkt} &= (b_0 + b_1 + b_2 + b_5 + b_6 + n b_3) R_{lk} + [b_4 + (n-1) b_7] g_{lk} R, \\ \mathcal{S}^t_{ltk} &= (b_1 - b_0 + b_3 + b_4 + b_6 + n b_2) R_{lk} + [b_5 - (n-1) b_7] g_{lk} R, \\ \mathcal{S}^t_{tkl} &= (n b_1 + b_2 + b_3 + b_4 + b_5) R_{kl} + b_6 g_{kl} R. \end{aligned}$$

Obviously

$$\mathcal{S}^t_{lkt} = \mathcal{S}^t_{klt}, \mathcal{S}^t_{ltk} = \mathcal{S}^t_{ktl}, \; \mathcal{S}^t_{tkl} = \mathcal{S}^t_{tlk}.$$

As mentioned in [20], the $\mathcal{S}-$tensor can be represented in the following form:

$$\mathcal{S}^m_{jkl} = \mathcal{B}^h_{jkl} + \delta^h_j C_{kl} + \delta^h_k \mathcal{D}_{jl} + \delta^h_l \mathcal{E}_{jk}, \tag{7}$$

where $\mathcal{B}^h_{jkl}$ is an unique traceless $(1,3)$ tensor and C_{kl}, $\mathcal{D}_{jl}$, $\mathcal{E}_{jk}$ are three unique $(0,2)$ tensors. These $(0,2)$ tensors can be achieved in the following manner:

$$\begin{aligned} C_{kl} &= \frac{1}{(n-1)(n+2)} \left[(n+1) \mathcal{S}^t_{tkl} - \mathcal{S}^t_{ktl} - \mathcal{S}^t_{lkt} \right], \\ &= \left[b_1 + \frac{n b_4 + n b_5 - 2 b_6}{(n-1)(n+2)} \right] R_{kl} + \left[\frac{(n+1) b_6 - b_5 - b_4}{(n-1)(n+2)} \right] g_{lk} R, \\ \mathcal{D}_{jl} &= \frac{1}{(n-1)(n+2)} \left[-\mathcal{S}^t_{tjl} - \mathcal{S}^t_{ljt} + (n+1) \mathcal{S}^t_{jtl} \right], \\ &= \left[b_2 - \frac{b_0}{(n-1)} + \frac{n(b_4 + b_6) - 2 b_5}{(n-1)(n+2)} \right] R_{jl} + \left[\frac{(n+1) b_5 - b_4 - b_6}{(n-1)(n+2)} - b_7 \right] g_{jl} R, \end{aligned}$$

$$\begin{aligned}\mathcal{E}_{jk} &= \frac{1}{(n-1)(n+2)}\left[(n+1)\mathcal{S}^t_{jkt} - \mathcal{S}^t_{tkj} - \mathcal{S}^t_{ktj}\right],\\ &= \left[b_3 + \frac{b_0}{(n-1)} + \frac{n(b_5+b_6)-2b_4}{(n-1)(n+2)}\right]R_{jk} + \left[\frac{(n+1)b_4 - b_5 - b_6}{(n-1)(n+2)} + b_7\right]g_{jk}R.\end{aligned}$$

Assume that the $\mathcal{S}-$tensor is traceless, therefore

$$C_{kl} = \mathcal{D}_{kl} = \mathcal{E}_{kl} = 0.$$

As a result, we obtain

$$\begin{aligned}\left[b_1 + \frac{nb_4 + nb_5 - 2b_6}{(n-1)(n+2)}\right]R_{kl} &= -\left[\frac{(n+1)b_6 - b_5 - b_4}{(n-1)(n+2)}\right]g_{lk}R,\\ \left[b_2 - \frac{b_0}{(n-1)} + \frac{n(b_4+b_6)-2b_5}{(n-1)(n+2)}\right]R_{jl} &= -\left[\frac{(n+1)b_5 - b_4 - b_6}{(n-1)(n+2)} - b_7\right]g_{jl}R,\\ \left[b_3 + \frac{b_0}{(n-1)} + \frac{n(b_5+b_6)-2b_4}{(n-1)(n+2)}\right]R_{jk} &= -\left[\frac{(n+1)b_4 - b_5 - b_6}{(n-1)(n+2)} + b_7\right]g_{jk}R.\end{aligned}$$

These three equations show that the manifold is Einstein.

Three different contractions of the previous three equations with g^{kl}, g^{jl}, and g^{jk} imply

$$\begin{aligned}(n-1)(n+2)[b_1+b_6]R &= 0,\\ (n-1)(n+2)\left[b_2 + b_5 - nb_7 - \frac{b_0}{n-1}\right]R &= 0,\\ (n-1)(n+2)\left[b_3 + b_4 + nb_7 + \frac{b_0}{n-1}\right]R &= 0.\end{aligned}$$

Consequently, either the scalar curvature vanishes or

$$\begin{aligned}b_1 &= -b_6,\\ b_2 + b_5 &= -(b_3+b_4) = nb_7 + \frac{b_0}{n-1}.\end{aligned}$$

Thus, we can state

Theorem 2.1. *A semi-Riemannian manifold of dimension ≥ 3 with traceless $\mathcal{S}-$tensor is Einstein. Moreover, either the scalar curvature vanishes or*

$$\begin{aligned}b_1 &= -b_6\\ b_2 + b_5 &= -(b_3+b_4) = nb_7 + \frac{b_0}{n-1}.\end{aligned}$$

3. The impact of the flatness of the $\mathcal{S}-$tensor on semi-Riemannian manifolds

This section is dedicated to the investigation of $\mathcal{S}-$curvature flat semi-Riemannian manifolds. A semi-Riemannian manifold M is referred to as $\mathcal{S}-$curvature flat when the $\mathcal{S}-$tensor vanishes at each point of the manifold.

Putting $\mathcal{S}_{ijkl} = 0$ in Eq. (1), one can find

$$\begin{aligned}-b_0R_{ijkl} &= b_1g_{ij}R_{kl} + b_2g_{ik}R_{jl} + b_3g_{il}R_{jk} + b_4g_{jk}R_{il}\\ &\quad + b_5g_{jl}R_{ik} + b_6g_{kl}R_{ij} + b_7R\left(g_{il}g_{jk} - g_{jl}g_{ik}\right).\end{aligned}$$

Suppose that b_0 is non-zero and define $\bar{b}_i = \frac{b_i}{b_0}$. Therefore,

$$-R_{ijkl} = \bar{b}_1g_{ij}R_{kl} + \bar{b}_2g_{ik}R_{jl} + \bar{b}_3g_{il}R_{jk} + \bar{b}_4g_{jk}R_{il}$$

$$+\bar{b}_5 g_{jl} R_{ik} + \bar{b}_6 g_{kl} R_{ij} + \bar{b}_7 R \left(g_{il} g_{jk} - g_{jl} g_{ik} \right). \tag{8}$$

Contracting with g^{il} and g^{jk} respectively, we acquire that

$$\begin{aligned} \left(1 + \bar{b}_1 + \bar{b}_2 + n\bar{b}_3 + \bar{b}_5 + \bar{b}_6\right) R_{jk} &= -\left[(n-1)\bar{b}_7 + \bar{b}_4\right] g_{jk} R, & (9)\\ \left(1 + \bar{b}_1 + \bar{b}_2 + n\bar{b}_4 + \bar{b}_5 + \bar{b}_6\right) R_{il} &= -\left[(n-1)\bar{b}_7 + \bar{b}_3\right] g_{il} R. & (10) \end{aligned}$$

Transvecting Eq. (8) with g^{ik} and g^{jl} respectively, we realize that

$$\begin{aligned} \left(-1 + \bar{b}_1 + n\bar{b}_2 + \bar{b}_3 + \bar{b}_4 + \bar{b}_6\right) R_{jl} &= \left[(n-1)\bar{b}_7 - \bar{b}_5\right] g_{jl} R, & (11)\\ \left(-1 + \bar{b}_1 + n\bar{b}_5 + \bar{b}_3 + \bar{b}_4 + \bar{b}_6\right) R_{ik} &= \left[(n-1)\bar{b}_7 - \bar{b}_2\right] g_{ik} R. & (12) \end{aligned}$$

Contracting Eq. (8) with g^{ij} and g^{kl} respectively, we get

$$\begin{aligned} \left(n\bar{b}_1 + \bar{b}_2 + \bar{b}_3 + \bar{b}_4 + \bar{b}_5\right) R_{kl} &= -\bar{b}_6 g_{kl} R, & (13)\\ \left(n\bar{b}_6 + \bar{b}_2 + \bar{b}_3 + \bar{b}_4 + \bar{b}_5\right) R_{ij} &= -\bar{b}_1 g_{ij} R. & (14) \end{aligned}$$

The equations presented above demonstrate the following:

$$\begin{aligned} n\left(\bar{b}_3 - \bar{b}_4\right) R_{jk} &= \left(\bar{b}_3 - \bar{b}_4\right) g_{jk} R, & (15)\\ n\left(\bar{b}_2 - \bar{b}_5\right) R_{jk} &= \left(\bar{b}_2 - \bar{b}_5\right) g_{jk} R, & (16)\\ n\left(\bar{b}_1 - \bar{b}_6\right) R_{jk} &= \left(\bar{b}_1 - \bar{b}_6\right) g_{jk} R. & (17) \end{aligned}$$

If any of the conditions $\bar{b}_3 \neq \bar{b}_4$, $\bar{b}_2 \neq \bar{b}_5$ or $\bar{b}_1 \neq \bar{b}_6$ is satisfied, then

$$R_{ij} = \frac{R}{n} g_{ij},$$

which illustrates that the manifold is Einstein.

Otherwise, if $\bar{b}_3 = \bar{b}_4$, $\bar{b}_2 = \bar{b}_5$, and $\bar{b}_1 = \bar{b}_6$. Then

$$\begin{aligned} \left(1 + 2\bar{b}_1 + 2\bar{b}_2 + n\bar{b}_3\right) R_{jk} &= -\left[(n-1)\bar{b}_7 + \bar{b}_3\right] g_{jk} R, & (18)\\ \left(-1 + 2\bar{b}_1 + n\bar{b}_2 + 2\bar{b}_3\right) R_{jl} &= \left[(n-1)\bar{b}_7 - \bar{b}_2\right] g_{jl} R, & (19)\\ \left(n\bar{b}_1 + 2\bar{b}_2 + 2\bar{b}_3\right) R_{kl} &= -\bar{b}_1 g_{kl} R. & (20) \end{aligned}$$

Performing contractions of the foregoing three equations, we get

$$\begin{aligned} \left[1 + 2\bar{b}_1 + 2\bar{b}_2 + 2n\bar{b}_3 + n(n-1)\bar{b}_7\right] R &= 0, & (21)\\ \left[-1 + 2\bar{b}_1 + 2n\bar{b}_2 + 2\bar{b}_3 - n(n-1)\bar{b}_7\right] R &= 0, & (22)\\ \left[2n\bar{b}_1 + 2\bar{b}_2 + 2\bar{b}_3\right] R &= 0. & (23) \end{aligned}$$

If $R \neq 0$, then Eqs. (21), (22), and (23) provide us with the following information

$$\begin{aligned} 2\bar{b}_1 + 2\bar{b}_2 + 2n\bar{b}_3 + n(n-1)\bar{b}_7 &= -1,\\ 2\bar{b}_1 + 2n\bar{b}_2 + 2\bar{b}_3 - n(n-1)\bar{b}_7 &= 1,\\ 2n\bar{b}_1 + 2\bar{b}_2 + 2\bar{b}_3 &= 0. \end{aligned}$$

Solving the previous three equations, we find that

$$\bar{b}_1 = 0,$$

$$\bar{b}_2 = -\bar{b}_3 = \frac{1}{2(n-1)} + \frac{n}{2}\bar{b}_7.$$

Consequently, Eq. (18) becomes

$$R_{jk} = \frac{R}{n} g_{jk},$$

this shows that the manifold is Einstein.

Lemma 3.1. *A $\mathcal{S}-$curvature flat semi-Riemannian manifold with non-zero scalar curvature is Einsteinian.*

It is important to show that for $\bar{b}_3 = \bar{b}_4$, $\bar{b}_2 = \bar{b}_5$, and $\bar{b}_1 = \bar{b}_6$ we have

$$\begin{aligned}
\bar{b}_3 + \bar{b}_4 &= \frac{-1}{(n-1)} - n\bar{b}_7, \\
\bar{b}_2 + \bar{b}_5 &= \frac{1}{(n-1)} + n\bar{b}_7, \\
\bar{b}_1 + \bar{b}_6 &= 0.
\end{aligned}$$

Consequently, Eq. (8) becomes

$$R_{ijkl} = \frac{R}{n(n-1)} \left(g_{jk} g_{il} - g_{jl} g_{ik} \right),$$

which clarifies that the manifold is of constant sectional curvature.

Thus, we can derive

Theorem 3.2. *A $\mathcal{S}-$curvature flat semi-Riemannian manifold with non-zero scalar curvature is of constant sectional curvature.*

Since the spacetime is of constant sectional curvature, therefore the spacetime becomes conformally flat and hence the 4−dimensional spacetime is of Petrov type O.

Corollary 3.3. *A 4−dimensional $\mathcal{S}-$curvature spacetime is of Petrov type O.*

Corollary 3.4. *A $\mathcal{S}-$curvature flat space-time corresponds to the de Sitter space-time for $R > 0$, whereas it corresponds to the anti-de Sitter space-time for $R < 0$.*

It is widely realized that a 4−dimensional space-time with constant sectional curvature possesses a maximum of 10−parameter group of isometries, indicating its homogeneous nature.

Thus, we have

Corollary 3.5. *A 4−dimensional $\mathcal{S}-$curvature flat space-time with positive scalar curvature transforms into a homogenous space-time.*

As previously stated, a $\mathcal{S}-$curvature flat manifold is Einstein. This finding can be employed in Eq (5) to introduce

$$T_{jk} = \frac{(2-n)R}{2n\kappa} g_{jk}. \tag{24}$$

Applying the operator ∇_h to Eq. (24) yields the following results

$$\nabla_h T_{jk} = 0,$$

this indicates that T_{jk} is covariantly constant.

Thus, we get

Corollary 3.6. *The energy-momentum tensor of $\mathcal{S}-$curvature flat space-time is covariantly constant.*

Remark 3.7. *Chaki and Ray [1] conducted a study on space-times in which T_{jk} is covariantly constant.*

Using Eq. (24) in Eq. (3), it arises

$$(\sigma + p)\,\lambda_j\lambda_k + pg_{jk} = \frac{(2-n)\,R}{2n\kappa}g_{jk}.$$

By contracting the preceding equation with g^{jk} and λ^k, we can derive that

$$-\sigma + (n-1)\,p = \frac{2-n}{2\kappa}R, \tag{25}$$

$$\sigma = \frac{n-2}{2n\kappa}R. \tag{26}$$

Employing Eq. (26) in Eq. (25), one sees that

$$p = -\frac{n-2}{2n\kappa}R.$$

By merging the previous two equations, we can obtain the following outcome:

$$\sigma + p = 0,$$

this illustrates that the space-time represents dark matter era [31].

Theorem 3.8. *A $\mathcal{S}-$curvature flat PFS spacetime of non-zero scalar curvature denotes dark matter era.*

From the foregoing Lemma, it follows that in a $\mathcal{S}-$curvature flat semi-Riemannian manifold $C^h_{ijk,h} = 0$, C^h_{ijk} denotes the Weyl tensor. In [23], Mantica et al. proved that a PFS with $C^h_{ijk,h} = 0$ is a GRW space-time. Therefore, the $\mathcal{S}-$curvature flat PFS spacetime is a GRW space-time. Also, it is known that [16] a $4-$dimensional GRW spacetime is a PFS if and only if it is RW spacetime.

Hence, we can state

Theorem 3.9. *A $\mathcal{S}-$curvature flat PFS spacetime is a GRW spacetime and in dimension* 4, *the spacetime becomes RW spacetime.*

4. Semi-Riemannain manifold with divergence-free $\mathcal{S}-$tensor

In this section, characterizations of semi-Riemannain manifolds with divergence-free $\mathcal{S}-$tensor are investigated.

The $\mathcal{S}-$tensor can be written as

$$\begin{aligned}\mathcal{S}^h_{jkl} \;=\;& b_0R^h_{jkl} + b_1\delta^h_jR_{kl} + b_2\delta^h_kR_{jl} + b_3\delta^h_lR_{jk} + b_4g_{jk}R^h_l \\ &+b_5g_{jl}R^h_k + b_6g_{kl}R^h_j + b_7R\left(\delta^h_lg_{jk} - \delta^h_kg_{jl}\right).\end{aligned}$$

The divergence of the $\mathcal{S}-$tensor is

$$\begin{aligned}\nabla_h\mathcal{S}^h_{jkl} \;=\;& b_0\nabla_hR^h_{jkl} + b_1\nabla_jR_{kl} + b_2\nabla_kR_{jl} + b_3\nabla_lR_{jk} + b_4g_{jk}\nabla_hR^h_l \\ &+b_5g_{jl}\nabla_hR^h_k + b_6g_{kl}\nabla_hR^h_j + b_7g_{jk}\nabla_lR - b_7g_{jl}\nabla_kR.\end{aligned}$$

It is known that $\nabla_h R^h_{jkl} = \nabla_l R_{jk} - \nabla_k R_{jl}$ and $\nabla_h R^h_j = \frac{1}{2}\nabla_j R$, thus

$$\begin{aligned} \nabla_h \mathcal{S}^h_{jkl} &= b_1 \nabla_j R_{kl} + (b_2 - b_0)\nabla_k R_{jl} + (b_3 + b_0)\nabla_l R_{jk} + \frac{b_6}{2} g_{kl}\nabla_j R \\ &\quad + \left(\frac{b_4}{2} + b_7\right) g_{jk}\nabla_l R + \left(\frac{b_5}{2} - b_7\right) g_{jl}\nabla_k R. \end{aligned} \tag{27}$$

If the divergence $\mathcal{S}-$tensor is zero, then

$$\begin{aligned} 0 &= b_1 \nabla_j R_{kl} + (b_2 - b_0)\nabla_k R_{jl} + (b_3 + b_0)\nabla_l R_{jk} + \frac{b_6}{2} g_{kl}\nabla_j R \\ &\quad + \left(\frac{b_4}{2} + b_7\right) g_{jk}\nabla_l R + \left(\frac{b_5}{2} - b_7\right) g_{jl}\nabla_k R. \end{aligned} \tag{28}$$

Transvecting with g^{kl}, g^{jl}, and g^{jk}, one uncovers that

$$\begin{aligned} [2b_1 + b_2 + b_3 + b_4 + b_5 + nb_6]\nabla_j R &= 0, \\ [b_1 + 2b_2 + b_3 + b_4 + nb_5 + b_6 + 2(1-n)b_7 - b_0]\nabla_k R &= 0, \\ [b_1 + b_2 + 2b_3 + nb_4 + b_5 + b_6 + 2(n-1)b_7 + b_0]\nabla_l R &= 0. \end{aligned}$$

Consequently, the scalar curvature is constant or

$$\begin{aligned} 2b_1 + b_2 + b_3 + b_4 + b_5 + nb_6 &= 0, \\ b_1 + 2b_2 + b_3 + b_4 + nb_5 + b_6 + 2(1-n)b_7 &= b_0, \\ b_1 + b_2 + 2b_3 + nb_4 + b_5 + b_6 + 2(n-1)b_7 &= -b_0. \end{aligned}$$

Hence, we can state

Theorem 4.1. *In a pseudo-Riemannian manifold M with divergence-free of the $\mathcal{S}-$tensor, either the scalar curvature R remains constant or the following conditions are satisfied*

$$\begin{aligned} 2b_1 + b_2 + b_3 + b_4 + b_5 + nb_6 &= 0, \\ b_1 + 2b_2 + b_3 + b_4 + nb_5 + b_6 + 2(1-n)b_7 &= b_0, \\ b_1 + b_2 + 2b_3 + nb_4 + b_5 + b_6 + 2(n-1)b_7 &= -b_0. \end{aligned}$$

Utilizing Eqs. (5) and (6) in (27), we reveal that

$$\begin{aligned} \nabla_h \mathcal{S}^h_{jkl} &= \kappa\{\left[b_1 \nabla_j T_{kl} + (b_2 - b_0)\nabla_k T_{jl} + (b_3 + b_0)\nabla_l T_{jk}\right] \\ &\quad - \frac{1}{n-2}[(b_1 + b_6) g_{kl}\nabla_j T + (b_2 - b_0 + b_5 - 2b_7) g_{jl}\nabla_k T \\ &\quad + (b_3 + b_0 + b_4 + 2b_7) g_{jk}\nabla_l T]\}. \end{aligned}$$

Suppose that $\nabla_h \mathcal{S}^h_{jkl} = 0$, thus the preceding equation simplify to

$$\begin{aligned} 0 &= \left[b_1 \nabla_j T_{kl} + (b_2 - b_0)\nabla_k T_{jl} + (b_3 + b_0)\nabla_l T_{jk}\right] \\ &\quad - \frac{1}{n-2}[(b_1 + b_6) g_{kl}\nabla_j T + (b_2 - b_0 + b_5 - 2b_7) g_{jl}\nabla_k T \\ &\quad + (b_3 + b_0 + b_4 + 2b_7) g_{jk}\nabla_l T]. \end{aligned} \tag{29}$$

Contracting with g^{kl}, one acquires that

$$[2b_1 + b_2 + b_3 + b_4 + b_5 + nb_6]\nabla_j T = 0.$$

If $2b_1 + b_2 + b_3 + b_4 + b_5 + nb_6 \neq 0$, thus

$$\nabla_j T = 0. \tag{30}$$

Consequently, Eq. (29) reduces to

$$b_1 \nabla_j T_{kl} + (b_2 - b_0) \nabla_k T_{jl} + (b_3 + b_0) \nabla_l T_{jk} = 0. \tag{31}$$

Inserting Eq. (3) in Eq. (31), we infer that

$$\begin{aligned} 0 \;=\;& b_1 \left[\lambda_l \lambda_k \nabla_j (\sigma + p) + (\sigma + p) \lambda_l \nabla_j \lambda_k + (\sigma + p) \lambda_k \nabla_j \lambda_l + g_{kl} \nabla_j p \right] \\ &+ (b_2 - b_0) \left[\lambda_j \lambda_l \nabla_k (\sigma + p) + (\sigma + p) \lambda_j \nabla_k \lambda_l + (\sigma + p) \lambda_l \nabla_k \lambda_j + g_{jl} \nabla_k p \right] \\ &+ (b_3 + b_0) \left[\lambda_j \lambda_k \nabla_l (\sigma + p) + (\sigma + p) \lambda_j \nabla_l \lambda_k + (\sigma + p) \lambda_k \nabla_l \lambda_j + g_{jk} \nabla_l p \right]. \end{aligned}$$

Transvecting with λ^j, it arises

$$\begin{aligned} 0 \;=\;& b_1 \left[\lambda_l \lambda_k (\sigma + \dot{p}) + (\sigma + p) \lambda_l \dot{\lambda}_k + (\sigma + p) \lambda_k \dot{\lambda}_l + g_{kl} \dot{p} \right] \\ &+ (b_2 - b_0) \left[\lambda_l \nabla_k p - \lambda_l \nabla_k (\sigma + p) - (\sigma + p) \nabla_k \lambda_l \right] \\ &+ (b_3 + b_0) \left[\lambda_k \nabla_l p - \lambda_k \nabla_l (\sigma + p) - (\sigma + p) \nabla_l \lambda_k \right]. \end{aligned} \tag{32}$$

Two different contractions with λ^k and g^{lk} imply

$$(\sigma + p) \dot{\lambda}_l = -\nabla_l p + \dot{p} \lambda_l, \tag{33}$$

$$\dot{\sigma} = -(\sigma + p) \eta, \tag{34}$$

where $\eta = \nabla_l \lambda^l$ denotes the expansion scalar.

Using Eq. (33) in Eq. (32), one deduces that

$$\begin{aligned} 0 \;=\;& b_1 \left[\lambda_l \lambda_k \dot{\sigma} + 3 \lambda_l \lambda_k \dot{p} - \lambda_l \nabla_k p - \lambda_k \nabla_l p + g_{kl} \dot{p} \right] \\ &+ (b_2 - b_0) \left[\lambda_l \nabla_k p - \lambda_l \nabla_k (\sigma + p) - (\sigma + p) \nabla_k \lambda_l \right] \\ &+ (b_3 + b_0) \left[\lambda_k \nabla_l p - \lambda_k \nabla_l (\sigma + p) - (\sigma + p) \nabla_l \lambda_k \right]. \end{aligned}$$

Multiplying with g^{kl}, with the assistance of Eq. (34), we conclude that

$$b_1 \left[-\dot{\sigma} + (n - 5) \dot{p} \right] = 0,$$

provides $b_1 \neq 0$, thus

$$(n - 5) \dot{p} - \dot{\sigma} = 0. \tag{35}$$

By applying ∇_l to Eq. (4), we can derive

$$\nabla_l T = -\nabla_l \sigma + (n - 1) \nabla_l p. \tag{36}$$

Inserting Eq. (30) in Eq. (36), one gets

$$(n - 1) \nabla_l p - \nabla_l \sigma = 0.$$

Transvecting with λ^l, we conclude that

$$(n - 1) \dot{p} - \dot{\sigma} = 0. \tag{37}$$

Merging Eq. (35) in Eq. (36), one infers that

$$\dot{p} = 0,$$

$$\dot{\sigma} = 0.$$

Consequently

$$p = \text{constant},$$
$$\sigma = \text{constant}.$$

In particular, if we choose $b_1 = 1$, $b_2 = 3$, $b_0 = 1$, and $b_3 = 1$, then Eq. (31) turns into

$$\nabla_j T_{kl} + \nabla_k T_{jl} + \nabla_l T_{jk} = 0,$$

this suggests that T_{ij} is Killing [35].

In [30], the authors proved that in a PFS if T_{ij} is Killing, then the spacetime is expansion-free and shear-free and its flow is geodesic, but not necessary vorticity-free.

Thus, we can derive

Theorem 4.2. *A perfect fluid space-time with $\nabla_h \mathcal{S}^h_{jkl} = 0$ is expansion-free and shear-free and its flow is geodesic, but not necessary vorticity-free, under certain restriction.*

5. On pseudo $\mathcal{S}-$symmetric semi-Riemannian manifolds

Here, we investigate some general properties of pseudo $\mathcal{S}-$symmetric semi-Riemannian manifolds. Utilizing Eq. (1) in Eq. (2) and assuming that b_0 is non-zero and $\bar{b}_i = \frac{b_i}{b_0}$, thus

$$\begin{aligned}
&\left[\nabla_h R_{ijkl} - 2\beta_h R_{ijkl} - \beta_i R_{hjkl} - \beta_j R_{ihkl} - \beta_k R_{ijhl} - \beta_l R_{ijkh}\right] \\
= \; & \bar{b}_1 \left[g_{ij}\left(2\beta_h - \nabla_h\right) R_{kl} + \beta_i g_{hj} R_{kl} + \beta_j g_{ih} R_{kl} + \beta_k g_{ij} R_{hl} + \beta_l g_{ij} R_{kh}\right] \\
& + \bar{b}_2 \left[g_{ik}\left(2\beta_h - \nabla_h\right) R_{jl} + \beta_i g_{hk} R_{jl} + \beta_j g_{ik} R_{hl} + \beta_k g_{ih} R_{jl} + \beta_l g_{ik} R_{jh}\right] \\
& + \bar{b}_3 \left[g_{il}\left(2\beta_h - \nabla_h\right) R_{jk} + \beta_i g_{hl} R_{jk} + \beta_j g_{il} R_{hk} + \beta_k g_{il} R_{jh} + \beta_l g_{ih} R_{jk}\right] \\
& + \bar{b}_4 \left[g_{jk}\left(2\beta_h - \nabla_h\right) R_{il} + \beta_i g_{jk} R_{hl} + \beta_j g_{hk} R_{il} + \beta_k g_{jh} R_{il} + \beta_l g_{jk} R_{ih}\right] \\
& + \bar{b}_5 \left[g_{jl}\left(2\beta_h - \nabla_h\right) R_{ik} + \beta_i g_{jl} R_{hk} + \beta_j g_{hl} R_{ik} + \beta_k g_{jl} R_{ih} + \beta_l g_{jh} R_{ik}\right] \\
& + \bar{b}_6 \left[g_{kl}\left(2\beta_h - \nabla_h\right) R_{ij} + \beta_i g_{kl} R_{hj} + \beta_j g_{kl} R_{ih} + \beta_k g_{hl} R_{ij} + \beta_l g_{kh} R_{ij}\right] \\
& + \bar{b}_7 \{\left(g_{il} g_{jk} - g_{jl} g_{ik}\right)\left(2\beta_h - \nabla_h\right) R + \beta_i R \left(g_{hl} g_{jk} - g_{jl} g_{hk}\right) \\
& + \beta_j R \left(g_{il} g_{hk} - g_{hl} g_{ik}\right) + \beta_k R \left(g_{il} g_{jh} - g_{jl} g_{ih}\right) + \beta_l R \left(g_{ih} g_{jk} - g_{jh} g_{ik}\right)\}.
\end{aligned} \tag{38}$$

Multiplying with g^{il}, we deduce that

$$\begin{aligned}
&\left[\nabla_h R_{jk} - 2\beta_h R_{jk} - \beta_j R_{hk} - \beta_k R_{jh} - \beta^l R_{hjkl} - \beta^i R_{ijkh}\right] \\
= \; & \bar{b}_1 \left[\left(2\beta_h - \nabla_h\right) R_{jk} + 2\beta_j R_{hk} + \beta_k R_{jh} + \beta^l g_{hj} R_{kl}\right] \\
& + \bar{b}_2 \left[\left(2\beta_h - \nabla_h\right) R_{jk} + \beta^l g_{hk} R_{jl} + \beta_j R_{hk} + 2\beta_k R_{jh}\right] \\
& + \bar{b}_3 \left[\left(2n\beta_h + 2\beta_h - n\nabla_h\right) R_{jk} + n\beta_j R_{hk} + n\beta_k R_{jh}\right] \\
& + \bar{b}_4 \left[\left(2\beta_h - \nabla_h\right) g_{jk} R + \left(\beta_j g_{hk} + \beta_k g_{jh}\right) R + \beta^l g_{jk} R_{hl} + \beta^i g_{jk} R_{ih}\right] \\
& + \bar{b}_5 \left[\left(2\beta_h - \nabla_h\right) R_{jk} + 2\beta_j R_{hk} + \beta_k R_{jh} + \beta^i g_{jh} R_{ik}\right] \\
& + \bar{b}_6 \left[\left(2\beta_h - \nabla_h\right) R_{jk} + 2\beta_k R_{jh} + \beta_j R_{hk} + \beta^i g_{kh} R_{ij}\right] \\
& + \bar{b}_7 \left[\left(1 - n\right) g_{jk} \nabla_h R + 2n\beta_h g_{jk} R + \left(n - 2\right)\left(\beta_j g_{hk} R + \beta_k g_{jh} R\right)\right].
\end{aligned}$$

Transvecting with g^{jk}, we obtain

$$\begin{aligned}
&\left[1+\bar{b}_1+\bar{b}_2+\bar{b}_5+\bar{b}_6+n\bar{b}_3+n\bar{b}_4+n(n-1)\bar{b}_7\right]\nabla_h R \\
&-2\left[2+2\bar{b}_1+2\bar{b}_2+2\bar{b}_5+2\bar{b}_6+n\bar{b}_3+n\bar{b}_4\right]\beta^k R_{hk} \\
=\ & 2\left[1+\bar{b}_1+\bar{b}_2+\bar{b}_3+\bar{b}_4+\bar{b}_5+\bar{b}_6+n\left(\bar{b}_3+\bar{b}_4\right)+(n-1)(n+2)\bar{b}_7\right]\beta_h R.
\end{aligned}$$

We define γ as follows:

$$\gamma=-\frac{1+\bar{b}_1+\bar{b}_2+\bar{b}_3+\bar{b}_4+\bar{b}_5+\bar{b}_6+n\left(\bar{b}_3+\bar{b}_4\right)+(n-1)(n+2)\bar{b}_7}{2+2\bar{b}_1+2\bar{b}_2+2\bar{b}_5+2\bar{b}_6+n\left(\bar{b}_3+\bar{b}_4\right)},$$

where the denominator is non-zero.

Assume that R is constant, hence

$$\beta^k R_{hk}=\gamma R\beta_h, \tag{39}$$

this indicates that β^k performs as an eigenvector of R_{hk}, with a corresponding eigenvalue equals γR.

Consequently, we get

Theorem 5.1. *In a $(PSS)_n$ manifold, β^k is an eigenvector of R_{hk} and its eigenvalue equals γR.*

As mentioned above, a semi-Riemannian manifold M with traceless $\mathcal{S}-$tensor is an Einstein manifold. As a result, Eq. (38) simplifies to the following expression:

$$\begin{aligned}
&\left[\nabla_h R_{ijkl}-2\beta_h R_{ijkl}-\beta_i R_{hjkl}-\beta_j R_{ihkl}-\beta_k R_{ijhl}-\beta_l R_{ijkh}\right] \\
=\ & \{\left(\bar{b}_1+\bar{b}_6\right)g_{kl}g_{ij}+\left(\bar{b}_2+\bar{b}_5\right)g_{ik}g_{jl}+\left(\bar{b}_3+\bar{b}_4\right)g_{jk}g_{il}+n\bar{b}_7 g_{il}g_{jk} \\
&-n\bar{b}_7 g_{jl}g_{ik}\}\left(2\beta_h-\nabla_h\right)\frac{R}{n}+\beta_i\frac{R}{n}\{\left(\bar{b}_1+\bar{b}_6\right)g_{hj}g_{kl}+\left(\bar{b}_2+\bar{b}_5\right)g_{hk}g_{jl} \\
&+\left(\bar{b}_3+\bar{b}_4\right)g_{hl}g_{jk}+n\bar{b}_7\left(g_{hl}g_{jk}-g_{jl}g_{hk}\right)\}+\beta_j\frac{R}{n}\{\left(\bar{b}_1+\bar{b}_6\right)g_{ih}g_{kl} \\
&+\left(\bar{b}_2+\bar{b}_5\right)g_{ik}g_{hl}+\left(\bar{b}_3+\bar{b}_4\right)g_{il}g_{hk}+n\bar{b}_7\left(g_{il}g_{hk}-g_{hl}g_{ik}\right)\} \\
&+\beta_k\frac{R}{n}\{\left(\bar{b}_1+\bar{b}_6\right)g_{ij}g_{hl}+\left(\bar{b}_2+\bar{b}_5\right)g_{ih}g_{jl}+\left(\bar{b}_3+\bar{b}_4\right)g_{il}g_{jh}+n\bar{b}_7 g_{il}g_{jh} \\
&-n\bar{b}_7 g_{jl}g_{ih}\}+\beta_l\frac{R}{n}\{\left(\bar{b}_1+\bar{b}_6\right)g_{ij}g_{kh}+\left(\bar{b}_2+\bar{b}_5\right)g_{ik}g_{jh}+\left(\bar{b}_3+\bar{b}_4\right)g_{ih}g_{jk} \\
&+n\bar{b}_7\left(g_{ih}g_{jk}-g_{jh}g_{ik}\right)\}.
\end{aligned}$$

Hence, we can state

Theorem 5.2. *A $(PSS)_n$ manifold with traceless $\mathcal{S}-$tensor reduces to $(PS)_n$ manifold iff the scalar curvature vanishes.*

Adding Eq. (3) to Eq. (5), one sees that

$$R_{hk}-\frac{1}{2}g_{hk}R=\kappa(\sigma+p)\beta_h\beta_k+\kappa p g_{hk}. \tag{40}$$

Transvecting with β^k and employing Eq. (39), we uncover that

$$\left(\frac{1}{2}-\gamma\right)\frac{R}{\kappa}=\sigma. \tag{41}$$

Again, contracting Eq. (40) with g^{hk}, one discovers that

$$\left(1-\frac{n}{2}\right)\frac{R}{\kappa}=-\sigma+(n-1)\,p. \tag{42}$$

Substituting from Eq. (41) in Eq. (42), we get

$$\frac{(3-n-2\gamma)}{2\,(n-1)\,\kappa}R=p. \tag{43}$$

Hence, we can derive

Theorem 5.3. *The isotropic pressure p and the energy density σ for a perfect fluid $(PSS)_n$ manifold given by Eqs. (41) and (43).*

Remark 5.4. *Since p and σ are not constant, therefore the spacetime under consideration is conformity with the present state of the universe.*

In virtue of Eqs. (41) and (43), one deduces that

$$\frac{p}{\sigma}=\frac{\frac{3-n}{2}-\gamma}{\left[\frac{n-1}{2}-(n-1)\,\gamma\right]}.$$

Consequently we have the following table:

The space-time	EoS	γ
represents quintessence era	$\sigma+3p=0$	$\frac{4-n}{n+2}$
represents dust matter era	$p=0$	$\frac{3-n}{2}$
represents radiation era	$\sigma-3p=0$	$\frac{5-n}{4-n}$
represents dark matter era	$\sigma+p=0$	$\frac{1}{n}$

6. Example

In this section, a example of pseudo $\mathcal{S}$−symmetric semi-Riemannian manifolds is introduced. Let us now make the assumption that the manifold M is of dimension 4 and equipped with a metric g, which can be expressed in the following manner:

$$ds^2=g_{hk}dx^hdx^k=\left(dx^1\right)^2+\left(x^1\right)^2\left(dx^2\right)^2+\left(x^2\right)^2\left(dx^3\right)^2-\left(dx^4\right)^2,\quad \forall h,k=1,2,3,4.$$

Christoffel symbols Γ^i_{jk} have the following non-zero components:

$$\Gamma^3_{23}=\frac{1}{x^2},\quad \Gamma^2_{12}=\frac{1}{x^1},\quad \Gamma^2_{33}=\frac{-x^2}{(x^1)^2},\quad \Gamma^1_{22}=-x^1.$$

Consequently, it arises

$$R_{12}=\frac{-1}{x^1x^2},\qquad R_{1332}=-\frac{x^2}{x^1}.$$

It is observed that

$$R=0.$$

The non-zero component of the $\mathcal{S}-$tensor is

$$\mathcal{S}_{1332} = -\frac{(b_0 + b_4)\,x^2}{x^1}, \tag{44}$$

and its covariant derivatives are expressed as

$$\nabla_1\mathcal{S}_{1332} = \frac{(b_0 + b_4)\,x^2}{(x^1)^2}, \qquad \nabla_2\mathcal{S}_{1332} = -\frac{(b_0 + b_4)}{x^1}. \tag{45}$$

Choosing the associated covector β_i as given in the subsequent form:

$$\beta_i\,(x) = \begin{array}{cc} \frac{-1}{3x^1} & i = 1 \\ \frac{1}{3x^2} & i = 2 \\ 0 & \text{otherwise} \end{array}. \tag{46}$$

In view of Eq. (2), we get

$$\nabla_1\mathcal{S}_{1332} = 2\beta_1\mathcal{S}_{1332} + \beta_1\mathcal{S}_{1332} + \beta_3\mathcal{S}_{1132} + \beta_3\mathcal{S}_{1312} + \beta_2\mathcal{S}_{1331}. \tag{47}$$

$$\nabla_2\mathcal{S}_{1332} = 2\beta_2\mathcal{S}_{1332} + \beta_1\mathcal{S}_{2332} + \beta_3\mathcal{S}_{1232} + \beta_3\mathcal{S}_{1322} + \beta_2\mathcal{S}_{1332}. \tag{48}$$

Using Eqs. (44) and (46) in Eqs. (47) and (48), one infers

$$\nabla_1\mathcal{S}_{1332} = 3\beta_1\mathcal{S}_{1332} = \frac{(b_0 + b_4)\,x^2}{(x^1)^2}, \tag{49}$$

$$\nabla_2\mathcal{S}_{1332} = 3\beta_2\mathcal{S}_{1332} = -\frac{(b_0 + b_4)}{x^1}. \tag{50}$$

Thus, the considered manifold is a 4−dimensional $(\mathrm{PSS})_4$ semi-Riemannian manifold.

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References

[1] M. C. Chaki and Sarbari Ray, *Space-times with covariant-constant energy-momentum tensor*, International Journal of Theoretical Physics **35** (1996): 1027-1032.
[2] M. C. Chaki, *On pseudosymmetric manifolds*, Ann St Univ, Al I Cuza Iasi **33** (1987): 53–58.
[3] B.-Y. Chen, U. C. De, N. B. Turki, and A. A. Syied, *Warped product manifolds: Characterizations through the symmetry of the semiconformal curvature tensor and applications, https://doi.org/10.1142/S0219887824502815*.
[4] B.-Y. Chen, *Pseudo-Riemannian Geometry, δ-invariants and Applications*, World Scientific, Hackensack (2011).
[5] B.-Y. Chen, S. Shenawy, U. C. De, A. Rabie, and N. B. Turki, *The Impact of Quasi-Conformal Curvature Tensor on Warped Product Manifolds*, Axioms **13(8)** (2024): 500.
[6] B.-Y. Chen, *A simple characterization of generalized Robertson–Walker space-times*, Gen. Relat. Grav. **46** (2014): 1833.
[7] S. K. Chaubey and R. H. Ojha, *On the m-projective curvature tensor of a Kenmotsu manifold*, Differential Geometry-Dynamical Systems **12** (2010): 52-60.
[8] P. H. Chavanis, *Cosmology with a stiff matter era*, Physical Review D **92(10)** (2015): 103004.
[9] U. C. De and S. Mallick, *m-Projective curvature tensor on N (k)-quasi-Einstein manifolds*, Differential Geometry Dynamical Systems **16** (2014): 98-112.
[10] U. C. De, Y. J. Suh, S. K. Chaubey and S. Shenawy, *On pseudo H-symmetric Lorentzian manifolds with applications to relativity*, Filomat **34(10)** (2020): 3287-3297.
[11] K. De, U. C. De, A. A. Syied, N. B. Turki, and S. Alsaeed, *Perfect fluid spacetimes and gradient solitons*, Journal of Nonlinear Mathematical Physics **29 (4)** (2022): 843-858.
[12] U. C. De, S. Shenawy, H. M. Abu-Donia, N. Bin Turki, S. Alsaeed, and A. A. Syied, *Spacetimes Admitting Concircular Curvature Tensor in f (R) Gravity*, Front. Phys. **9** (2022), 800060.
[13] F. Defever, R. Deszcz, M. Hotlo´s, M. Kucharski and Z. Sent¨urk, *Generalizations of Robertson-Walker spaces*, Annu. Univ. Sci. Budapest E¨otv¨os Sect. Math. **43** (2000), 13–24.

[14] D. Hazra, and Avijit Sarkar, *Quasi-conformal curvature tensor on N (k)-quasi einstein manifolds*, Korean Journal of Mathematics **29(4)** (2021): 801-810.

[15] S. Güler and S. A. Demirbağ, *Riemannian manifolds satisfying certain conditions on pseudo-projective curvature tensor*, Filomat **30(3)** (2016): 721-731.

[16] M. Gutiérrez and B. Olea, *Global decomposition of a Lorentzian manifold as a generalized Robertson–Walker space*, Differential Geometry and its Applications **27(1)** (2009): 146-156.

[17] Y.Ishii, *On conharmonic transformations*, Tensor NS **7 (**1957), 73–80.

[18] J. Kim, *A type of conformal curvature tensor*, Far East Journal of Mathematical Sciences **99(1)** (2016), 61.

[19] J. Kim, *On pseudo semiconformally symmetric manifolds*, Bull Korean Math Soc. **54** (2017): 177–186.

[20] D. Krupka, *The trace decomposition of tensors of type (1, 2) and (1, 3)*, *New Developments in Differential Geometry, Math. Appl. 350, Kluwer Acad. Publ. Dordrecht*, (1996), 243-253.

[21] C. A. Mantica, and L. G. Molinari , *On the Weyl and Ricci tensors of Generalized Robertson-Walker space-times*, J. Math. Phys., **57** (2016): 102502.

[22] C. A. Mantica, and L. G. Molinari, *Generalized Robertson–Walker spacetimes—a survey*, International Journal of Geometric Methods in Modern Physics, **14** (2017): 1730001.

[23] C. A. Mantica, U. C. De, Y. J. Suh, and L. G. Molinari, *Perfect fluid spacetimes with harmonic generalized curvature tensor*, Osaka Journal of Mathematics **56(1)** (2019): 173-182.

[24] S. Mallick and Uday Chand De, *Spacetimes admitting W2-curvature tenso*, Intern. J. Geom. Meth. Mod. Phys. **11(4)** (2014), 1450030.

[25] S. Mallick and U. C. De, *On pseudo $\varphi-$symmetric spacetimes*, Analysis and Mathematical physics **9** (2019):1333-1345.

[26] S. Mallick, Y. J. Suh, and U. C. De, *A spacetime with pseudo-projective curvature tensor*, Journal of Mathematical Physics **57(6)** (2016): 062501.

[27] R. S. Mishra, *Structures on a differentiable manifold and their applications*, Chandrama Prakasana. Allahabad, India (1984).

[28] B. O'Neill, *Semi-Riemannian Geometry,* Academic Press, New York, 1983.

[29] G. P. Pokhariyal and R. S. Mishra, *Curvature tensors and their relativistic significance*, *Yokohama Math. J.* **19(2)** (1971): 97-103.

[30] R. Sharma and A. Ghosh, *Perfect fluid space-times whose energy-momentum tensor is conformal Killing*, Journal of mathematical physics **51(2)** (2010).

[31] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Herlt, *Exact solutions of Einstein's field equations*, Cambridge university press, 2009.

[32] A. A. Syied, U. C. De, and N. B. Turki, *Conharmonically flat and conharmonically symmetric warped product manifolds*, AIP Advances **14(3)** (2024): 035322.

[33] A. A. Syied, U. C. De, N. B. Turki, and R. Kumar, *Doubly warped product manifolds: Investigations through the projective curvature tensor and relativistic applications*, AIP Advances, **14(6) (**2024**)**: 065324.

[34] N. B. Turki, U. C. De, A. A. Syied, and G. E. Vîlcu, *Investigation of space-times through W_2-curvature tensor in $f(R, G)$ gravity*, Journal of Geometry and Physics **194** (2023): 104987.

[35] M. Walker and R. Penrose, *On quadratic first integrals of the geodesic equations for type* {*2,2*} *spacetimes*, Communications in Mathematical Physics **18** (1970): 265-274.

[36] K. Yano, *Concircular geometry I. Concircular transformations*, Proc. Imperial Academy (Tokyo) **16**(6) (1940), 195-200.

[37] F. O. Zengin, and S. A. Demirbag, *On weakly and pseudo concircular symmetric structures on a Riemannian manifold*, Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica **47(1)** (2008): 129-138.

[38] F. O. Zengin, Y. Yavuz, *Pseudo conharmonically symmetric manifolds*, Eur J Pure and Appl Math **7** (2014): 246–255.