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A study of hybrid fractional differential equations via measure of noncompactness

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Abstract. In this paper, we study a nonlinear boundary value problems (BVPs) of hybrid Hadamard fractional differential equations with nonlocal hybrid Hadamard integral boundary conditions. We use the technique of the measure of noncompactness and degree theory under some suitable conditions to investigate the existence of a solution for our problem. Further, we study the uniqueness results and stability analysis of the considered problem. To show the applicability of our obtained results, we give a numerical example.

1. Introduction

Recently, as a new branch of applied mathematics, fractional differential equations (FDEs) have played a very important role due their significations in mathematical modeling of many phenomena in real world related to engineering and scientific disciplines such as biology, chemistry, economics and numerous branches of physical sciences [2, 9, 10, 20–22].

Boundary value problems of (FDEs) implicit several kinds of fractional derivatives like Riemann-Liouville-type, Caputo-type, Hadamard-type, Caputo-Hadamard-type, Erdélyi-Kober, and Hilfer-Hadamard-type fractional derivative with different sorts of boundary conditions have studied by many authors (see [1, 3, 5–7, 12–14, 16]). In many papers , when the authors use fixed point theorem like Schauder fixed point theorem for the existence of solutions, this it needs more powerful condition in sense of compactness on the nonlinear function that involves in the problem which restricts the area to very particular boundary value problems (BVPs). In order to extend the tools to a class of BVPs to use less restricted conditions, one needs to explore more refined tools of functional analysis, such as the topological degree theory combined with notions of noncompactness and convexity. The topological degree method proved to be dominant tool in the study of many mathematical models, we refer to [8, 15, 23, 26, 28, 29, 31].

Inspired and motivated by the works mentioned in this paper, we study the following nonlinear hybrid fractional differential equations

$$D^{\sigma}\left[\frac{\upsilon(\tau)-\sum_{i=1}^{m} I^{\theta_i}\chi_i(\tau,\upsilon(\tau))}{\Psi(\tau,\upsilon(\tau),I^{\gamma}\upsilon(\tau))}\right]=g(t,\upsilon(\tau),I^{\gamma}\upsilon(\tau)),\quad 1<t<e, \tag{1}$$

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subject to the boundary conditions

$$\begin{cases} \upsilon(1) = 0, \\ D^{\sigma-1}\left(\dfrac{\upsilon(\tau) - \sum_{i=1}^{m} I^{\theta_i}\chi_i(\tau,\upsilon(\tau))}{\Psi(\tau,\upsilon(\tau), I^{\gamma}\upsilon(\tau))}\right)\Bigg|_{\tau=1} = 0 \\ \left(\dfrac{\upsilon(\tau) - \sum_{i=1}^{m} I^{\theta_i}\chi_i(\tau,\upsilon(\tau))}{\Psi(\tau,\upsilon(\tau), I^{\gamma}\upsilon(\tau))}\right)\Bigg|_{\tau=e} = \lambda\left(I^{p}\upsilon\right)(\eta), \end{cases} \tag{2}$$

where D^{σ} denotes the Hadamard fractional derivative of order $\sigma \in]2,3]$. I^{γ}, I^{θ_i} and I^{p} are the Hadamard fractional integrals of order $\gamma, \theta_i, p > 0$ ($i = 1,2,...,m$), $\Psi \in C\left([1,e]\times\mathbb{R}^2, \mathbb{R}\setminus\{0\}\right)$, $g \in C([1,e]\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, and $\chi_i \in C([1,e]\times\mathbb{R},\mathbb{R})$ with $\chi_i(1,0) = 0$, for $i = 1,2,...,m$. λ, η are two real parameters with $\lambda > 0$, $1 < \eta < e$ and $\frac{\lambda\Gamma(\sigma-1)}{\Gamma(p+\sigma-1)}(\log\eta)^{p+\sigma-2} \neq 1$. Moreover, we establish some conditions about Ulam-type stability for the problem (1)-(2).

2. Basic definitions and preliminaries

Here, we give certain definitions and results which are needed to prove our main results.

Let $C(I,\mathbb{R})$ be the Banach space of all continuous functions from I into $\mathbb{R}$.

We begin by defining Hadamard fractional integrals and derivatives, and we introduce some properties that can be used thereafter.

Definition 2.1. *[17] The Hadamard fractional integral of order $\sigma \in \mathbb{R}^+$ for a function $f \in C[a,b]$, $0 \leq a \leq \tau \leq b \leq \infty$, is defined as*

$$I^{\sigma}f(\tau) = \frac{1}{\Gamma(\sigma)}\int_a^{\tau}\left(\log\frac{\tau}{s}\right)^{\sigma-1} f(s)\frac{ds}{s},$$

where $\Gamma(.)$ is the Euler Gamma function, which is defined by $\Gamma(r) = \int_0^{\infty} e^{-\tau}\tau^{r-1}d\tau$ and $\log(.) = \log_e(.)$.

Definition 2.2. *[17] Let $0 < a < b < \infty$ and $\delta = \tau\frac{d}{d\tau}$. The Hadamard derivative of fractional order $\sigma \in \mathbb{R}^+$ for a function $f \in C^{n-1}([a,b],\mathbb{R})$ is defined as*

$$D^{\sigma}f(\tau) = \delta^n\left(I^{n-\sigma}f\right)(\tau) = \frac{1}{\Gamma(n-\sigma)}\left(\tau\frac{d}{d\tau}\right)^n\int_a^{\tau}\left(\log\frac{\tau}{s}\right)^{n-\sigma-1}\frac{f(s)}{s}ds,$$

where $n-1 < \sigma \leq n \in \mathbb{Z}^+$, $n = [\sigma]+1$ denotes the integer part of the real number q.

Lemma 2.3. *([17], Property 2.24) If $a, \alpha, \beta > 0$, then*

$$\left(D^{\sigma}\left(\log\frac{\tau}{a}\right)^{\beta-1}\right)(\tau) = \frac{\Gamma(\beta)}{\Gamma(\beta-\sigma)}\left(\log\frac{\tau}{a}\right)^{\beta-\sigma-1},$$

$$\left(I^{\sigma}\left(\log\frac{\tau}{a}\right)^{\beta-1}\right)(\tau) = \frac{\Gamma(\beta)}{\Gamma(\beta+\sigma)}\left(\log\frac{\tau}{a}\right)^{\beta+\sigma-1}.$$

Lemma 2.4. *([17]) Let $\sigma > 0$ and $\upsilon \in C[1,\infty) \cap L^1[1,\infty)$. Then the solution of Hadamard fractional differential equation $D^{\sigma}\upsilon(\tau) = 0$ is given by*

$$\upsilon(\tau) = \sum_{i=1}^{n} c_i(\log\tau)^{\sigma-i},$$

and the following formula holds:

$$I^{\sigma}D^{\sigma}\upsilon(\tau)=\upsilon(\tau)+\sum_{i=1}^{n}c_i(\log\tau)^{\sigma-i},$$

for some $c_i\in\mathbb{R}$, $i=1,2,...,n$, *where* $n=[\sigma]+1$.

Now, we put $\mathcal{Y}=C(I,\mathbb{R})$ where $I=[1,e]$ endowed with the norm $\|\upsilon\|=\sup_{1\leq\tau\leq e}|\upsilon(\tau)|$. Consider $\mathbb{B}\in\mathcal{P}(\mathcal{Y})$ the family of all its bounded sets. We recall the following definition [18].

Definition 2.5. *The Kuratowski measure of noncompactness* $\varpi:\mathbb{B}\longrightarrow\mathbb{R}_+$ *is defined as*

$$\varpi(S)=\inf\{d>0:\ S\in\mathbb{B}\,admits\ a\ finite\ cover\ by\ sets\ of\ diameter\leq d\}.$$

Now, we recall the following notions which can be found in [4].

Proposition 2.6. *The Kuratowski measure* ϖ *satisfies the following properties:*
(*i*) $\varpi(S)=0$ *if and only if* S *is relatively compact.*
(*ii*) ϖ *is seminorm, i.e.,* $\varpi(\lambda S)=|\lambda|\varpi(S)$, $\lambda\in\mathbb{R}$ *and* $\varpi(S_1+S_2)\leq\varpi(S_1)+\varpi(S_2)$.
(*iii*) $S_1\subset S_2$ *implies* $\varpi(S_1)\leq\varpi(S_2)$; $\varpi(S_1\cup S_2)=\max\{\varpi(S_1),\varpi(S_2)\}$.
(*iv*) $\varpi(conv\,S)=\varpi(S)$.
(*v*) $\varpi(\overline{S})=\varpi(S)$.

Definition 2.7. *Consider* $\Omega_0\subset\mathcal{Y}$ *and* $\mathcal{F}:\Omega_0\longrightarrow\mathcal{Y}$ *a continuous bounded map. Then,* $\mathcal{F}$ *is called* ϖ*-Lipschitz, if there exists* $\mu\geq0$ *such that*

$$\varpi(\mathcal{F}(S))\leq\mu\varpi(S),\quad for\ all\ bounded\ S\subset\Omega_0.$$

If, in addition, $\mu<1$ *then we say that* $\mathcal{F}$ *is a strict-contraction.*

Definition 2.8. *The map* $\mathcal{F}$ *is called* ϖ*-condensing if*

$$\varpi(F(S))<\varpi(S),\ for\ all\ bounded\ S\subset\Omega_0\ with\ \varpi(S)>0.$$

In other words, $\varpi(\mathcal{F}(S))\geq\varpi(S)$ *implies* $\varpi(S)=0$.

Proposition 2.9. *If* $\mathcal{F},C:\Omega_0\longrightarrow\mathcal{Y}$ *are* ϖ*-Lipschitz maps with constants* μ *respectively* μ', *then* $\mathcal{F}+C:\Omega_0\longrightarrow\mathcal{Y}$ *is* ϖ*-Lipschitz with constant* $\mu+\mu'$.

Proposition 2.10. *If* $\mathcal{F}:\Omega_0\longrightarrow\mathcal{Y}$ *is compact, then* $\mathcal{F}$ *is* ϖ*-Lipschitz with constant* $\mu=0$

Proposition 2.11. *If* $C:\Omega_0\longrightarrow\mathcal{Y}$ *is Lipschitz with constant* μ, *then* C *is* ϖ*-Lipschitz with the same constant.*

The following theorem [11], will be used to prove the existence result.

Theorem 2.12. *Let* $\mathcal{T}:\mathcal{Y}\longrightarrow\mathcal{Y}$ *be a* ϖ *condensing map and*

$$\mathcal{W}=\{\upsilon\in\mathcal{Y}:\ there\ exists\ \varsigma\in[0,1],\ such\ that\ \upsilon=\varsigma\mathcal{T}\upsilon\}.$$

If $\mathcal{W}$ *is a bounded set in* $\mathcal{Y}$, *that is, there exists* $r>0$ *such that* $\mathcal{W}\subset B_r(0)$, *then the topological degree*

$$\mathbb{D}(I-\varsigma\mathcal{T},\ B_r(0),0)=1,\quad for\ all\ \varsigma\in[0,1].$$

Consequently, $\mathcal{T}$ *has at least one fixed point and the set of the fixed points of* $\mathcal{T}$ *lies in* $B_r(0)$.

3. Existence results

In this section, we will establish the existence and uniqueness of solution of the boundary value problem (1)-(2) by using the fixed point approaches. To do this, we first transform our problem into an integral form of the problem.

For convenience we put

$$\Omega = 1 - \frac{\lambda\Gamma(\sigma-1)}{\Gamma(p+\sigma-1)}(\log\eta)^{p+\sigma-2}, \tag{3}$$

$$\Upsilon = \frac{1}{\Gamma(\sigma+1)} + \frac{1}{\Omega}\left(\frac{\lambda}{\Gamma(\sigma+p+1)}(\log\eta)^{p+\sigma} - \frac{1}{\Gamma(\sigma+1)}\right), \tag{4}$$

Lemma 3.1. *Let $h \in C([1,e],\mathbb{R})$. The solution function υ_0 of the hybrid Hadamard equation*

$$D^{\sigma}\left[\frac{\upsilon(\tau) - \sum_{i=1}^{m} I^{\theta_i}\chi_i(\tau,\upsilon(\tau))}{\Psi(\tau,\upsilon(\tau),I^{\gamma}\upsilon(\tau))}\right] = h(\tau), \quad 1<t<e,\ 2<\sigma\leq 3, \tag{5}$$

subject to the boundary conditions

$$\begin{aligned}
&\upsilon(1) = 0,\\
&D^{\sigma-1}\left(\frac{\upsilon(\tau) - \sum_{i=1}^{m} I^{\theta_i}\chi_i(\tau,\upsilon(\tau))}{\Psi(\tau,\upsilon(\tau),I^{\gamma}\upsilon(\tau))}\right)\Bigg|_{\tau=1} = 0\\
&\left(\frac{\upsilon(\tau) - \sum_{i=1}^{m} I^{\theta_i}\chi_i(\tau,\upsilon(\tau))}{\Psi(\tau,\upsilon(\tau),I^{\gamma}\upsilon(\tau))}\right)\Bigg|_{\tau=e} = \lambda(I^{p}\upsilon)(\eta),
\end{aligned} \tag{6}$$

if and only if the function υ_0 is a solution for the following Hadamard integral equation:

$$\begin{aligned}
\upsilon(\tau) = {}& \Psi(\tau,\upsilon(\tau),I^{\gamma}\upsilon(\tau))\Bigg[\frac{1}{\Gamma(\sigma)}\int_1^{\tau}\left(\log\frac{\tau}{s}\right)^{\sigma-1}\frac{h(s)}{s}ds\\
&+\frac{(\log\tau)^{\sigma-2}}{\Omega}\left(\frac{\lambda}{\Gamma(\sigma+p)}\int_1^{\eta}\left(\log\frac{\eta}{s}\right)^{p+\sigma-1}\frac{h(s)}{s}ds - \frac{1}{\Gamma(\sigma)}\int_1^{e}\left(\log\frac{e}{s}\right)^{\sigma-1}\frac{h(s)}{s}ds\right)\Bigg]\\
&+\sum_{i=1}^{m} I^{\theta_i}\chi_i(t,\upsilon(\tau)).
\end{aligned} \tag{7}$$

Proof. Let υ_0 be a solution for hybrid equation (5). By virtue of the lemma 2.4, there exist constants c_1, c_2, $c_3 \in \mathbb{R}$ provided that

$$\begin{aligned}
\left[\frac{\upsilon(\tau) - \sum_{i=1}^{m} I^{\theta_i}\chi_i(\tau,\upsilon(\tau))}{\Psi(\tau,\upsilon(\tau),I^{\gamma}\upsilon(\tau))}\right] = {}& \frac{1}{\Gamma(\sigma)}\int_1^{\tau}\left(\log\frac{\tau}{s}\right)^{\sigma-1}\frac{h(s)}{s}ds\\
&+ c_1(\log\tau)^{\sigma-1} + c_2(\log\tau)^{\sigma-2} + c_3(\log\tau)^{\sigma-3}.
\end{aligned} \tag{8}$$

Since $\chi_i(1,0) = 0$, $i = 1,2,...,m$ and $\Psi(1,0,0) \neq 0$, the use of boundary conditions $\upsilon(1) = 0$ and $D^{\sigma-1}\left(\frac{\upsilon(\tau)-\sum_{i=1}^{m} I^{\theta_i}\chi_i(\tau,\upsilon(\tau))}{\Psi(\tau,\upsilon(\tau),I^{\gamma}\upsilon(\tau))}\right)\Big|_{\tau=1} = 0$ gives $c_1 = c_3 = 0$. Applying Hadamard fractional integral operator of order $p > 0$ on both sides of equality (8) and using Lemmas 2.3, we get that

$$\begin{aligned}
I^{p}\left(\frac{\upsilon(\tau) - \sum_{i=1}^{m} I^{\theta_i}\chi_i(\tau,\upsilon(\tau))}{\Psi(\tau,\upsilon(\tau),I^{\gamma}\upsilon(\tau))}\right) = {}& \frac{1}{\Gamma(\sigma+p)}\int_1^{\tau}\left(\log\frac{\tau}{s}\right)^{p+\sigma-1}\frac{h(s)}{s}ds\\
&+ c_2\frac{\Gamma(\sigma-1)}{\Gamma(p+\sigma-1)}(\log\tau)^{p+\sigma-2}.
\end{aligned}$$

By using the Hadamard integral boundary condition $\left(\frac{v(\tau)-\sum_{i=1}^{m} I^{\theta_i}\chi_i(\tau,v(\tau))}{\Psi(\tau,v(\tau),I^{\gamma}v(\tau))}\right)\Big|_{t=e} = \lambda\left(I^{p}v\right)(\eta)$, we get

$$c_2 = \frac{1}{\Omega}\left(\frac{\lambda}{\Gamma(\sigma+p)}\int_1^{\eta}\left(\log\frac{\eta}{s}\right)^{p+\sigma-1}\frac{h(s)}{s}ds - \frac{1}{\Gamma(\sigma)}\int_1^{e}\left(\log\frac{e}{s}\right)^{\sigma-1}\frac{h(s)}{s}ds\right),$$

where Ω is defined in (3).

By inserting the values c_i for $i = 1,2,3$ in (8), we get

$$\begin{aligned} v_0(t) = \Psi(\tau, v_0(\tau), I^{\gamma}v_0(\tau))&\left[\frac{1}{\Gamma(\sigma)}\int_1^{\tau}\left(\log\frac{\tau}{s}\right)^{\sigma-1}\frac{h(s)}{s}ds\right. \\ &\left.+\frac{(\log\tau)^{\sigma-2}}{\Omega}\left(\frac{\lambda}{\Gamma(\sigma+p)}\int_1^{\eta}\left(\log\frac{\eta}{s}\right)^{p+\sigma-1}\frac{h(s)}{s}ds - \frac{1}{\Gamma(\sigma)}\int_1^{e}\left(\log\frac{e}{s}\right)^{\sigma-1}\frac{h(s)}{s}ds\right)\right] \\ &+\sum_{i=1}^{m} I^{\theta_i}\chi_i(t, v_0(t)). \end{aligned}$$

That is v_0 a solution for integral equation (7). Conversely, one can easily see that v_0 is a solution function for the hybrid boundary value problem of fractional order (5)-(6) whenever v_0 is a solution function for the fractional integral equation (7). □

Define operators $\mathcal{A}, \mathcal{B} : \mathcal{Y} \longrightarrow \mathcal{Y}$ by

$$\begin{aligned} \mathcal{A}v(\tau) = \Psi(\tau, v(\tau), I^{\gamma}v(\tau))&\left[\frac{1}{\Gamma(\sigma)}\int_1^{\tau}\left(\log\frac{\tau}{s}\right)^{\sigma-1}\frac{g(s, v(s), I^{\gamma}v(s))}{s}ds\right. \\ &+\frac{(\log\tau)^{\sigma-2}}{\Omega}\left(\frac{\lambda}{\Gamma(\sigma+p)}\int_1^{\eta}\left(\log\frac{\eta}{s}\right)^{p+\sigma-1}\frac{g(s, v(s), I^{\gamma}v(s))}{s}ds\right. \\ &\left.\left.-\frac{1}{\Gamma(\sigma)}\int_1^{e}\left(\log\frac{e}{s}\right)^{\sigma-1}\frac{g(s, v(s), I^{\gamma}v(s))}{s}ds\right)\right], \end{aligned} \tag{9}$$

$\mathcal{B}v(\tau) = \sum_{i=1}^{m} I^{\theta_i}\chi_i(\tau, v(\tau))$,
then a function $v \in \mathcal{Y}$ is a solution for problem (1)-(2) if and only if v is a fixed point of the operator equation

$$v(\tau) = \mathcal{A}v(\tau) + \mathcal{B}v(\tau) = \mathcal{T}v(\tau). \tag{10}$$

For developing the existence result, we list the following hypothesis.
(H_1) There are positive continuous functions $\kappa, \xi : [1, e] \longrightarrow \mathbb{R}$, parameters $0 < q, w < 1$, positive constants $C_{\chi_i}, M_{\chi_i}, \delta$ and $0 < a_i < 1$ such that for $v \in \mathcal{Y}$, we have

$$|\chi_i(\tau, v(\tau))| \le C_{\chi_i}|v|^{a_i} + M_{\chi_i},$$

$$\left|g(\tau, v(\tau), I^{\gamma}v(\tau))\right| \le \kappa(\tau) + \delta\left(|v|^{q} + |I^{\gamma}v|^{q}\right),$$

and

$$|\Psi(\tau, v(\tau), I^{\gamma}v(\tau))| \le \xi(\tau) + \beta\left(|v(\tau)|^{w} + |I^{\gamma}v(\tau)|^{w}\right).$$

(H_2) For arbitrary $\tau \in [1, e]$, $v, \overline{v} \in C([1, e], \mathbb{R})$, there are positive constants K_{χ_i} for $i = 1, 2, ..., m$, $\kappa_0 = \max_{1\le\tau\le e}\kappa(\tau)$ and $\xi_0 = \max_{1\le\tau\le e}\xi(\tau)$,

$$|\chi_i(\tau, v(\tau)) - \chi_i(\tau, \overline{v}(\tau))| \le K_{\chi_i}|v - \overline{v}|,$$

$$|\Psi(\tau, v(\tau), I^{\gamma}v(\tau)) - \Psi(\tau, \overline{v}(\tau), I^{\gamma}\overline{v}(\tau))| \le \xi_0|v - \overline{v}|,$$

and

$$\left|g(\tau, v(\tau), I^{\gamma}v(\tau)) - g(\tau, \overline{v}(\tau), I^{\gamma}\overline{v}(\tau))\right| \le \kappa_0|v - \overline{v}|.$$

Lemma 3.2. *With suppostions* (H_1) *and* (H_2), *the operator* $\mathcal{B}$ *is* ϖ*-Lipschitz with constant* $\mu = \frac{mK_{\chi_i}}{\Gamma(\sigma+1)}$. *Further,* $\mathcal{B}$ *satisfies the following growth condition:*

$$\|\mathcal{B}\upsilon(\tau)\| \leq \frac{mC}{\Gamma(\sigma+1)}. \tag{11}$$

where $C = \max_{1\leq i\leq m}\{C_{\chi_i}\|\upsilon\|^{a_i} + M_{\chi_i}\}$.

Proof. For $\upsilon, \overline{\upsilon} \in \mathcal{Y}$ such that $\upsilon < \overline{\upsilon}$, using (H_2), we obtain

$$\begin{aligned}\|\mathcal{B}(\upsilon) - \mathcal{B}(\overline{\upsilon})\| &\leq \sum_{i=1}^{m} I^{\theta_i}|\chi_i(\tau,\upsilon(\tau)) - \chi_i(\tau,\overline{\upsilon}(\tau))|\\ &\leq \frac{K_{\chi_i}}{\Gamma(\sigma+1)}|\upsilon - \overline{\upsilon}|,\end{aligned}$$

hence

$$\|\mathcal{B}(\upsilon) - \mathcal{B}(\overline{\upsilon})\| \leq \mu\|\upsilon - \overline{\upsilon}\|. \tag{12}$$

By Proposition 2.11, $\mathcal{B}$ is ϖ-Lipschitz with constant μ. For growth condition, using the assumption (H_1), we get

$$\begin{aligned}|\mathcal{B}(\upsilon)| &\leq \sum_{i=1}^{m}(C_{\chi_i}|\upsilon|^{a_i} + M_{\chi_i})I^{\theta_i}(\tau)\\ &= \frac{1}{\Gamma(\sigma+1)}\sum_{i=1}^{m}(C_{\chi_i}|\upsilon|^{a_i} + M_{\chi_i})(\log\tau)^{\theta_i}\\ &\leq \frac{1}{\Gamma(\sigma+1)}\sum_{i=1}^{m}(C_{\chi_i}\|\upsilon\|^{a_i} + M_{\chi_i})\\ &\leq \frac{mC}{\Gamma(\sigma+1)}.\end{aligned}$$

Then

$$\|\mathcal{B}\upsilon(\tau)\| \leq \frac{mC}{\Gamma(\sigma+1)}.$$

□

Lemma 3.3. *Under conditions* (H_1) *and* (H_2), *the operator* $\mathcal{A}$ *is* ϖ *-Lipschitz with zero constant. Further,* $\mathcal{A}$ *satisfies the following growth condition:*

$$\|\mathcal{A}\upsilon\| \leq f_0 + f_1\|\upsilon\|^q + f_2\|\upsilon\|^{w+q},\ \upsilon \in \mathcal{Y}. \tag{13}$$

Proof. The continuity of the operator $\mathcal{A}$ for each fixed $\tau \in [1,e]$ follows from the continuity of Ψ, g with respect to υ for each fixed $\tau \in [1,e]$. Then, for each $\upsilon \in \mathcal{Y}$, using (H_1), we have

$$\begin{aligned}|\mathcal{A}\upsilon(\tau)| &\leq (\xi(\tau) + \beta(|\upsilon(\tau)|^w + |I^\gamma\upsilon(\tau)|^w))\Bigg[\frac{(\kappa(s) + \delta(|\upsilon(s)|^q + |I^\gamma\upsilon(s)|^q))}{\Gamma(\sigma+1)}(\log\tau)^\sigma\\ &\quad + \frac{1}{\Omega}\Bigg(\frac{\lambda(\kappa(\tau) + \delta(|\upsilon(\tau)|^q + |I^\gamma\upsilon(\tau)|^q))}{\Gamma(\sigma+p+1)}(\log\eta)^{p+\sigma}\\ &\quad + \frac{(\kappa(s) + \delta(|\upsilon(s)|^q + |I^\gamma\upsilon(s)|^q))}{\Gamma(\sigma+1)}\Bigg)\Bigg]\\ &\leq \left(\xi_0 + \beta\|\upsilon\|^w\left(1 + \frac{1}{(\Gamma(\gamma+1))^w}\right)\right)\left(\kappa_0 + \delta\|\upsilon\|^q\left(1 + \frac{1}{(\Gamma(\gamma+1))^q}\right)\right)|\Upsilon|.\end{aligned}$$

This gives us

$$\|\mathcal{A}v\| \leq f_0 + f_1 \|v\|^q + f_2 \|v\|^{w+q}, \ v \in \mathcal{Y}, \tag{14}$$

where $f_0 = |\Upsilon| \xi_0 \kappa_0$, $f_1 = \xi_0 \delta |\Upsilon| \left(1 + \frac{1}{(\Gamma(\gamma+1))^q}\right)$ and $f_2 = |\Upsilon| \kappa_0 \beta \delta \left(1 + \frac{1}{(\Gamma(\gamma+1))^w}\right)\left(1 + \frac{1}{(\Gamma(\gamma+1))^q}\right)$.
From (14), we conclude that $\mathcal{A}$ is uniformly bounded on any bounded subset Φ of $\mathcal{Y}$.
Now, we use the following notations

$$\Psi_v(\tau) = \Psi(\tau, v(\tau), I^\gamma v(\tau)), \ \ g_v(\tau) = g(\tau, v(\tau), I^\gamma v(\tau)). \tag{15}$$

Next, we show that $\mathcal{A}$ is equicontinuous. Take $v \in \mathcal{Y}$ and $1 \leq \tau_1 < \tau_2 \leq e$, then we have

$$\begin{aligned}
|\mathcal{A}v(\tau_2) - \mathcal{A}v(\tau_1)| = &\left| \frac{1}{\Gamma(\sigma)} \int_1^{\tau_2} \left(\log \frac{\tau_2}{s}\right)^{\sigma-1} \Psi_v(\tau_2) \frac{g_v(s)}{s} ds \right.\\
&+ \frac{\Psi_v(\tau_2)(\log \tau_2)^{\sigma-2}}{\Omega} \left(\frac{\lambda}{\Gamma(\sigma+p)} \int_1^{\eta} \left(\log \frac{\eta}{s}\right)^{p+\sigma-1} \frac{g_v(s)}{s} ds \right.\\
&\left. - \frac{1}{\Gamma(\sigma)} \int_1^{e} \left(\log \frac{e}{s}\right)^{\sigma-1} \frac{g_v(s)}{s} ds \right)\\
&- \frac{1}{\Gamma(\sigma)} \int_1^{\tau_1,} \left(\log \frac{\tau_1}{s}\right)^{\sigma-1} \Psi_v(\tau_1) \frac{g_v(s)}{s} ds\\
&- \frac{\Psi_v(\tau_1)(\log \tau_1,)^{\sigma-2}}{\Omega} \left(\frac{\lambda}{\Gamma(\sigma+p)} \int_1^{\eta} \left(\log \frac{\eta}{s}\right)^{p+\sigma-1} \frac{g_v(s)}{s} ds \right.\\
&\left.\left. - \frac{1}{\Gamma(\sigma)} \int_1^{e} \left(\log \frac{e}{s}\right)^{p+\sigma-1} \frac{g_v(s)}{s} ds \right) \right|\\
\leq &\frac{1}{\Gamma(\sigma)} \int_1^{\tau_1} \left(\log \frac{\tau_2}{s}\right)^{\sigma-1} |\Psi_v(\tau_2) - \Psi_v(\tau_1)| + \Psi_v(\tau_1) \left| \left(\log \frac{\tau_2}{s}\right)^{\sigma-1} - \left(\log \frac{\tau_1}{s}\right)^{\sigma-1} \right| \frac{g_v(s)}{s} ds\\
&+ \left. \int_{\tau_1}^{\tau_2} \left(\log \frac{\tau_2}{s}\right)^{\sigma-1} |\Psi_v(\tau_2)| \right] \frac{g_v(s)}{s} ds\\
&+ \frac{1}{\Omega} \left[|\Psi_v(\tau_2)| \left| (\log \tau_2)^{\sigma-2} - (\log \tau_1)^{\sigma-2} \right| + (\log \tau_1)^{\sigma-2} |\Psi_v(\tau_2) - \Psi_v(\tau_1)| \right]\\
&\times \left| \frac{\lambda}{\Gamma(\sigma+p)} \int_1^{\eta} \left(\log \frac{\eta}{s}\right)^{p+\sigma-1} \frac{g_v(s)}{s} ds - \frac{1}{\Gamma(\sigma)} \int_1^{e} \left(\log \frac{e}{s}\right)^{\sigma-1} \frac{g_v(s)}{s} ds \right|.
\end{aligned} \tag{16}$$

Then, the right-hand side of the above inequality tends to zero when $\tau_1 \longrightarrow \tau_2$.
Therefore $\mathcal{A}$ is equicontinuous. Hence, by using the Arzela-Ascoli theorem, $\mathcal{A}$ is compact. By proposition 2.10, the operator $\mathcal{A}$ is ϖ-Lipschitz with zero constant. □

Theorem 3.4. *Assume that* (H_1) *and* (H_2) *hold. Then problem (1)-(2) has at least one solution* $v \in \mathcal{Y}$ *provided that* $q + w \leq 1$. *Further, the set of solutions of (1)-(2) is bounded in* $\mathcal{Y}$.

Proof. By Lemma 3.2, the operator $\mathcal{B}$ is ϖ-Lipschitz for $0 \leq \mu < 1$, and by Lemma 3.3, the operator $\mathcal{A}$ is ϖ-Lipschitz with zero constant. From proposition 2.9, we claim that $\mathcal{T}$ is ϖ-Lipschitz with constant $\mu \in [0, 1)$.
Define

$$Q = \{v \in \mathcal{Y} : \textit{ there exists } \zeta \in [0,1], \ v = \zeta T v\}.$$

We shall to prove that Q is bounded in $\mathcal{Y}$.
Let $v \in Q$. Using growth conditions (11) and (13), we obtain

$$\begin{aligned}\|v\| &\le \zeta\left(\|\mathcal{A}v\| + \|\mathcal{B}v\|\right)\\ &\le \zeta\left(f_0 + f_1\|v\|^q + f_2\|v\|^{w+q} + \frac{mC}{\Gamma(\sigma+1)}\right)\\ &= \zeta\left(f_1\|v\|^q + f_2\|v\|^{w+q}\right) + \zeta\left(f_0 + \frac{mC}{\Gamma(\sigma+1)}\right).\end{aligned}$$

Since $q + w \le 1$ and $f_2 < 1$, it follows that the set Q is bounded. Hence, by Theorem 2.12, problem (1)-(2) has at least one solution. □

Choose $0 < R < 1$ and consider a closed bounded and convex subset $\overline{B}_R = \{u \in \mathcal{Y} : \|u\| \le R\}$.

Theorem 3.5. *Assume that* (H_1) *and* (H_2) *hold. If*

$$\mu + \left[\left(\xi_0 + \beta R^w\left(1 + \frac{1}{(\Gamma(\gamma+1))^w}\right)\right)\kappa_0 + \left(\kappa_0 + \delta R^q\left(1 + \frac{1}{(\Gamma(\gamma+1))^q}\right)\right)\xi_0\right]|\Upsilon| < 1,$$

then the problem (1)-(2) has a unique solution.

Proof. We use the Banach contraction principal. For $v, \overline{v} \in \overline{B}_R$, we have

$$\begin{aligned}\left|\Psi_v(\tau)g_v(s) - \Psi_{\overline{v}}(\tau)g_{\overline{v}}(s)\right| &\le |\Psi_v(\tau)|\left|g_v(s) - g_{\overline{v}}(s)\right| + \left|g_{\overline{v}}(s)\right||\Psi_v(\tau) - \Psi_{\overline{v}}(\tau)|\\ &\le \left[\left(\xi_0 + \beta|v|^w\left(1 + \frac{1}{(\Gamma(\gamma+1))^w}\right)\right)\kappa_0\right.\\ &\quad\left.+\left(\kappa_0 + \delta|\overline{v}|\left(1 + \frac{1}{(\Gamma(\gamma+1))^q}\right)\right)\xi_0\right]|v - \overline{v}|.\end{aligned} \tag{17}$$

By using (9) and (15), we have

$$\begin{aligned}\|\mathcal{A}(v) - \mathcal{A}(\overline{v})\| &\le \frac{1}{\Gamma(\sigma)}\int_1^\tau\left(\log\frac{\tau}{s}\right)^{\sigma-1}\left|\Psi_v(\tau)g_v(s) - \Psi_{\overline{v}}(\tau)g_{\overline{v}}(s)\right|\frac{ds}{s}\\ &+ \frac{(\log\tau)^{\sigma-2}}{\Omega}\left(\frac{\lambda}{\Gamma(\sigma+p)}\int_1^\eta\left(\log\frac{\eta}{s}\right)^{p+\sigma-1}\left|\Psi_v(\tau)g_v(s) - \Psi_{\overline{v}}(\tau)g_{\overline{v}}(s)\right|\frac{ds}{s}\right.\\ &\left.+\frac{1}{\Gamma(\sigma)}\int_1^e\left(\log\frac{e}{s}\right)^{p+\sigma-1}\left|\Psi_v(\tau)g_v(s) - \Psi_{\overline{v}}(\tau)g_{\overline{v}}(s)\right|\frac{ds}{s}\right).\end{aligned}$$

Then, from (17), we get

$$\begin{aligned}\|\mathcal{A}(v) - \mathcal{A}(\overline{v})\| &\le |\Upsilon|\left[\left(\xi_0 + \beta R^w\left(1 + \frac{1}{(\Gamma(\gamma+1))^w}\right)\right)\kappa_0\right.\\ &\left.+\left(\kappa_0 + \delta R^q\left(1 + \frac{1}{(\Gamma(\gamma+1))^q}\right)\right)\xi_0\right]\|v - \overline{v}\|\\ &= l\|v - \overline{v}\|,\end{aligned} \tag{18}$$

where

$$l = \left[\left(\xi_0 + \beta R^w\left(1 + \frac{1}{(\Gamma(\gamma+1))^w}\right)\right)\kappa_0 + \left(\kappa_0 + \delta R^q\left(1 + \frac{1}{(\Gamma(\gamma+1))^q}\right)\right)\xi_0\right]|\Upsilon|$$

From (12) and (18), we conclude

$$\begin{aligned}\|\mathcal{T}(v) - \mathcal{T}(\overline{v})\| &\le \|\mathcal{A}(v) - \mathcal{A}(\overline{v})\| - \|\mathcal{B}(v) - \mathcal{B}(\overline{v})\|\\ &\le (l + \mu)\|v - \overline{v}\|,\end{aligned}$$

which implies that problem (1)-(2) has a unique solution in $\overline{B}_R$. □

4. Hyers–Ulam stability

This section is devoted to the study of the Hyers-Ulam stability analysis for the problem (1)-(2). For more related problems to the Hyers-Ulam stability, the reader can consult the works [24, 25, 27, 30] and the literature.

Definition 4.1. *The fractional integral equation (10) is said to be Hyers-Ulam stable if there exists a constant ϱ such that, for given Φ and for each solution υ of the inequality*

$$\|\upsilon - (\mathcal{A} - \mathcal{B})\,\upsilon(\tau)\| < \Phi, \tag{19}$$

there exists a solution $\overline{\upsilon}(\tau)$ of the problem (1)-(2)

$$\overline{\upsilon}(\tau) = (\mathcal{A} - \mathcal{B})\,\overline{\upsilon}(\tau),$$

such that

$$\|\upsilon(\tau) - \overline{\upsilon}(\tau)\| < \Phi\varrho.$$

Theorem 4.2. *Assume that (H_1) and (H_2) hold. If $l + \mu < 1$, then problem (1)-(2) is Hyers-Ulam stable.*

Proof. Let $\upsilon \in \mathcal{Y}$ satisfy the inequalty (19) and $\overline{\upsilon}$ a solution of the problem (1)-(2) satisfying the integral equation (10). We have

$$\begin{aligned}\|\upsilon(\tau) - \overline{\upsilon}(\tau)\| &= \|\upsilon(\tau) - (\mathcal{A} + \mathcal{B})\,\overline{\upsilon}(\tau)\| \\ &\le \|\upsilon(\tau) - (\mathcal{A} + \mathcal{B})\,\upsilon(\tau)\| + \|(\mathcal{A} + \mathcal{B})\,\upsilon(\tau) - (\mathcal{A} + \mathcal{B})\,\overline{\upsilon}(\tau)\| \\ &\le \Phi + \|(\mathcal{A} + \mathcal{B})\,\upsilon(\tau) - (\mathcal{A} + \mathcal{B})\,\overline{\upsilon}(\tau)\|.\end{aligned} \tag{20}$$

On one side, we have

$$\|(\mathcal{A} + \mathcal{B})\,\upsilon(\tau) - (\mathcal{A} + \mathcal{B})\,\overline{\upsilon}(\tau)\| \le \|\mathcal{A}\upsilon(\tau) - \mathcal{A}\overline{\upsilon}(\tau)\| + \|\mathcal{B}\upsilon(\tau) - \mathcal{B}\overline{\upsilon}(\tau)\|.$$

In other side and by using (12), (18) and (20), we get

$$\|\upsilon(\tau) - \overline{\upsilon}(\tau)\| < \Phi + (l + \mu)\,\|\upsilon(\tau) - \overline{\upsilon}(\tau)\|,$$

which implies that

$$\|\upsilon(\tau) - \overline{\upsilon}(\tau)\| < \Phi\varrho,$$

where $\varrho = \frac{1}{1-(l+\mu)}$.
□

5. Example

In this section, we present an example to illustrate our obtained results.
Consider the following hybrid fractional differential equations

$$D^{\frac{5}{2}}\left[\frac{\upsilon(\tau) - \sum_{i=1}^{4} I^{\frac{2i-1}{2}}\chi_i(\tau, \upsilon(t))}{\Psi\left(\tau, \upsilon(\tau), I^{\frac{9}{2}}\upsilon(\tau)\right)}\right] = g\left(\tau, \upsilon(\tau), I^{\frac{9}{2}}\upsilon(\tau)\right), \quad \tau \in [1, e], \tag{21}$$

with the boundary conditions

$$\begin{aligned}&\upsilon(1) = 0, \\ &D^{\sigma-1}\left(\frac{\upsilon(\tau) - \sum_{i=1}^{m} I^{\theta_i}\chi_i(\tau, \upsilon(\tau))}{\Psi\left(\tau, \upsilon(\tau), I^{\frac{9}{2}}\upsilon(\tau)\right)}\right)\Bigg|_{\tau=1} = 0 \\ &\left(\frac{\upsilon(\tau) - \sum_{i=1}^{m} I^{\theta_i}\chi_i(\tau, \upsilon(\tau))}{\Psi\left(\tau, \upsilon(\tau), I^{\frac{9}{2}}\upsilon(\tau)\right)}\right)\Bigg|_{\tau=e} = 0,02\left(I^{2,5}\upsilon\right)(2),\end{aligned} \tag{22}$$

where

$$\chi_i(\tau, \upsilon(\tau)) = \frac{2\tau + 3|\upsilon(\tau)|}{4e}, \quad i = 1, 2, 3,$$

$$\Psi\left(\tau, \upsilon(\tau), I^{\frac{9}{2}}\upsilon(\tau)\right) = \frac{6\tau + 7\upsilon(\tau) + I^{\frac{9}{2}}\upsilon(\tau)}{4}$$

and $g\left(\tau, \upsilon(\tau), I^{\frac{9}{2}}\upsilon(\tau)\right) = \frac{1+4\tau+\upsilon(\tau)+I^{\frac{9}{2}}\upsilon(\tau)}{5}$
Here, $\sigma = \frac{5}{2}$, $m = 3$, $\gamma = \frac{9}{2}$, $p = \frac{5}{2}$, $\eta = 2$ and $\lambda = 0,02$.
Functions χ_i, g and Ψ satisfy hypothesis (H_1) and (H_2) for $a_1 = 0,2$, $a_2 = 0,4$, $a_3 = 0,6$, $C\chi_1 = \frac{3}{4}$, $C_{\chi_2} = \frac{3}{2}$, $C_{\chi_3} = \frac{9}{4}$, $M_{\chi_i} = \frac{1}{2}$, $\kappa(\tau) = \frac{1+4\tau}{5}$, $\delta = \frac{1}{10}$, $q = 0,5$, $K_{\chi_i} = \frac{3}{4e}$, $\kappa_0 = \frac{1+4e}{5}$, $\xi_0 = \frac{3e}{2}$ and $w = 0,2$.
By a simple calculation, we find that $f_2 = 0,0000676992 < 1$ and $q + w < 1$. By using Theorem 3.4, the problem (21)-(22) has at least one solution and the set of solutions is bounded in $\mathcal{Y}$.
Taking $R = 0,8$. We have $\mu = 0,0830215$ and $l = 0,0871847$, which give $l + \mu < 1$, then by Theorem 3.5, the problem (21)-(22) has a unique solution in $\overline{B}_R$ and by Theorem 4.2, the problem (21)-(22) is Hyers-Ulam stable.

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