Filomat 39:26 (2025), 9087–9100
https://doi.org/10.2298/FIL2526087B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Existence of mild solutions for ξ-Caputo fractional integro-differential evolution problems

Walid Benhadda[a,*], Abderrazak Kassidi[a], M'hamed Elomari[a], Ali El Mfadel[a,b]

[a] *Laboratory of Applied Mathematics and Scientific Computing, Sultan Moulay Slimane University, Beni Mellal , Morocco*
[b] *Higher School of Technology, Sultan Moulay Slimane University, Khenifra, Morocco*

Abstract. The aim of this paper is to investigate the existence of mild solutions to a nonlocal ξ-Caputo fractional semilinear integro-differential evolution equation in any arbitrary Banach space. The existence of the results is proved by using the fixed point theorem for condensing maps. To illustrate our theoretical results, a non-trivial example is given as an application.

1. Introduction

The traditional calculus of integers is extended by the introduction of the calculus of fractions, which has infinite memory and inherits certain properties, we suggest that you refer to the special monographs for fundamental insights into the theory of fractional calculus and fractional models [2, 4, 10, 16, 18, 20–24].

Recently, significant discoveries have been made by researchers on the existence and uniqueness of solutions to boundary value fractional differential equations involving different fractional derivatives, such as Riemann-Liouville, Caputo, Hilfer, and Hadamard. For more details, see [1, 12, 16, 18]. In 2016, Almeida [3] introduced the ξ-Caputo fractional derivative, a specific form of fractional derivative in which the exponent of the kernel involves a strictly increasing function. For certain choices of ξ, we get several well-known fractional derivatives, including Rieman-Liouville, Caputo, and Caputo-Hadamard. This flexibility makes the ξ-Caputo operator a robust modeling framework in fractional calculus, see [6–8, 13, 14].

In contrast, another major focus within the field of fractional calculus is the fractional evolution equations. This field provides an abstract, conceptual framework that proves invaluable in tackling a variety of engineering and physics challenges characterized by systems that evolve dynamically over time. Numerous studies have been carried out with the aim of confirming both the existence and the uniqueness of mild solutions to the fractional evolution equations. These investigations rely heavily on the mathematical foundations provided by semigroup theory and fixed point theory, which shed light on the intricate behavior of systems governed by fractional dynamics [8, 25–27].

2020 *Mathematics Subject Classification*. Primary 34A08; Secondary 34A12, 35F25, 35R11.
Keywords. Existence of mild solutions, ξ-Caputo fractional derivative, ξ-fractional integral, C_0-semigroup, measure of noncompactness.
Received: 14 November 2023; Revised: 04 August 2025; Accepted: 07 August 2025
Communicated by Maria Alessandra Ragusa

* Corresponding author: Walid Benhadda
Email addresses: walid.benhadda@usms.ma (Walid Benhadda), a.kassidi@usms.ma (Abderrazak Kassidi), m.elomari@usms.ma (M'hamed Elomari), a.elmfadel@usms.ma (Ali El Mfadel)
ORCID iDs: https://orcid.org/0009-0005-0401-9715 (Walid Benhadda), https://orcid.org/0000-0002-9105-1123 (Abderrazak Kassidi), https://orcid.org/0000-0002-4256-1434 (M'hamed Elomari), https://orcid.org/0000-0002-2479-1762 (Ali El Mfadel)

In 2010, Yong Zhou et al. [26] studied the existence and uniqueness of mild solutions for the following nonlocal fractional evolution differential equation in a Banach space $\mathcal{Z}$.

$$\begin{cases} {}^{C}\mathcal{D}^{\beta}_{0^+}z(t) = \mathcal{A}z(t) + g(t, z(t))), & t \in \mathcal{J}, \\ z(0) + \nu(z) = z_0, \end{cases}$$

where ${}^{C}\mathcal{D}^{\beta}_{0^+}$ is the Caputo fractional derivative at order $0 < \beta < 1$, $z_0 \in \mathcal{Z}$, $\mathcal{J} = (0, T]$, $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$ of operators on $\mathcal{Z}$, $g : [0, \infty) \times \mathcal{Z} \to \mathcal{Z}$ and the nonlocal term $\nu : C([0, \infty), \mathcal{Z}) \to \mathcal{Z}$ are given functions.

In 2020, Suechoei et al. [25] studied the local and global existence of mild solution to the following initial value fractional semilinear evolution equations in a Banach space $\mathcal{Z}$

$$\begin{cases} {}^{C}\mathcal{D}^{\beta,\xi}_{0^+}z(t) = \mathcal{A}z(t) + g(t, z(t)), & t \in \mathcal{J}, \\ z(0) = z_0, \end{cases}$$

where ${}^{C}\mathcal{D}^{\beta,\xi}_{0^+}$ is the ξ-Caputo fractional derivative at order $0 < \beta < 1$, $z_0 \in \mathcal{Z}$, $\mathcal{A}$ is the infinitesimal generator of a C_0 -semigroup of uniformly bounded linear operators$\{\mathcal{T}(t)\}_{t\geq 0}$ on $\mathcal{Z}$ and $g : [0, \infty) \times \mathcal{Z} \to \mathcal{Z}$ is given function satisfying some assumptions.

Inspired by the aforementioned studies, our goal is to establish the existence of mild solutions to the following fractional evolution equation in an arbitrary Banach space $\mathcal{Z}$

$$\begin{cases} {}^{C}\mathcal{D}^{\beta,\xi}_{0^+}z(t) + \mathcal{A}z(t) = g(t, z(t), \mathcal{Q}z(t))), & t \in \mathcal{J}, \\ z(0) + \nu(z) = z_0, \end{cases} \tag{1}$$

where $\beta \in (0, 1)$, $z_0 \in \mathcal{Z}$, $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{Z} \longrightarrow \mathcal{Z}$ is a closed linear operator, $-\mathcal{A}$ generates an uniformly bounded and equicontinuous C_0-semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$ in $\mathcal{Z}$ and the given functions $g : \mathcal{J} \times \mathcal{Z} \times \mathcal{Z} \to \mathcal{Z}$ and $\nu : C(\mathcal{J}, \mathcal{Z}) \to \mathcal{Z}$ satisfy certain assumptions. The term $\mathcal{Q}z(t)$ can be interpreted as a system control, it is defined as follows:

$$\mathcal{Q}z(t) = \int_0^t \mathcal{K}(t, s)z(s)ds,$$

$$\text{where } \mathcal{K} \in C(\Delta, \mathbb{R}^+), \text{ with } \Delta = \{(t, s) \in \mathbb{R}^2 : 0 \leq s \leq t\} \text{ and } \mathcal{Q}^* = \sup_{t\in\mathcal{J}} \int_0^t \mathcal{K}(t, s)ds < \infty.$$

The organization of this work is as follows: in Section 2, we introduced some tools that will help us to prove our main purpose. In Section 3, we establish the existence of mild solutions to the problem (1). Then, in Section 4, we provide a specific example to illustrate our main results, followed by a conclusion in the last section.

2. Preliminaries

Let $\mathcal{Z}$ be a Banach space with the norm $\|\cdot\|_{\mathcal{Z}}$ and $C(\mathcal{J}, \mathcal{Z})$ be the Banach space of all continuous functions $z : \mathcal{J} \to \mathcal{Z}$ endowed with the norm

$$\|z\| = \sup_{t\in\mathcal{J}} \|z(t)\|_{\mathcal{Z}},$$

and $L^p(\mathcal{J}, \mathcal{Z})(1 \leq p < \infty)$ be the Banach space of all Bochner integrable functions $z : \mathcal{J} \to \mathcal{Z}$ with the norm

$$\|z\|_{L^p} = \Big(\int_0^t \|z(t)\|^p dt\Big)^{\frac{1}{p}}.$$

We Consider $\mathcal{M} = \sup_{t\in\mathcal{J}} \|\mathcal{T}(t)\|_{\mathcal{L}(\mathcal{Z})} \geq 1$, where $\mathcal{L}(\mathcal{Z})$ denotes the Banach space of all linear and bounded operators on $\mathcal{Z}$.

Definition 2.1. *[3] Let $f \in L^1(\mathcal{J}, \mathbb{R})$ and $\xi \in C^1(\mathcal{J}, \mathbb{R})$ with $\xi'(t) > 0$ for every $t \in \mathcal{J}$. The ξ-Riemann-Liouville fractional integral of the function f at order $\alpha > 0$ is given by*

$$\mathcal{I}^{\alpha,\xi}_{0^+} f(t) = \int_0^t \frac{(\xi(t) - \xi(s))^{\alpha-1}}{\Gamma(\alpha)} \xi'(s) f(s) ds. \tag{2}$$

Definition 2.2. *[3] Let $f, \xi \in C^n(\mathcal{J}, \mathbb{R})$ with $\xi'(t) > 0$ for every $t \in \mathcal{J}$. The ξ-Caputo fractional derivative of the function f at order $\alpha > 0$ is given by*

$${}^C\mathcal{D}^{\alpha,\xi}_{0^+} f(t) = \int_0^t \frac{(\xi(t) - \xi(s))^{n-\alpha-1}}{\Gamma(n-\alpha)} \xi'(s) f^{[n]}_{\xi}(s) ds,$$

where $f^{[n]}_{\xi}(s) = \left(\frac{1}{\xi'(s)} \frac{d}{ds}\right)^n f(s)$ and $n = [\alpha] + 1$, such that $[\alpha]$ denotes the integer part of α.

Remark 2.3. *Integrals and derivatives appearing in previous definitions are taken in Bochner sense when f is a function with values in $\mathcal{Z}$.*

Proposition 2.4. *[3] Let $\alpha > 0$, if $f \in C^{n-1}(\mathcal{J}, \mathbb{R})$, we find that*

1) ${}^C\mathcal{D}^{\alpha,\xi}_{0^+} \mathcal{I}^{\alpha,\xi}_{0^+} f(t) = f(t)$.
2) $\mathcal{I}^{\alpha,\xi}_{0^+} {}^C\mathcal{D}^{\alpha,\xi}_{0^+} f(t) = f(t) - \displaystyle\sum_{i=0}^{n-1} \frac{f^{[i]}_{\xi}(0)}{i!} (\xi(t) - \xi(0))^i$.

Definition 2.5. *[17] Let $f : \mathbb{R}^+ \to \mathbb{R}$ the generalized Laplace transform of the function f is given by*

$$\widehat{f}(\omega) := \mathcal{L}_{\xi}\{f(t)\}(\omega) = \int_0^{\infty} e^{-\omega(\xi(t)-\xi(0))} f(t) \xi'(t) dt, \quad \text{for all } \omega.$$

Lemma 2.6. *[17] Let $\alpha > 0$, f be a piecewise continuous function on each $[0, t]$ and $\xi(t)$-exponential order, then*

$$\mathcal{L}_{\xi}\{\mathcal{I}^{\alpha,\xi}_{0^+} f(t)\}(\omega) = \frac{\widehat{f(\omega)}}{\omega^{\alpha}}.$$

Definition 2.7. *[18, 23] The Wright function is given by*

$$\upsilon_{\beta}(y) = \sum_{i=0}^{\infty} \frac{(-y)^i \Gamma(\beta(i+1)) \sin(\pi(i+1)\beta)}{i!} \quad \text{for } y \in \mathbb{C} \text{ and } 0 < \beta < 1.$$

Proposition 2.8. *[18, 23] The Wright function υ_{β} satisfies the following proprieties*

1. *For all $t \in \mathbb{R}^+$, $\upsilon_{\beta}(t) \geq 0$*
2. $\displaystyle\int_0^{\infty} \upsilon_{\beta}(t) dt = 1$
3. $\displaystyle\int_0^{\infty} \upsilon_{\beta}(y) y^{\theta} dy = \frac{\Gamma(1+\theta)}{\Gamma(1+\beta\theta)}$ *for $\theta > -1$,*
4. $\displaystyle\int_0^{\infty} \upsilon_{\beta}(t) e^{-yt} dt = E_{\beta}(-y)$, $y \in \mathbb{C}$,
5. $\displaystyle\beta \int_0^{\infty} t \upsilon_{\beta}(t) e^{-yt} dt = E_{\beta,\beta}(-y)$, $y \in \mathbb{C}$,

where $E_{\beta}(.)$ and $E_{\beta,\beta}(.)$ are Mittag-Leffler functions.

Definition 2.9. *[19] Let $\rho \geq 0$. The one-sided stable probability density is defined by*

$$\ell_\beta(\rho) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\rho^{-(\beta n+1)}\frac{\Gamma(\beta n+1)}{n!}\sin(n\pi\beta).$$

Lemma 2.10. *[19] Let $\omega > 0$ and $\rho \geq 0$, then we have*

$$\int_0^{\infty}\ell_\beta(\rho)e^{-\omega\rho}d\rho = e^{-\omega^\beta}, \quad \text{for every } \beta \text{ in } (0,1).$$

Definition 2.11. *[11] Let O bounded subset of $\mathcal{Z}$. The map λ: $O \to \mathbb{R}_+$ given by*

$$\lambda(O) = \inf\{\delta > 0 : O \subseteq \cup_{i=1}^{n} O_i \text{ and } diam(O_i) \leqslant \delta\}.$$

is called Kuratowski measure of non-compactness.

Proposition 2.12. *[11] Let $O, O_1, O_2 \subset \mathcal{Z}$ be bounded, then*

1. $\lambda(O) = 0 \Leftrightarrow O$ *is relatively compact.*
2. $\lambda(\kappa O) = |\kappa|\lambda(O), \quad \kappa \in \mathbb{R}$.
3. $\lambda(O_1 + O_2) \leq \lambda(O_1) + \lambda(O_2)$.
4. $O_1 \subset O_2 \Rightarrow \lambda(O_1) \leq \lambda(O_2)$.
5. $\lambda(O_1 \cup O_2) = \max\{\lambda(O_1),\ \lambda(O_2)\}$.
6. $\lambda(O) = \lambda(\overline{O}) = \lambda(convO)$, *where convO means the convex hull of O.*
7. $\lambda(O + z) = \lambda(O), \quad$ *for all* $z \in \mathcal{Z}$.

Lemma 2.13. *[9] Let $\mathcal{U} \subset \mathcal{Z}$ such that $\mathcal{U}$ is bounded, then there exists a countable set $\mathcal{U}_1 = \{z_k\}_{k=1}^{\infty} \subset \mathcal{U}$ such that $\lambda(\mathcal{U}) \leq 2\lambda(\mathcal{U}_1)$.*

Lemma 2.14. *[5] Let $\mathcal{U}$ be a subset of $C(\mathcal{J}, \mathcal{Z})$ such that $\mathcal{U}(t) = \{z(t) : z \in \mathcal{U} \text{ and } t \in \mathcal{J}\}$. If $\mathcal{U}$ is equicontinuous and bounded in $C(\mathcal{J}, \mathcal{Z})$, then $\lambda(\mathcal{U}(t))$ is continuous on $\mathcal{J}$, and*

$$\lambda(\mathcal{U}) = \max_{t\in\mathcal{J}}\lambda(\mathcal{U}(t)) = \lambda(\mathcal{U}(\mathcal{J})).$$

Lemma 2.15. *[15] Let $\mathcal{U} = \{z_k\}_{k=1}^{\infty} \subset C(\mathcal{J}, \mathcal{Z})$ be a countable and bounded set. Then $\lambda(\mathcal{U}(t))$ is a Lebesgue integral on $\mathcal{J}$, and*

$$\lambda\Big(\Big\{\int_0^t z_k(s)ds\Big\}_{k=1}^{\infty}\Big) \leq 2\int_0^t \lambda(\{z_k(s)\}_{k=1}^{\infty})ds.$$

Lemma 2.16. *[5] Let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be two Banach spaces and $\mathcal{F} : dom(\mathcal{F}) \subset \mathcal{Z}_1 \to \mathcal{Z}_2$ Lipschitz continuous having constant κ, then $\lambda(\mathcal{F}(\mathcal{U})) \leq \kappa\lambda(\mathcal{U})$ for every bounded $\mathcal{U} \subset dom(\mathcal{F})$.*

Theorem 2.17. *[11] Let $B \subset \mathcal{Z}$ be a bounded closed and convex set and $\mathcal{F} : B \to B$ be a condensing operator which means that $\lambda(\mathcal{T}(B)) < \lambda(B)$, then $\mathcal{F}$ has a fixed point in B.*

Theorem 2.18. *[11] Let $B \subset \mathcal{Z}$ be open and bounded with $0 \in B$. If $\mathcal{F} : \bar{B} \to \mathcal{Z}$ is a condensing function and satisfies.*

$$z \neq \gamma\mathcal{F}(z) \text{ for } z \in \partial B \text{ and } \gamma \in (0,1),$$

then $\mathcal{F}$ has a fixed point in $\bar{B}$.

3. Main results

Definition 3.1. *A function $z \in C(\mathcal{J}, \mathcal{Z})$ is called a solution of problem* (1) *if it satisfies both the equation ${}^C\mathcal{D}_{0^+}^{\beta,\xi} z(t) + \mathcal{A}z(t) = g(t, z(t), \mathcal{Q}z(t))$ and the condition $z(0) + \nu(z) = z_0$ on $\mathcal{J}$.*

Lemma 3.2. *Let $z \in C(\mathcal{J}, \mathcal{Z})$. Then, z is a solution of* (1) *if and only if it satisfies the following fractional integral equation*

$$z(t) = z_0 - \nu(z) + \int_0^t \frac{\xi'(s)(\xi(t) - \xi(s))^{\beta-1}}{\Gamma(\beta)} (-\mathcal{A}z(s) + g(s, z(s), \mathcal{Q}z(s)))ds. \tag{3}$$

Proof. Suppose that z is a solution of (1), then by using Proposition 2.4 we obtain

$$\mathcal{I}_{0^+}^{\beta,\xi}\, {}^C\mathcal{D}_{0^+}^{\beta,\xi} z(t) = \mathcal{I}_{0^+}^{\beta,\xi}(-\mathcal{A}z(t) + g(t, z(t), \mathcal{Q}z(t))).$$

It follows

$$z(t) = z(0) + \mathcal{I}_{0^+}^{\beta,\xi}(-\mathcal{A}z(t) + g(t, z(t), \mathcal{Q}z(t))).$$

Therefore

$$z(t) = z_0 - \nu(z) + \int_0^t \frac{\xi'(s)(\xi(t) - \xi(s))^{\beta-1}}{\Gamma(\beta)} (-\mathcal{A}z(s) + g(s, z(s), \mathcal{Q}z(t))))ds.$$

Therefore, the integral equation (3) is satisfied.
Conversely, it is obvious that if z satisfies the equation (3), then it also satisfies problem (1). □

Lemma 3.3. *If $z \in C(\mathcal{J}, \mathcal{Z})$ satisfies the fractional equation* (3), *then we have*

$$z(t) = \mathcal{S}_{1,\xi}^{\beta}(t, 0)(z_0 - \nu(z)) + \int_0^t \mathcal{S}_{2,\xi}^{\beta}(t, s)(\xi(t) - \xi(s))^{\beta-1} g(s, z(s), \mathcal{Q}z(s))\xi'(s)ds,$$

where

$$\mathcal{S}_{1,\xi}^{\beta}(t, s)z(t) := \int_0^\infty \upsilon_\beta(\rho)\mathcal{T}((\xi(t) - \xi(s))^\beta \rho)z(t)d\rho \text{ and } \mathcal{S}_{2,\xi}^{\beta}(t, s)z(t) := \beta \int_0^\infty \rho\upsilon_\beta(\rho)\mathcal{T}((\xi(t) - \xi(s))^\beta \rho)z(t)d\rho.$$

Proof. Let $\omega > 0$. Then, by means of the generalized Laplace transform applied to the equation (3), we get

$$\widehat{z}(\omega) = \frac{1}{\omega}(z_0 - \nu(z)) + \frac{1}{\omega^\beta}(-\mathcal{A}\widehat{z}(\omega) + \widehat{g}(\omega)),$$

with $\widehat{z}(\omega) = \mathcal{L}_\xi\Big\{z(t)\Big\}(\omega)$ and $\widehat{g}(\omega) = \mathcal{L}_\xi\Big\{g(t, z(t), \mathcal{Q}z(t)))\Big\}(\omega)$.
Then, we have

$$\widehat{z}(\omega) = \omega^{\beta-1}(\omega^\beta I + \mathcal{A})^{-1}(z_0 - \nu(z)) + (\omega^\beta I + \mathcal{A})^{-1}\widehat{g}(\omega).$$

It follows that

$$\begin{aligned}
\widehat{z}(\omega) &= \omega^{\beta-1} \int_0^\infty e^{-\omega^\beta s}\mathcal{T}(s)(z_0 - \nu(z))ds + \int_0^\infty e^{-\omega^\beta s}\mathcal{T}(s)\widehat{g}(\omega)ds, \\
&= \beta \int_0^\infty (\omega\eta)^{\beta-1} e^{-(\omega\eta)^\beta}\mathcal{T}(\eta^\beta)(z_0 - \nu(z))d\eta + \beta \int_0^\infty \eta^{\beta-1} e^{-(\omega\eta)^\beta}\mathcal{T}(\eta^\beta)\widehat{g}(\omega)d\eta. \\
&:= \widehat{z_1}(\omega) + \widehat{z_2}(\omega)
\end{aligned}$$

Choosing $\eta = \xi(t) - \xi(0)$, we obtain

$$\widehat{z_1}(\omega) = \int_0^\infty \frac{-1}{\omega} \frac{d}{dt}\left(e^{-(\omega(\xi(t)-\xi(0)))^\beta}\right)\mathcal{T}((\xi(t) - \xi(0))^\beta)(z_0 - \nu(z))dt,$$

and

$$\widehat{z_2}(\omega) = \beta \int_0^\infty (\xi(t) - \xi(0))^{\beta-1} e^{-(\omega(\xi(t)-\xi(0)))^\beta} \mathcal{T}((\xi(t) - \xi(0))^\beta)\widehat{g}(\omega)\xi'(t)dt.$$

From Lemma 2.10, we obtain

$$\widehat{z_1}(\omega) = \int_0^\infty e^{-\omega(\xi(t)-\xi(0))} \Big(\int_0^\infty \ell_\beta(\rho)\mathcal{T}(\frac{(\xi(t) - \xi(0))^\beta}{\rho^\beta})(z_0 - \nu(z))d\rho\Big)\xi'(t)dt. \tag{4}$$

On the other hand, we have

$$\widehat{z_2}(\omega) = \int_0^\infty \int_0^\infty \int_0^\infty \beta e^{-\omega(\xi(t)+\xi(s)-2\xi(0))} \frac{(\xi(t) - \xi(0))^{\beta-1}}{\rho^\beta} \ell_\beta(\rho)\mathcal{T}\Big(\frac{(\xi(t) - \xi(0))^\beta}{\rho^\beta}\Big)g(s, z(s), \mathcal{Q}z(s))\xi'(s)\xi'(t)d\rho ds dt,$$

It follows that

$$\widehat{z_2}(\omega) = \int_0^\infty e^{-\omega(\xi(\tau)-\xi(0))} \Big(\int_0^\tau \int_0^\infty \beta\ell_\beta(\rho) \frac{(\xi(\tau) - \xi(s))^{\beta-1}}{\rho^\beta} \mathcal{T}(\frac{(\xi(\tau) - \xi(s))^\beta}{\rho^\beta})g(s, z(s), \mathcal{Q}z(s))\xi'(s)d\rho ds\Big)\xi'(\tau)d\tau. \tag{5}$$

According to (4) and (5), we get

$$\begin{aligned}\widehat{z}(\omega) &= \int_0^\infty e^{-\omega(\xi(t)-\xi(0))} \left(\int_0^\infty \ell_\beta(\rho)\mathcal{T}(\frac{(\xi(t) - \xi(0))^\beta}{\rho^\beta})(z_0 - \nu(z))d\rho \right) \xi'(t)dt \\ &+ \int_0^\infty e^{-\omega(\xi(\tau)-\xi(0))} \Big(\int_0^\tau \int_0^\infty \beta\ell_\beta(\rho) \frac{(\xi(\tau) - \xi(s))^{\beta-1}}{\rho^\beta} \mathcal{T}(\frac{(\xi(\tau) - \xi(s))^\beta}{\rho^\beta})g(s, z(s), \mathcal{Q}z(s))\xi'(s)d\rho ds\Big)\xi'(\tau)d\tau.\end{aligned}$$

By means of the inverse of the generalized Laplace transform, we get

$$\begin{aligned}z(t) &= \int_0^\infty \upsilon_\beta(\rho)\mathcal{T}((\xi(t) - \xi(0))^\beta \rho)(z_0 - \nu(z))d\rho \\ &+ \beta \int_0^t \int_0^\infty \rho\upsilon_\beta(\rho)(\xi(t) - \xi(s))^{\beta-1} \mathcal{T}((\xi(t) - \xi(0))^\beta \rho)g(s, z(s), \mathcal{Q}z(s))\xi'(s)d\rho ds,\end{aligned}$$

where $\upsilon_\beta(\rho) = \frac{1}{\beta}\rho^{-1-\frac{1}{\beta}}\ell_\beta\big(\rho^{-\frac{1}{\beta}}\big)$.
Then

$$z(t) = \mathcal{S}^\beta_{1,\xi}(t, 0)(z_0 - \nu(z)) + \int_0^t \mathcal{S}^\beta_{2,\xi}(t, s)(\xi(t) - \xi(s))^{\beta-1} g(s, z(s), \mathcal{Q}z(s))\xi'(s)ds.$$

□

Lemma 3.4. *[27] The operators $\mathcal{S}^\beta_{1,\xi}$ and $\mathcal{S}^\beta_{2,\xi}$ satisfy the following :*

1. *$\mathcal{S}^\beta_{1,\xi}(t, s)$ and $\mathcal{S}^\beta_{2,\xi}(t, s)$ are bounded linear operators for each fixed $t \geq s \geq 0$. Moreover*
$$\|\mathcal{S}^\beta_{1,\xi}(t, s)z\| \leq \mathcal{M}\|z\| \text{ and } \|\mathcal{S}^\beta_{2,\xi}(t, s)z\| \leq \frac{\mathcal{M}}{\Gamma(\beta)}\|z\|, \text{ for every } z \in \mathcal{Z}.$$
2. *The operators $\mathcal{S}^\beta_{1,\xi}(t, s)z, \mathcal{S}^\beta_{2,\xi}(t, s)z : [0, +\infty) \to \mathcal{Z}$ are continuous, for all $z \in \mathcal{Z}$.*
3. *If $\{\mathcal{T}(t)\}_{t\geq 0}$ is an equicontinuous semigroup, Then the operators $\mathcal{S}^\beta_{1,\xi}(t, s)$ and $\mathcal{S}^\beta_{2,\xi}(t, s)$ are continuous in $(0, +\infty)$ by the operator norm, i. e., for each $0 < s < t_1 < t_2 \leq T$, we have*
$$\left\|\mathcal{S}^\beta_{1,\xi}(t_2, s) - \mathcal{S}^\beta_{1,\xi}(t_1, s)\right\| \to 0 \text{ and } \left\|\mathcal{S}^\beta_{2,\xi}(t_2, s) - \mathcal{S}^\beta_{2,\xi}(t_1, s)\right\| \to 0 \text{ as } t_2 \to t_1.$$

We introduce the operators $\mathcal{F}^{\beta}_{1,\xi}$, $\mathcal{F}^{\beta}_{2,\xi}$, and $\mathcal{F}$ as follows

$$\begin{aligned}\mathcal{F}^{\beta}_{1,\xi} :& C(\mathcal{J},\mathcal{Z}) \to C(\mathcal{J},\mathcal{Z})\\ &\mathcal{F}^{\beta}_{1,\xi}z(t) = \mathcal{S}^{\beta}_{1,\xi}(t,0)(z_0 - \nu(z)),\end{aligned}$$

$$\begin{aligned}\mathcal{F}^{\beta}_{2,\xi} :& C(\mathcal{J},\mathcal{Z}) \to C(\mathcal{J},\mathcal{Z})\\ &\mathcal{F}^{\beta}_{2,\xi}z(t) = \int_0^t \mathcal{S}^{\beta}_{2,\xi}(t,s)(\xi(t)-\xi(s))^{\beta-1}g(s,z(s),\mathcal{Q}z(s))\xi'(s)ds,\end{aligned}$$

and

$$\begin{aligned}\mathcal{F} :& C(\mathcal{J},\mathcal{Z}) \to C(\mathcal{J},\mathcal{Z})\\ &\mathcal{F}z(t) = \mathcal{F}^{\beta}_{1,\xi}z(t) + \mathcal{F}^{\beta}_{2,\xi}z(t).\end{aligned}$$

Based on Lemma 3.3, we can deduce that a fixed point of the operator $\mathcal{F}$ corresponds to a mild solution of problem (1).
To establish the main results, we need to use the following assumptions

$(\mathcal{H}_1)$
1. The function $g(t,.,.) : \mathcal{Z}\times\mathcal{Z} \to \mathcal{Z}$ is continuous, for each $t \in \mathcal{J}$.
2. The function $g(.,z,\bar{z}) : \mathcal{J} \to \mathcal{Z}$ is Lebesgue measurable, for each $(z,\bar{z}) \in \mathcal{Z}\times\mathcal{Z}$.

$(\mathcal{H}_2)$ There is an increasing continuous function $\Lambda : \mathbb{R}^+ \to \mathbb{R}^+$, and $h \in L^{\frac{1}{\beta_1}}(\mathcal{J},\mathbb{R}^+)$ with $\beta_1 \in (0,\beta)$, such that

$$\|g(t,z,\bar{z})\| \leq h(t)\Lambda(\|z\|) \quad \text{for each } z,\bar{z} \in \mathcal{Z} \text{ and } t \in \mathcal{J}.$$

$(\mathcal{H}_3)$ The function ν is continuous and there exist a constant $\mathcal{L}_\nu > 0$ such that

$$\|\nu(z_1) - \nu(z_2)\| \leq \mathcal{L}_\nu\|z_1 - z_2\| \quad \text{for all } z_1, z_2 \in C(\mathcal{J},\mathcal{Z}).$$

$(\mathcal{H}_4)$ There are two positive constants $\mathcal{L}_1$ and $\mathcal{L}_2$ such that

$$\lambda\left(g(t,\mathcal{D}_1,\mathcal{D}_2)\right) \leq \mathcal{L}_1\lambda\left(\mathcal{D}_1\right) + \mathcal{L}_2\lambda\left(\mathcal{D}_2\right) \quad \text{for every } t \in \mathcal{J} \text{ and } \mathcal{D}_1, \mathcal{D}_2 \in \mathcal{Z}.$$

Theorem 3.5. *If assumptions* $(\mathcal{H}_1)$-$(\mathcal{H}_4)$ *hold, then the problem* (1) *has a mild solution provided that*

$$\left(\mathcal{L}_\nu + \frac{2(\mathcal{L}_1 + \mathcal{Q}^*\mathcal{L}_2)}{\Gamma(\beta+1)}(\xi(T)-\xi(0))^{\beta}\right) \in \left(0, \frac{1}{2\mathcal{M}}\right). \tag{6}$$

Proof. We consider $B_{\mathcal{R}} = \{z \in C(\mathcal{J},\mathcal{Z}) : \|z\| \leq \mathcal{R}\}$ with

$$\mathcal{R} := \frac{\mathcal{M}}{1-\mathcal{M}\mathcal{L}_\nu}\left(\|z_0\| + \|\nu(0)\| + \frac{(\xi(T)-\xi(0))^{(\beta_2+1)((1-\beta_1))}}{\Gamma(\beta)(\beta_2+1)^{(1-\beta_1)}}\Lambda(\mathcal{R})\|h\|_{L^{\frac{1}{\beta_1}}}\right).$$

The proof consists of the following steps:

Step 1. We will show that $\mathcal{F}(B_{\mathcal{R}}) \subset B_{\mathcal{R}}$.
Let $z \in B_{\mathcal{R}}$ and $\beta_2 = \dfrac{\beta-1}{1-\beta_1} \in (-1,0)$, we have

$$\begin{aligned}\|\mathcal{F}z\| &\leq \|\mathcal{F}_{1,\xi}z(t)\| + \|\mathcal{F}_{2,\xi}z(t)\|\\ &\leq \|\mathcal{S}^{\beta}_{1,\xi}(t,0)(z_0-\nu(z))\| + \left\|\int_0^t \mathcal{S}^{\beta}_{2,\xi}(t,s)(\xi(t)-\xi(s))^{\beta-1}g(s,z(s),\mathcal{Q}z(s))\xi'(s)ds\right\|.\end{aligned}$$

By Lemma 3.4 and $(\mathcal{H}_3)$, we get

$$\|\mathcal{S}^{\beta}_{1,\xi}(t,0)(z_0-\nu(z))\| \leq \mathcal{M}\left(\|z_0\|+\mathcal{L}_\nu\|z\|_C+\|\nu(0)\|\right).$$

By applying Hölder's inequality and $(\mathcal{H}_2)$, we obtain

$$\begin{aligned}\int_0^t \|(\xi(t)-\xi(s))^{\beta-1}g(s,z(s),\mathcal{Q}z(s))\xi'(s)\|ds &\leq \Big(\int_0^t(\xi(t)-\xi(s))^{\beta_2}ds\Big)^{1-\beta_1}\Lambda(\mathcal{R})\|h\|_{L^{\frac{1}{\beta_1}}}\\ &\leq \frac{(\xi(T)-\xi(0))^{(\beta_2+1)(1-\beta_1)}}{(\beta_2+1)^{(1-\beta_1)}}\Lambda(\mathcal{R})\|h\|_{L^{\frac{1}{\beta_1}}}.\end{aligned}$$

Then, by Lemma 3.4 we obtain

$$\|\int_0^t \mathcal{S}^{\beta}_{2,\xi}(t,s)(\xi(t)-\xi(s))^{\beta-1}g(s,z(s),\mathcal{Q}z(s))\xi'(s)ds\| \leq \frac{(\xi(T)-\xi(0))^{(\beta_2+1)(1-\beta_1)}}{\Gamma(\beta)(\beta_2+1)^{(1-\beta_1)}}\mathcal{M}\Lambda(\mathcal{R})\|h\|_{L^{\frac{1}{\beta_1}}}.$$

Therefore

$$\begin{aligned}\|\mathcal{F}z(t)\| &\leq \mathcal{M}\left(\|z_0\|+\mathcal{L}_\nu\|z\|_c+\|\nu(0)\|\right)+\frac{(\xi(T)-\xi(0))^{(\beta_2+1)((1-\beta_1))}}{\Gamma(\beta)(\beta_2+1)^{(1-\beta_1)}}\mathcal{M}\Lambda(\mathcal{R})\|h\|_{L^{\frac{1}{\beta_1}}},\\ &\leq \mathcal{M}\left(\|z_0\|+\mathcal{L}_\nu R+\|\nu(0)\|\right)+\frac{(\xi(T)-\xi(0))^{(\beta_2+1)((1-\beta_1))}}{\Gamma(\beta)(\beta_2+1)^{(1-\beta_1)}}\mathcal{M}\Lambda(\mathcal{R})\|h\|_{L^{\frac{1}{\beta_1}}},\\ &= \mathcal{R}.\end{aligned}$$

Hence, $\|\mathcal{F}z\| \leq \mathcal{R}$ for every $z \in B_{\mathcal{R}}$.

Step 2. We will prove that $\mathcal{F}^{\beta}_{1,\xi}$ is Lipschitz continuous.
Let $z,\bar{z} \in B_{\mathcal{R}}$, by using $(\mathcal{H}_3)$, we have

$$\begin{aligned}\left\|\mathcal{F}^{\beta}_{1,\xi}z-\mathcal{F}^{\beta}_{1,\xi}\bar{z}\right\| &= \left\|\mathcal{S}^{\beta}_{1,\xi}(t,0)\,(z_0-\nu(z))-\mathcal{S}^{\beta}_{1,\xi}(t,0)\,(z_0-\nu(\bar{z}))\,\right\|\\ &\leq \mathcal{M}\|\nu(z)-\nu(\bar{z})\|\\ &\leq \mathcal{M}\mathcal{L}_\nu\|z-\bar{z}\|,\end{aligned}$$

thus $\mathcal{F}^{\beta}_{1,\xi}$ is Lipschitz continuous with constant $\mathcal{M}\mathcal{L}_\nu$.

Step 3. We will show that $\mathcal{F}^{\beta}_{2,\xi}$ is continuous.
Let $\{z_n\}_{n=1}^{\infty} \subset B_{\mathcal{R}}$ with $\lim_{n\to+\infty} z_n(t) = z(t)$ in $B_{\mathcal{R}}$, for each $t \in \mathcal{J}$.
Then, by using $(\mathcal{H}_1)$ we get

$$\lim_{n\to+\infty} g\left(t,z_n(t),\mathcal{Q}z_n(t)\right) = g(t,z(t),\mathcal{Q}z(t)) \quad \text{for every } t \in \mathcal{J}.$$

Therefore

$$\lim_{n\to+\infty}\sup_{t\in\mathcal{J}}\|g\left(t,z_n(t),\mathcal{Q}z_n(t)\right)-g(t,z(t),\mathcal{Q}z(t))\| = 0$$

In other hand, for $t \in \mathcal{J}$

$$\begin{aligned}\|\mathcal{F}^{\beta}_{2,\xi}z_n-\mathcal{F}^{\beta}_{2,\xi}z\| &\leq \frac{\mathcal{M}}{\Gamma(\beta)}\int_0^t(\xi(t)-\xi(s))^{\beta-1}\xi'(s)\|g(s,z_n,\mathcal{Q}z_n)-g(s,z,\mathcal{Q}z)\|ds\\ &\leq \frac{\mathcal{M}((\xi(T)-\xi(0))^{\beta})}{\Gamma(\beta+1)}\sup_{t\in\mathcal{J}}\|g(s,z_n(s),\mathcal{Q}z_n(s))-g(s,z(s),\mathcal{Q}z(s))\|.\end{aligned}$$

Thus

$$\|\mathcal{F}_{2,\xi}^{\beta} z_n(t) - \mathcal{F}_{2,\xi}^{\beta} z(t)\| \to 0 \text{ as } n \to +\infty.$$

Therefore $\mathcal{F}_{2,\xi}^{\beta}$ is a continuous on $B_{\mathcal{R}}$.

Step 4. We will demonstrate that $\mathcal{F}$ is equicontinuous in $B_{\mathcal{R}}$.
Let $z \in B_{\mathcal{R}}$ and $0 \le t_1 < t_2 \le T$, we get

$$\begin{aligned}
\|\mathcal{F}z(t_2) - \mathcal{F}z(t_1)\| &\le \left\| \mathcal{S}_{1,\xi}^{\beta}(t_2,0)(z_0 - \nu(z)) - \mathcal{S}_{1,\xi}^{\beta}(t_1,0)(z_0 - \nu(z)) \right\| \\
&+ \left\| \int_0^{t_2} (\xi(t_2) - \xi(s))^{\beta-1} \mathcal{S}_{2,\xi}^{\beta}(t_2,s) g(s,z(s),Qz(s)) \xi'(s) ds \right. \\
&\qquad \left. - \int_0^{t_1} (\xi(t_1) - \xi(s))^{\beta-1} \mathcal{S}_{2,\xi}^{\beta}(t_1,s) g(s,z(s),Qz(s)) \xi'(s) ds \right\| \\
&\le \left\| \mathcal{S}_{1,\xi}^{\beta}(t_2,0)(z_0 - \nu(z)) - \mathcal{S}_{1,\xi}^{\beta}(t_1,0)(z_0 - \nu(z)) \right\| \\
&+ \left\| \int_{t_1}^{t_2} (\xi(t_2) - \xi(s))^{\beta-1} \mathcal{S}_{2,\xi}^{\beta}(t_2,s) g(s,z(s),Qz(s)) ds \right\| \\
&+ \left\| \int_0^{t_1} \left[(\xi(t_2) - \xi(s))^{\beta-1} - (\xi(t_1) - \xi(s))^{\beta-1} \right] \mathcal{S}_{2,\xi}^{\beta}(t_2,s) g(s,z(s),Qz(s)) ds \right\| \\
&+ \left\| \int_0^{t_1} (\xi(t_1) - \xi(s))^{\beta-1} \left(\mathcal{S}_{2,\xi}^{\beta}(t_2,s) - \mathcal{S}_{2,\xi}^{\beta}(t_1,s) \right) g(s,z(s),Qz(s)) ds \right\| \\
&:= \Upsilon_1 + \Upsilon_2 + \Upsilon_3 + \Upsilon_4.
\end{aligned}$$

By Lemma 3.4, we obtain

$$\Upsilon_1 \le \left\| \left(\mathcal{S}_{1,\xi}^{\beta}(t_2,0) - \mathcal{S}_{1,\xi}^{\beta}(t_1,0) \right) z_0 \right\| + \left\| \left(\mathcal{S}_{1,\xi}^{\beta}(t_2,0) - \mathcal{S}_{1,\xi}^{\beta}(t_1,0) \right) \nu(z) \right\|.$$

Then, $\Upsilon_1 \to 0$ as $t_2 \to t_1$.
For Υ_2, we have

$$\Upsilon_2 \le \frac{\mathcal{M}\Lambda(\mathcal{R}) \|h\|_{L^{\frac{1}{\beta_1}}}}{\Gamma(\beta)(1+\beta_2)^{1-\beta_1}} \left(\xi(t_2) - \xi(t_1) \right)^{(1+\beta_2)(1-\beta_1)}.$$

Then, $\Upsilon_2 \to 0$ as $t_2 \to t_1$.
Now, for Υ_3, by using Lemma 3.4, we have

$$\begin{aligned}
\Upsilon_3 &\le \frac{\mathcal{M}\Lambda(\mathcal{R}) \|h\|_{L^{\frac{1}{\beta_1}}}}{\Gamma(\beta)} \left(\int_0^{t_1} \left((\xi(t_1) - \xi(s))^{\beta-1} - (\xi(t_2) - \xi(s))^{\beta-1} \right)^{\frac{1}{1-\beta_1}} ds \right)^{1-\beta_1} \\
&= \frac{\mathcal{M}\Lambda(\mathcal{R}) \|h\|_{L^{\frac{1}{\beta_1}}}}{\Gamma(\beta)(1+\beta_2)^{1-\beta_1}} \left(\xi(t_1)^{1+\beta_2} - \xi(t_2)^{1+\beta_2} + (\xi(t_2) - \xi(t_1))^{1+\beta_2} \right)^{1-\beta_1} \\
&\le \frac{\mathcal{M}\Lambda(\mathcal{R}) \|h\|_{L^{\frac{1}{\beta_1}}}}{\Gamma(\beta)(1+\beta_2)^{1-\beta_1}} (\xi(t_2) - \xi(t_1))^{(1+\beta_2)(1-\beta_1)}.
\end{aligned}$$

Since ξ is a continuous function, then we get $\Upsilon_3 \to 0$ as $t_2 \to t_1$.
It is clear that $\Upsilon_4 = 0$, for $t_1 = 0$ and $0 < t_2 \le T$. Now, for $t_1 > 0$ and $\epsilon > 0$ small enough,

by using Lemma 3.4, we get

$$\begin{aligned}
\Upsilon_4 \leq & \left\| \int_0^{t_1-\epsilon} (\xi(t_1)-\xi(s))^{\beta-1} \left(\mathcal{S}^{\beta}_{2,\xi}(t_2,s)-\mathcal{S}^{\beta}_{2,\xi}(t_1,s)\right) g(s,z(s),\mathcal{Q}z(s))\xi'(s)ds \right\| \\
& + \left\| \int_{t_1-\epsilon}^{t_1} (\xi(t_1)-\xi(s))^{\beta-1} \left(\mathcal{S}^{\beta}_{2,\xi}(t_2,s)-\mathcal{S}^{\beta}_{2,\xi}(t_1,s)\right) g(s,z(s),\mathcal{Q}z(s))\xi'(s)ds \right\| \\
\leq & \int_0^{t_1-\epsilon} \left\|(\xi(t_1)-\xi(s))^{\beta-1} g(s,z(s),\mathcal{Q}z(s))\xi'(s)\right\| ds \sup_{s\in(0,t_1-\epsilon)} \left\|\left(\mathcal{S}^{\beta}_{2,\xi}(t_2,s)-\mathcal{S}^{\beta}_{2,\xi}(t_1,s)\right)\right\| \\
& + \frac{2\mathcal{M}}{\Gamma(\beta)} \int_{t_1-\epsilon}^{t_1} \left\|(\xi(t_1)-\xi(s))^{\beta-1} g(s,z(s),\mathcal{Q}z(s))\xi'(s)\right\| ds \\
\leq & \frac{\Lambda(\mathcal{R})\|h\|_{L^{\frac{1}{\beta_1}}} \left(((\xi(t_1)-\xi(0))^{(1+\beta_2)} - (\xi(t_1)-\xi(t_1-\epsilon))^{(1+\beta_2)}\right)^{1-\beta_1}}{(1+\beta_2)^{1-\beta_1}} \\
& \times \sup_{s\in[0,t_1-\epsilon]} \left\|\left(\mathcal{S}^{\beta}_{2,\xi}(t_2,s)-\mathcal{S}^{\beta}_{2,\xi}(t_1,s)\right)\right\| + \frac{2\mathcal{M}\Lambda(\mathcal{R})\|h\|_{L^{\frac{1}{\beta_1}}}}{\Gamma(\beta)\,(1+\beta_2)^{1-\beta_1}} (\xi(t_1)-\xi(t_1-\epsilon))^{(1+\beta_2)(1-\beta_1)}.
\end{aligned}$$

Since ξ is a continuous function, then $\Upsilon_4 \to 0.$ as $t_2 \to t_1$ and $\epsilon \to 0$.
Then, $\|\mathcal{F}z(t_2) - \mathcal{F}z(t_1)\| \to 0$ as $t_2 \to t_1$ for every $z \in B_{\mathcal{R}}$ and $0 \leq t_1 < t_2 \leq T$.
Therefore, $\mathcal{F}$ is equicontinuous in $B_{\mathcal{R}}$.

Step 5. We will prove that $\mathcal{F}$ is a condensing operator.
Let $B \subset B_{\mathcal{R}}$, we have $\mathcal{F}(B)$ is equicontinuous and bounded. Then by using Lemma 2.13, there exists a set $B_1 = \{z_n\}_{n=1}^{\infty} \subset B$, with

$$\lambda(\mathcal{F}(B)) \leq 2\lambda\left(\mathcal{F}(B_1)\right). \tag{7}$$

Since $\mathcal{F}(B_1) \subset \mathcal{F}(B_{\mathcal{R}})$ is equicontinuous, then by Lemma 2.14 we have

$$\lambda\left(\mathcal{F}(B_1)\right) = \max_{t\in J} \lambda\left(\mathcal{F}(B_1)(t)\right). \tag{8}$$

For $t \in J$, by using Lemmas 3.4 and 2.14, also $(\mathcal{H}_3)$ and $(\mathcal{H}_4)$, we obtain

$$\begin{aligned}
\lambda\Big(\mathcal{F}(B_1)(t)\Big) & \leq \lambda\left(\mathcal{F}^{\beta}_{1,\xi}(B_1)(t)\right) + \lambda\left(\mathcal{F}^{\beta}_{2,\xi}(B_1)(t)\right) \\
& \leq \lambda\Big(\mathcal{F}^{\beta}_{1,\xi}(B_1)\Big) + \lambda\left(\left\{\int_0^t \mathcal{S}^{\beta}_{2,\xi}(t,s)(\xi(t)-\xi(s))^{\beta-1} g(s,z_n(s),\mathcal{Q}z_n(s))\xi'(s)ds\right\}_{n=1}^{\infty}\right).
\end{aligned}$$

By Lemma 2.16, we have $\lambda\left(\mathcal{F}^{\beta}_{1,\xi}(B_1)\right) \leq \mathcal{M}\mathcal{L}_v\lambda(B_1)$ and by Lemma 2.15, we obtain

$$\begin{aligned}
\lambda(\mathcal{F}(B_1)(t)) & \leq \mathcal{M}\mathcal{L}_v\lambda(B_1) + 2\int_0^t \lambda\Big(\Big\{\mathcal{S}^{\beta}_{2,\xi}(t,s)(\xi(t)-\xi(s))^{\beta-1} g(s,z_n(s),\mathcal{Q}z_n(s))\xi'(s)\Big\}_{n=1}^{\infty}\Big)ds \\
& \leq \mathcal{M}\mathcal{L}_v\lambda(B) + \frac{2\mathcal{M}}{\Gamma(\beta)} \int_0^t (\xi(t)-\xi(s))^{\beta-1} \lambda\Big\{g(s,z_n(s),\mathcal{Q}z_n(s))\Big\}_{n=1}^{\infty} \xi'(s)ds \\
& \leq \mathcal{M}\mathcal{L}_v\lambda(B) + \frac{2\mathcal{M}}{\Gamma(\beta)} \int_0^t (\xi(t)-\xi(s))^{\beta-1} \Big(\mathcal{L}_1\lambda((B_1)(s)) + \mathcal{L}_2\lambda(\mathcal{Q}(B_1)(s))\Big)\xi'(s)ds
\end{aligned}$$

Meanwhile, we have

$$\lambda\left(\mathcal{Q}B_1(s)\right) \leq \lambda(\mathcal{Q}(B_1)) \leq \|\mathcal{Q}\|\lambda(B_1) \leq \mathcal{Q}^*\lambda(B_1) \leq \mathcal{Q}^*\lambda(B).$$

Hence

$$\begin{aligned}\lambda(\mathcal{F}(B_1)(t)) &\leq \mathcal{M}\mathcal{L}_\nu\lambda(B) + \frac{2\mathcal{M}}{\Gamma(\beta)}\int_0^t (\xi(t)-\xi(s))^{\beta-1}\Big(\mathcal{L}_1\lambda(B) + \mathcal{L}_2 Q^*\lambda(B)\Big)\xi'(s)ds \\ &\leq \mathcal{M}\mathcal{L}_\nu\lambda(B) + \frac{2\mathcal{M}(\mathcal{L}_1 + Q^*\mathcal{L}_2)}{\Gamma(\beta+1)}(\xi(T)-\xi(0))^\beta\lambda(B) \\ &\leq \mathcal{M}\left(\mathcal{L}_\nu + \frac{2(\mathcal{L}_1 + Q^*\mathcal{L}_2)}{\Gamma(\beta+1)}(\xi(T)-\xi(0))^\beta\right)\lambda(B).\end{aligned}$$

From (7)) and (8), we have

$$\lambda(\mathcal{F}(B)) < 2\mathcal{M}\left(\mathcal{L}_\nu + \frac{2(\mathcal{L}_1 + Q^*\mathcal{L}_2)}{\Gamma(\beta+1)}(\xi(T)-\xi(0))^\beta\right)\lambda(B).$$

By the condition (6), it follows that

$$\lambda(\mathcal{F}(B)) < \lambda(B).$$

Thus, the operator $\mathcal{F}$ is a condensing operator.
Since all the conditions of Theorem 2.17 are satisfied, it follows that $\mathcal{F}$ has a fixed point in $B_{\mathcal{R}}$, which corresponds to a mild solution of problem (1). □

In another case, if condition ($\mathcal{H}_3$) does not hold, we make the following assumption.

($\mathcal{H}_5$) The function ν is completely continuous and there are two constants $K_1 \in \left(0, \frac{1}{\mathcal{M}}\right)$ and $K_2 > 0$ such that

$$\|\nu(z)\| \leq K_1\|z\| + K_2 \text{ for every } z \in B_r, \quad \text{where } r > 0.$$

Theorem 3.6. *If assumptions ($\mathcal{H}_1$), ($\mathcal{H}_2$), ($\mathcal{H}_4$), ($\mathcal{H}_5$), and the following condition hold:*

$$\frac{(\mathcal{L}_1 + Q^*\mathcal{L}_2)}{\Gamma(\beta+1)}(\xi(T)-\xi(0))^\beta \in \left(0, \frac{1}{4\mathcal{M}}\right). \tag{9}$$

Then the problem (1) *has a mild solution in B_r where r satisfying*

$$\frac{(1-K_1\mathcal{M})r}{\mathcal{M}\Big(\|z_0\| + K_2 + \frac{\mathcal{M}_1}{(1+\beta_2)^{1-\beta_1}}(\xi(T)-\xi(0))^{(1+\beta_2)(1-\beta_1)}\Big)} > 1 \tag{10}$$

Proof. Let

$$r > \frac{\mathcal{M}\Big(\|z_0\| + K_2 + \frac{\mathcal{M}_1}{(1+\beta_2)^{1-\beta_1}}(\xi(T)-\xi(0))^{(1+\beta_2)(1-\beta_1)}\Big)}{(1-K_1\mathcal{M})}$$

As in the proof of theorem 3.5, we can prove that $\mathcal{F}$ is equicontinuous in B_r.
Now, let $D \subset B_r$, we have $\mathcal{F}(D)$ is equicontinuous and bounded , by Lemma 2.13 there exists $D_1 = \{z_n\}_{n=1}^{\infty} \subset D$ a countable set such that

$$\lambda(\mathcal{F}(D)) \leq 2\lambda(\mathcal{F}(D_1)). \tag{11}$$

Since $\mathcal{F}(D_1) \subset \mathcal{F}(B_r)$ is equicontinuous and by Lemma 2.14 we get

$$\lambda(\mathcal{F}(D_1)) = \max_{t\in J}\lambda(\mathcal{F}(D_1)(t)).$$

By employing ($\mathcal{H}_4$) and ($\mathcal{H}_5$), we can conclude that

$$\lambda(\mathcal{F}(D)) \leq \frac{4\mathcal{M}(\mathcal{L}_1 + Q^*\mathcal{L}_2)}{\Gamma(\beta+1)}(\xi(T)-\xi(0))^\beta\lambda(D). \tag{12}$$

The combination of (9) and (12), we get

$$\lambda\left(\mathcal{F}(D)\right) < \lambda(D),$$

thus, $\mathcal{F}$ is a condensing operator.
Now, we consider $\gamma \in (0,1)$ and $z = \gamma\mathcal{F}(z)$, then we have

$$z(t) = \gamma\mathcal{S}^{\beta}_{1,\xi}(t,0)(z_0 - \nu(z)) + \gamma\int_0^t \mathcal{S}^{\beta}_{2,\xi}(t,s)(\xi(t)-\xi(s))^{\beta-1}g(s,z(s),\mathcal{Q}z(s))\xi'(s)ds.$$

It follows that

$$\|z(t)\| \leq \mathcal{M}(\|z_0\| + K_1 r + K_2) + \frac{\mathcal{M}\mathcal{M}_1}{\Gamma(\beta)(1+\beta_2)^{1-\beta_1}}.$$

Then, we get

$$\frac{(1-K_1\mathcal{M})r}{\mathcal{M}\Big(\|z_0\| + K_2 + \frac{\mathcal{M}_1}{\Gamma(\beta)(1+\beta_2)^{1-\beta_1}}(\xi(T)-\xi(0))^{(1+\beta_2)(1-\beta_1)}\Big)} \leq 1.$$

Thus, by using (10), there exists some constant r such that $\|z\| \neq r$. By the choice of B_r, does not exist $z \in \partial B_r$ such that $z = \gamma\mathcal{F}(z)$ for some $\gamma \in (0,1)$.

Hence, by Theorem 2.18 we conclude that $\mathcal{F}$ has a fixed point in B_r, which is a mild solution to the problem (1). □

4. An illustrative example

As an example, we have the following fractional evolution problem

$$\begin{cases} {}^{C}\mathcal{D}^{\frac{1}{2},\sqrt{1+t}}_{0^+}u(t,z) + \dfrac{\partial^2}{\partial z^2}u(t,z) = \dfrac{1}{25}\dfrac{e^{-t}}{(1+t)}u(t,z) + \displaystyle\int_0^t \frac{1}{50}e^{-s}u(s,z)ds, \quad (t,z)\in[0,1]^2, \\ u(t,0) = u(t,1) = 0, \\ u(0,z) + \displaystyle\sum_{j=1}^{10}\frac{1}{40}log\Big(1+\|u(t_j,z)\|\Big) = u_0(z), \quad t_j\in(0,1), \ j=1,2,...,10. \end{cases} \tag{13}$$

Let $\mathcal{Z} := L^2([0,1])$ and $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{Z} \to \mathcal{Z}$ be an operator such that $\mathcal{A}u = u''$ with the domain

$$D(\mathcal{A}) = \Big\{u\in\mathcal{Z} \ : \ u, u' \text{ are absolutely continuous and } \ u''\in\mathcal{Z} \text{ with } u(0)=u(1)=0\Big\}.$$

Thus, $-\mathcal{A}$ generates an equicontinuous C_0-semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$ on $\mathcal{Z}$, and $\|\mathcal{T}(t)\| \leq 1$, for all $t \geq 0$.

By putting $u(t) = u(t,\cdot)$, the problem (13) can be rewritten into the form of the main problem(1) such that $\beta = \frac{1}{2}$, $\mathcal{J} = [0,1]$, $\xi(t) = \sqrt{1+t}$, $g(t,u,\mathcal{Q}u) = \dfrac{1}{25}\dfrac{e^{-t}}{(1+t)}u(t) + \displaystyle\int_0^t \frac{1}{50}e^{-s}u(s)ds$ with $\mathcal{Q}u(t) = \displaystyle\int_0^t \frac{1}{50}e^{-s}u(s)ds$ and $\nu(u) = \dfrac{1}{40}\displaystyle\sum_{j=1}^{10} log\Big(1+\|u(t_j)\|\Big)$.

Let u, $v \in \mathcal{Z}$, we have

$$\|\nu(u)-\nu(v)\| = \Big\|\frac{1}{40}\sum_{j=1}^{10}log\Big(1+\|u(t_j)\|\Big) - \frac{1}{40}\sum_{j=1}^{10}log\Big(1+\|v(t_j)\|\Big)\Big\| \leq \frac{1}{4}\|u-v\|.$$

And

$$\|\ g(t,u,\mathcal{Q}u)\ \| \ = \ \Big\|\ \frac{1}{25}\frac{e^{-t}}{(1+t)}u(t) + \int_0^t \frac{1}{50}e^{-s}u(s)ds\ \Big\| \leq h(t)\Lambda(\|u\|),$$

with

$$h(t) = \left(\frac{1}{25}\frac{e^{-t}}{(1+t)} + \frac{1}{50}\right) \text{ and } \Lambda(R) = R.$$

Hence,

$$\mathcal{L}_1 = \frac{1}{25}\,\mathcal{L}_2 = \frac{1}{50},\, Q^* = \frac{1}{50} \text{ and } \mathcal{L}_\nu = \frac{1}{4}.$$

Furthermore, we have

$$2\mathcal{M}\left(\mathcal{L}_\nu + \frac{2(\mathcal{L}_1 + Q^*\mathcal{L}_2)}{\Gamma(\beta+1)}(\xi(T) - \xi(0))^\beta\right) \approx 0.5636 < 1$$

Since all the conditions of the theorem 3.5 hold, which means that the problem (13) has a mild solution in $C([0,1], L^2([0,1]))$.

5. Conclusion

In this work, we investigated the existence of mild solutions to a semilinear fractional integro-differential evolution equation involving ξ-Caputo derivatives with nonlocal conditions in an arbitrary Banach space. In the first step, we constructed the form of the mild solutions of our problem using generalized ξ-Laplace transforms, semigroups, and some techniques of the ξ-fractional calculus. In the second, we proved the existence of mild solutions using some fixed point theorems for condensing maps. Finally, we provided a convenient example to illustrate the investigation of our theoretical results.

References

[1] R.P. Agarwal, S.K. Ntouyas, B. Ahmad and A.K. Alzahrani,*Hadamard-type fractional functional differential equations and inclusions with retarded and advanced arguments*,Advances in Difference Equations, **1** (2006)1–15.

[2] R.P. Agarwal, S. Hristova and D.O'Regan,*A survey of Lyapunov functions, stability and impulsive Caputo fractional differential equations*, *Fract. Calc. Appl. Anal*, **19** (2016) 290–318.

[3] R. Almeida *Caputo fractional derivative of a function with respect to another function*. Commun. Nonlinear Sci. Numer. Simul, **44** (2017) 460–481.

[4] S. Aslan, A.O. Akdemir, *New estimations for quasi-convex functions and (h,m)-convex functions with the help of Caputo-Fabrizio fractional integral operators, Electronic Journal of Applied Mathematics*, 1 (3), (2023).

[5] J. Banas and K. Goebel, *Measure of Noncompactness in Banach Spaces*, Lect. Notes Pure Appl. Math., vol. 60, Marcel Dekker, New York, (1980).

[6] W. Benhadda ,M. Elomari , A. Kassidi and A. El Mfadel, *Existence of Anti-Periodic Solutions for ξ-Caputo Fractional p-Laplacian Problems via Topological Degree Methods*. Asia Pacific Journal of Mathematics, 10. 11. 10.28924/APJM/(2023),10-13.

[7] W. Benhadda,A. Kassidi, A. El Mfadel and M. Elomari, *Existence Results for an Implicit Coupled System Involving ξ-Caputo and p-Laplacian Operators*. Sahand Communications in Mathematical Analysis, 21(4), (2024) 137-153.

[8] W. Benhadda, M. Elomari, A. El Mfadel and A. Kassidi, *Existence of mild solutions for non-instantaneous impulsive ξ-Caputo fractional integro-differential equations* . Proyecciones (Antofagasta), 43(5), (2024) 1207-1228.

[9] D. Bothe, *Multivalued perturbations of m-accretive differential inclusions*, Israel J. Math. 108 (1998) 109-138.

[10] N. Chems Eddine, M.A. Ragusa, D.D. Repovs, On the concentration-compactness principle for anisotropic variable exponent Sobolev spaces and its applications, Fractional Calculus and Applied Analysis, 27 (2), 725–756, doi:10.1007/s13540-024-00246-8, (2024).

[11] K. Deimling ,*Nonlinear Functional Analysis* , Springer-Verlag, New York, (1985).

[12] A. El Mfadel, S. Melliani, and M. Elomari, *Existence and uniqueness results for Caputo fractional boundary value problems involving the p-Laplacian operator*. U.P.B. Sci. Bull. Series A, **84**(1) (2022) 37–46.

[13] A. El Mfadel , S. Melliani, M. Elomari ,*Existence results for nonlocal Cauchy problem of nonlinear ξ-Caputo type fractional differential equations via topological degree methods*. Advances in the Theory of Nonlinear Analysis and its Application. 6. . 10.31197/atnaa.1059793.(2022),270 - 279.

[14] A. El Mfadel , S. Melliani and M. Elomari, *Existence of solutions for nonlinear $\xi-$Caputo-type fractional hybrid differential equations with periodic boundary conditions*, Asia Pac. J. Math. (2022).

[15] H.P. Heinz, *On the behaviour of measure of noncompactness with respect to differentiation and integration of rector-valued functions*, Nonlinear Anal. 7 (1983), 1351-1371.

[16] R. Hilfer, *Applications of Fractional Calculus in Physics* , Singapore, (2000).

[17] F. Jarad , T. Abdeljawad, *Generalized fractional derivatives and Laplace transform* , Discrete Contin. Dyn. Syst. Ser, **13**(3)(2019) 709–722.

[18] A.A Kilbas, H.M. Srivastava , J.J. Trujillo , *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematical studies 204. Ed van Mill. Amsterdam, Elsevier Science B.V. Amsterdam (2006).

[19] F. Mainardi , *On the initial value problem for the fractional diffusion-wave equation*, Waves and Stability in Continuous Media, World Scientific, Singapore. (1994) 246-251.

[20] Y. Meng, X.R. Du, H.H. Pang, Iterative positive solutions to a coupled Riemann-Liouville fractional q-difference system with the Caputo fractional q-derivative boundary conditions, Journal of Function Spaces, vol.2023, (2023).

[21] A.T. Nguyen, N.H. Tuan, L.X. Dai, N.H. Can, On an inverse problem for a tempered fractional diffusion equation, Filomat, 38 (19),(2024) 6809–6827.

[22] K. Oldham , *Fractional differential equations in electrochemistry*, Adv. Eng. Softw, 41 (1), 912, (2010).

[23] I. Podlubny, *Fractional Differential Equations: An Introduction to Fractional Derivatives. Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications* , vol. 198. Elsevier, Amsterdam , (1998).

[24] J. Sabatier , O. Agrawal , J. Machado, *Advances in Fractional Calculus Theoretical Developments and Applications in Physics and Engineering*. Springer, Dordrecht, (2007).

[25] A. Suechoei , P. Sa Ngiamsunthorn, *Existence uniqueness and stability of mild solutions for semilinear ξ-Caputo fractional evolution equations, Advances in Difference Equations*. **(1)**(2020)1–28.

[26] Y. Zhou , F. Jiao, *Nonlocal Cauchy problem for fractional evolution equations. Nonlinear analysis: real world applications*. **11**(5)(2010) 4465–4475.

[27] J. Wang ,Y. Zhou, *A class of fractional evolution equations and optimal controls*, Nonlinear Anal. Real World Appl. 12 (2011) 263–272.