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Renormalized solutions for some nonlinear degenerated elliptic problems in weighted Sobolev space with variable exponents

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Abstract. In this article, we prove the existence of a renormalised solution to the problem of the nonlinear elliptic equation:

$$\begin{cases} -\operatorname{div}(a(x,u,Du)) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $a(x,u,Du)$ is blow-up for a finite value m of the unknown u and the data $f \in L^1(\Omega)$.

1. Introduction

Recently, interest in weighted Sobolev space with variable exponent has grown rapidly because they have many physical applications such image processing (underline the borders, eliminate the noise) and electro-rheological fluids, for more detail, we invite the reader to see([3, 4]).

Our objective in this paper is to establish that a renormalized solution exists for a class of elliptic-type problems with the following form.

$$\begin{cases} -\operatorname{div}(a(x,u,Du)) = f \ \text{ in } \Omega, \\ \qquad u = 0 \text{ on } \partial\Omega, \end{cases} \tag{2}$$

where Ω is an open bounded subset of $\mathbb{R}^N (N \geq 2)$, and the data f in $L^1(\Omega)$. The operator $-\operatorname{div}(a(x,u,Du))$ is a Leray–Lions operator defined on weighted Sobolev space with variable exponent and blow-up when $u \to m^-$, with m is strictly positive real number.

This problem is motivated by physical applications where the internal variable u is constrained to satisfy $u \leq m$. Comprehensive discussions on such internal constraints are provided in Frémond's studies on shape memory materials [13] and on constitutive laws in phase-change models [14]. Further related contributions can be found in the context of turbulence modeling [21, 22].

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For the analysis of Problem (2), we will use the notion of renormalized solution (see Definition (3.1). The reason for using this kind of solution is to capture the behavior of the solution on the set $\{u = m\}$, which will be suitably replaced by $T_k(u)$, since the datum is only L^1-integrable. Moreover, the flux $a(x, u, Du)$ on $\{u = m\}$ must make sense (see 15).

For convenience, we recall that the notion of renormalized solution was first introduced by DiPerna [17] in the study of the Boltzmann equation and later developed by DiPerna and Lions in [17]. This concept has since been extended to a wide range of nonlinear PDEs, including applications in fluid mechanics [26]. For the elliptic case, we refer to [9, 23, 24], while the parabolic case is discussed in [10, 12, 18].

The problem (2), has been investigated in a few cases by several authors. In [**?**], H.Redwane has studied the problem (2) in weighted Sobolev spaces and they proved the existence of renormalized solutions $u \in W_0^{1,p}(\Omega, \omega)$. Next, In (see [11, 12]), the authors proved the existence of renormalized solutions $u \in W_0^{1,p}(\Omega)$, in the case where the operator a is replaced by a symmetric matrix, and blow-up for a finite value of the unknown.

This paper is broken down as follows: section 2, we present the weighted Sobolev space with variable exponent and some of its properties. Section 3, we make assumptions and provide definition of renormalized solution. Section 4, we etablish the proof of main result.

2. Preliminaries

We should remember the weight Lebesgue and Sobolev with variable exponents.

Let $p : \bar{\Omega} \to [1, \infty]$ be a measurable function, where Ω is a domain of $\mathbb{R}^n$, satisfy the log-Hölder continuity condition

$$|p(x) - p(y)| \leq \frac{A}{\log \frac{1}{|x-y|}}, \quad \text{for all } x, y \in \Omega \text{ with } |x - y| < r, \tag{3}$$

where $A > 0$ and $0 < r < 1$, and $1 < p_- < p_+ < N$, where

$$p_- = \min_{x\in\bar{\Omega}} p(x) \text{ and } p_+ = \max_{x\in\bar{\Omega}} p(x)$$

and let ω be a mesurable positive a.e. finite function defined in $\mathbb{R}^n$ and satisfied the following integrability conditions

$$\omega \in L^1_{loc}(\Omega) \text{ and } \omega \in L_{loc}^{\frac{-1}{p(x)-1}}(\Omega),$$

$$\omega \in L_{loc}^{-s(x)}(\Omega), \text{ where } s(x) \in (\frac{N}{p(x)}, \infty) \cap [\frac{1}{p(x)-1}, \infty].$$

We define the variable exponent Lebesgue space by

$$L^{p(x)}(\Omega, \omega) = \{u : \Omega \to R : \ u \text{ is measurable in } \Omega \text{ and } \int_\Omega \omega(x)|u(x)|^{p(x)} dx < \infty \}.$$

The space $L^{p(x)}(\Omega, \omega)$ equipped with the Luxemburg-type norm

$$||u||_{p(.),\omega} = \inf\left\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)} \omega(x) dx \leq 1\right\}$$

becomes a Banach space [16]. We denote by $L^{p'(.)}(\Omega, \omega^*)$ the conjugate space of $L^{p(.)}(\Omega, \omega)$, where

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1,$$

and where

$$\omega^*(x) = \omega^*(x)^{1-p'(x)} \ \text{ for all } \ x \in \Omega.$$

The relation between the modular $\int_\Omega |f|^{p(x)}\,\omega(x)dx$ and the norm follows from

$$\min(\|f\|^{p^-}_{p(.),\omega}, \|f\|^{p^+}_{p(.),\omega}) \leq \int_\Omega |f|^{p(x)}\,\omega(x)dx \leq \max(\|f\|^{p^-}_{p(.),\omega}, \|f\|^{p^+}_{p(.),\omega}).$$

For all $f \in L^{p(\cdot)}(\Omega)$, $g \in L^{p'(\cdot)}(\Omega)$ with

$$p(x) \in (1, \infty), \quad p'(x) = \frac{p(x)}{p(x)-1},$$

the generalized Hölder inequality holds,

$$\int_\Omega |f\,g|\,dx \leq \Big(\frac{1}{p^-} + \frac{1}{p'^-}\Big)\|f\|_{p(x)}\|g\|_{p'(x)} \leq 2\|f\|_{p(x)}\|g\|_{p'(x)}\,.$$

The weighted Sobolev space with variable exponent is defined by

$$W^{1,p(\cdot)}(\Omega,\omega) = \{u \in L^{p(\cdot)}(\Omega,\omega) : \nabla u \text{ exists and } |\nabla u| \in L^{p(\cdot)}(\Omega,\omega)\},$$

with respect to the norm

$$\|u\|_{1,p(.),\omega} = \|u\|_{p(.),\omega} + \|\nabla u\|_{p(.)}.$$

The space $W^{1,p(\cdot)}_0(\Omega,\omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega,\omega)$ with respect to the norm $\|u\|_{1,p(.),\omega}$. In addition the spaces $W^{1,p(x)}(\Omega,\omega)$ and $W^{1,p(x)}_0(\Omega,\omega)$ are separable and reflexive Banach spaces.

For more detail, we invite the reader to see [1, 6–8, 16, 19, 20, 27].

3. Assumptions on the Data and notations

Let Ω be a bounded open set of $\mathbb{R}^N (N \geq 2)$, and

$$a : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N \tag{4}$$

is a Carathéodory function, such that there exists a positive function $b \in C^0((-\infty, m))$ which satisfies

$$b^{p(x)-1}(s) \geq \alpha > 0 \quad \forall s \in (-\infty, m)\,; \lim_{s \longrightarrow m^-} b(s) = +\infty, \tag{5}$$

$$\int_0^m b(s)ds < +\infty, \tag{6}$$

and

$$a(x,s,\xi)\xi \geq \omega(x)b(s)^{p(x)-1}\,|\xi|^{p(x)}\,,\ a(x,s,0) = 0. \tag{7}$$

$$|a(x,s,\xi)| \leq \omega(x)^{\frac{1}{p(x)}}\beta\left[L(x) + \omega(x)^{\frac{1}{p'(x)}}b(s)^{p(x)-1}\,|\xi|^{p(x)-1}\right], \tag{8}$$

$$[a(x,s,\xi) - a(x,s,\xi')]\,[\xi - \xi'] \geq 0, \tag{9}$$

where L is a non negative function in $L^{p'(\cdot)}(\Omega)$, and $\beta > 0$, for almost every $x \in \Omega$, for every $s \in \Omega$ and $\xi \in \mathbb{R}^N$.

Finally the data

$$f \quad \text{is an element of } L^1(\Omega). \tag{10}$$

Definition 3.1. *A mesuarble function u defined on* Ω *is a renormalized solution of problem (2) if*

$$T_k(u) \in W_0^{1,p(.)}(\Omega, \omega), \tag{11}$$

$$u \leq m \ \ a.e. \ in \ \Omega, \tag{12}$$

$$a(x, T_m^k(u), DT_m^k(u))\chi_{\{u<m\}} \in (L^{p'(.)}(\Omega, \omega^{1-p'(.)}))^N, \tag{13}$$

$$\lim_{s\to+\infty} \int\limits_{\{-s-1\leq u(x)\leq -s\}} a(x, u, Du)Dudx = 0, \tag{14}$$

$$\lim_{\delta\to 0}\frac{1}{\delta} \int\limits_{\{m-2\delta\leq u(x)\leq m-\delta\}} a(x, u, Du)Dudx = \int\limits_{\{u(x)=m\}} f dx, \tag{15}$$

and, for every function S in $W^{1,\infty}(\mathbb{R})$ *such that supp*(S) *is compact and* $S(m) = 0$, *u satisfies*

$$\int\limits_{\Omega} a(x, u, Du)D(S(u)\varphi)dx = \int\limits_{\Omega} fS(u)\varphi dx, \tag{16}$$

for every $\varphi \in W_0^{1,p(.)}(\Omega, \omega) \cap L^{\infty}(\Omega)$.

Remark 3.2. *Conditions (11) and (13) to show that all term in (16) are well defined. The assumption (15), has been establissed in [11].*

The following notations will be used throughout the paper. For any $k > 0$ and $\varepsilon > 0$, we define

$$T_\varepsilon^k(s) = \begin{cases} -k, & \text{if } s \leq -k, \\ r, & \text{if } -k \leq r \leq \varepsilon, \\ \varepsilon, & \text{if } \ r \geq \varepsilon. \end{cases}$$

For $l \geq 1$ fixed, we define

$$\theta_l(s) = T_1\left(r - T_l(r)\right)$$

and $h_l(s) = 1 - |\theta_l(s)|$,for all $s \in \mathbb{R}$.

4. The existence theorem

Theorem 4.1. *Assume that (4)-(10) hold true there existe a renormalized solution u of problem (2).*

Proof of theorem

4.1. *Step 1: Approximate problem and a priori estimates*

Let us introduce the following regularisation of the data: for a fixed $n \geq 1$. let

$$b_n(r) = b(T_{m-1/n}^n(r)),$$

$$a^n(x, s, \xi) = a(x, T_{m-1/n}^n(s), \xi),$$

$$f_n \in L^{\infty}(\Omega), \text{ such that } f_n \to f \text{ strongly in } L^1(\Omega), \text{ as } n \text{ tends to } +\infty, \tag{17}$$

Let us now consider the following regularized problem

$$\begin{cases} -\operatorname{div}\left(a^n(x, u^n, Du^n)\right) = f_n \text{ in } \Omega, \\ \qquad\qquad u^n = 0 \text{ on } \partial\Omega, \end{cases} \tag{18}$$

As a result, demonstrating the existence of a weak solution $u^n \in W_0^{1,p(.)}(\Omega, \omega)$ of (18) is an easy task (see [25]).

We choose $T_k(u^n)$ as a test function in (18), we get

$$\int_\Omega a^n(x, u^n, Du^n) DT_k(u^n) dx = \int_\Omega f_n T_k(u^n) dx.$$

Since (5) and (7) , we have

$$\int_\Omega \omega(x) \left| DT_k(u^n) \right|^{p(x)} dx < C, \tag{19}$$

where C does not depend on n and k .

By a classical argument (see e.g [2, 5]), for a subsequence still indexed by n, from (19), we have

$$u^n \to u \ a.e. \text{ in } \Omega, \tag{20}$$

and

$$T_k\left(u^n\right) \rightharpoonup T_k\left(u\right) \text{ weakly in } W_0^{1,p(.)}(\Omega, \omega). \tag{21}$$

Taking now $Z^n = \int_0^{T_m^k(u^n)} b_n(s) ds$ as a test function in (18), we give

$$\int_\Omega a^n(x, u^n, Du^n) DZ^n(u^n) dx = \int_\Omega f_n Z^n(u^n) dx. \tag{22}$$

By the assumptions (7) of a^n and (5), we deduce that

$$\int_\Omega \omega(x) \left| DZ^n(u^n) \right|^{p(x)} dx < C. \tag{23}$$

For every $k > 0$, we write

$$\left| a^n(x, T_m^k(u^n), DT_m^k(u^n)) \right| \leq \beta \omega(x)^{\frac{1}{p(x)}} \left[L(x) + \omega(x)^{\frac{1}{p'(x)}} \left| DZ^n(u^n) \right|^{p(x)-1} \right]. \tag{24}$$

Putting together (23) and (24), we deduce that

$$a^n(x, T_m^k(u^n), DT_m^k(u^n) \text{ is bounded in } L^{p'(x)}(\Omega, \omega^{1-p'(x)}),$$

then there exists a function $\varphi_k \in L^{p'(x)}(\Omega)$ such that

$$a^n(x, T_m^k(u^n), DT_m^k(u^n) \rightharpoonup \varphi_k \text{ weakly in } L^{p'(x)}(\Omega, \omega^{1-p'(x)}). \tag{25}$$

Using $T_{2m}^+(u^n) - T_m^+(u^n)$ as a test function in (18) leads to

$$\int_\Omega a^n(x, u^n, Du^n).D\left(T_{2m}^+(u^n) - T_m^+(u^n)\right) dx = \int_\Omega f_n \left(T_{2m}^+(u^n) - T_m^+(u^n)\right) dx,$$

thanks to (7), we have

$$b(m - \frac{1}{n})^{p(x)-1} \int_\Omega \omega(x) \left| D\left(T_{2m}^+(u^n) - T_m^+(u^n)\right) \right|^{p(x)} dx \leq C, \tag{26}$$

we can pass to the limit in (26) as n tends to $+\infty$, to deduce that

$$\begin{aligned} T_{2m}^+(u) - T_m^+(u) &= 0 \quad a.e. \text{ in } Q, \\ u &\leq m \quad a.e. \text{ in } Q. \end{aligned} \tag{27}$$

Taking now $T_k(v^n)$ as a test function in (18), we obtain

$$\int_\Omega a^n(x,u^n,Du^n)DT_k(v^n)dx \leq C.$$

By (5) and (7), we have

$$\int_\Omega \omega(x)\left|DT_k(v^n)\right|^{p(x)} dx \leq C. \tag{28}$$

By a classical argument (see e.g [2, 5]), for a subsequence still indexed by n, from (28), we have

$$v^n \to v \text{ a.e. in } \Omega, \tag{29}$$

and

$$T_k(v^n) \rightharpoonup T_k(v) \text{ weakly in } W_0^{1,p(.)}(\Omega,\omega).$$

We choose $\theta_k(v^n)$ as a test function in (18), it gives

$$\lim_{n\to 0}\int_\Omega a^n(x,u^n,Du^n)D\left(\theta_k(v^n)\right)dx = \int_\Omega f_n\theta_k(v)dx.$$

Since $f_n \in L^\infty(\Omega)$, Lebesgue's convergence theorem, we have

$$\lim_{k\to 0}\lim_{n\to 0}\int_{\{n\leq|v^n|\leq n+1\}} a^n(x,u^n,Du^n)Dv^n dx = 0. \tag{30}$$

4.2. Step 2: the monotonicity estimate and the weak limit

Lemma 4.2. *For any $k \geq 0$, we have*

$$\lim_{n\to 0}\int_\Omega\left[\frac{a^n(x,T_m^k(u^n),DT_m^k(u^n))}{b_n(u^n)^{p(x)-1}} - \frac{a^n(x,T_m^k(u^n),DT_m^k(u))}{b_n(u^n)^{p(x)-1}}\right] \times\left(DT_m^k(u^n) - DT_m^k(u)\right)dx = 0. \tag{31}$$

Proof. Let $k \geq 0$ be fixed. Equality (31) is split into

$$\int_\Omega\left[\frac{a^n(x,T_m^k(u^n),DT_m^k(u^n))}{b_n(u^n)^{p(x)-1}} - \frac{a^n(x,T_m^k(u^n),DT_m^k(u))}{b_n(u^n)^{p(x)-1}}\right] \times\left(DT_m^k(u^n) - DT_m^k(u)\right)dx = I_1^n + I_2^n + I_3^n, \tag{32}$$

where

$$I_1^n = \int_\Omega \frac{a^n(x,T_m^k(u^n),DT_m^k(u^n))}{b_n(u^n)^{p(x)-1}}DT_m^k(u^n)dx,$$

$$I_2^n = -\int_\Omega \frac{a^n(x,T_m^k(u^n),DT_m^k(u^n))}{b_n(u^n)^{p(x)-1}}DT_m^k(u)dx,$$

$$I_3^n = -\int_\Omega \frac{a^n(x,T_m^k(u^n),DT_m^k(u))}{b_n(u^n)^{p(x)-1}}\left(DT_m^k(u^n) - DT_m^k(u)\right)dx.$$

In what follows we pass to the limit as n tends to $+\infty$ in (32).

Limit of I_1^n

We choose $h_l(v^n)\int_0^{T_m^k(u)}\frac{1}{b(s)^{p(x)-1}}ds$ as a test function in (18) to obtain

$$\begin{aligned}&\int_\Omega h_l(v^n)a^n(x,u^n,Du^n)\frac{DT_m^k(u)}{b(u)^{p(x)-1}}dx+\\&\int_\Omega a^n(x,u^n,Du^n)Dh_l(v^n).\left(\int_0^{T_m^k(u)}\frac{1}{b(s)^{p(x)-1}}ds\right)dx\\&=\int_\Omega f_nh_l(v^n)\int_0^{T_m^k(u)}\frac{1}{b(s)^{p(x)-1}}dsdx.\end{aligned}\tag{33}$$

Since h_k have a compact support, we have for a large n

$$|a^n(x,u^n,Du^n)h_l(v^n)|\le\beta\omega(x)^{\frac{1}{p(x)}}\left[L(x)+\omega(x)^{\frac{1}{p'(x)}}|DT_{l+1}(v^n)|^{p(x)-1}\right].\tag{34}$$

From (34) and (28), we deduce that

$$a^n(x,T_{(l+1)/\alpha}(u^n),DT_{(l+1)/\alpha}(u^n))h_l(v^n)\text{ is bounded in }L^{p'(x)}(\Omega,\omega^{1-p'(x)}),\tag{35}$$

for every large n.

We first use the estimate (35) to extract another subsequence, still indexed by l, such that

$$a^n(x,T_{(l+1)/\alpha}(u^n),DT_{(l+1)/\alpha}(u^n))h_l(v^n)\rightharpoonup\psi_l\text{ weakly in }L^{p'(x)}(\Omega,\omega^{1-p'(x)}),\tag{36}$$

as n tends to $+\infty$.

Now for $\max(k,m)\le l/\alpha$, we have

$$\begin{aligned}&a^n(x,T_{(l+1)/\alpha}(u^n),DT_{(l+1)/\alpha}(u^n))h_l(v^n)\,\chi_{\{-k<u^n<m\}}\\&=h_l(v^n)a^n(x,T_m^k(u^n),DT_m^k(u^n))\,\chi_{\{-k<u^n<m\}}\end{aligned}$$

a.e. in Ω. Using the covergences (36), (29), and (36) and letting n tends to $+\infty$, we have for

$$\psi_lDT_m^k(u)=h_l(v)\varphi_kDT_m^k(u)\ a.e.\text{in }\Omega.\tag{37}$$

Letting now n tends to $+\infty$ and l tends to $+\infty$. The first term in (33) yeilds

$$\lim_{l\to+\infty}\lim_{n\to+\infty}\int_\Omega h_l(v^n)a^n(x,u^n,Du^n)\frac{DT_m^k(u)}{b(u)^{p(x)-1}}dx=\int_\Omega\varphi_k\frac{DT_m^k(u)}{b(u)^{p(x)-1}}dx.\tag{38}$$

The second term of (33)

$$\left|a^n(x,u^n,Du^n)Dh_l(v^n).\left(\int_0^{T_m^k(u)}\frac{1}{b(s)^{p(x)-1}}ds\right)\right|\le\frac{\max(m,k)}{\alpha}|a^n(x,u^n,Du^n)Dv^n|.$$

Since (30), we deduce that

$$\lim_{l\to+\infty}\lim_{n\to+\infty}\int_\Omega a^n(x,u^n,Du^n)Dh_l(v^n).\left(\int_0^{T_m^k(u)}\frac{1}{b(s)^{p(x)-1}}ds\right)dx=0.\tag{39}$$

Due to (38) and (39) , we have

$$\int_\Omega\varphi_k\frac{DT_m^k(u)}{b(u)^{p(x)-1}}dx=\int_\Omega f\left(\int_0^{T_m^k(u)}\frac{1}{b(s)^{p(x)-1}}ds\right)dx.\tag{40}$$

Take $\int_0^{T_m^k(u^n)} \frac{1}{b_n(s)^{p(x)-1}} ds$ as a test fuction in (18), we get

$$\int_\Omega a^n(x,u^n,Du^n)\frac{DT_m^k(u^n)}{b_n(u^n)^{p(x)-1}}dx = \int_\Omega f_n \int_0^{T_m^k(u^n)} \frac{1}{b_n(s)^{p(x)-1}}dsdx. \tag{41}$$

Passing to the limit as n tends to $+\infty$ in (41) , in view (40) , we have

$$\lim_{n\to+\infty} I_1^n = \int_\Omega \varphi_k \frac{DT_m^k(u)}{b(u)^{p(x)-1}}dx. \tag{42}$$

Limit of I_2^n

By the assumptiom of b_n, we remark

$$\frac{1}{b_n(u^n)^{p(x)-1}} \to \frac{1}{b_n(u^n)^{p(x)-1}} \ a.e.\text{in } \Omega, \tag{43}$$

as n tends to $+\infty$. Since (21), (25), and (43), we have

$$\lim_{n\to+\infty} I_2^n = -\int_\Omega \varphi_k \frac{DT_m^k(u)}{b(u)^{p(x)-1}}dx. \tag{44}$$

Limit of I_3^n

We notice (4), (5), and (21), we show

$$\frac{a^n(x,T_m^k(u^n),DT_m^k(u))}{b_n(u^n)^{p(x)-1}} \to \frac{a(x,T_m^k(u),DT_m^k(u))}{b(u)^{p(x)-1}} \ a.e.\text{in } \Omega, \tag{45}$$

as n tends to $+\infty$, and

$$\left|\frac{a^n(x,T_m^k(u^n),DT_m^k(u))}{b_n(u^n)^{p(x)-1}}\right| \le \tag{46}$$

$$\omega(x)^{\frac{1}{p(x)}}\frac{1}{\alpha}\left[\beta\left[L(x)+\omega(x)^{\frac{1}{p'(x)}}\left|DT_m^k(u^n)\right|^{p(x)-1}\right]\right] \ a.e.\text{in } \Omega, \tag{47}$$

uniformly with respect to n. by (45), and (46), we deduce

$$\frac{a^n(x,T_m^k(u^n),DT_m^k(u))}{b_n(u^n)^{p(x)-1}} \to \frac{a(x,T_m^k(u),DT_m^k(u))}{b(u)^{p(x)-1}} \text{ weakly in } L^{p'(x)}(\Omega,\omega^{1-p'(x)}), \tag{48}$$

as n tends to $+\infty$.

From (21), we conclude

$$DT_m^k(u^n) - DT_m^k(u) \to 0 \text{ weakly in } L^{p(x)}(\Omega,\omega). \tag{49}$$

Due to (48), and (49) imply that

$$\lim_{n\to+\infty} I_3^n = 0. \tag{50}$$

Combining (32) with (42)–(50), we etablish 31

□

Lemma 4.3. *For fixed $k \ge 0$, one has*

$$\varphi_k = a(x,T_m^k(u),DT_m^k(u)) \ a.e.in \ \{x\in\Omega\,;\,u(x)<m\}. \tag{51}$$

And

$$\frac{a^n(x,T_m^k(u^n),DT_m^k(u^n))}{b_n(u^n)^{p(x)-1}}DT_m^k(u^n)$$
$$\to \frac{a(x,T_m^k(u),DT_m^k(u))}{b(u)^{p(x)-1}}DT_m^k(u) \text{ weakly in } L^1(\Omega), \tag{52}$$

when $n \to +\infty$.

Proof. Let $k \geq 0$ be fixed, by (20) and (48), we have

$$\lim_{n\to+\infty}\int_\Omega \frac{a^n(x,T_m^k(u^n),DT_m^k(u^n))}{b_n(u^n)^{p(x)-1}}DT_m^k(u^n)\,dx = \int_\Omega \frac{\varphi_k}{b(u)^{p(x)-1}}DT_m^k(u)\,dx.$$

Since (43) and (5), we have for every ψ

$$\begin{aligned}
0 &\leq \lim_{n\to+\infty}\int_\Omega \left[\frac{a^n(x,T_m^k(u^n),DT_m^k(u^n))}{b_n(u^n)^{p(x)-1}} - \frac{a^n(x,T_m^k(u^n),\psi)}{b_n(u^n)^{p(x)-1}}\right]\left[DT_m^k(u^n)-\psi\right]dx \\
&= \lim_{n\to+\infty}\int_\Omega \frac{a^n(x,T_m^k(u^n),DT_m^k(u^n))}{b_n(u^n)^{p(x)-1}}\left[DT_m^k(u^n)-\psi\right]dx \\
&- \lim_{n\to+\infty}\int_\Omega \frac{a^n(x,T_m^k(u^n),\psi)}{b_n(u^n)^{p(x)-1}}\left[DT_m^k(u^n)-\psi\right]dx \\
&= \int_\Omega \left(\frac{\varphi_k}{b(u)^{p(x)-1}} - \frac{a(x,T_m^k(u),\psi)}{b(u)^{p(x)-1}}\right)\left[DT_m^k(u^n)-\psi\right]dx.
\end{aligned} \tag{53}$$

By Minty trick lemma, we conclude that for any

$$\frac{\varphi_k}{b(u)^{p(x)-1}} = \frac{a(x,T_m^k(u),DT_m^k(u))}{b(u)^{p(x)-1}} \text{ a.e.in } \Omega. \tag{54}$$

Since (54) and (27), we deduce (51).

To prove (52), we observe that the monotone character of a and (31) give

$$\left[\frac{a^n(x,T_m^k(u^n),DT_m^k(u^n))}{b_n(u^n)^{p(x)-1}} - \frac{a^n(x,T_m^k(u^n),DT_m^k(u^n))}{b_n(u^n)^{p(x)-1}}\right]\times$$
$$\left[DT_m^k(u^n)-DT_m^k(u^n)\right] \to 0$$

strongly in $L^1(\Omega)$ as n tends to $+\infty$. From (21), (25), (48), and (51), we conclude when $n \to +\infty$

$$\frac{a^n(x,T_m^k(u^n),DT_m^k(u^n))}{b_n(u^n)^{p(x)-1}}DT_m^k(u) \to \frac{a(x,T_m^k(u),DT_m^k(u))}{b(u)^{p(x)-1}}DT_m^k(u) \text{ weakly in } L^1(\Omega), \tag{55}$$

and

$$\frac{a^n(x,T_m^k(u^n),DT_m^k(u))}{b_n(u^n)^{p(x)-1}}DT_m^k(u^n) \to \frac{a(x,T_m^k(u),DT_m^k(u))}{b(u)^{p(x)-1}}DT_m^k(u) \text{ weakly in } L^1(\Omega), \tag{56}$$

and

$$\frac{a^n(x,T_m^k(u^n),DT_m^k(u))}{b_n(u^n)^{p(x)-1}}DT_m^k(u) \to \frac{a(x,T_m^k(u),DT_m^k(u))}{b(u)^{p(x)-1}}DT_m^k(u) \text{ weakly in } L^1(\Omega). \tag{57}$$

By the convergences (55), (56), and (57), we obtain that for any $k \geq 0$

$$\frac{a^n(x,T_m^k(u^n),DT_m^k(u^n))}{b_n(u^n)^{p(x)-1}}DT_m^k(u^n) \to \frac{a(x,T_m^k(u),DT_m^k(u))}{b(u)^{p(x)-1}}DT_m^k(u) \text{ weakly in } L^1(\Omega),$$

as n tends to $+\infty$.

4.3. Step 3: End of the proof

Taking $T_m^{s+1}(u^n) - T_m^s(u^n)$ as a test function in (18) gives

$$\int_\Omega a^n(x,u^n,Du^n).D\left(T_m^{s+1}(u^n) - T_m^s(u^n)\right)dx = \int_\Omega f_n\left(T_m^{s+1}(u^n) - T_m^s(u^n)\right)dx. \tag{58}$$

Since $supp\left(T_m^{s+1}(.) - T_m^s(.)\right) \subset [-(s+1),-s]$, we obtain

$$\int_{\{-1-s\le u^n\le -s\}} a^n(x,u^n,Du^n).Du^n dx \tag{59}$$

$$= \int_\Omega a^n(x,u^n,Du^n)D\left(T_m^{s+1}(u^n) - T_m^s(u^n)\right)dx$$

$$= \int_\Omega \frac{a^n(x,u^n,Du^n)}{b_n(u^n)^{p(x)-1}}D\left(T_m^{s+1}(u^n) - T_m^s(u^n)\right)b_n(T_{m-1}^{s+1}(u^n))^{p(x)-1}dx$$

$$= \int_\Omega \frac{a^n(x,T_m^{s+1}(u^n),DT_m^{s+1}(u^n))}{b_n(T_m^{s+1}(u^n))^{p(x)-1}}D\left(T_m^{s+1}(u^n)\right)b_n(T_{m-1}^{s+1}(u^n))^{p(x)-1} \tag{60}$$

$$- \int_\Omega \frac{a^n(x,T_m^s(u^n),DT_m^s(u^n))}{b_n(u^n)^{p(x)-1}}D\left(T_m^s(u^n)\right)b_n(T_{m-1}^{s+1}(u^n))^{p(x)-1}$$

We deduce from (20) and (52) that

$$\lim_{n\to+\infty}\int_{\{-1-s\le u^n\le -s\}} a^n(x,u^n,Du^n).Du^n dx$$

$$= \int_\Omega a^n(x,u^n,Du^n)D\left(T_m^{s+1}(u^n) - T_m^s(u^n)\right)dx$$

$$= \int_\Omega \frac{a(x,T_m^{s+1}(u),DT_m^{s+1}(u))}{b_n(T_m^{s+1}(u^n))^{p(x)-1}}D\left(T_m^{s+1}(u)\right)b_n(T_{m-1}^{s+1}(u))^{p(x)-1}$$

$$- \int_\Omega \frac{a(x,T_m^s(u),DT_m^s(u))}{b(u)^{p(x)-1}}D\left(T_m^s(u)\right)b_n(T_{m-1}^{s+1}(u))^{p(x)-1} \tag{61}$$

$$= \int_{\{-1-s\le u\le -s\}} a(x,u,Du)Dudx.$$

Taking the limit as s tends to $+\infty$ in (58) and using the estimates (60) and (61) show that u satisfies (3.8).

Choosing $h_l(v^n)\frac{1}{\delta}(T_{m-\delta}^+(u) - T_{m-2\delta}^+(u))$ as a test function in (18), we have

$$\frac{1}{\delta}\int_\Omega a^n(x,u^n,Du^n)D(h_l(v^n)\left(T_{m-\delta}^+(u) - T_{m-2\delta}^+(u)\right))dx$$

$$= \frac{1}{\delta}\int_\Omega f_n h_l(v^n)(T_{m-\delta}^+(u) - T_{m-2\delta}^+(u))dx. \tag{62}$$

Since $supp\,(h_l) \subset [-(l+1), l+1]$, we obtain

$$\frac{1}{\delta}\int_\Omega h_l(v^n)a^n(x,u^n,Du^n)D(T_{m-\delta}^+(u) - T_{m-2\delta}^+(u))dx =$$

$$\frac{1}{\delta}\int_\Omega h_l(v^n)a^n(x,T_{(l+1)/\alpha}(u^n),DT_{(l+1)/\alpha}(u^n))D(T_{m-\delta}^+(u) - T_{m-2\delta}^+(u))dx. \tag{63}$$

In addition, using the same procedures as above, we deduce

$$\lim_{l\to+\infty}\lim_{n\to+\infty}\frac{1}{\delta}\int_\Omega h_l(v^n)a^n(x,u^n,Du^n)D(T_{m-\delta}^+(u) - T_{m-2\delta}^+(u))dx$$

$$= \frac{1}{\delta}\int_{\{m-2\delta\le|u|\le m-\delta\}} a(x,u,Du)Dudx. \tag{64}$$

Taking the limit as s tends to $+\infty$ in (62) and using the estimates (63) and (64) show that u satisfies (3.8).

Let S be a function in $W^{1,\infty}$ ($\mathbb{R}$) such that S has a compact support and $S(m) = 0$ and let $\varphi \in W_0^{1,p(.)}(\Omega, \omega^{1-p'(x)}) \cap L^\infty(\Omega)$. Take $S(u)\, h_l(v^n\,)\varphi$ as a test fuction in (18), we get

$$\int_\Omega h_l(v^n)a^n(x, u^n, Du^n)D\,(S(u)\varphi)\,dx + \int_\Omega S(u)\varphi a^n(x, u^n, Du^n)Dh_l(v^n)dx \tag{65}$$
$$= \int_\Omega f_n S(u)h_l(v^n)\varphi dx.$$

Taking the limit as n tends to $+\infty$ and l tends to $+\infty$ in (65).

Limit of first term in (65)

Since $supp\,(h_l) \subset [-(l+1), l+1]$, we obtain

$$a^n(x, T_{(l+1)/\alpha}\,(u^n)\,, DT_{(l+1)/\alpha}\,(u^n))h_l(v^n)) = h_l(v^n)a^n(x, u^n, Du^n)\ a.e.\text{in } \Omega.$$

From (27), (36), (37) and (51), we get

$$\begin{aligned}
&\lim_{l\to+\infty}\lim_{n\to+\infty}\int_\Omega h_l(v^n)a^n(x, u^n, Du^n)D\,(S(u)\varphi)\,dx\\
=\ &\lim_{l\to+\infty}\int_\Omega h_l(v)a(x, T_m^k\,(u^n)\,, DT_m^k\,(u^n))D\,(S(u)\varphi)\,dx\\
=\ &\int_\Omega a(x, u, Du)D\,(S(u)\varphi)\,dx.
\end{aligned}$$

Limit of second term in (65)

As a consequence of (30), we conclude

$$\lim_{l\to+\infty}\lim_{n\to+\infty}\int_\Omega S(u)\varphi a^n(x, u^n, Du^n)Dh_l(v^n)dx = 0.$$

Limit of the Right-Hand Side of (65)

From (17) and (29)

$$\lim_{l\to+\infty}\lim_{n\to+\infty}\int_\Omega f_n S(u)h_l(v^n)\varphi dx = \int_\Omega f S(u)\varphi dx.$$

Then, u satisfies (16). □

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