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Existence, uniqueness and stability of mild solutions for fractional hybrid semilinear equations with boundary conditions

Abdelmjid Benmerrous[a,*], Fatima Ezzahra Bourhim[a], M'hamed Elomari[a], Ali Elmfadel[a]

[a]*Laboratory of Applied Mathematics and Scientific Computing, Sultan Moulay Slimane University, PO Box 532, Beni Mellal, 23000, Morocco*

Abstract. In this paper, we study the existence of mild solutions for hybrid fractional semilinear evolution equations with boundary conditions. Additionally, we prove four different types of Mittag-Leffler-Ulam-Hyers stability results for mild solutions. The existence of mild solutions is proved by the Dhage fixed point theorem. Finally, we provide an example to demonstrate our findings.

1. Introduction

Fractional differential equations have found applications across a wide array of disciplines including economics, engineering, chemistry, aerodynamics and control of dynamical systems. The burgeoning interest in fractional calculus stems from its ability to provide more accurate descriptions of certain dynamical systems compared to those relying solely on integer-order derivatives. Notably, fractional calculus offers more realistic models that unveil hidden dynamics in systems such as spring pendulums, particle motion in circular cavities, and epidemic models (see e.g. [12], [13], [14], [15], [16] and references therein).
Researchers are currently investigating multiple facets of fractional differential equations, encompassing the exploration of solution existence and uniqueness, stability analysis, and techniques for deriving explicit and numerical solutions. Methods such as fixed point theorems, upper-lower solutions, iterative techniques, and numerical methods are frequently utilized to verify the existence and uniqueness of solutions in fractional differential equations. Stability analysis often focuses on exploring the dependence of solutions on initial conditions and parameters, with particular attention to concepts like Mittag-Leffler-Ulam-Hyers-Rassias stability (see e.g. [1], [8], [29], [23] and reference therein).
Almeida [6] extended the definition of the Caputo derivative by introducing dependencies on another function, thereby enhancing the accuracy of models with increased adaptability.
Lately, Jarad et al [30] presented a generalized Laplace transform and its inverse, which streamline the solution process for fractional differential equations involving generalized Caputo fractional derivatives. This innovation has facilitated more efficient approaches to analyzing and solving such equations in mathematical research.

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* Corresponding author: Abdelmjid Benmerrous
Email addresses: abdelmajid.benmerrous@usms.ma (Abdelmjid Benmerrous), fati.zahra.bourhim98@gmail.com (Fatima Ezzahra Bourhim), m.elomari@usms.ma (M'hamed Elomari), a.elmfadel@usms.ma (Ali Elmfadel)
ORCID iDs: https://orcid.org/0000-0001-9620-8161 (Abdelmjid Benmerrous), https://orcid.org/0000-0002-4256-1434 (M'hamed Elomari), https://orcid.org/0000-0002-2479-1762 (Ali Elmfadel)

Inspired by the research presented in [36], we delve into the analysis of the existence and different classifications of Ulam-Hyers stability outcomes for mild solutions for the semilinear fractional hybrid differential equations that include the $\rho-$Caputo fractional derivative of order $0 \leq \beta \leq 1$.

$$\begin{cases} {}^{C}D^{\beta,\rho}_{0+}\left(\frac{u(s)}{g(s,u(s))}\right) = \mathcal{A}\left(\frac{u(s)}{g(s,u(s))}\right) + h(s,u(s)), \ s \in [0,T], 0 < \beta < 1, \\ u(0) = 0. \end{cases} \tag{1.1}$$

Where $T > 0$, $\mathcal{A}$ is the infinitesimal generator of C_0-semigroup $(T(s))_{s\geq 0}$ on a Banach space X, $g \in C([0,T] \times X, X \setminus \{0\})$ and $h \in C([0,T] \times X, X)$.

The article is arranged as follows: Section 2 provides preliminaries, Section 3 presents the existence and uniqueness of the mild solution of the problem, Section 4 prove four different types of Mittag-Leffler-Ulam-Hyers stability results for mild solutions, after that section 5 provide an example to demonstrate our findings and finally, Section 6 offers a concise conclusion.

2. Preliminaries

Consider X as a Banach space, and let $C([0,T],X)$ denote the Banach space of continuous functions from $[0,T]$ to X. In this space, the norm $\|u\|$ is defined as $\|u\| = \sup_{s\in[0,T]} \|u(s)\|$.

Definition 2.1. *[13] Let $\beta > 0$, f be an integrable function defined on $[a,b]$ and $\rho : [a,b] \to \mathbb{R}$ that is an increasing differentiable function such that $\rho'(s) \neq 0$, for all $s \in [a,b]$.*
The ρ-Riemann-Liouville fractional integral operator of order β applied to a function f is defined as

$$I^{\beta,\rho}_{a+} f(s) = \frac{1}{\Gamma(\beta)} \int_a^s \rho'(t)(\rho(s) - \rho(t))^{\beta-1} f(t)dt.$$

Definition 2.2. *[13] Let $n \in \mathbb{N}$, $k, \rho \in C^n([a,b])$ be two functions such that ρ is increasing with $\rho'(s) \neq 0$, for all $s \in [a,b]$.*
ρ-Riemann-Liouville fractional derivative of order β applied to a function f is defined as

$$\mathcal{D}^{\beta,\rho}_{a+} f(s) = \frac{1}{\Gamma(n-\beta)} \left(\frac{1}{\rho'(s)} \frac{d}{ds}\right)^n \int_a^s \rho'(t)(\rho(s) - \rho(t))^{n-\beta-1} f(t)dt,$$

where $n = [\beta] + 1$ and $[\beta]$ denotes the integer part of β.

Definition 2.3. *[13]*
Let $n \in \mathbb{N}$, $f, \rho \in C^n([a,b])$ be two functions such that ρ is increasing with $\rho'(s) \neq 0$, for all $s \in [a,b]$.
ρ-Caputo fractional derivative of order β applied to a function f is defined as

$${}^{C}D^{\beta,\rho}_{a+} f(s) = \frac{1}{\Gamma(n-\beta)} \int_a^s \rho'(t)(\rho(s) - \rho(t))^{n-\beta-1} f^{[n]}_{\rho}(t)dt,$$

where $n = [\beta] + 1$, for $\beta \notin \mathbb{N}$. And $f^{[n]}_{\beta}(s) = \left(\frac{1}{\rho'(s)} \frac{d}{ds}\right)^n f(s)$ on $[a,b]$.

Remark 2.4. *1) The equality in Definition 2.3 between the ρ-Caputo derivative and the ρ-Riemann–Liouville derivative follows from the fact that the Caputo operator can be expressed as a fractional integral of the higher-order derivative of the function. More precisely, for $f \in C^n([a,b])$ and $n-1 < \beta < n$, we have*

$$CD^{\beta,\rho}_{a+} f(s) = I^{n-\beta,\rho}_{a+}\left(f^{[n]}_{\rho}(s)\right),$$

where $I^{\gamma,\rho}_{a+}$ denotes the ρ-Riemann–Liouville integral of order γ.

2) This representation shows that the ρ-Caputo derivative is obtained by applying the ρ-Riemann–Liouville integral to the n-th ρ-derivative of f. Consequently, the equality

$$CD^{\beta,\rho}_{a+}f(s) \;=\; D^{\beta,\rho}_{a+}\left(f(s)-\sum_{k=0}^{n-1}\frac{f^{[k]}_{\rho}(a)}{k!}(\rho(s)-\rho(a))^{k}\right)$$

is valid, since subtracting the polynomial involving the initial values ensures that the Caputo derivative vanishes on constants, which distinguishes it from the Riemann–Liouville derivative.

Theorem 2.5. *Given $f \in C^n([a,b])$ and $\beta > 0$. Then*

$$\mathcal{I}^{\beta,\rho}_{a+}\,{}^{C}\mathcal{D}^{\beta,\rho}_{a+}f(s) = f(s) - \sum_{k=0}^{n-1}\frac{f^{[k]}_{\rho}(a)}{k!}(\rho(s)-\rho(a))^{k}.$$

In particular, if $\beta \in (0,1)$ we have:

$$\mathcal{I}^{\beta,\rho}_{a+}\,{}^{C}\mathcal{D}^{\beta,\rho}_{a+}f(s) = f(s) - f(a).$$

Here, we introduce a generalized integral transform recently proposed by Jarad and Abdeljawad [30]. This transform is particularly applicable in solving linear Fractional Differential Equations (FDEs) that feature ρ-Riemann-Liouville and ρ-Caputo fractional derivatives. It provides a unified approach to handle these types of fractional derivatives within the context of linear FDEs, offering a powerful tool for theoretical analysis and numerical computations in fractional calculus.

Definition 2.6. *Let $v, \rho : [a,\infty) \to \mathbb{R}$ be a real-valued function and ρ be a non-negative increasing function such that $\rho'(0) > 0$. Then the Laplace transform of v with respect to ρ is defined by*

$$\mathcal{L}_{\rho}\{v(s)\} = \mathfrak{V}(\lambda) = \int_0^{\infty}\exp\{-\lambda(\rho(s)-\rho(0)\}\rho'(s)v(s)ds,$$

for every $\lambda \in \mathbb{C}$ for which this integral converges.
In this context, $\mathcal{L}_{\rho}$ denotes a generalized Laplace transform with respect to the function ρ.

Next, we present some information regarding the semigroups of linear operators. These findings are documented in references [24, 33].
For a strongly continuous semigroup, often denoted as C_0−semigroup, represented by $(T(s))_{s\geq 0}$ and the infinitesimal generator is:

$$\mathcal{A}y = \lim_{s\to 0^+}\frac{T(s)y-y}{y}, \quad y \in X.$$

The domain of $\mathcal{A}$, denoted as $\mathcal{D}(\mathcal{A})$, is such that

$$\mathcal{D}(\mathcal{A}) = \left\{y \in X : \lim_{s\to 0^+}\frac{T(s)y-y}{y} \text{ exists}\right\}.$$

Theorem 2.7. *[24] Let be $(T(s))_{s\geq 0}$ be a C_0−semigroup then there exist constants $w \in \mathbb{R}$ and $M \geq 1$, such that*

$$\|T(s)\| \leq M\exp\{wt\} \text{ for } 0 \leq t < +\infty.$$

Theorem 2.8. *[24] A linear operator $\mathcal{A}$ serves as the infinitesimal generator of a contraction C_0−semigroup if and only if the following conditions hold:*

1. *$\mathcal{A}$ is closed, and its closure of the domain $\overline{\mathcal{D}(\mathcal{A})} = X$.*
2. *The resolvent set of $\mathcal{A}$, denoted $\rho(A)$, contains $\mathbb{R}^{*+}$ and $\forall\beta > 0$*

$$\mathcal{R}(\beta,\mathcal{A}) \leq \frac{1}{\beta},$$

with $\mathcal{R}(\beta,\mathcal{A}) := (\beta^{\alpha}I - \mathcal{A})^{-1} = \int_0^{\infty}\exp\{-\beta^{\alpha}t\}T(t)ydt.$

Definition 2.9. *[31] The Wright type function is defined by*

$$\phi_\beta(s) = \sum_{k=0}^{\infty} \frac{(-s)^k}{k!\Gamma(-\beta k + 1 - \beta)} = \sum_{k=0}^{\infty} \frac{(-s)^k \Gamma(\beta(k+1)) \sin(\pi(k+1)\beta)}{k!},$$

for $0 < \beta < 1$ *and* $s \in \mathbb{C}$.

Theorem 2.10. *[20] Let $\mathcal{S}$ be a closed, bounded and convex subset of the Banach algebra X. We consider the two operators $\mathcal{I} : X \to X$ and $\mathcal{J} : \mathcal{S} \to X$ such that*

(i) *$\mathcal{I}$ is Lipschitzian with a Lipschitz constant α;*

(2i) *$\mathcal{J}$ is completely continuous;*

(3i) *$u = \mathcal{I}u\mathcal{J}v \Rightarrow u \in \mathcal{S}$, for all $v \in \mathcal{S}$;*

(4i) *$\alpha\mathcal{K} < 1$, where $\mathcal{K} = \|\mathcal{J}(\mathcal{S})\|$.*
Then the operator function $u = \mathcal{I}u\mathcal{J}u$ has a solution on $\mathcal{S}$.

3. The mild solution

Based on Definition (2.3) and Theorem (2.5), it is appropriate to reformulate the problem into

$$\begin{aligned} u(s) = g(s, u(s)) \Bigg\{ &\frac{1}{\Gamma(\beta)} \int_0^s \rho'(t)(\rho(s) - \rho(t))^{\beta-1} \mathcal{A}\left(\frac{u(t)}{g(s, u(t))}\right) dt \\ &+ \frac{1}{\Gamma(\beta)} \int_0^s \rho'(t)(\rho(s) - \rho(t))^{\beta-1} h(t, u(t)) dt \Bigg\}. \end{aligned} \tag{3.1}$$

Proof.

$$\mathcal{I}_{0+}^{\beta,\rho}\, {}^C\mathcal{D}_{0+}^{\beta,\rho} \left(\frac{u(s)}{g(s, u(s))}\right) = \mathcal{I}_{0+}^{\beta,\rho} \left\{ \mathcal{A}\left(\frac{u(s)}{g(s, u(s))}\right) + h(s, u(s)) \right\},$$

then

$$\frac{u(s)}{g(s, u(s))} - \frac{u(0)}{g(0, u(0))} = \mathcal{I}_{0+}^{\beta,\rho} \mathcal{A}\left(\frac{u(s)}{g(s, u(s))}\right) + \mathcal{I}_{0+}^{\beta,\rho} h(s, u(s)).$$

□

Lemma 3.1. *If* (3.1) *holds, then we have*

$$u(s) = g(s, u(s)) \left\{ \beta \int_0^s \int_0^\infty \theta \Phi_\beta(\theta)(\rho(s) - \rho(t))^{\beta-1} T(\rho(s) - \rho(t))\theta) h(t, u(t)) \rho'(t) d\theta dt \right\}.$$

Proof. Consider $\lambda > 0$. By applying the generalized Laplace transform to equation (3.1), we obtain

$$\mathfrak{U}(\lambda) = \frac{1}{\lambda^\beta}(\mathcal{A}\mathfrak{U}(\lambda) + \mathfrak{H}(\lambda)),$$

where

$$\mathfrak{U}(\lambda) = \int_0^\infty \exp\{-\lambda(\rho(s) - \rho(0))\} \frac{u(s)}{g(s, u(s))} \rho'(s) ds,$$

$$\mathfrak{H}(\lambda) = \int_0^\infty \exp\{-\lambda(\rho(s) - \rho(0))\} h(s, u(s)) \rho'(s) ds.$$

It follows that

$$\mathfrak{U}(\lambda) = \int_0^\infty \exp\{-\lambda^\beta s\} T(s)\mathfrak{H}(\lambda) ds = \int_0^\infty \beta w^{\beta-1} \exp\{-(\lambda w)^\beta\} T(w^\beta)\mathfrak{H}(\lambda) dw.$$

Taking $w = \rho(t) - \rho(0)$, we get

$$\begin{aligned}\mathfrak{U}(\lambda) &= \int_0^\infty \beta(\rho(t)-\rho(0))^{\beta-1} \exp\{-\lambda(\rho(t)-\rho(0))^\beta\} T((\rho(t)-\rho(0))^\beta)\mathfrak{H}(\lambda)\rho'(t)dt \\ &= \int_0^\infty \int_0^\infty \beta(\rho(t)-\rho(0))^{\beta-1} \exp\{-\lambda(\rho(t)-\rho(0))^\beta\} T((\rho(t)-\rho(0))^\beta) \\ &\times \exp\{-\lambda(\rho(s)-\rho(0))\} h(s,u(s))\rho'(s)\rho'(t)dsdt.\end{aligned}$$

We examine the following probability density function in [32]

$$\rho_\beta(\theta) = \frac{1}{\pi}\sum_{i=1}^{\infty}(-1)^{i-1}\theta^{-\beta i-1}\frac{\Gamma(\beta i+1)}{i!}\sin(i\pi\beta),\ \theta \in (0,\infty),$$

whose integration, is given by

$$\int_0^\infty \exp\{-\lambda\theta\}\rho_\beta(\theta)d\theta = \exp\{-\lambda^\beta\}, \quad \text{where } \beta \in (0,1). \tag{3.2}$$

Using (3.2), we get

$$\begin{aligned}\mathfrak{U}(\lambda) &= \int_0^\infty \int_0^\infty \int_0^\infty \beta\rho_\beta(\theta)(\rho(t)-\rho(0))^{\beta-1} \exp\{-\lambda(\rho(t)-\rho(0))\theta\} \\ &\times T((\rho(t)-\rho(0))^\beta) \exp\{-\lambda(\rho(s)-\rho(0))\} h(s,u(s))\rho'(s)\rho'(t)d\theta dsdt \\ &= \int_0^\infty \int_0^\infty \int_0^\infty \beta\rho_\beta(\theta)\frac{(\rho(t)-\rho(0))^{\beta-1}}{\theta^\beta} \exp\{-\lambda(\rho(t)+\rho(s)-2\rho(0))\} \\ &\times T\left(\frac{(\rho(t)-\rho(0))^\beta}{\theta^\beta}\right) h(s,u(s))\rho'(s)\rho'(t)d\theta dsdt \\ &= \int_0^\infty \int_t^\infty \int_0^\infty \beta\rho_\beta(\theta)\frac{(\rho(t)-\rho(0))^{\beta-1}}{\theta^\beta} \exp\{-\lambda(\rho(\tau)-\rho(0))\} \\ &\times T\left(\frac{(\rho(t)-\rho(0))^\beta}{\theta^\beta}\right) \\ & h(\rho^{-1}(\rho(\tau)-\rho(t)+\rho(0)), u(\rho^{-1}(\rho(\tau)-\rho(t)+\rho(0))))\rho'(t)\rho'(\tau)d\theta d\tau dt,\end{aligned}$$

by Fubini's theorem, we have

$$\begin{aligned}\mathfrak{U}(\lambda) &= \int_0^\infty \exp\{-\lambda(\rho(\tau)-\rho(0))\}\left\{\int_0^\tau \int_0^\infty \beta\rho_\beta(\theta)\frac{(\rho(t)-\rho(0))^{\beta-1}}{\theta^\beta} T\left(\frac{(\rho(t)-\rho(0))^\beta}{\theta^\beta}\right)\right. \\ &\left.\times\, h(\rho^{-1}(\rho(\tau)-\rho(t)+\rho(0)), u(\rho^{-1}(\rho(\tau)-\rho(t)+\rho(0))))\rho'(t)d\theta dt\right\}\rho'(\tau)d\tau \\ &= \int_0^\infty \exp\{-\lambda(\rho(\tau)-\rho(0))\}\left\{\int_0^\tau \int_0^\infty \beta\rho_\beta(\theta)\frac{(\rho(\tau)-\rho(s))^{\beta-1}}{\theta^\beta}\right. \\ &\left.\times\, T\left(\frac{(\rho(\tau)-\rho(s))^\beta}{\theta^\beta}\right) h(s,u(s))\rho'(s)d\theta ds\right\}\rho'(\tau)d\tau.\end{aligned}$$

Now, we proceed to perform the inversion of the Laplace transform in order to obtain

$$\frac{u(s)}{g(s,u(s)} = \beta\int_0^s \int_0^\infty \theta\phi_\beta(\theta)(\rho(s)-\rho(t))^{\beta-1} \times T((\rho(s)-\rho(t))^\beta\theta)h(t,u(t))\rho'(t)d\theta dt,$$

where $\phi_\beta(\theta) = \frac{1}{\beta}\theta^{-1-\frac{1}{\beta}}\rho_\beta(\theta^{-\frac{1}{\beta}})$ is the probability density in $(0,\infty)$.

Thus
$u(s) = g(s,u(s))\left\{\beta\int_0^s\int_0^\infty \theta\phi_\beta(\theta)(\rho(s)-\rho(t))^{\beta-1}\times T((\rho(s)-\rho(t))^\beta\theta)h(t,u(t))\rho'(t)d\theta dt\right\}$. □

For any $u \in X$, define the operator $\omega_\rho^\beta(s,t)$ by

$$\omega_\rho^\beta(s,t)u = \beta\int_0^\infty \theta\phi_\beta(\theta)T((\rho(s)-\rho(t))^\beta\theta)\, u\, d\theta \quad \text{for} \quad 0 \le t \le s \le T.$$

Definition 3.1. *We say that $u \in C([0,T],X)$ is a mild solution of equation* (1.1) *If we have*

$$u(s) = g(s,u(s))\left\{\int_0^s (\rho(s)-\rho(t))^{\beta-1}\omega_\rho^\beta(s,t)h(t,u(t))\rho'(t)dt\right\}.$$

Prior to commencing the proof of the main result, we introduce the following hypothesis

(C_1) $T(s)$ is compact operator $\forall s > 0$.

(C_2) Let $r > 0$, $\exists h_r \in L^\infty([0,T],X)$ such that

$$\sup_{\|u\|\le r}\|h(s,u)\| \le h_r(s),\ s \in [0,T],$$

and there is a constant ζ >0 such that

$$\lim_{r\to\infty}\sup\frac{\|h_r(s)\|_{L^\infty}}{r} = \zeta.$$

(C_3) The function $g \in C_c([0,T]\times X, X\setminus\{0\})$ is bounded and there exist constants $\mu > 0$ and $L > 0$ such that for all $u,v \in X$ and $s \in [0,T]$, we have

$$|g(s,u)-g(s,v)| \le \mu|u-v| \text{ and } |g(s,u)| < L.$$

Theorem 3.2. *Assume that condition* $(C_1)-(C_3)$ *hold. Then the problem* (1.1) *has at least mild solution provided that*

$$\frac{\mu M\|h_r\|_{L^\infty}}{\Gamma(\beta+1)}(\rho(T)-\rho(0))^\beta < 1.$$

Proof. Let $\mathcal{V} = \{u \in X, \|u\| \le b\}$, where $b = \frac{LM\|h_r\|_{L^\infty}}{\Gamma(\beta+1)}(\rho(t)-\rho(0))^\beta$.
We have

$$u(s) = g(s,u(s))\left\{\int_0^s (\rho(s)-\rho(t))^{\beta-1}\omega_\rho{}^\beta(s,t)h(t,u(t))\rho'(t)dt\right\}.$$

Then we can transform into $u(s) = \mathcal{I}u(t)\mathcal{J}u(s),\ s \in [0,T]$.
Now we prove that all conditions of Theorem (2.10) are satisfied.
Step 1. Let $u,v \in X$ then

$$\begin{aligned}|\ \mathcal{I}u(s)-\mathcal{I}v(s)\ | &=|\ g(s,u(s))-g(s,v(s))\ |\\ &\le \mu\ |\ u(s)-v(s)\ |,\ s \in [0,T].\end{aligned}$$

Step 2. Firstly, we prove that $\mathcal{J}$ is completely continuous.
Let $u_n, u \in \mathcal{V}$ with $\lim_{n\to+\infty}\|u_n-u\| = 0$. Then

$$h(s,u_n(s)) \to h(s,u(s)), \text{ as } n \to \infty.$$

Therefore

$$\|\mathcal{J}u_n(s) - \mathcal{J}u(s)\| \le \int_0^s (\rho(s)-\rho(t))^{\beta-1}\frac{M}{\Gamma(\beta)}\|h(t,u_n(t)) - h(t,u(t))\|\rho'(t)dt.$$

Via Lebesgue dominated convergence theorem, we get

$$\|\mathcal{J}u_n(s) - \mathcal{J}u(s)\| \longrightarrow 0 \text{ as } \quad n \to \infty.$$

•$\mathcal{J}(\mathcal{V})$ is uniformly bounded
Let $u \in \mathcal{V}$,

$$\begin{aligned}\|\mathcal{J}u(s)\| &\le \int_0^s (\rho(s)-\rho(t))^{\beta-1}\frac{M}{\Gamma(\beta)}\|h_r\|_{L^\infty}\rho'(t)dt\\ &= \frac{M\|h_r\|_{L^\infty}}{\Gamma(\beta+1)}(\rho(s)-\rho(0))^{\beta}, \ \ 0 \le s \le T\\ &\le \frac{M\|h_r\|_{L^\infty}}{\Gamma(\beta+1)}(\rho(T)-\rho(0))^{\beta}.\end{aligned}$$

•$\mathcal{J}(\mathcal{V})$ is equicontinuous.
Let $s_1, s_2 \in [0,T]$ with $s_1 < s_2$ and $u \in \mathcal{V}$.

$$\begin{aligned}&\|\mathcal{J}u(s_2) - \mathcal{J}u(s_1)\| \le \int_{s_1}^{s_2} (\rho(s_1)-\rho(t))^{\beta-1}\frac{M}{\Gamma(\beta)}\|h_r\|_{L^\infty}\rho'(t)dt\\ &+ \int_0^{s_1} \|\omega_\rho^\beta(s_2,t)\|\|\{(\rho(s_2)-\rho(t))^{\beta-1} - (\rho(s_1)-\rho(t))^{\beta-1}\}\|h(t,u(t))\|\rho'(t)dt\\ &+ \int_0^{s_1} (\rho(s_1)-\rho(t))^{\beta-1}\|(\omega_\rho^\beta(s_2,t) - \omega_\rho^\beta(s_1,t))\| \times \|h(t,u(t))\|\rho'(t)dt\\ &\le \frac{M\|h_r\|_{L^\infty}}{\Gamma(\beta+1)}(\rho(s_1)-\rho(s_2))^{\beta} + \frac{M\|h_r\|_{L^\infty}}{\Gamma(\beta+1)}\left\{(\rho(s_1)-\rho(0))^{\beta} - (\rho(s_2)-\rho(s_1))^{\beta}\right.\\ &\left. - (\rho(s_1)-\rho(0))^{\beta}\right\} + \frac{\|h_r\|_{L^\infty}}{\beta}(\rho(s_1)-\rho(0))^{\beta}\,\|\omega_\rho^\beta(s_2,t) - \omega_\rho^\beta(s_1,t)\|.\end{aligned}$$

Thus

$$\|\mathcal{J}u(s_2) - \mathcal{J}u(s_1)\| \to 0 \text{ as } s_2 \to s_1.$$

Step 3. Let $u \in X$ and $v \in \mathcal{V}$ such that $\mathcal{I}u\mathcal{J}v = u$, prove that $u \in \mathcal{V}$.

$$\|u(t)\| = \|\mathcal{I}u\mathcal{J}v\| = \|g(t,u(t))\| \times \|\mathcal{J}v(t)\| \qquad \le L \times \frac{M\|h_r\|_{L^\infty}}{\Gamma(\beta+1)}(\rho(T)-\rho(0))^{\beta}.$$

Step 4. Suppose that $\mathcal{S} = \sup\{\|\mathcal{J}u\|, u \in \mathcal{V}\} \le b$. Then

$$\mu \times \mathcal{S} \le \mu \times b < 1.$$

□

4. Mittag-Leffler-Ulam-Hyers Stability

For $g \in C([0,T] \times X, X \setminus \{0\})$, $h \in C_c([0,T] \times X, X)$, $\psi \in C([0,T], \mathbb{R}^+)$ and $\epsilon > 0$. We consider the equation

$$^{C}D_{0+}^{\beta,\rho}\left(\frac{u(s)}{g(s,u(s))}\right) = \mathcal{A}\left(\frac{u(s)}{g(s,u(s))}\right) + h(s,u(s)), \ s \in [0,T], 0 < \beta < 1. \tag{4.1}$$

And the inequalities

$$\left\| {}^{C}D_{0+}^{\beta,\rho}\left(\frac{u(s)}{g(s,u(s))}\right) - \mathcal{A}\left(\frac{u(s)}{g(s,u(s))}\right) - h(s,u(s))\right\| \leq \epsilon,\ s \in [0,T]; \tag{4.2}$$

$$\left\| {}^{C}D_{0+}^{\beta,\rho}\left(\frac{u(s)}{g(s,u(s))}\right) - \mathcal{A}\left(\frac{u(s)}{g(s,u(s))}\right) - h(s,u(s))\right\| \leq \psi,\ s \in [0,T]; \tag{4.3}$$

$$\left\| {}^{C}D_{0+}^{\beta,\rho}\left(\frac{u(s)}{g(s,u(s))}\right) - \mathcal{A}\left(\frac{u(s)}{g(s,u(s))}\right) - h(s,u(s))\right\| \leq \epsilon\psi,\ s \in [0,T]. \tag{4.4}$$

Definition 4.1. *We define equation* (4.1) *as Mittag-Leffler-Ulam-Hyers stable with respect to E_β if there exists a real number $\delta > 0$ such that for every $\epsilon > 0$ and for each solution $v \in C^1([0,T],X)$ of inequality* (4.2)*, there exists a mild solution $u \in C([0,T],X)$ of equation* (4.1) *satisfying $\|v(s) - u(s)\| \leq \delta\epsilon E_\beta(s)$, $s \in [0,T]$.*

We can repeat the same definition in other cases.

Remark 4.2. *A function $u \in C^1([0,T],X)$ is a solution of inequality* (4.2) *if and only if there exists a function $\varphi \in C^1([0,T],X)$ (which depend on u) such that*

1. $\|\varphi(s)\| \leq \epsilon$ *for* $s \in [0,T]$.
2. ${}^{C}D_{0+}^{\beta,\rho}\left(\frac{u(s)}{g(s,u(s))}\right) = \mathcal{A}\left(\frac{u(s)}{g(s,u(s))}\right) + h(s,u(s)) + \varphi(s),\ s \in [0,T].$

Lemma 4.1. *If $v \in C^1([0,T],X)$ is a solution of* (4.2)*, v is a solution of the following integral inequality*

$$\left\| v(s) - g(s,v(s))\left\{\int_0^s (\rho(s)-\rho(t))^{\beta-1}\omega_\rho^\beta(s,t)h(t,v(t))\rho'(t)dt\right\}\right\| \leq \frac{LM\epsilon}{\Gamma(\beta+1)}(\rho(T)-\rho(0))^\beta.$$

Proof. By the previous remark, we have

$${}^{C}D_{0+}^{\beta,\rho}\left(\frac{u(s)}{g(s,u(s))}\right) = \mathcal{A}\left(\frac{u(s)}{g(s,u(s))}\right) + h(s,u(s)) + \varphi(s),\ s \in [0,T].$$

And from Theorem (3.2), we get

$$\begin{aligned}\left\| v(s) - g(s,v(s))\left\{\int_0^s (\rho(s)-\rho(t))^{\beta-1}\mathcal{W}_\rho^\beta(s,t)h(t,v(t))\rho'(t)dt\right\}\right\| &\leq L \times \epsilon \\ &\times \frac{M}{\Gamma(\beta)}\{\rho(T)-\rho(0)\}^\beta.\end{aligned}$$

□

Theorem 4.3. *Assume that $h \in C([0,T] \times X, X$ and there exists $L_h > 0$ such that $|h(s,u_1) - h(s,u_2)| \leq L_h|u_1 - u_2|$, for all $s \in [0,T]$ and $u_1, u_2 \in X$.*
Then equation (4.1) *is Mittag-Leffler-Ulam-Hyers stable.*

Proof. Let $v \in C^1([0,T],X)$ be a solution of inequality (4.2) and let us denote by $u \in C([0,T],X)$ the unique mild solution of the Cauchy problem

$$\begin{cases} {}^{C}D_{0+}^{\beta,\rho}\left(\frac{u(s)}{g(s,u(s))}\right) = \mathcal{A}\left(\frac{u(s)}{g(s,u(s))}\right) + h(s,u(s)),\ s \in [0,T], \\ u(0) = v(0) = 0. \end{cases} \tag{4.5}$$

We have

$$
\begin{aligned}
\|\nu(s)-u(s)\| &\le \left\|\nu(s)-g(s,\nu(s))\left\{\int_0^s(\rho(s)-\rho(t))^{\beta-1}\omega_\rho^\beta(s,t)h(t,\nu(t))\rho'(t)dt\right\}\right\| \\
&+\left\|g(s,\nu(s))\int_0^s(\rho(s)-\rho(t))^{\beta-1}\omega_\rho^\beta(s,t)h(t,\nu(t))\rho'(t)dt\right. \\
&\left.-\ g(s,\nu(s))\int_0^s(\rho(s)-\rho(t))^{\beta-1}\omega_\rho^\beta(s,t)h(t,u(t))\rho'(t)dt\right\| \\
&+\left\|g(s,\nu(s))\int_0^s(\rho(s)-\rho(t))^{\beta-1}\omega_\rho^\beta(s,t)h(t,u(t))\rho'(t)dt\right. \\
&\left.-\ g(s,u(s))\int_0^s(\rho(s)-\rho(t))^{\beta-1}\omega_\rho^\beta(s,t)h(t,u(t))\rho'(t)dt\right\| \\
&\le \frac{LM\epsilon}{\Gamma(\beta+1)}(\rho(T)-\rho(0))^\beta+\frac{LL_hM}{\Gamma(\beta)}\int_0^s(\rho(s)-\rho(t))^{\beta-1}\|\nu(t)-u(t)\|\rho'(t)dt \\
&+\frac{M\mu\|h_r\|_{L^\infty}}{\Gamma(\beta+1)}(\rho(T)-\rho(0))^\beta\|\nu(s)-u(s)\| \\
&\le \frac{LM\epsilon}{\Gamma(\beta+1)}(\rho(T)-\rho(0))^\beta\times\frac{1}{d} \\
&+\frac{LL_hM}{d\Gamma(\beta)}\int_0^s(\rho(s)-\rho(t))^{\beta-1}\|\nu(t)-u(t)\|\rho'(t)dt,
\end{aligned}
$$

with

$$d=1-\frac{M\mu\|h_r\|_{L^\infty}}{\Gamma(\beta+1)}(\rho(T)-\rho(0))^\beta>0 \ \ \text{(under the hypothesis of Theorem (3.2)).}$$

By Gronwall's inequality, we get

$$\|\nu(s)-u(s)\|\le\frac{LM\epsilon}{\Gamma(\beta+1)}\times\frac{1}{d}(\rho(T)-\rho(0))^\beta E_\beta\left(\frac{LL_hM}{d}(\rho(t)-\rho(0))^\beta\right)\ \ (d>0).$$

□

Theorem 4.4. *Suppose the following conditions are satisfied:*

i. $h\in C([0,\infty)\times X,X)$;

ii. *The function* $\psi\in C([0,\infty],\mathbb{R}^+)$ *is increasing and there exists* $\lambda>0$ *such that*

$$\frac{LM\epsilon}{\Gamma(\beta+1)}(\rho(T)-\rho(0))^\beta\le\lambda\psi(t),\quad t\in[0,\infty);$$

iii. $\mu(s)$ *is nonnegative, nondecreasing continuous function defined on* $s\in[0,\infty)$

$$\text{and } \ |h(s,u_1)-h(s,u_2)|\le\mu(s)|u_1-u_2|,\quad \text{for all } s\ge0 \text{ and } \ u_1,u_2\in X.$$

Then, equation (4.1) *is generalized Mittag-Leffler-Ulam-Hyers-Rassias stable with respect to* ψE_β.

Proof. Let $\nu\in C^1([0,T],\infty)$ be a solution of equation (4.3). Then, we get

$$\left\|\nu(s)-g(s,\nu(s))\left\{\int_0^s(\rho(s)-\rho(t))^{\beta-1}\omega_\rho^\beta(s,t)h(t,\nu(t))\rho'(t)dt\right\}\right\| \le\frac{LM\epsilon}{\Gamma(\beta+1)}(\rho(T)-\rho(0))^\beta\le\lambda\psi(s),\quad s\in[0,\infty).$$

Let us denote by $u \in C([0,T],\infty)$ the unique mild solution of the cauchy problem

$$\begin{cases} {}^{C}D_{0+}^{\beta,\rho}\left(\frac{u(s)}{g(s,u(s))}\right) = \mathcal{A}\left(\frac{u(s)}{g(s,u(s))}\right) + h(s,u(s)),\ s \in [0,\infty), \\ u(0) = \nu(0) = 0. \end{cases}$$

We have

$$u(s) = g(s,u(s))\left\{\int_0^s (\rho(s)-\rho(t))^{\beta-1}\omega_\rho^\beta(s,t)h(t,u(t))\rho'(t)dt\right\},\ s \in [0,\infty).$$

It follows that

$$\begin{aligned}
\|\nu(s)-u(s)\| \le{}& \left\|\nu(s) - g(s,\nu(s))\left\{\int_0^s (\rho(s)-\rho(t))^{\beta-1}\omega_\rho^\beta(s,t)h(t,\nu(t))\rho'(t)dt\right\}\right\| \\
&+ \left\|g(s,\nu(s))\int_0^s (\rho(s)-\rho(t))^{\beta-1}\omega_\rho^\beta(s,t)h(t,\nu(t))\rho'(t)dt\right. \\
&\left.- g(s,u(s))\int_0^s (\rho(s)-\rho(t))^{\beta-1}\omega_\rho^\beta(s,t)h(t,u(t))\rho'(t)dt\right\| \\
\le{}& \lambda\psi(s) + \|g(s,\nu(s))\|\int_0^s (\rho(s)-\rho(t))^{\beta-1}\omega_\rho^\beta(s,t)\|h(t,\nu(t))-h(t,u(t))\|\rho'(t)dt \\
&+ \|g(s,\nu(s))-g(s,u(s))\|\,\|h_r\|_{L^\infty}\frac{M}{\Gamma(\beta+1)}(\rho(T)-\rho(t))^{\beta-1} \\
\le{}& \lambda\psi(s) + L\times\mu(s)\frac{M}{\Gamma(\beta)}\int_0^s (\rho(s)-\rho(t))^{\beta-1}\times\|\nu(t)-u(t)\|\rho'(t)dt \\
&+ \mu\|\nu(s)-u(s)\|\|\|h_r\|_{L^\infty}\frac{M}{\Gamma(\beta+1)}(\rho(T)-\rho(0))^\beta \\
\le{}& \frac{\lambda}{d}\psi(s) + \frac{LM\mu(s)}{\Gamma(\beta)}\frac{1}{d}\int_0^s (\rho(s)-\rho(t))^{\beta-1}\times\|\nu(t)-u(t)\|\rho'(t)dt, d>0.
\end{aligned}$$

By Gronwall's inequality, we get

$$\|\nu(s)-u(s)\| \le \frac{\lambda}{d}\psi(s)E_\beta\left(\frac{LM\mu(s)}{d}(\rho(s)-\rho(0))^\beta\right),$$

with $d = 1-\mu\|h_r\|_{L^\infty}\frac{M}{\Gamma(\beta+1)}(\rho(T)-\rho(0))^\beta$. □

5. Example

Let $X = L^2([0,\pi])$ with the norm and inner product defined as follows for any $u,\nu \in L^2([0,\pi])$:

$$\|u\| = \left(\int_0^\pi |u(y)|^2dy\right)^{\frac{1}{2}} \quad \text{and} \quad <u,\nu>= \int_0^\pi u(y)\,\overline{\nu(y)}\,dy.$$

Let's examine the initial-boundary value problem of a fractional parabolic partial differential equation with a nonlinear source term concerning tire dynamics

$$\begin{cases} {}^{C}\mathcal{D}_{0^+}^{\beta,\rho}\left(\frac{u(y,s)}{e^{-s}\,u(y,s)}\right) = \frac{\partial^2}{\partial y^2}\left(\frac{u(y,s)}{e^{-s}\,u(y,s)}\right) + \frac{1}{2}\,e^{-s}\,u(y,s),\ (s,y)\in[0,1]\times[0,\pi] \\ u(0,s) = u(\pi,s) = 0, \quad s\in[0,1] \\ u(y,0) = 0, \quad y \in [0,\pi]. \end{cases} \tag{5.1}$$

Where $\beta = \frac{2}{3}$, $T = 1$, $\rho = s$.
We define an operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset X \to X$ by

$$\mathcal{D}(\mathcal{A}) := \{v \in \mathcal{E};\, v, v' \text{ are absolutely continuous and } v'',\ v(0) = v(\pi) = 0\},$$

and

$$\mathcal{A}u = \frac{\partial^2}{\partial y^2}u.$$

It is well known that $\mathcal{A}$ has a discrete spectrum, the eigenvalue are $-j^2$, $j \in \mathbb{N}$, with corresponding normalized eigenvectors $e_j(z) = \sqrt{\frac{2}{\pi}}\sin(jz)$. Then

$$\mathcal{A}y = \sum_{j=1}^{\infty} -j^2 < y, e_j > e_j,\ \ y \in \mathcal{D}(A).$$

Thus, $\mathcal{A}$ generates a uniformly bounded analytic semigroup $\{T(s)\}_{s\geq 0}$ in X and it is given by

$$T(s)y = \sum_{j=1}^{\infty} e^{-j^2 s} < y, e_j > e_j,\ \ y \in X$$

with

$$\|T(s)\| \leq e^{-s},\ \ \forall s \geq 0.$$

Hence, we take $M = 1$ which implies that $\sup\limits_{t\in[0,\infty)} \|T(s)\| = 1$ and (C_1) is satisfied.
Then for all $s \in [0, 1]$, we have

$$\begin{cases} \|h(s,u)\| = \frac{1}{2}\, e^{-s}\|u\|, \\ \sup\limits_{\|u\|\leq r}\|h(s,u)\| \leq \frac{1}{2}\, e^{-s}\, r := h_r(s), \\ \lim\limits_{r\to\infty}\sup \frac{\|h_s(s)\|_{L^\infty}}{r} = \frac{1}{2} := L. \end{cases}$$

And

$$\|g(s,u_1) - g(s,u_2)\| \leq |u_1 - u_2|,\ \ \ u_1, u_2 \in X.$$

Therefore (C_2) and (C_3) are satisfied, which is given us

$$\frac{M\mu\|h_r\|_{L^\infty}}{\Gamma(\beta+1)}(\rho(T) - \rho(0))^{\beta} = \frac{1}{2\Gamma\left(\frac{5}{3}\right)} \simeq 0.45137 < 1.$$

According to Theorem (3.2). The problem (5.1) has a unique mild solution on [0,1].

6. Conclusion

In this paper, we have developed the theory of existence for mild solutions to initial value problems concerning hybrid fractional semi-linear evolution equations based on the $\psi-$Caputo fractional derivative. Our approach is rooted in the Dhage fixed point theorem. Furthermore, we have examined four distinct types of Mittag-Leffler-Ulam-Hyers stability regarding solutions to a specified problem. Additionally, an illustrative example has been provided to elucidate our principal findings.

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References

[1] M.I. Abbas, M.A. Ragusa, Nonlinear fractional differential inclusions with non-singular Mittag-Leffler kernel, AIMS Math., 7 (11), 20328-20340, (2022).

[2] Y. Adjabi, F. Jarad, T. Abdeljawad, On generalized fractional operators and a Gronwall type inequality with applications, Filomat, 31(17), 5457–5473, (2017).

[3] R.P. Agarwal, S. Hristova, D. O'Regan, A survey of Lyapunov functions stability and impulsive Caputo fractional differential equations, Fract. Calc. Appl. Anal., 19, 290–318, (2016).

[4] R.P. Agarwal, S.K. Ntouyas, B. Ahmad, A.K. Alzahrani, Hadamard-type fractional functional differential equations and inclusions with retarded and advanced arguments, Adv. Differ. Equ., 1, 1–15, (2016).

[5] R.P. Agarwal, Y. Zhou, J. Wang, X. Luo, Fractional functional differential equations with causal operators in Banach spaces, Math. Comput. Modelling, 54, 1440–1452, (2011).

[6] R. Almeida, Caputo fractional derivative of a function with respect to another function, Commun. Nonlinear Sci. Numer. Simul., 44, 460–481, (2017).

[7] R. Almeida, A.B. Malinowska, M.T.T. Monteiro, Fractional differential equations with a Caputo derivative with respect to a kernel function and their applications, Math. Methods Appl. Sci., 41(1), 3336–352, (2018).

[8] S. Aslan, A.O. Akdemir, New estimations for quasi-convex functions and (h,m)-convex functions with the help of Caputo-Fabrizio fractional integral operators, Electron. J. Appl. Math., 1 (3), (2023).

[9] D. Baleanu, H. Jafari, H. Khan, S.J. Johnston, Results for mild solution of fractional coupled hybrid boundary value problems, Open Math., 13, 601–608, (2015).

[10] D. Baleanu, H. Khan, H. Jafari, R.A. Alipour, On existence results for solutions of a coupled system of hybrid boundary value problems with hybrid condition, Adv. Differ. Equ., 1, 1–14, (2015).

[11] A. Belarbi, M. Benchohra, A. Ouahab, Uniqueness results for fractional functional differential equations with infinite delay in Fréchet spaces, Appl. Anal., 58, 1459–1470, (2006).

[12] A. Benmerrous, F.E. Bourhim, A. El Mfadel, M.H. Elomari, Solving a time-fractional semilinear hyperbolic equations by Fourier truncation with boundary conditions, Chaos Solitons Fractals, 185, 115086, (2024).

[13] A. Benmerrous, L.S. Chadli, A. Moujahid, S. Melliani, Generalized solutions for time ψ-fractional heat equation, Filomat, 37(27), 9327–9337, (2023).

[14] A. Benmerrous, L.S. Chadli, A. Moujahid, M. Elomari, S. Melliani, Conformable cosine family and nonlinear fractional differential equations, Filomat, 38(9), 3193–3206, (2024).

[15] A. Benmerrous, L.S. Chadli, A. Moujahid, M. Elomari, S. Melliani, Generalized Cosine Family, J. Elliptic Parabol. Equ., 8(1), 367–381, (2022).

[16] A. Benmerrous, L.S. Chadli, A. Moujahid, M. Elomari, S. Melliani, Generalized solution of Schrödinger equation with singular potential and initial data, Int. J. Nonlinear Anal. Appl., 13(1), 3093–3101, (2022).

[17] J. Da Vanterler, C. Sousa, E. Capelas de Oliveira, A Gronwall inequality and the Cauchy-type problem by means of ψ-Hilfer operator, Differ. Equ. Appl., 11(1), 87–106, (2019).

[18] K. Deimling, Nonlinear Functional Analysis, Springer-Verlag, New York, 1985.

[19] B.C. Dhage, V. Lakshmikantham, Basic results on hybrid differential equations, Nonlinear Anal. Hybrid Syst., 4, 414–424, (2010).

[20] B.C. Dhage, On a fixed point theorem in Banach algebras with applications, Appl. Math. Lett., 18, 273–280, (2005).

[21] B.C. Dhage, On α-condensing mappings in Banach algebras, Math. Student, 63, 146–152, (1994).

[22] K. Diethelm, The Analysis of Fractional Differential Equations, Lect. Notes Math., 3–12, (2010).

[23] G. Lu, W.L. Sun, Y. Jin, Q. Liu, Ulam stability of Jensen functional inequality on a class of noncommutative groups, J. Funct. Spaces, 2023, (2023).

[24] K. Engel, R. Nagel, One-Parameter Semigroups for Linear Evolution Equations, Springer, New York, 2000.

[25] J.W. Green, F.A. Valentine, On the Arzelá–Ascoli theorem, Math. Mag., 34, 199–202, (1961).

[26] T.L. Guo, M. Jiang, Impulsive fractional functional differential equations, Comput. Math. Appl., 64, 3414–3424, (2012).

[27] K. Hilal, A. Kajouni, Boundary value problems for hybrid differential equations with fractional order, Adv. Differ. Equ., 1–19, (2015).

[28] R. Hilfer, Applications of Fractional Calculus in Physics, Springer, Singapore, 2000.

[29] M. Houas, M.I. Abbas, F. Martínez, Existence and Mittag-Leffler-Ulam-stability results of sequential fractional hybrid pantograph equations, Filomat, 37 (20), 6891–6903, (2023).

[30] F. Jarad, T. Abdeljawad, Generalized fractional derivatives and Laplace transform, Discrete Contin. Dyn. Syst. Ser. S, 13, 709–722, (2020).

[31] F. Mainardi, On the initial value problem for the fractional diffusion-wave equation, In: Rionero, S., Ruggeri, T. (eds.) Waves and Stability in Continuous Media, pp. 246–251, (1994).

[32] F. Mainardi, P. Paradisi, R. Gorenflo, Probability distributions generated by fractional diffusion equations, In: Kertesz, J., Kondor, I. (eds.) Econophysics: An Emerging Science, Springer, Dordrecht, (2000).

[33] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer, New York, 1983.

[34] I. Podlubny, Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications, vol. 198, Elsevier, Amsterdam, 1998.

[35] Z. Smith, Fixed point methods in nonlinear analysis, (2014).

[36] S. Zerbib, K. Hilal, A. Kajouni, On the hybrid fractional semi-linear evolution equations, Int. J. Nonlinear Anal. Appl., In Press, 1–12, (2023).

[37] Y. Zhou, F. Jiao, Existence of mild solutions for fractional neutral evolution equations, Comput. Math. Appl., 59, 1063–1077, (2010).