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Some properties of Pareto H-eigenvalues on tensors and hypergraphs

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Abstract. A Pareto H-eigenvalue of a tensor $\mathbb{A}$ is a real number λ satisfying the complementarity system: $0 \leq x \perp (\mathbb{A}x - \lambda \mathbb{I}x) \geq 0$. The Pareto H-spectrum is the set of all Pareto H-eigenvalues. In this note, we first obtain some invariance on Pareto H-spectrum of tensors under tensor permutational similar (resp. diagonal similar) translation. As their application, we know that Pareto H-spectrum of hypergraphs is independent of the ordering of their vertices. Furthermore, we attain some nice properties on Pareto H-spectrum of hypergraphs. At the same time, we improve some results in [4] and [5].

1. Introduction

Let R (resp. R_+, C) be the set of real (resp. nonnegative real, complex) numbers. Let $R_+^n = \{(x_1, \ldots, x_n)^T : x_i \geq 0, i \in [n] = \{1, 2, \ldots, n\}\}$ and $R_{++}^n = \{(x_1, \ldots, x_n)^T : x_i > 0, i \in [n]\}$. Let $\mathbb{A} = (a_{i_1 \ldots i_m})$ be a real tensor with order m dimension n, where $a_{i_1 \ldots i_m} \in R$, $i_j \in [n]$ and $j \in [m]$. Obviously, if $m = 2$ (resp. $m = 1$), $\mathbb{A}$ is exactly the matrix (resp. vector) of order n. For simplicity, we will write $a_{i_1 \ldots i_m}$ as $a_{i_1 \alpha}$, where $\alpha = i_2 \cdots i_m \in [n]^{m-1}$. For an order $m(\geq 2)$ dimension n tensor $\mathbb{A}$ and an order $k(\geq 1)$ dimension n tensor $\mathbb{B}$, their general product $\mathbb{AB}$, which is introduced by Shao in [3], is the tensor $\mathbb{C} = (c_{i\alpha_1 \ldots \alpha_{m-1}})$ of order $(m-1)(k-1)+1$ and dimension n, for $i \in [n]$,

$$(\mathbb{AB})_{i\alpha_1 \ldots \alpha_{m-1}} = c_{i\alpha_1 \ldots \alpha_{m-1}} = \sum_{i_2, \ldots, i_m = 1}^{n} a_{ii_2 \ldots i_m} b_{i_2 \alpha_1} \cdots b_{i_m \alpha_{m-1}} \tag{1}$$

where $\alpha_1, \ldots, \alpha_{m-1} \in [n]^{k-1}$. Especially, when $k = 1$ and $\mathbb{B} = x \in R^n = \{(x_1, \ldots, x_n)^T : x_i \in R, i \in [n]\}$,

$$(\mathbb{A}x)_i = c_i = \sum_{i_2, \ldots, i_m = 1}^{n} a_{ii_2 \ldots i_m} x_{i_2} \cdots x_{i_m},$$

which is equivalent to the definition of $\mathbb{A}x^{m-1}$ introduced by Qi in [2].

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A Pareto H-eigenvalue of a tensor $\mathbb{A}$, which is first introduced by Song and Qi in [4], is a real number λ satisfying the complementarity system:

$$0 \leq x \perp (\mathbb{A}x - \lambda \mathbb{I}x) \geq 0 \tag{2}$$

where $\perp$ stands for orthogonality, $x \geq 0$ means that every entry of vector x is nonnegative, $\mathbb{I}_n = (\delta_{i_1 i_2 \cdots i_m})$ is an identity tensor with demension n (for simiplicity, we write it as $\mathbb{I}$ unless otherwise specified), where $\delta_{i_1 i_2 \cdots i_m}$ is equal to 1 for $i_1 = i_2 = \cdots = i_m$, and 0 otherwise. The Pareto H-spectrum (that is, the set of Pareto H-eigenvalues) of a tensor $\mathbb{A}$ is denoted by $\Pi(\mathbb{A})$. The vector x satisfying (2) is called Pareto H-eigenvector corresponding to λ. Especially, if $\mathbb{A}x = \lambda \mathbb{I}x$ and x is not required to be nonnegative in (2), then λ is the well-known eigenvalue of tensor $\mathbb{A}$ and x is its corresponding eigenvector, which is introduced by Qi in [2]. Therefore, Pareto H-eigenvalue of a tensor is an extension of eigenvalue of a tensor that contains a complementary condition on nonnegative variables.

Let $H = (V(H), E(H))$ be an m-uniform hypergraph with vertex set $V(H)$ and edge set $E(H)$, where $E(H)$ is a family of m-subsets (that is, each of elements in $E(H)$ has cardinality m) of $V(H)$, $|V(H)|$ is the order of H. The adjacency tensor $\mathbb{A}(H) = (a_{i_1 i_2 \cdots i_m})$ of H is defined as

$$(m-1)! a_{i_1 \cdots i_m} = \begin{cases} 1, & if \{i_1, \ldots, i_m\} \text{ is an edge of } H, \\ 0, & otherwise. \end{cases}$$

The recent work of Zheng and Zhou [5] replaced $\mathbb{A}$ in (2) as the adjacency tensor $\mathbb{A}(H)$, treated Pareto H-spectrum of the adjacency tensor $\mathbb{A}(H)$ as Pareto H-spectrum of the hypergraph H and denoted it by $\Pi(H)$. This is the first one that deals specifically with Pareto H-spectrum of hypergraphs. There have been few further results by now and we urgently hope for more research on this topic. In this note, we first obtain some invariance on Pareto H-spectrum of tensors under some tensor translations and then attain some nice properties on Pareto H-spectrum of hypergraphs.

2. Preliminaries

In this section, we first list some results that will be used in the sequel.

Lemma 2.1. *[4] If $\mathbb{A}$ is a symmetric tensor of order m and dimension n, then $\mathbb{A}$ has at least one Pareto H-eigenvalue and the smallest eigenvale of $\mathbb{A}$ is a Pareto H-eigenvalue.*

Lemma 2.2. *[3] The tensor product defined as (1) has the following properties.*

(i) $(\mathbb{A}_1 + \mathbb{A}_2)\mathbb{B} = \mathbb{A}_1\mathbb{B} + \mathbb{A}_2\mathbb{B}$;
(ii) $A(\mathbb{B}_1 + \mathbb{B}_2) = A\mathbb{B}_1 + A\mathbb{B}_2$, *when A is a matrix;*
(iii) $(\lambda\mathbb{A})\mathbb{B} = \lambda(\mathbb{A}\mathbb{B})$, *where $\lambda \in C$;*
(iv) $(\mathbb{A}\mathbb{B})\mathbb{C} = \mathbb{A}(\mathbb{B}\mathbb{C})$.

Definition 2.3. *Let $\mathbb{A}$ and $\mathbb{B}$ be two order k tensors with dimension n and m, respectively. Define the direct product $\mathbb{A} \otimes \mathbb{B}$ be the following tensor of order k and dimension nm:*

$$\mathbb{A} \otimes \mathbb{B}_{(i_1,j_1),\ldots,(i_k,j_k)} = A_{i_1 i_2 \ldots i_k} B_{j_1 j_2 \ldots j_k}.$$

Lemma 2.4. *[3] The direct product of tensors defined in definition 2.3 has the following properties.*

(i) $(\mathbb{A}_1 + \mathbb{A}_2) \otimes \mathbb{B} = \mathbb{A}_1 \otimes \mathbb{B} + \mathbb{A}_2 \otimes \mathbb{B}$;
(ii) $\mathbb{A} \otimes (\mathbb{B}_1 + \mathbb{B}_2) = \mathbb{A} \otimes \mathbb{B}_1 + \mathbb{A} \otimes \mathbb{B}_2$;
(iii) $(\lambda\mathbb{A}) \otimes \mathbb{B} = \mathbb{A} \otimes (\lambda\mathbb{B}) = \lambda(\mathbb{A} \otimes \mathbb{B})$, *where $\lambda \in C$;*
(iv) $(\mathbb{A} \otimes \mathbb{B})(\mathbb{C} \otimes \mathbb{D}) = (\mathbb{A}\mathbb{C}) \otimes (\mathbb{B}\mathbb{D})$.

Definition 2.5. *Let G and H be two m-uniform hypergraphs. Define the Cartesian product $G \square H$ of G and H as: $V(G \square H) = V(G) \times V(H)$, and $\{(i_1, j_1), \ldots, (i_m, j_m)\} \in E(G \square H)$ if*

(i) $i_1 = \cdots = i_m$ *and* $\{j_1, \ldots, j_m\} \in E(H)$;
(ii) $j_1 = \cdots = j_m$ *and* $\{i_1, \ldots, i_m\} \in E(G)$.

Definition 2.6. *Let G and H be two m-uniform hypergraphs. Define the direct product $G \times H$ of G and H as: $V(G \times H) = V(G) \times V(H)$, and $\{(i_1, j_1), \ldots, (i_m, j_m)\} \in E(G \times H)$ if and only if $\{i_1, \ldots, i_m\} \in E(G)$ and $\{j_1, \ldots, j_m\} \in E(H)$.*

Lemma 2.7. *[3] Let G (resp. H) be an m-uniform hypergraph of order n (resp. order k) and $\mathbb{A}(G)$ (resp. $\mathbb{A}(H)$) be its adjacency tensor. Then*

(i) $\mathbb{A}(G \square H) = \mathbb{A}(G) \otimes \mathbb{I}_k + \mathbb{I}_n \otimes \mathbb{A}(H)$;
(ii) $\mathbb{A}(G \times H) = (m-1)!\mathbb{A}(G) \otimes \mathbb{A}(H)$.

3. Main results

3.1. Some properties on Pareto H-eigenvalues of tensors

In this subsection, we will give some invariances of Pareto H-spectrum of tensors under tensor permutational similar (resp. diagonal similar) translation, which are important on studying Pareto H-spectrum. At the same time, some other properties on Pareto H-spectrum of tensors are obtained.

Definition 3.1. *Let $\sigma \in S_n$ be a permutation on the set $[n]$, $P = P_\sigma = (P_{ij})$ be the corresponding permutation matrix of σ (where $P_{ij} = 1 \Leftrightarrow j = \sigma(i)$). Two order m dimension n tensors $\mathbb{A}$ and $\mathbb{B}$ are called permutational similar if $\mathbb{B} = P\mathbb{A}P^T$.*

Lemma 3.2. *Let $\mathbb{A}$ and $\mathbb{B}$ be two tensors with order m dimension n.*

(i) *If tensors $\mathbb{A}$ and $\mathbb{B}$ are permutational similar, then $\Pi(\mathbb{A}) = \Pi(\mathbb{B})$.*
(ii) *$\Pi(\mathbb{A} - r\mathbb{I}) = \Pi(\mathbb{A}) - r$ for all $r \in R$, where $\Pi(\mathbb{A}) - r = \{a - r : a \in \Pi(\mathbb{A})\}$.*
(iii) *$\Pi(\beta\mathbb{A}) = \beta\Pi(\mathbb{A})$ for all $\beta > 0$, where $\beta\Pi(\mathbb{A}) = \{\beta a : a \in \Pi(\mathbb{A})\}$.*

Proof. (i) For any $\lambda \in \Pi(\mathbb{A})$, by the definition of Pareto H-eigenvalue and Pareto H-eigenvector, there exists a nonzero vector $x \in R^n$ satisfying the complementarity system:

$$0 \leq x \perp (\mathbb{A}x - \lambda\mathbb{I}x) \geq 0.$$

Since $\mathbb{A}$ and $\mathbb{B}$ are permutational similar, it must have a permutation matrix $P = P_\sigma$ satisfying $\mathbb{B} = P\mathbb{A}P^T$. Let $y = Px$, it has

$$y_i = (Px)_i = \sum_{j=1}^{n} P_{ij}x_j = P_{i\sigma(i)}x_{\sigma(i)} = x_{\sigma(i)},$$

this is to say that $y \geq 0$ and $y \neq 0$. By (1) and Lemma 2.2, we know that

$$(P\mathbb{A}P^T)_{ii_2\ldots i_m} = \sum_{j_1,j_2,\ldots,j_m \in [n]} a_{j_1 j_2 \ldots j_m} P_{ij_1} P_{i_2 j_2} \cdots P_{i_m j_m} = a_{\sigma(i)\sigma(i_2)\ldots\sigma(i_m)}.$$

Furthermore

$$\begin{aligned}
(\mathbb{B}y - \lambda\mathbb{I}y)_i &= [(P\mathbb{A}P^T)(Px)]_i - \lambda(Px)_i^{m-1} \\
&= \sum_{i_2=1,\ldots,i_m=1}^{n} (P\mathbb{A}P^T)_{ii_2\ldots i_m}(Px)_{i_2}\cdots(Px)_{i_m} - \lambda x_{\sigma(i)}^{m-1} \\
&= \sum_{i_2=1,\ldots,i_m=1}^{n} a_{\sigma(i)\sigma(i_2)\ldots\sigma(i_m)} x_{\sigma(i_2)}\cdots x_{\sigma(i_1)} - \lambda x_{\sigma(i)}^{m-1} \\
&= (\mathbb{A}x - \lambda\mathbb{I}x)_{\sigma(i)} \geq 0; \\
y^T(\mathbb{B}y - \lambda\mathbb{I}y) &= \sum_{i=1}^{n} x_{\sigma(i)}(\mathbb{A}x - \lambda\mathbb{I}x)_{\sigma(i)} = x^T(\mathbb{A}x - \lambda\mathbb{I}x) = 0.
\end{aligned}$$

Then $\lambda \in \Pi(\mathbb{B})$. Further we have $\Pi(\mathbb{A}) \subseteq \Pi(\mathbb{B})$. Similarly, we can prove that $\Pi(\mathbb{A}) \supseteq \Pi(\mathbb{B})$. Hence $\Pi(\mathbb{A}) = \Pi(\mathbb{B})$.

(ii) For any $\lambda \in \Pi(\mathbb{A} - r\mathbb{I})$, by the definitions of Pareto H-eigenvalue and Pareto H-eigenvector, there exists $u \in R^n$ satisfying

$$0 \le u \perp ((\mathbb{A} - r\mathbb{I})u - \lambda \mathbb{I}u) \ge 0.$$

By Lemma 2.2, it has,

$$0 \le u \perp ((\mathbb{A}u - (\lambda + r)\mathbb{I}u) \ge 0,$$

then $\lambda + r \in \Pi(\mathbb{A})$, that is, $\lambda \in \Pi(\mathbb{A}) - r$.

For any $\lambda \in \Pi(\mathbb{A}) - r$, it has $t \in \Pi(\mathbb{A})$ and $u \in R^n$ satisfying

$$\begin{aligned} \lambda &= t - r. \\ 0 \le u \perp ((\mathbb{A}u - t\mathbb{I}u) \ge 0 &\Rightarrow 0 \le u \perp ((\mathbb{A} - r\mathbb{I})u - (t - r)\mathbb{I}u) \ge 0. \end{aligned}$$

Then $\lambda = t - r \in \Pi(\mathbb{A} - r\mathbb{I})$. Hence $\Pi(\mathbb{A} - r\mathbb{I}) = \Pi(\mathbb{A}) - r$.

(iii) For any $\lambda \in \Pi(\beta\mathbb{A})$, there exists $u \in R^n$ satisfying

$$0 \le u \perp ((\beta\mathbb{A})u - \lambda \mathbb{I}u) \ge 0.$$

Further by $\beta > 0$ and Lemma 2.2, we have

$$0 \le u \perp (\mathbb{A}u - \frac{\lambda}{\beta}\mathbb{I}u) \ge 0,$$

that is, $\frac{\lambda}{\beta} \in \Pi(\mathbb{A})$ and $\lambda \in \beta\Pi(\mathbb{A})$. Then $\Pi(\beta\mathbb{A}) \subseteq \beta\Pi(\mathbb{A})$. Similarly, we can prove $\Pi(\beta\mathbb{A}) \supseteq \beta\Pi(\mathbb{A})$. Hence $\Pi(\beta\mathbb{A}) = \beta\Pi(\mathbb{A})$. □

As we know that the adjacency tensor of a hypergraph depends on the ordering of its vertices. Thus a natural question arises: Is the Pareto H-spectrum of a hypergraph independent of the ordering of its vertices? From Lemma 3.2 (1), we can give an affirmative answer to this question. Furthermore, we have the following corollaries.

Corollary 3.3. *The Pareto H-spectrum of a hypergraph is independent of the ordering of its vertices and isomorphic hypergraphs have the same Pareto H-spectrum.*

Corollary 3.4. *If the two order m dimension n tensors $\mathbb{A}$ and $\mathbb{B}$ are permutational similar, that is, $\mathbb{B} = P\mathbb{A}P^T$ for some permutational matrix P, then x is a Pareto H-eigenvector of $\mathbb{A}$ corresponding to the Pareto H-eigenvalue λ if and only if $y = Px$ is a Pareto H-eigenvector of $\mathbb{B}$ corresponding to the same Pareto H-eigenvalue λ.*

Definition 3.5. *Two order m dimension n tensors $\mathbb{A}$ and $\mathbb{B}$ are diagonal similar if there exists some positive diagonal matrix D satisfying $\mathbb{B} = D^{-(m-1)}\mathbb{A}D$.*

Lemma 3.6. *If the two order m dimension n tensors $\mathbb{A}$ and $\mathbb{B}$ are diagonal similar, then $\Pi(\mathbb{A}) = \Pi(\mathbb{B})$.*

Proof. For any $\lambda \in \Pi(\mathbb{A})$, by the definition of Patreto H-eigenvalue and Patreto H-eigenvector, there exists a nonzero vector $x \in R^n$ satisfying the complementarity system:

$$0 \le x \perp (\mathbb{A}x - \lambda \mathbb{I}x) \ge 0. \tag{3}$$

Since $\mathbb{A}$ and $\mathbb{B}$ are diagonal similar, it must have a positive diagonal matrix D satisfying $\mathbb{B} = D^{-(m-1)}\mathbb{A}D$. Let $y = D^{-1}x$, it has

$$y_i = (D^{-1}x)_i = \sum_{j=1}^{n} D_{ij}^{-1} x_j = D_{ii}^{-1} x_i.$$

That is, $y \geq 0$ and $y \neq 0$. Furthermore

$$\begin{aligned}
& (\mathbb{B}y - \lambda \mathbb{I}y)_i \\
= \; & [(D^{-(m-1)}\mathbb{A}D)(D^{-1}x)]_i - \lambda (D^{-1}x)_i^{m-1} \\
= \; & \sum_{i_2=1,\dots,i_m=1}^{n} (D^{-(m-1)}\mathbb{A}D)_{ii_2\dots i_m}(D^{-1}x)_{i_2}\cdots(D^{-1}x)_{i_m} - \lambda D_{ii}^{-(m-1)}x_i^{m-1} \\
= \; & \sum_{i_2=1,\dots,i_m=1}^{n} [a_{ii_2\dots i_m}D_{ii}^{-(m-1)}D_{i_2i_2}\cdots D_{i_mi_m}](D_{i_2i_2}^{-1}x_{i_2})\cdots(D_{i_mi_2}^{-1}x_{i_m}) - \lambda D_{ii}^{-(m-1)}x_i^{m-1} \\
= \; & D_{ii}^{-(m-1)}(\mathbb{A}x - \lambda \mathbb{I}x)_i \geq 0.
\end{aligned}$$

From (3), it has

$$x^T(\mathbb{A}x - \lambda \mathbb{I}x) = \sum_{i=1}^{n} x_i(\mathbb{A}x - \lambda \mathbb{I}x)_i = 0 \Rightarrow x_i(\mathbb{A}x - \lambda \mathbb{I}x)_i = 0, i \in [n].$$

Then

$$y^T(\mathbb{B}y - \lambda \mathbb{I}y) = \sum_{i=1}^{n} D_{ii}^{-m}x_i(\mathbb{A}x - \lambda \mathbb{I}x)_i = 0.$$

Then $\lambda \in \Pi(\mathbb{B})$ and $\Pi(\mathbb{A}) \subseteq \Pi(\mathbb{B})$. Similarly, we can prove that $\Pi(\mathbb{A}) \supseteq \Pi(\mathbb{B})$. Hence $\Pi(\mathbb{A}) = \Pi(\mathbb{B})$. □

From the proof of the above lemma, we have the following corollary.

Corollary 3.7. *If the two order m dimension n tensors $\mathbb{A}$ and $\mathbb{B}$ are diagonal similar, that is, $\mathbb{B} = D^{-(m-1)}\mathbb{A}D$ for some positive diagonal matrix D, then x is a Pareto H-eigenvector of $\mathbb{A}$ corresponding to the Pareto H-eigenvalue λ if and only if $y = D^{-1}x$ is a Pareto H-eigenvector of $\mathbb{B}$ corresponding to the same Pareto H-eigenvalue λ.*

3.2. Some results of Pareto H-eigenvalues on hypergraphs

In this subsection, we will give some properties of Pareto H-eigenvalues with respect to hypergraphs.

Let $\mathbb{A}$ be a tensor of order m and dimension n. For $\emptyset \neq J \subseteq [n]$, the principal subtensor $\mathbb{A}^J$ of $\mathbb{A}$ is a tensor of order m and dimension $|J|$ with entries $a_{i_1i_2\cdots i_m}$ and $i_1, i_2, \cdots, i_m \in J$. We can reformulate Theorem 3.1 [4] as follows. Perhaps it is more convenient to use for studying on hypergraphs.

Theorem 3.8. *Let $\mathbb{A} = (a_{i_1i_2\cdots i_m})$ be a tensor of order m and dimension n. A real number λ is Pareto H-eigenvalue of $\mathbb{A}$ if and only if there exists a nonempty subset $J \subseteq [n]$ and a vector $w \in R_+^{|J|}$ satisfying*

(i) $\mathbb{A}^J w = \lambda \mathbb{I}_{|J|} w$;
(ii) $\sum_{j_2,j_3,\dots,j_m \in J} a_{jj_2\cdots j_m} w_{j_2} w_{j_3} \cdots w_{j_m} > 0$ *for* $j \in \bar{J}$.

In such a case, the vector $x = (x_1, \dots, x_n)^T \in R_+^n$ is a Pareto H-eigenvector of $\mathbb{A}$ corresponding to the real number λ, where $x_i = w_i$ for $i \in J$ and $x_i = 0$ otherwise.

Proof. (⇐) It is trivial.

(⇒) Let λ be a Pareto H-eigenvalue of $\mathbb{A}$ and x be its corresponding Pareto H-eigenvector, then

$$0 \leq x \perp (\mathbb{A}x - \lambda \mathbb{I}x) \geq 0.$$

It is easy to see that $x_j(\mathbb{A}x - \lambda \mathbb{I}x)_j = 0$ for all $j \in [n]$. Let

$$\begin{aligned}
J &= \{j : x_j > 0\} \cup \{j : x_j = (\mathbb{A}x - \lambda \mathbb{I}x)_j = 0\}, \\
\bar{J} &= \{j : x_j = 0, (\mathbb{A}x - \lambda \mathbb{I}x)_j > 0\}.
\end{aligned}$$

Let $w \in R_+^{|J|}$ be defined by $w_j = x_j$ for all $j \in J$. Obviously, for all $j \in J$, it has $(\mathbb{A}x - \lambda \mathbb{I}x)_j = 0$, that is, $\mathbb{A}^J w = \lambda \mathbb{I}_{|J|} w$; and for all $j \in \bar{J}$, it has

$$\begin{aligned}(\mathbb{A}x - \lambda \mathbb{I}x)_j &= \sum_{j_2,\cdots,j_m \in [n]} a_{jj_2\cdots j_m} x_{j_2} \cdots x_{j_m} - \lambda x_j^{m-1} \\ &= \sum_{j_2,\cdots,j_m \in J} a_{jj_2\cdots j_m} x_{j_2} \cdots x_{j_m} = \sum_{j_2,\cdots,j_m \in J} a_{jj_2\cdots j_m} w_{j_2} \cdots w_{j_m} > 0.\end{aligned}$$

□

From Theorem 3.8, we can give a new form of Theorem 4.1 in [5] as follows.

Theorem 3.9. *Let H be an m-uniform hypergraph with n vertices, then $\Pi(\mathbb{A}(H)) = \{\rho(G) : G$ is a induced subhypergraph of $H\}$, where $\rho(G)$ is the largest modulus of eigenvalues of G.*

Note that the induced subhypergraph G of H in Theorem 3.9 may be not connected, which is different from Theorem 4.1 in [5].

Theorem 3.10. *Let H be a connected m-uniform hypergraph with n vertices and*

$$\mathbb{S}(H) = \{\mathbb{A} = (a_{i_1 i_2 \cdots i_m}) : \mathbb{A} \ \text{ is symmetric and } \ a_{i_1 i_2 \cdots i_m} \neq 0 \ \ iff \ \{i_1, i_2, \cdots, i_m\} \in E(H)\}$$

be the set of tensors with respect to H. If the nondiagonal entries of $\mathbb{A} \in \mathbb{S}(H)$ are non positive, then $\mathbb{A}$ has a unique Pareto H-eigenvalue, which is the smallest eigenvale of $\mathbb{A}$.

Proof. By Lemma 2.1, let λ be a Pareto H-eigenvalue of $\mathbb{A}$ and x be its corresponding Pareto H-eigenvector. Let $J_1 = \{j : x_j > 0\}$, $J_2 = \{j : x_j = (\mathbb{A}x - \lambda \mathbb{I}x)_j = 0\}$ and $J = J_1 \cup J_2$, by Theorem 3.8, it has

$$\sum_{j_2, j_3, \ldots, j_m \in J} a_{jj_2\cdots j_m} x_{j_2} x_{j_3} \cdots x_{j_m} > 0, \quad for \ \ j \in \bar{J} = [n] \backslash J. \tag{4}$$

Claim 1 $J = [n]$.

Otherwise, $\bar{J} \neq \emptyset$. Then there exists at least one edge $e = \{j_1, j_2, \cdots, j_m\}$ between J and $\bar{J}$, without loss of generality, let $j_1 \in \bar{J}$ and $j_2 \in J$. Then $a_{j_1 j_2 \cdots j_m} < 0$ and

$$\sum_{j_2,\cdots,j_m \in J} a_{j_1 j_2 \cdots j_m} x_{j_2} \cdots x_{j_m} \leq 0,$$

which is contradict to (4). So we have

$$\mathbb{A}x = \lambda \mathbb{I}x, x \in R_+^n.$$

Claim 2 $J_2 = \emptyset$.

For any $j \in J_2$, it has

$$(\mathbb{A}x - \lambda \mathbb{I}x)_j = \sum_{j_2,\cdots,j_m \in J_1} a_{jj_2\cdots j_m} x_{j_2} \cdots x_{j_m} = 0 \Rightarrow a_{jj_2\cdots j_m} = 0,$$

that is, there is no edge between J_1 and J_2, and this is contradict to the connectivity of H.

Claim 3 λ is the unique Pareto H-eigenvalue of $\mathbb{A}$, which is the smallest eigenvale of $\mathbb{A}$.

By Claim 1 and Claim 2, we have $\mathbb{A}x = \lambda \mathbb{I}x, x \in R_{++}^n$. Note that $\mathbb{A}$ is a Z-tensor, by Lemma 4.2 in [1] and Lemma 2.1, we can obtain the desired results. □

The following we will give some properties on Pareto H-eigenvalue with respect to some hypergraph operators.

Theorem 3.11. *Let G_1 be a m-uniform hypergraph of order n_1 and G_2 be an m-uniform hypergraph of order n_2. Let λ be a Pareto H-eigenvalue of G_1 with corresponding Pareto H-eigenvector u, and μ be a Pareto H-eigenvalue of G_2 with corresponding Pareto H-eigenvector v, respectively. Then*

(i) *$\lambda + \mu$ is a Pareto H-eigenvalue of $G_1 \Box G_2$ with corresponding Pareto H-eigenvector $u \otimes v$;*
(ii) *$(m-1)!\lambda\mu$ is a Pareto H-eigenvalue of $G_1 \times G_2$ with corresponding Pareto H-eigenvector $u \otimes v$.*

Proof. By the definitions of Pareto H-eigenvalue and Pareto H-eigenvector, it has

$$0 \le u \perp (\mathbb{A}(G_1)u - \lambda \mathbb{I}_{n_1} u) \ge 0; \tag{5}$$
$$0 \le v \perp (\mathbb{A}(G_2)v - \mu \mathbb{I}_{n_2} v) \ge 0. \tag{6}$$

Let $w = u \otimes v$, since $(u \otimes v)_{(i,j)} = u_i v_j \ge 0$, then $w \ge 0$ and $w \ne 0$.

(i) By Lemma 2.7, we know that the adjacency tensor $\mathbb{A}(G_1 \Box G_2)$ of $G_1 \Box G_2$ is $\mathbb{A}(G_1) \otimes \mathbb{I}_{n_2} + \mathbb{I}_{n_1} \otimes \mathbb{A}(G_2)$. By Lemmas 2.2 and 2.4, we have

$$\begin{aligned}
\mathbb{A}(G_1 \Box G_2)(u \otimes v) &= (\mathbb{A}(G_1) \otimes \mathbb{I}_{n_2} + \mathbb{I}_{n_1} \otimes \mathbb{A}(G_2))(u \otimes v) \\
&= (\mathbb{A}(G_1) \otimes \mathbb{I}_{n_2})(u \otimes v) + (\mathbb{I}_{n_1} \otimes \mathbb{A}(G_2))(u \otimes v) \\
&= (\mathbb{A}(G_1)u \otimes \mathbb{I}_{n_2} v) + (\mathbb{I}_{n_1} u \otimes \mathbb{A}(G_2)v).
\end{aligned}$$

Further we have

$$\begin{aligned}
& \mathbb{A}(G_1 \Box G_2)(u \otimes v) - (\lambda + \mu)\mathbb{I}_{n_1 n_2}(u \otimes v) \\
= \; & (\mathbb{A}(G_1)u) \otimes \mathbb{I}_{n_2} v + \mathbb{I}_{n_1} u \otimes (\mathbb{A}(G_2)v) - (\lambda + \mu)(\mathbb{I}_{n_1} u \otimes \mathbb{I}_{n_2} v) \\
= \; & [(\mathbb{A}(G_1)u \otimes \mathbb{I}_{n_2} v) - \lambda(\mathbb{I}_{n_1} u \otimes \mathbb{I}_{n_2} v)] + [(\mathbb{I}_{n_1} u \otimes \mathbb{A}(G_2)v) - \mu(\mathbb{I}_{n_1} u \otimes \mathbb{I}_{n_2} v)] \\
= \; & [(\mathbb{A}(G_1)u - \lambda \mathbb{I}_{n_1} u] \otimes \mathbb{I}_{n_2} v + \mathbb{I}_{n_1} u \otimes [\mathbb{A}(G_2)v - \mu \mathbb{I}_{n_2} v] \ge 0.
\end{aligned}$$

From (5) and (6), we have $u_i\left(\mathbb{A}(G_1)u - \lambda \mathbb{I}_{n_1} u\right)_i = 0$ and $v_j\left(\mathbb{A}(G_2)v - \mu \mathbb{I}_{n_2} v\right)_j = 0$ for $i \in [n_1]$, $j \in [n_2]$, then

$$\begin{aligned}
& ((u \otimes v))^T\left[\mathbb{A}(G_1 \Box G_2)(u \otimes v) - (\lambda + \mu)\mathbb{I}_{n_1 n_2}(u \otimes v)\right] \\
= \; & \sum_{(i,j) \in [n_1] \times [n_2]} (u \otimes v)_{(i,j)} \left\{[(\mathbb{A}(G_1)u - \lambda \mathbb{I}_{n_1} u] \otimes \mathbb{I}_{n_2} v + \mathbb{I}_{n_1} u \otimes [\mathbb{A}(G_2)v - \mu \mathbb{I}_{n_2} v]_{(i,j)}\right\} \\
= \; & \sum_{(i,j) \in [n_1] \times [n_2]} (u_i v_j)\left\{[(\mathbb{A}(G_1)u - \lambda \mathbb{I}_{n_1} u]_i v_j^{m-1} + u_i^{m-1}[\mathbb{A}(G_2)v - \mu \mathbb{I}_{n_2} v]_j\right\} \\
= \; & \sum_{(i,j) \in [n_1] \times [n_2]} \left\{\left[u_i\left(\mathbb{A}(G_1)u - \lambda \mathbb{I}_{n_1} u\right)_i\right] v_j^m + u_i^m\left[v_j\left(\mathbb{A}(G_2)v - \mu \mathbb{I}_{n_2} v\right)_j\right]\right\} = 0.
\end{aligned}$$

(ii) By Lemma 2.7, we know that the adjacency tensor $\mathbb{A}(G_1 \times G_2)$ of $G_1 \times G_2$ is $(m-1)!(\mathbb{A}(G_1) \otimes \mathbb{A}(G_2))$. Note that $\lambda \ge 0$, $\mu \ge 0$. By Lemmas 2.2 and 2.4, we have

$$\begin{aligned}
\mathbb{A}(G_1 \times G_2)(u \otimes v) &= \left[(m-1)!(\mathbb{A}(G_1) \otimes \mathbb{A}(G_2))\right](u \otimes v) \\
&= (m-1)!\left[(\mathbb{A}(G_1) \otimes \mathbb{A}(G_2))\right](u \otimes v) \\
&= (m-1)![\mathbb{A}(G_1)u] \otimes [\mathbb{A}(G_2)v]. \\
\frac{1}{(m-1)!}\mathbb{A}(G_1 \times G_2)(u \otimes v) - (\lambda\mu)\mathbb{I}_{n_1 n_2}(u \otimes v) &= [\mathbb{A}(G_1)u] \otimes [\mathbb{A}(G_2)v] - [(\lambda \mathbb{I}_{n_1} u) \otimes (\mu \mathbb{I}_{n_2} v)] \\
&= [\mathbb{A}(G_1)u - \lambda \mathbb{I}_{n_1} u] \otimes [\mathbb{A}(G_2)v - \mu \mathbb{I}_{n_2} v] + \\
&\quad\; \mu[\mathbb{A}(G_1)u - \lambda \mathbb{I}_{n_1} u] \otimes \mathbb{I}_{n_2} v + \\
&\quad\; \lambda \mathbb{I}_{n_1} u \otimes [\mathbb{A}(G_2)v - \mu \mathbb{I}_{n_2} v] \ge 0.
\end{aligned}$$

Further we have

$$
\begin{aligned}
& (u\otimes v)^T\Big\{\mathbb{A}(G_1\times G_2)(u\otimes v)-((m-1)!\lambda\mu)\mathbb{I}_{n_1n_2}(u\otimes v)\Big\}\\
= \;& (m-1)!(u\otimes v)^T\Big\{\frac{1}{(m-1)!}\mathbb{A}(G_1\times G_2)(u\otimes v)-(\lambda\mu)\mathbb{I}_{n_1n_2}(u\otimes v)\Big\}\\
= \;& (m-1)!(u\otimes v)^T\Big[\mathbb{A}(G_1)u]\otimes[\mathbb{A}(G_2)v]-[(\lambda\mathbb{I}_{n_1}u)\otimes(\mu\mathbb{I}_{n_2}v)\Big]\\
= \;& (m-1)!\sum_{(i,j)\in[n_1]\times[n_2]}(u\otimes v)_{(i,j)}\Big[\mathbb{A}(G_1)u]\otimes[\mathbb{A}(G_2)v]-[(\lambda\mathbb{I}_{n_1}u)\otimes(\mu\mathbb{I}_{n_2}v)\Big]_{(i,j)}\\
= \;& (m-1)!\sum_{(i,j)\in[n_1]\times[n_2]}\Big\{u_i[\mathbb{A}(G_1)u-\lambda\mathbb{I}_{n_1}u]_i v_j[\mathbb{A}(G_2)v-\mu\mathbb{I}_{n_2}v]_j+\\
& \mu u_i[\mathbb{A}(G_1)u-\lambda\mathbb{I}_{n_1}u]_i v_j^m+\lambda u_i^m v_j[\mathbb{A}(G_2)v-\mu\mathbb{I}_{n_2}v]_j\Big\}=0.
\end{aligned}
$$

This completes the proof of (i) and (ii). □

Corollary 3.12. *Let G_1 be an m-uniform hypergraph of order n_1 and G_2 be a m-uniform hypergraph of order n_2.*

(i) $\Pi(G_1\square G_2)\supseteq\{\lambda+\mu:\lambda\in\Pi(G_1),\mu\in\Pi(G_2)\}$.
(ii) $\Pi(G_1\times G_2)\supseteq\{(m-1)!\lambda\mu:\lambda\in\Pi(G_1),\mu\in\Pi(G_2)\}$.

Statements and Declarations

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